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Mathematical modelling of transient shear wave elastography in the heart

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Abstract

The aim of this work is to provide a mathematical model and analysis of the excitation and the resulting shear wave propagation in Acoustic Radiation Force (ARF)-based shear wave cardiac elastography. Our approach is based on asymptotic analysis; more precisely, it consists in considering a family of problems, parametrised by a small parameter inversely proportional to the excitation frequency of the probes, the viscosity and the velocity of pressure wave propagation. We derive a simplified model for the expression of the ARF by investigating the limit behaviour of the solution when the small parameter goes to zero. By formal asymptotic analysis - an asymptotic expansion of the solution is used - and energy analysis of the nonlinear elastodynamic problem, we show that the leading-order term of the expansion is solution of the underlying, incompressible, nonlinear cardiac mechanics. Subsequently, two corrector terms are derived. The first is a fast-oscillating pressure wave generated by the probes, solution of a Helmholtz equation at every time. The second corrector term consists in an elastic field with prescribed divergence, having a function of the first corrector as a source term. This field corresponds to the shear acoustic wave induced by the ARF. We also confirm that, in cardiac mechanics, the presence of viscosity in the model is essential to derive an expression of the shear wave propagation from the ARF, and that this phenomenon is related to the nonlinearity of the partial differential equation.

Keywords: Shear Wave Elastography, Acoustic Radiation Force, Asymptotic analysis

2010 MSC: 35L05, 35Q74, 74L15

1. Introduction

Shear acoustic waves remotely induced by the acoustic radiation force (ARF) of a focused ultrasound beam have raised a growing interest in biomedical applications, mainly in elasticity imaging. In fact, by measuring the velocity of propagation of the generated shear waves, it is possible to locally assess biomechanical properties highly sensitive to structural changes associated with physiological and pathological processes. The development of the first ARF-based shear wave elastography (SWE) technique dates back to late 1990s and was proposed under the name of Shear Wave Elastography Imaging (SWEI) [1]. Thereafter, several techniques based on impulsive acoustic radiation force were introduced, e.g. Vibro-acoustic Spectrography [2], Acoustic Radiation Force Imaging (ARFI) [3], Supersonic Shear Imaging (SSI) [4], Spatially Modulated Ultrasound Radiation Force (SMURF) [5] and Comb-push Ultrasound Shear Elastography (CUSE) [6].

ARF-driven SWE techniques are based on three main steps. First, an acoustic radiation force source is used to remotely induce shear wave propagation in the tissue (Figure 1). ARF is obtained from the emission of a high-intensity acoustic pressure pulse at a specific tissue depth by a conventional ultrasound transducer. Shear motion is then tracked

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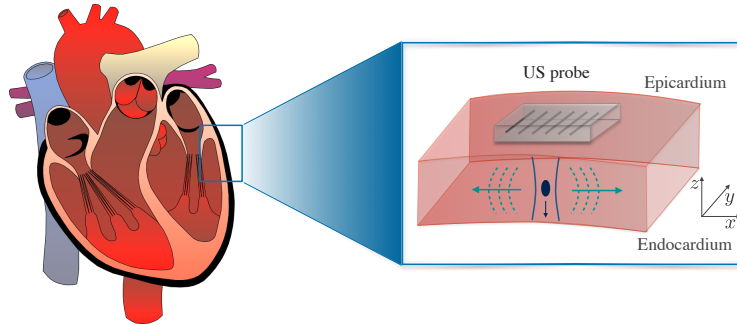


Figure 1: Schematic description of shear wave generation.

by ultrasound imaging, by emission of low-intensity pulses. Finally, tissue properties are quantitatively estimated by post-processing techniques and inversion algorithms. We recall that in nearly-incompressible soft media, such as biological tissues, the propagation velocity of shear waves (1 m s^{-1} to 10 m s^{-1}) is much smaller than the velocity of pressure waves (1500 m s^{-1}), and this difference is fundamental for elastography. In more detail, the ARF phenomenon is caused by the transfer of momentum from the focused ultrasound wave to the medium, mainly due to wave absorption [7, 8, 9]. It is generally defined as a time-averaged force applied to a medium by a sound wave. The main orientation of this body force (also known as radiation pressure) is perpendicular to the probe surface. Tissue displacements are first localised close to the focus, then start propagating in the medium. Particle displacements of these shear waves are then oriented perpendicular to the direction of propagation of the wave. Two relevant advantages of ARF-driven SWE are the highly localised motion and the full attenuation of shear waves in a few wavelengths distance from the focal point of the ultrasonic beam. This induces a narrowing of the induced strain, and excludes coupling phenomena at the surface of the tissue. Consequently, the dynamics of the localised motion is only defined by the parameters of acoustic excitation and the mechanical properties of the tissue. Furthermore, it is not necessary to incorporate complicated modelling ingredients (e.g. specific boundary conditions), that take into account far-field interactions, to model this phenomenon. This represents a great simplification in reconstruction methods, since the quality of reconstruction can be significantly affected by errors in the definition of boundary conditions [10].

Various models to analyse the physical mechanisms responsible for ARF and remote generation of shear waves in water-like media have been proposed. The first theoretical models of shear oscillations remotely induced by radiation force were presented in [1] and [11]. The models are derived in the paraxial regime from the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation [12]. In that context, we refer to [13, 14] for a derivation of an evolution equation for shear waves in a nonlinear soft medium. A substantial contribution in this subject is also due to E. Dontsov and B. B. Guzina. In more detail, they have derived an expression of the body force generated by a high-intensity, focused ultrasound signal, modulated by a low-frequency signal, induced in a Newtonian fluid [15], or in an isotropic, viscoelastic, homogeneous solid with heat conduction [16]. To do so, they split the temporal variable into a “fast” (associated with the high-frequency pulse) and “slow” component (related to the ultrasound modulation), they consider that the pressure wave has a specific structure, it is a quasi-plane wave or a focused-beam and they perform a linearisation around an equilibrium state. Moreover, assuming some a-priori properties of the solution, they derive the nonlinear acoustic ARF source term. Furthermore, they have analysed the discrepancy of the solution with respect to the classical solution of the KZK equation, due to the effects of modulation [17]. The role of inhomogeneity and viscosity in the generation of shear waves is analysed in [18], and a theoretical model for shear stress and shear motion in an isotropic solid is presented. It is also proved that an acoustic field in a purely elastic (non-dissipative), homogeneous and nonlinear solid cannot generate shear motion.

Nonetheless, there is still a lack of a complete mathematical analysis of all the phenomena involved in the ARF and the related shear wave generation, in particular when complex soft tissues are considered, such as a fibered muscle tissue. The aim of this work is to partially fill this gap. In this article, we introduce a new methodology to analyse the ARF and provide a new perspective on the understanding and practical simulation of this phenomenon.

In more detail, the approach adopted is based on 2-scale homogenisation techniques – that are standard for the space variable – for the time variable. The use of 2 time scales is not original for this problem (see [16]). However, our approach relies on an asymptotic ansatz, that leads to a series expansion – related to a small parameter ε – of the solution, and each term of this expansion is systematically studied. We assume, indeed, that two time variables can be distinguished: one slow, related to the nonlinear mechanics and the shear wave propagation, and a fast one, associated with the pressure wave induced by the ARF excitation. In order to derive a simplified model for the expression of ARF, we approximate the solution by the first terms of the asymptotic expansion, and we insert this approximation in the elastodynamics equation. Then, we identify and solve the equations proportional to the same order of ε . However, such approach is not sufficient to be conclusive and we need to take into account a thorough energy analysis to derive our results.

Moreover, the results are obtained by considering a family of problems parametrised by ε , that is inversely proportional to the excitation frequency of the probes, the viscosity and the velocity of pressure wave propagation. To our knowledge, this corresponds to the minimal set of assumptions necessary to understand, by asymptotic methods, the ARF phenomenon. By doing so, we avoid the standard simplifying hypotheses on the properties of the emitted pressure waves. For instance, in [1, 11, 13, 14] the expression of the ARF is deduced under the assumption that paraxial approximations are valid. Furthermore, we consider a general constitutive law for hyperelastic, nonlinear soft tissues that can be heterogeneous and anisotropic, and the formulation proposed in this work is valid for media undergoing large deformation, independently from the nature of the prestress. Since one of our applications is the cardiac setting, we also consider the electric activation of the muscle fibres responsible for the heart contraction. Finally, we also derive an original expression of the ARF that is compatible with standard Finite Element (FE) computations, allowing to overcome the standard assumptions that are made on the resulting body force: typically, a gaussian profile centred around a focalisation point (see [19, 20]). More precisely, we show that the computation of the ARF source term involves the solution of a scalar Helmholtz problem in the frequency domain at each time instant of the shear wave propagation.

The article is organised as follows. First, some preliminary notions on the elastodynamic problem and our mathematical approach are provided in Section 2. In more detail, the constitutive laws of the medium under consideration are introduced. Then, a family of problems is proposed and a stability estimate is proved. In Section 3 a regular asymptotic expansion of the solution is defined. Moreover, the nonlinear equation for the limit term of the expansion and the equation for the higher-order correctors are given in a general framework. Subsequently, the main contributions of this work are summarised and the results are discussed. The methodological approach used to retrieve our formulation is detailed in Section 5. In particular, we derive the equation satisfied by the leading-order term of the expansion, that is associated with the nonlinear heart mechanics, and the governing equations of two corrector terms. The first consists in a fast-oscillating pressure wave excited by the probes, and we show that it is the solution of a Helmholtz equation at every time. The second corrector term is shown to be an elastic field with prescribed divergence, having as a source term a function of the first corrector. This field corresponds to the shear acoustic wave induced by the ARF. A detailed expression of the source term responsible for shear wave propagation is also provided. As a by-product of our analysis we prove that, in prestressed media, the presence of viscosity is essential to produce shear waves with ARF. Furthermore, they are related to the nonlinearity of the partial differential equation (PDE). Appendix A is devoted to further considerations and detail on some theoretical results.

2. A 3D nonlinear model for ARF and elastic wave propagation

The starting point of our analysis is the fundamental equation of elastodynamics in the cardiac setting. For simplicity of analysis, we consider a smooth, bounded, convex, domain $\Omega_0 \subset \mathbb{R}^3$, that corresponds to some region of the domain under investigation. We consider that forces are imposed at the boundary of the domain

$$\partial\Omega_0 = \Gamma_0.$$

As given in [21], we use the following variational formulation to describe the nonlinear cardiac mechanics in the Lagrangian framework: Given an admissible functional space \mathcal{X} , for all $t \in [0, T]$, find $\underline{y}(t) \in \mathcal{X}$ such that for all $\underline{w} \in \mathcal{X}$

$$\int_{\Omega_0} \rho_0 \partial_t^2 \underline{y} \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} \underline{\Sigma} : d_{\underline{y}} \underline{e} \cdot \underline{w} \, d\Omega_0 = \int_{\Gamma_0} \underline{t} \cdot \underline{w} \, d\Gamma_0 + \int_{\Omega_0} \rho_0 \underline{f} \cdot \underline{w} \, d\Omega_0, \quad (1)$$

with vanishing initial conditions $\underline{y}(0) = \underline{0}$ and $\partial_t \underline{y}(0) = \underline{0}$. We recall that \underline{e} is the Green-Lagrange strain tensor and it reads

$$\underline{e} = \frac{1}{2} \left((\underline{\nabla} \underline{y} + (\underline{\nabla} \underline{y})^T) + (\underline{\nabla} \underline{y})^T \cdot \underline{\nabla} \underline{y} \right).$$

In (1), $d_{\underline{y}} \underline{e}$ is the derivative of the Green-Lagrange tensor with respect to the displacement and is given by

$$d_{\underline{y}} \underline{e} \cdot \underline{w} = \frac{1}{2} \left((\underline{\nabla} \underline{w})^T \cdot \underline{F} + \underline{F}^T \cdot \underline{\nabla} \underline{w} \right),$$

where \underline{F} is the deformation gradient $\underline{F} = \underline{\mathbb{1}} + \underline{\nabla} \underline{y}$. Moreover $\underline{\Sigma}$ is the Second Piola-Kirchhoff stress tensor accounting for total stress, and \underline{t} accounts for the surface excitation on the boundary, due to the piezoelectric probes. Note that \underline{f} represents generic external volume forces (e.g. gravity) per unit volume, henceforth we set $\underline{f} = \underline{0}$. The operator “ \cdot ” corresponds to scalar product, whereas “ $:$ ” stands for dyadic product.

2.1. Constitutive law

For cardiac muscle tissue, it is standard to consider a constitutive law that is composed of two contributions, accounting for passive and active stress. Therefore, we define

$$\underline{\Sigma} = \underline{\Sigma}^P + \underline{\Sigma}^A.$$

As far as the passive term is concerned, we consider a visco-hyperelastic medium, therefore we can express

$$\underline{\Sigma}^P = \frac{\partial W^e}{\partial \underline{e}} + \frac{\partial W^{\text{VS}}}{\partial \underline{e}}, \quad (2)$$

with W^e the hyperelastic potential and W^{VS} the viscous pseudo-potential. In cardiac modelling (see [21] or [22]) it is standard to take into account a hyperelastic potential composed of a term accounting for a transversely isotropic (TI) law due to the presence of muscle fibers, and a penalisation term related to nearly-incompressibility (NI). More precisely, we set

$$W^e = W^{\text{TI}} + W^{\text{NI}}. \quad (3)$$

The potential associated with nearly-incompressibility reads

$$W^{\text{NI}} := \frac{\kappa}{2} \left((J^2 - 1) - \log(J^2) \right), \quad (4)$$

where $J = \det \underline{F}$ stands for the jacobian of the deformation and κ is the (strictly positive) bulk modulus. Given some strictly positive parameters $\{\kappa_i\}$, the potential associated with the transversely isotropic constitutive law reads

$$W^{\text{TI}} := \kappa_1 e^{\kappa_2 (I_1 - 3)^2} + \kappa_3 e^{\kappa_4 (I_4 - 1)^2} + \kappa_5 (I_2 - 2 \log(J^2) - 3), \quad (5)$$

where, defining the right *Cauchy-Green deformation tensor* $\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}} = \underline{\underline{1}} + 2 \underline{\underline{e}}$, we have

$$I_1 = \text{tr}(\underline{\underline{C}}), \quad I_2 = |\text{adj } \underline{\underline{F}}|^2, \quad I_4 = \underline{\tau}_1 \cdot \underline{\underline{C}} \cdot \underline{\tau}_1, \quad (6)$$

where $\underline{\tau}_1(\underline{\xi})$ is the – space-dependent – fibre direction. Note that the norm operator $|\cdot|$ denotes the euclidean norm in \mathbb{R}^d or the Frobenius norm $\mathbb{R}^{d \times d}$, accordingly. Finally the viscous pseudo-potential reads

$$W^{\text{VS}} := \frac{\zeta}{2} \text{tr}(\underline{\underline{C}}^{-1} \cdot \underline{\underline{\dot{e}}})^2 = \frac{\zeta}{2} (\partial_t \log(J^2))^2, \quad \text{with} \quad \underline{\underline{\dot{e}}} := d_{\underline{y}} \underline{\underline{e}} \cdot \partial_t \underline{y} = \frac{1}{2} \left((\underline{\underline{\nabla}} \partial_t \underline{y})^T \cdot \underline{\underline{F}} + \underline{\underline{F}}^T \cdot \underline{\underline{\nabla}} \partial_t \underline{y} \right). \quad (7)$$

We emphasise that the expression of W^{VS} in Eq. (7) implies that the viscous term only involves J . Our analysis could be extended to more complex viscosity terms, but this will not change the main physical phenomena we want to illustrate. If we define

$$\underline{\underline{\Sigma}}^{\text{TI}} := \frac{\partial W^{\text{TI}}}{\partial \underline{\underline{e}}}, \quad \underline{\underline{\Sigma}}^{\text{NI}} := \frac{\partial W^{\text{NI}}}{\partial \underline{\underline{e}}} \quad \text{and} \quad \underline{\underline{\Sigma}}^{\text{VS}} := \frac{\partial W^{\text{VS}}}{\partial \underline{\underline{\dot{e}}}},$$

then Eq. (2) can be recast as $\underline{\underline{\Sigma}}^P = \underline{\underline{\Sigma}}^{\text{TI}} + \underline{\underline{\Sigma}}^{\text{NI}} + \underline{\underline{\Sigma}}^{\text{VS}}$. For the active term $\underline{\underline{\Sigma}}^A$, we define a simple constitutive law, and suppose that the stress tensor is only oriented along the fibre direction and is a given function of space and time. Namely, we set

$$\underline{\underline{\Sigma}}^A = \sigma_a \underline{\tau}_1 \otimes \underline{\tau}_1, \quad (8)$$

where σ_a is a smooth space and time-dependent function. Note that $\underline{\underline{\Sigma}}^A$ does not depend on the displacement \underline{y} and

$$\int_{\Omega_0} \underline{\underline{\Sigma}}^A : d_{\underline{y}} \underline{\underline{e}} \cdot \underline{w} \, d\Omega_0$$

represents a source term for Problem (1).

2.2. Definition of a parametric family of problems

In nearly-incompressible soft media, such as biological tissues, the propagation velocity of shear waves (1 m s^{-1} to 10 m s^{-1}) is much smaller than the velocity of pressure waves (1500 m s^{-1}). This difference is related to the value of the bulk modulus that is very high in such media. As a result, we can consider a family of problems, parametrised by a small parameter ε such that we can rewrite the bulk modulus as

$$\kappa = \varepsilon^{-2} \hat{\kappa}, \quad (9)$$

where $\hat{\kappa}$ is a normalised parameter and ε is dimensionless. Since experimental evidence suggests that pressure waves are also localised, we hypothesise that the viscosity coefficient satisfies

$$\zeta = \varepsilon^{-1} \hat{\zeta},$$

where $\hat{\zeta}$ is a normalised parameter. Moreover, our asymptotic analysis of the ARF relies on the assumption that the source term - generated by the piezoelectric sensors - is of high frequency ω . Therefore, in agreement with [18], we set the ultrasound frequency proportional to the viscosity coefficient. In particular, given a normalised frequency $\hat{\omega}$, such that $\hat{\omega} = \varepsilon \omega > 0$, we define

$$\underline{\underline{t}}(\underline{\xi}, t) := g(\underline{\xi}, t) \cos(\hat{\omega} t / \varepsilon) J \left| \underline{\underline{F}}^{-T} \cdot \underline{n}_0 \right| \underline{\underline{F}}^{-T} \cdot \underline{n}_0, \quad (10)$$

where \underline{n}_0 is the outward unitary normal and g is a smooth space and time-dependent function representing the action of the probes on the boundary. The remaining terms account for geometric effects and are justified later in the text.

We denote by $\underline{y}_{-\varepsilon}$ the family of solutions associated with Eq. (1) with the definitions above, and we define the deformation gradient $\underline{F}_{-\varepsilon}$, the Cauchy-Green stress tensor $\underline{C}_{-\varepsilon}$, the reduced invariant J_ε and the second Piola-Kirchhoff stress tensor $\underline{\Sigma}_{-\varepsilon}$ associated with the solution $\underline{y}_{-\varepsilon}$. Then, the variational formulation reads:

Find, for all $t \in [0, T]$, a function $\underline{y}_{-\varepsilon}(t) \in \mathcal{X}$ such that for all $\underline{w} \in \mathcal{X}$

$$\int_{\Omega_0} \rho_0 \partial_t^2 \underline{y}_{-\varepsilon} \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} \left(\underline{\Sigma}_{-\varepsilon}^{\text{PI}} + \underline{\Sigma}_{-\varepsilon}^{\text{NI}} + \underline{\Sigma}_{-\varepsilon}^{\text{VS}} \right) : d_{\underline{y}_{-\varepsilon}} \underline{e} \cdot \underline{w} \, d\Omega_0 = - \int_{\Omega_0} \underline{\Sigma}_{-\varepsilon}^A : d_{\underline{y}_{-\varepsilon}} \underline{e} \cdot \underline{w} \, d\Omega_0 + \int_{\Gamma_0} \underline{t}_{-\varepsilon} \cdot \underline{w} \, d\Gamma_0 \quad (11)$$

with vanishing initial conditions $\underline{y}_{-\varepsilon}(0) = \underline{0}$ and $\partial_t \underline{y}_{-\varepsilon}(0) = \underline{0}$.

Assumption 1. For all $\varepsilon > 0$ sufficiently small, there exists a unique smooth solution $\underline{y}_{-\varepsilon}$ to Eq. (11) satisfying

$$\det(\underline{\mathbb{1}} + \underline{\nabla} \underline{y}_{-\varepsilon}) > 0.$$

2.3. Stability estimates

We now retrieve some energy estimates allowing to derive some interesting properties of the solution $\underline{y}_{-\varepsilon}$. Choosing the actual velocity $\partial_t \underline{y}_{-\varepsilon}$ as a test function, we obtain the following energy balance (see [21])

$$\frac{d\mathcal{E}_\varepsilon^{\text{tot}}}{dt} + \mathcal{E}_\varepsilon^{\text{VS}} = \mathcal{P}_\varepsilon^{\text{ext}} + \mathcal{P}_\varepsilon^A, \quad (12)$$

where the total energy $\mathcal{E}_\varepsilon^{\text{tot}}$ reads

$$\mathcal{E}_\varepsilon^{\text{tot}} = \mathcal{K}_\varepsilon + \mathcal{E}_\varepsilon^{\text{PI}} + \mathcal{E}_\varepsilon^{\text{NI}},$$

and the kinetic and potential energy associated with the hyperelastic term and nearly-incompressibility are given by

$$\mathcal{K}_\varepsilon = \frac{\rho_0}{2} \int_{\Omega_0} |\partial_t \underline{y}_{-\varepsilon}|^2 \, d\Omega_0, \quad \mathcal{E}_\varepsilon^{\text{PI}} = \int_{\Omega_0} W_\varepsilon^{\text{PI}} \, d\Omega_0, \quad \mathcal{E}_\varepsilon^{\text{NI}} = \int_{\Omega_0} W_\varepsilon^{\text{NI}} \, d\Omega_0.$$

where $W_\varepsilon^{\text{PI}}$ and $W_\varepsilon^{\text{NI}}$ are straightforwardly deduced from 5 and (4). In particular

$$W_\varepsilon^{\text{NI}} = \varepsilon^{-2} \frac{\hat{K}}{2} \left((J_\varepsilon^2 - 1) - \log(J_\varepsilon^2) \right).$$

The contribution related to viscous losses reads

$$\mathcal{E}_\varepsilon^{\text{VS}} = \varepsilon^{-1} \hat{\zeta} \int_{\Omega_0} \text{tr} \left(\underline{C}_{-\varepsilon}^{-1} \cdot \underline{\dot{e}}_{-\varepsilon} \right)^2 \, d\Omega_0 = \varepsilon^{-1} \hat{\zeta} \int_{\Omega_0} \text{tr} \left(\underline{F}_{-\varepsilon}^{-1} \cdot \underline{\nabla} \partial_t \underline{y}_{-\varepsilon} \right)^2 \, d\Omega_0,$$

whereas the source term contributions are given by

$$\mathcal{P}_\varepsilon^{\text{ext}} = \int_{\Gamma_0} \underline{t}_{-\varepsilon} \cdot \partial_t \underline{y}_{-\varepsilon} \, d\Gamma_0 = \int_{\Gamma_0} g \cos(\hat{\omega} t / \varepsilon) J_\varepsilon \left| \underline{F}_{-\varepsilon}^{-T} \cdot \underline{n}_0 \right| \partial_t \underline{y}_{-\varepsilon} \cdot \underline{F}_{-\varepsilon}^{-T} \cdot \underline{n}_0 \, d\Gamma_0, \quad \mathcal{P}_\varepsilon^A = - \int_{\Omega_0} \sigma_a \underline{\tau}_1 \cdot \underline{\dot{e}}_{-\varepsilon} \cdot \underline{\tau}_1 \, d\Omega_0.$$

Lemma 1. There exists a constant $C > 0$ such that for all ε small enough

$$\mathcal{E}_\varepsilon^{\text{tot}}(t) \leq C \quad \forall t \in [0, T].$$

In particular, $\forall t \in [0, T]$,

$$\int_{\Omega_0} \left((J_\varepsilon^2 - 1) - \log(J_\varepsilon^2) \right) \, d\Omega_0 \leq \varepsilon^2 C, \quad (13)$$

$$\int_0^t \int_{\Omega_0} \left| \text{tr} \left(\underline{C}_{-\varepsilon}^{-1} \cdot \underline{\dot{e}}_{-\varepsilon} \right) \right|^2 \, d\Omega_0 \, ds \leq \varepsilon C. \quad (14)$$

Proof of Proposition 1 can be found in Appendix. As a consequence of this proposition we deduce that, at the limit $\varepsilon \rightarrow 0$, the invariant J_ε tends to one in Ω_0 .

3. Asymptotic expansion of the solution

3.1. Definition of a power series expansion

We look for a regular asymptotic expansion of the solution of Eq. (11), i.e. we assume that $\underline{y}_{-\varepsilon}$ can be written using a power series expansion in ε . More precisely, we suppose that there exists an integer $N \geq 4$ such that the solution $\underline{y}_{-\varepsilon}$ can be decomposed, for all ε sufficiently small, as

$$\underline{y}_{-\varepsilon}(\underline{\xi}, t) = \sum_{i=0}^N \varepsilon^i \underline{y}_{-i}(\underline{\xi}, t, t/\varepsilon) + \varepsilon^N \underline{r}_{-\varepsilon}, \quad \forall \underline{\xi} \in \Omega_0, \forall t \in [0, T], \quad (15)$$

with $\underline{r}_{-\varepsilon}$ is smooth and $\lim_{\varepsilon \rightarrow 0} \|\underline{r}_{-\varepsilon}\| = 0$ for an adequate norm $\|\cdot\|$. We assume that every term $\underline{y}_{-i}(\underline{\xi}, t, \tau)$ of the power series is periodic in τ , with period $2\pi/\hat{\omega}$ (this is suggested by the structure of the source term). Furthermore, we assume that all the components \underline{y}_{-i} inherit the regularity properties – in $\underline{\xi}$ and t – of $\underline{y}_{-\varepsilon}$ and are regular enough in τ : as a consequence all $\partial \underline{y}_{-i}/\partial \tau$ are periodic in τ .

The idea is to approximate the solution $\underline{y}_{-\varepsilon}$ by the first terms of expansion (15). By doing so, we recover the limit (asymptotic) behaviour of the solution $\underline{y}_{-\varepsilon}$ for $\varepsilon \rightarrow 0$. In practice, we substitute the expression of $\underline{y}_{-\varepsilon}$ presented in Eq. (15) in Problem (11). To do so, all mechanical quantities must be rewritten in series form, accordingly. Furthermore, we shall use the following expansion of the second derivative in time for every term in Eq. (15):

$$\partial_{tt}^2 \underline{y}_{-i}(\underline{\xi}, t, t/\varepsilon) = \left(\varepsilon^{-2} \partial_{tt}^2 \underline{y}_{-i} + 2\varepsilon^{-1} \partial_{t\tau}^2 \underline{y}_{-i} + \varepsilon^0 \partial_{\tau\tau}^2 \underline{y}_{-i} \right) (\underline{\xi}, t, t/\varepsilon).$$

Once all the terms of the expansion are obtained and inserted in Eq. (11), the equations proportional to the same order of ε are identified. Note that this approach represents an application of a methodology for homogenisation which is standard in the spatial domain [23].

We show that the ARF is a nonlinear phenomenon that occurs at a higher order of approximation. More precisely, we show that the leading-order term in Eq. (15) corresponds to the underlying nonlinear mechanical behaviour of the solution. Then, the first two correctors are analysed. The first corrector is zero, whereas the second one can be decomposed into two contributions:

- a fast-oscillating pressure wave generated by the piezoelectric probes, solution of a Helmholtz equation at every time;
- a slow-varying elastic field with prescribed divergence, solution of a linear equation having as a source term a quadratic function of the first corrector. This field corresponds to the shear acoustic wave induced by the ARF (and it is divergence-free in case of no prestress).

3.2. Expansion of mechanical quantities

Henceforth, we introduce several definitions that are related to the linearisation of nonlinear mechanics, in order to state our main results and according to our hypothesis defined in Eq. (15). In particular, we derive a series expansion for all the deformation and stress tensors involved in Eq. (11), accordingly.

3.2.1. Deformation tensors.

From Eq. (15), we can rewrite the deformation gradient as

$$\underline{\underline{F}}_{-\varepsilon} := \underline{\underline{1}} + \underline{\underline{\nabla}} \underline{y}_{-\varepsilon} = \underline{\underline{F}}_{=0} + \sum_{i=1}^N \varepsilon^i \underline{\underline{\nabla}} \underline{y}_{-i} + o(\varepsilon^N), \quad \text{with} \quad \underline{\underline{F}}_{=0} := \underline{\underline{1}} + \underline{\underline{\nabla}} \underline{y}_{=0}.$$

Consequently, the Green-Lagrange tensor reads

$$\underline{e}_{-\varepsilon} := \frac{1}{2} \left(\underline{\nabla} y + (\underline{\nabla} y)^T + (\underline{\nabla} y)^T \cdot \underline{\nabla} y \right) = \underline{e}_{\underline{0}} + \sum_{i=1}^N \varepsilon^i \underline{e}_{\underline{0}-i} + \sum_{\substack{2 \leq i+j \leq N \\ i, j > 0}} \varepsilon^{i+j} (\underline{\nabla} y_i)^T \cdot \underline{\nabla} y_j + o(\varepsilon^N),$$

with

$$\underline{e}_{\underline{0}} := \frac{1}{2} \left(\underline{F}^T \cdot \underline{F} - \underline{\mathbb{1}} \right) = \frac{1}{2} \left(\underline{\nabla} y_0 + (\underline{\nabla} y_0)^T + (\underline{\nabla} y_0)^T \cdot \underline{\nabla} y_0 \right),$$

and, for any vector field \underline{w} ,

$$\underline{e}_{\underline{0}}(\underline{w}) := \frac{1}{2} \left(\underline{F}^T \cdot \underline{\nabla} w + (\underline{\nabla} w)^T \cdot \underline{F} \right) = \frac{1}{2} \left(\underline{\nabla} w + (\underline{\nabla} w)^T + (\underline{\nabla} y_0)^T \cdot \underline{\nabla} w + (\underline{\nabla} w)^T \cdot \underline{\nabla} y_0 \right).$$

Again using the expansion (15), we obtain for any vector field \underline{w}

$$d_{y_{-\varepsilon}} \underline{e} \cdot \underline{w} := \frac{1}{2} \left(\underline{F}^T \cdot \underline{\nabla} w + (\underline{\nabla} w)^T \cdot \underline{F} \right) = \underline{e}_{\underline{0}}(\underline{w}) + \sum_{i=1}^N \varepsilon^i \underline{e}_{\underline{0}-i}(\underline{w}) + o(\varepsilon^N),$$

where

$$\underline{e}_{\underline{0}-i}(\underline{w}) := \frac{1}{2} \left((\underline{\nabla} y_i)^T \cdot \underline{\nabla} w + (\underline{\nabla} w)^T \cdot \underline{\nabla} y_i \right).$$

Analogously, the Cauchy-Green deformation tensor and its inverse read

$$\underline{C}_{-\varepsilon} := \underline{\mathbb{1}} + 2 \underline{e}_{-\varepsilon} = \underline{C}_{\underline{0}} + \sum_{i=1}^N \varepsilon^i \underline{C}_{\underline{0}-i} + o(\varepsilon^N), \quad \underline{C}_{-\varepsilon}^{-1} = \underline{C}_{\underline{0}}^{-1} + \sum_{i=1}^N \varepsilon^i \underline{G}_{\underline{0}-i} + o(\varepsilon^N),$$

with

$$\underline{C}_{\underline{0}} = \left(\underline{\mathbb{1}} + 2 \underline{e}_{\underline{0}} \right), \quad \underline{C}_{\underline{0}-1} := 2 \underline{e}_{\underline{0}}(y_1), \quad \underline{G}_{\underline{0}-1} = -2 \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_1) \cdot \underline{C}_{\underline{0}}^{-1},$$

and we do not detail for the moment the expression of $\underline{C}_{\underline{0}-i}$ and $\underline{G}_{\underline{0}-i}$ for $i \geq 2$. The reduced invariant J_ε can be expanded as

$$J_\varepsilon = J_0 + \sum_{i=1}^N \varepsilon^i \mathcal{J}_i + o(\varepsilon^N), \tag{16}$$

with

$$J_0 = \left(\det \underline{C}_{\underline{0}} \right)^{\frac{1}{2}}, \quad \mathcal{J}_1 = J_0 \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_1) \right),$$

and the expression of \mathcal{J}_i is not given for $i \geq 2$ for the moment. Finally, we also have

$$\underline{t}_{-\varepsilon} = \cos(\hat{\omega} t / \varepsilon) \underline{t}_0 + \cos(\hat{\omega} t / \varepsilon) \sum_{i=0}^N \varepsilon^i \underline{t}_i + o(\varepsilon^N),$$

where the leading-order boundary source term is given by

$$\underline{t}_0(\underline{\xi}, t) := g(\underline{\xi}, t) J_0 \left| \underline{F}_{\underline{0}}^{-T} \cdot \underline{n}_0 \right| \underline{F}_{\underline{0}}^{-T} \cdot \underline{n}_0.$$

Note that \underline{t}_0 does not depend on ε . However, the expansion of the source term $\underline{t}_{-\varepsilon}$ is not polynomial in ε .

3.2.2. Stress tensors.

The second Piola-Kirchhoff stress tensor accounting for transverse isotropy, nearly-incompressibility and viscosity can be rewritten in terms of a power series expansion, following Eq. (15). In more detail, since $\partial J_\varepsilon / \partial \underline{e} = J_\varepsilon \underline{C}_\varepsilon^{-1}$ the tensor $\underline{\underline{\Sigma}}_\varepsilon^{\text{NI}}$ reads, from Eq. (4),

$$\underline{\underline{\Sigma}}_\varepsilon^{\text{NI}} := \varepsilon^{-2} \frac{\hat{k}}{2} (J_\varepsilon^2 - 1) \underline{C}_\varepsilon^{-1} = \frac{\hat{k}}{2} \sum_{i=0}^N \varepsilon^{i-2} \underline{\underline{\Sigma}}_i^{\text{NI}} + o(\varepsilon^N), \quad (17)$$

with

$$\underline{\underline{\Sigma}}_0^{\text{NI}} := (J_0^2 - 1) \underline{C}_0^{-1}.$$

In a similar way, we derive the power series expansion in ε of $\underline{\underline{\Sigma}}_\varepsilon^{\text{TI}}$, that reads

$$\underline{\underline{\Sigma}}_\varepsilon^{\text{TI}} = \underline{\underline{\Sigma}}_0^{\text{TI}} + \sum_{i=1}^N \varepsilon^i \underline{\underline{\Sigma}}_i^{\text{TI}} + o(\varepsilon^N),$$

where $\underline{\underline{\Sigma}}_0^{\text{TI}}$ is given in Appendix B. The expansion of the second Piola-Kirchhoff stress tensor $\underline{\underline{\Sigma}}_\varepsilon^{\text{VS}}$ related to viscosity gives

$$\underline{\underline{\Sigma}}_\varepsilon^{\text{VS}} := \varepsilon^{-1} \hat{\zeta} \operatorname{tr} \left(\underline{C}_\varepsilon^{-1} \cdot \dot{\underline{e}}_\varepsilon \right) \underline{C}_\varepsilon^{-1} = \hat{\zeta} \sum_{i=0}^N \varepsilon^{i-2} \underline{\underline{\Sigma}}_i^{\text{VS}} + o(\varepsilon^N), \quad \text{with} \quad \underline{\underline{\Sigma}}_0^{\text{VS}} := \underline{C}_0^{-1} \operatorname{tr} \left(\underline{C}_0^{-1} \cdot \partial_\tau \underline{e}_0 \right). \quad (18)$$

Note that the series above behaves in $O(\varepsilon^{-2})$, due to the fact that differentiation with respect to the fast time variable τ introduces another factor ε^{-1} in the expression. Finally, the stress stress tensor $\underline{\underline{\Sigma}}^A$ given by (8) is independent of ε .

The full expressions of the tensors $\underline{\underline{\Sigma}}_i^{\text{NI}}$, $\underline{\underline{\Sigma}}_i^{\text{TI}}$ and $\underline{\underline{\Sigma}}_i^{\text{VS}}$ are not detailed here, for the sake of compactness. In fact, we will show that their expression can be considerably simplified, due to the properties of the first terms in the expansion (15). More precisely, we will show that $\partial_\tau y_0 = \underline{0}$, $J_0 = 1$ and $\underline{y}_1 = \underline{0}$.

3.3. The incompressible, nonlinear heart mechanics and its linearisation

In Shear Wave Elastography, the propagation of elastic waves in a prestressed soft medium like the myocardium can be modelled as a perturbation of the underlying nonlinear incompressible heart mechanics. Therefore, we expect the limit term of our expansion to be the solution of this nonlinear problem. Henceforth, we formulate the nonlinear problem and its linearisation that govern some of the terms in the asymptotic expansion (15). More precisely, we will show that the perturbation associated with the shear wave propagation satisfies an equation that can be recovered by linearisation of Eq. (11) around a reference, time-dependent deformation state. The well-posedness properties of the two problems considered (existence/uniqueness/regularity of solutions) are not analysed but assumed instead.

3.3.1. Nonlinear mechanical model of heart deformation.

The nonlinear, incompressible cardiac mechanics is described as follows, given an adequate functional space \mathcal{L} : For all $t \in [0, T]$, find $(\underline{y}_0(t), p_0(t)) \in \mathcal{X} \times \mathcal{L}$ such that for all $\underline{w} \in \mathcal{X}$

$$\int_{\Omega_0} \rho_0 \partial_t^2 \underline{y}_0 \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} p_0 \underline{C}_0^{-1} : \underline{e}_0(\underline{w}) \, d\Omega_0 + \int_{\Omega_0} \underline{\underline{\Sigma}}_0^{\text{TI}} : \underline{e}_0(\underline{w}) \, d\Omega_0 = - \int_{\Omega_0} \underline{\underline{\Sigma}}^A : \underline{e}_0(\underline{w}) \, d\Omega_0, \quad (19)$$

satisfying $J_0 = 1$ and with initial conditions $\underline{y}_0(0) = \underline{0}$ and $\partial_t \underline{y}_0(0) = \underline{0}$ and where p_0 is a Lagrange multiplier to enforce incompressibility. Typically, the space \mathcal{L} is a space of zero-average scalar functions. The analysis of the existence and uniqueness of solution to Eq. (19) is a well-known difficult problem; as a consequence, we will state the following assumption.

Assumption 2. *There exists a unique smooth couple of function (\underline{y}_0, p_0) solution to Eq. (19).*

3.3.2. Linearised problem for the elastic perturbation.

The linearisation of (11) around a time-dependent solution $\underline{y}_{\underline{0}}(t)$ leads to the following linear problem: Given admissible functional spaces for the linearised problem \mathcal{X}_ℓ and \mathcal{L}_ℓ , find, for all $t \in [0, T]$, a couple $(\underline{\tilde{y}}(t), \underline{\tilde{p}}(t)) \in \mathcal{X}_\ell \times \mathcal{L}_\ell$ such that for all $\underline{w} \in \mathcal{X}_\ell$ and for all $q \in \mathcal{L}_\ell$

$$\begin{cases} \int_{\Omega_0} \rho_0 \partial_t^2 \underline{\tilde{y}} \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} \underline{\tilde{p}} \underline{C}_{\underline{0}}^{-1} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 + a(\underline{\tilde{y}}, \underline{w}) = s(t, \underline{w}), \\ \int_{\Omega_0} \underline{C}_{\underline{0}}^{-1} : \underline{e}_{\underline{0}}(\underline{\tilde{y}}) q \, d\Omega_0 = r(t, q), \end{cases} \quad (20)$$

with initial conditions $\underline{\tilde{y}}(0) = \underline{0}$ and $\partial_t \underline{\tilde{y}}(0) = \underline{0}$, where $s(t, \cdot)$ and $r(t, \cdot)$ are some linear forms yet unspecified, and where

$$\begin{aligned} a(\underline{\tilde{y}}, \underline{w}) &= \int_{\Omega_0} \underline{e}_{\underline{0}}(\underline{\tilde{y}}) : \underline{\mathbf{A}}_{\underline{0}} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 - 2 \int_{\Omega_0} p_0 \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{\tilde{y}}) : \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 \\ &+ \int_{\Omega_0} (\underline{\Sigma}_{\underline{0}}^{\text{II}} + \underline{\Sigma}_{\underline{0}}^{\text{A}} + p_0 \underline{C}_{\underline{0}}^{-1}) : \underline{\nabla} \underline{\tilde{y}}^T \cdot \underline{\nabla} \underline{w} \, d\Omega_0, \end{aligned}$$

the unknown $\underline{\tilde{p}}$ being a Lagrange multiplier enforcing incompressibility. $\underline{\mathbf{A}}_{\underline{0}}$ is a fourth-order tensor corresponding to the derivative of the second Piola-Kirchoff stress tensor $\underline{\Sigma}_{\underline{0}}^{\text{II}}$ with respect to the Green-Lagrange strain tensor $\underline{e}_{\underline{0}}$, evaluated at $\underline{y}_{\underline{0}}$. We refer the reader to Appendix B for the detailed expression of the tensor $\underline{\mathbf{A}}_{\underline{0}} : \underline{e}_{\underline{0}}(\underline{w})$. Finally, note that we typically have

$$\mathcal{L} \subset \mathcal{L}_\ell = \left\{ q \in L^2(\Omega) \mid \int_{\Omega} q \, d\Omega = 0 \right\} \quad \text{and} \quad \mathcal{X} \subset \mathcal{X}_\ell = H^1(\Omega_0)^3.$$

Assumption 3. Assume that $s \equiv 0$ and $r \equiv 0$, then $(\underline{\tilde{y}}, \underline{\tilde{p}}) = (0, 0)$ is the unique solution of Eq. (20).

One of the aims of this work is to demonstrate that the shear wave generated by ARF is solution of Eq. (20), with a source term $s(t, \underline{w})$ that depends on the viscous effects and the tissue nonlinearity. The presented analysis is formal, a complete analysis would require to turn most of the assumptions we made into mathematical results and to specify in which functional spaces the various quantities introduced before belong. However, this is a difficult task, since it involves the analysis of the nonlinear elastodynamics problem, that we did not want to address.

4. Statement of the main results

Preliminarily, we define the deformation map as the bijective map $\underline{\phi}_0 : \overline{\Omega}_0 \rightarrow \overline{\Omega}(t) \subset \mathbb{R}^3$ from the *reference* to the *deformed* configuration that reads $\underline{\phi}_0 : \underline{\xi} \mapsto \underline{x} = \underline{\phi}_0(\underline{\xi}, t)$. Then, the displacement $\underline{y}_{\underline{0}}$ reads $\underline{y}_{\underline{0}}(\underline{\xi}, t) = \underline{x} - \underline{\xi} = \underline{\phi}_0(\underline{\xi}, t) - \underline{\xi}$. For the sake of simplicity, we assume that the operators $\underline{\nabla}$, $\underline{\nabla}$, Δ , div , $\underline{\text{div}}$ and \underline{H} (Hessian matrix) correspond to differential operators in the reference configuration, and we use the same symbols with the subscript \underline{x} when considering differentiation in the deformed configuration. Finally, note that by definition $\underline{F}_{\underline{0}} := \underline{\nabla} \underline{\phi}_0$ and, for any function $h \in L^2(\Gamma_0)$, we have

$$\int_{\Gamma_0} h J_0 \left| \underline{F}_{\underline{0}}^{-T} \cdot \underline{n}_0 \right| \underline{F}_{\underline{0}}^{-T} \cdot \underline{n}_0 \, d\Gamma_0 = \int_{\Gamma(t)} (h \circ \underline{\phi}_0^{-1}) \underline{n}(t) \, d\Gamma,$$

where $\underline{n}(t)$ is the outward unitary normal of $\Omega(t)$ and $\Gamma(t) = \underline{\phi}_0(\Gamma_0)$. Henceforth, if not specified, we will implicitly assume the composition with the deformation map $\underline{\phi}_0$ when we consider a field in the reference configuration that is defined in the deformed configuration, and reciprocally for $\underline{\phi}_0^{-1}$.

Theorem 2. Assume that all the hypotheses mentioned above hold. Then,

- the leading-order term \underline{y}_0 is independent of τ and the couple $(\underline{y}_0, \tilde{p}_0)$ satisfies Eq. (19);
- the first-order corrector term \underline{y}_1 vanishes;
- the second-order corrector \underline{y}_2 can be decomposed into two terms $\underline{y}_2(\underline{\xi}, t, \tau) = \underline{y}_{-p}(\underline{\xi}, t, \tau) + \underline{y}_s(\underline{\xi}, t)$, where

- \underline{y}_{-p} is a smooth fast-oscillating, irrotational (pressure) wave and is solution of the mixed problem (in the deformed configuration)

$$\begin{cases} \rho_0 \partial_\tau^2 \underline{y}_{-p} - (\hat{\kappa} + \hat{\zeta} \partial_\tau) \underline{\nabla}_x p = 0 & \text{in } \Omega(t), \\ p = \operatorname{div}_x \underline{y}_{-p} & \text{in } \Omega(t), \\ (\hat{\kappa} + \hat{\zeta} \partial_\tau) p = g \cos(\hat{\omega} \tau) & \text{on } \Gamma(t); \end{cases} \quad (21)$$

- \underline{y}_s is a slowly varying component with prescribed divergence, corresponding to the nonlinear contribution associated with the ARF; the couple (\underline{y}_s, p) satisfies Eq. (20), with $s(t, \underline{w})$ and $r(t, q)$ given by

$$s(t, \underline{w}) = \int_{\Omega(t)} \underline{\sigma}_{\text{ARF}} : \underline{\nabla}_x \underline{w} \, d\Omega, \quad r(t, q) = \frac{1}{\hat{\kappa}} \int_{\Omega_0} p_0 q \, d\Omega_0, \quad (22)$$

where $\underline{\sigma}_{\text{ARF}}$ is given by

$$\underline{\sigma}_{\text{ARF}} = \frac{\hat{\omega}}{2\pi} \int_0^{2\pi} \operatorname{div}_x \left((\hat{\kappa} + \hat{\zeta} \partial_\tau) \underline{y}_{-p} \right) (\underline{\nabla}_x \underline{y}_{-p})^T \, d\tau. \quad (23)$$

Note that the field \underline{y}_{-p} , solution of (21), satisfies at each time $t \in [0, T]$ a periodic problem in τ . The field \underline{y}_{-p} can be computed from the solution of a frequency problem, since we have assumed the source term of Problem (21) to be a cosine at a single frequency, for simplicity. Furthermore, since the field \underline{y}_{-p} is irrotational, its computation can be reduced to solving a scalar problem, namely a Helmholtz problem. As shown in the following corollary, these observations can be used to provide a better interpretation of the source term accounting for the ARF phenomenon.

Corollary 3. The scalar field p solution of (21) satisfies

$$p = \operatorname{Re} \left(e^{-i\hat{\omega}\tau} \hat{p} \right),$$

where \hat{p} is the smooth solution at each time t of the Helmholtz equation

$$\begin{cases} \Delta_x \hat{p} + \alpha \hat{p} = 0 & \text{in } \Omega(t), \\ (\hat{\kappa} - i \hat{\omega} \hat{\zeta}) \hat{p}(x, t) = g(x, t) & \text{on } \Gamma(t), \end{cases} \quad \text{with } \alpha := \frac{\rho_0 \hat{\omega}^2}{\hat{\kappa} - i \hat{\omega} \hat{\zeta}} = \frac{\rho_0 \omega^2}{\kappa - i \omega \zeta} \in \mathbb{C}. \quad (24)$$

Moreover, there exists another scalar field \tilde{p} such that the couple $(\underline{y}_s, \tilde{p})$ satisfies Eq. (20) with

$$s(t, \underline{w}) = \int_{\Omega(t)} \underline{F}_{\text{ARF}} \cdot \underline{w} \, d\Omega + \int_{\Gamma_N(t)} \underline{w} \cdot \underline{\sigma}_{\text{bndy}} \cdot \underline{n} \, d\Gamma, \quad r(t, q) = \frac{1}{\hat{\kappa}} \int_{\Omega_0} p_0 q \, d\Omega_0,$$

where $\underline{F}_{\text{ARF}}$ is defined as

$$\underline{F}_{\text{ARF}} = -\frac{\hat{\zeta} \hat{\omega}}{2} \operatorname{Im} (\hat{p} \underline{\nabla}_x \bar{\hat{p}}), \quad (25)$$

and $\underline{\sigma}_{\text{bndy}}$ is given by

$$\underline{\sigma}_{\text{bndy}} = \frac{1}{4|\alpha|^2} (\rho_0 \hat{\omega}^2 |\underline{\nabla}_x \hat{p}|^2 - \hat{\kappa} |\alpha|^2 |\hat{p}|^2) \underline{\mathbb{1}} - \frac{\rho_0 \hat{\omega}^2}{2|\alpha|^2} \operatorname{Re} (\hat{p} \underline{H}_x(\bar{\hat{p}})). \quad (26)$$

Description of the source term. Note that \hat{p} corresponds to the pushing pressure associated with the ARF. It is computed as the solution of the Helmholtz equation (24) with a time-dependent surface source term. The linear form $s(t, \underline{w})$ – that acts as a source term for the shear wave \underline{y}_s – depends quadratically on \hat{p} , and it is composed of a volume integral – that corresponds to the ARF excitation – and a surface term. Note also that the pressure term p in Theorem 2 differs from the term \tilde{p} in Corollary 3. Indeed, the definition of $s(t, \underline{w})$ given in Eq. (22) only differs from the one given in Eq. (25) by a contribution that is of the form

$$- \int_{\Omega_0} \check{p} \underline{C}_0^{-1} : \underline{e}_0(\underline{w}) d\Omega_0,$$

and therefore $\tilde{p} = p + \check{p}$.

The prestress-free configuration. If we assume that $\underline{y}_0 = \underline{0}$, then we simply retrieve that $\Omega(t) \equiv \Omega_0$ and $\tilde{p}_0 = 0$, hence $r(t, q) = 0$ for all time. The Helmholtz equation (24) and the expression (25) are unchanged and the divergence-free field \underline{y}_s and \tilde{p} are solutions of a linear elastodynamics problem in an incompressible transversely isotropic medium.

The dissipation-free case. Let us assume that $\hat{\zeta} = 0$ and $\hat{\omega}$ is not close to a resonant frequency of the domain $\Omega(t)$. Then, the solution of equation (24) is real and $\underline{F}_{ARF} = \underline{0}$. Therefore, viscosity is essential for the generation of shear waves from ARF.

Effects of a modulation of the acoustic pressure pulse. The effect of modulation (see e.g. [16] or [17]) – that is characterised by the dependence in $\partial_t g$ in the \underline{F}_{ARF} – does not appear in our result. This can be explained since it is typically a higher-order effect that can be recovered by analysing y_i for $i \geq 3$. The derivation of such effects was not the main goal of this work but could be seen as an extension of our results.

5. Proof of Theorem 2 and Corollary 3

In this section, we detail the procedure used to state Theorem 2 and Corollary 3. As we have already introduced above, our approach is based on the approximation of $\underline{y}_{-\varepsilon}$ by the first terms of the series expansion in Eq. (15). Then, by injecting this expression in Eq. (11), we obtain a cascade of equations, and by identification of the equations proportional to the same order of ε , we retrieve the desired results. Henceforth, we will give a list of propositions, assuming the same all the hypotheses of Sections 2 and 3 hold, that show the following results:

- the leading term \underline{y}_0 corresponds to a purely incompressible displacement (i.e. $J_0 = 1$) and does not contribute to the viscous term at the leading order (Proposition 5);
- the leading term \underline{y}_0 does not depend on τ (Proposition 6);
- the term \underline{y}_1 does not depend on τ as well, and it is divergence-free (Proposition 7);
- the leading term \underline{y}_0 and a scalar field p_0 satisfy the incompressible nonlinear mechanical system (19) (Proposition 8);
- the term \underline{y}_1 and a scalar field p_1 satisfy Eq. (20), with zero source terms and null initial conditions, hence $\underline{y}_1 = \underline{0}$ (Proposition 9);
- the term $\underline{y}_2 = \underline{y}_p + \underline{y}_s$ and \underline{y}_p satisfies Eq. (21) and it represents the pressure field associated with the ARF (Proposition 10);
- the term \underline{y}_s and a scalar field p are solution of Eq. (20), with source terms as in Eqs. (22) and (23), and it represents the shear wave propagation generated by the ARF (Proposition 11).

In a final step we prove Corollary 3.

5.1. Preliminaries

Before detailing our approach, we introduce a decomposition that will be used in what follows to split each term y_i of the expansion (15) into simpler terms. Moreover, we derive some first properties based on the energy estimates (13) and (14).

Helmholtz decomposition in the deformed configuration.

Property 1. For every $t \in [0, T]$, let \underline{w} be a vector field in $L^2(\Omega(t))^3$. Then, there exist

$$\Phi \in H_0^1(\Omega(t)), \quad \underline{\Psi} \in \left\{ \underline{v} \in H(\text{curl}, \Omega(t)) \mid \text{div}_{\underline{x}}(\underline{v}) = 0 \text{ in } \Omega(t), \underline{n} \cdot \underline{v} = 0 \text{ on } \partial\Omega \right\},$$

such that

$$\underline{w} = \underline{\nabla}_{\underline{x}} \Phi + \underline{\nabla}_{\underline{x}} \times \underline{\Psi}. \quad (27)$$

Proposition 1 is a standard result in functional analysis. The reader may refer e.g. to [24] for a proof of the proposition. An analogous decomposition can be derived with respect to the reference domain Ω_0 , as asserted in the following Corollary. Such decomposition is useful to interpret the divergence-free property in the reference configuration.

Helmholtz decomposition in the reference configuration.

Corollary 4. For every $t \in [0, T]$, let $\underline{\tilde{w}}(t)$ be a vector field in $L^2(\Omega_0)^3$. Then, there exist

$$\tilde{\Phi}(t) \in H_0^1(\Omega_0), \quad \tilde{\underline{\Psi}}(t) \in H(\text{curl}, \Omega_0)$$

such that

$$\underline{\tilde{w}} = \underline{F}_{\underline{0}}^{-T} \cdot \underline{\nabla} \tilde{\Phi} + J_0^{-1} \underline{F}_{\underline{0}} \cdot (\underline{\nabla} \times \tilde{\underline{\Psi}}). \quad (28)$$

Moreover, if we define \underline{w} such that $\underline{w} \circ \phi_0 = \underline{\tilde{w}}$, then

$$\text{tr} \left(\underline{F}_{\underline{0}}^{-1} \cdot \underline{\nabla} \underline{\tilde{w}} \right) = 0 \text{ in } \Omega_0 \iff \text{div}_{\underline{x}}(\underline{w}) = 0 \text{ in } \Omega(t) \iff \tilde{\Phi} = 0 \text{ in } \Omega_0.$$

Proof. From Proposition 1 it is straightforward to derive Eq. (28) by applying a change of variable and by setting $\tilde{\Phi} \circ \phi_0 = \tilde{\Phi}$ and $\tilde{\underline{\Psi}} \circ \phi_0 = \underline{F}_{\underline{0}}^{-T} \tilde{\underline{\Psi}}$. Then, it is straightforward to prove that (see [24], Section 3.9)

$$\text{div}_{\underline{x}}(\underline{w}) \circ \phi_0 = J_0^{-1} \text{div} \underline{\tilde{w}} = J_0^{-1} \text{div} \left(J_0 \underline{F}_{\underline{0}}^{-1} \cdot \underline{\tilde{w}} \right).$$

This divergence can be expressed as

$$\text{div} \left(J_0 \underline{F}_{\underline{0}}^{-1} \cdot \underline{\tilde{w}} \right) = J_0 \underline{F}_{\underline{0}}^{-1} : \underline{\nabla} \underline{\tilde{w}} + \underline{\text{div}} \left(J_0 \underline{F}_{\underline{0}}^{-1} \right) \cdot \underline{\tilde{w}}.$$

However, using the fact that $J_0 \underline{F}_{\underline{0}}^{-1} = \text{Cof } \underline{F}_0^T$ has divergence-free columns (see [25], Section 8.1), we can infer that $\underline{\text{div}} \left(J_0 \underline{F}_{\underline{0}}^{-1} \right) = 0$ and, as a consequence,

$$\text{div}_{\underline{x}}(\underline{w}) \circ \phi_0 = \underline{F}_{\underline{0}}^{-1} : \underline{\nabla} \underline{\tilde{w}} = \text{tr} \left(\underline{F}_{\underline{0}}^{-1} \cdot \underline{\nabla} \underline{\tilde{w}} \right),$$

which concludes the proof. □

Decomposition into fast- and slow-varying terms.

Let us decompose $y_i(\underline{\xi}, t, \tau)$, for $i \in \{0, 1, \dots, N\}$ into two components, namely

$$y_i(\underline{\xi}, t, \tau) = y_i^f(\underline{\xi}, t, \tau) + y_i^s(\underline{\xi}, t), \quad \text{with} \quad y_i^s(\underline{\xi}, t) = \frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} y_i(\underline{\xi}, t, \tau) d\tau. \quad (29)$$

The superscript f refers to fast-varying components (with zero average in τ over its period) whereas s stands for slow-varying terms.

5.2. A first property of the limit displacement field from the energy estimate.

Proposition 5. *The field y_0 satisfies*

$$J_0 = 1, \quad \text{tr} \left(\underline{\underline{C}}_0^{-1} \cdot \partial_t \underline{\underline{e}}_0 \right) = 0, \quad \text{tr} \left(\underline{\underline{C}}_0^{-1} \cdot \partial_\tau \underline{\underline{e}}_0 \right) = 0,$$

hence $\underline{\underline{\Sigma}}_0^{\text{NI}} = \underline{\underline{\Sigma}}_0^{\text{VS}} = 0$.

Proof. If we substitute in Eq. (13) the expansion of J_ε given in Eq. (16), we obtain

$$\int_{\Omega_0} ((J_0^2 - 1) - 2 \log(J_0)) \, d\Omega_0 + 2\varepsilon \int_{\Omega_0} \mathcal{J}_1(J_0 - J_0^{-1}) \, d\Omega_0 + o(\varepsilon^2) \leq \varepsilon^2 C.$$

Note that $J_0 > 0$ due to Assumption 2, hence $\log(J_0)$ and J_0^{-1} are well defined. Thus, we can assert

$$(J_0^2 - 1) - 2 \log(J_0) = 0 \implies J_0 = 1. \quad (30)$$

Consequently, the leading term \underline{y}_0 is incompressible. Furthermore, from Eq. (30) and Jacobi's formula, we obtain

$$J_0^2 = \det(\underline{\underline{C}}_0) = 1 \implies 0 = \partial_t(J_0^2) = \text{tr}(\underline{\underline{C}}_0^{-1} \cdot \partial_t \underline{\underline{C}}_0) = 2 \text{tr}(\underline{\underline{C}}_0^{-1} \cdot \partial_t \underline{\underline{e}}_0). \quad (31)$$

Analogously,

$$0 = \partial_\tau(J_0^2) = \text{tr}(\underline{\underline{C}}_0^{-1} \cdot \partial_\tau \underline{\underline{C}}_0) = 2 \text{tr}(\underline{\underline{C}}_0^{-1} \cdot \partial_\tau \underline{\underline{e}}_0). \quad (32)$$

Finally, due to Definition (18) we have $\underline{\underline{\Sigma}}_0^{\text{VS}} = 0$. \square

5.3. Time dependence of the leading term

First, we take into account the equation proportional to ε^{-2} in problem (11) after injecting the decomposition of Eq. (15). It reads

$$\int_{\Omega_0} \rho_0 \partial_\tau^2 \underline{y}_0 \cdot \underline{w} \, d\Omega_0 + \frac{\hat{\kappa}}{2} \int_{\Omega_0} \underline{\underline{\Sigma}}_0^{\text{NI}} : \underline{\underline{e}}_0(\underline{w}) \, d\Omega_0 + \hat{\zeta} \int_{\Omega_0} \underline{\underline{\Sigma}}_0^{\text{VS}} : \underline{\underline{e}}_0(\underline{w}) \, d\Omega_0 = 0. \quad (33)$$

From Eq. (33) and using the result of Proposition 5, it is possible to prove that the term \underline{y}_0 does not depend on the fast time variable τ , as stated in the following proposition.

Proposition 6. *The leading-order term of the expansion \underline{y}_0 satisfies*

$$\partial_\tau \underline{y}_0 = 0. \quad (34)$$

Proof. Using Eq. (17) and Proposition 5, we obtain

$$\underline{\underline{\Sigma}}_0^{\text{NI}} = (J_0^2 - 1) \underline{\underline{C}}_0^{-1} = \underline{\underline{0}} \quad \text{and} \quad \underline{\underline{\Sigma}}_0^{\text{VS}} = \underline{\underline{0}}.$$

Hence, Eq. (33) reduces to

$$\int_{\Omega_0} \partial_\tau^2 \underline{y}_0 \cdot \underline{w} \, d\Omega_0 = 0.$$

Since \underline{w} is arbitrary, \underline{y}_0 is regular and periodic, and $\partial_\tau \underline{y}_0$ is also periodic, we obtain the desired result. \square

5.4. Properties of the first-order corrector

We consider the equation associated with ε^{-1} in problem (11). After some simplifications due to the properties that we have proved on \underline{y}_0 , namely, $\partial_\tau \underline{y}_0 = \underline{0}$, $\underline{\Sigma}_0^{\text{NI}} = \underline{0}$ and $\underline{\Sigma}_0^{\text{VS}} = \underline{0}$, it reads

$$\int_{\Omega_0} \rho_0 \partial_\tau^2 \underline{y}_{-1} \cdot \underline{w} \, d\Omega_0 + \frac{\hat{k}}{2} \int_{\Omega_0} \underline{\Sigma}_{-1}^{\text{NI}} : \underline{e}_{-0}(\underline{w}) \, d\Omega_0 + \hat{\zeta} \int_{\Omega_0} \underline{\Sigma}_{-1}^{\text{VS}} : \underline{e}_{-0}(\underline{w}) \, d\Omega_0 = 0. \quad (35)$$

Proposition 7. *The first-order corrector \underline{y}_{-1} satisfies $\partial_\tau \underline{y}_{-1} = \underline{0}$ and $\mathcal{J}_1 = \text{tr}\left(\underline{F}_{-0}^{-1} \cdot \underline{\nabla} \underline{y}_{-1}\right) = 0$, furthermore, $\underline{\Sigma}_{-1}^{\text{NI}} = \underline{\Sigma}_{-1}^{\text{VS}} = \underline{0}$.*

Proof. It is possible to show that the term $\underline{\Sigma}_{-1}^{\text{VS}}$ reads

$$\underline{\Sigma}_{-1}^{\text{VS}} = \underline{C}_{-0}^{-1} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}\right) + \underline{C}_{-0}^{-1} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}(\underline{y}_{-1}) + \underline{G}_{-1} \cdot \partial_\tau \underline{e}_{-0}\right) + \underline{G}_{-1} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}\right),$$

with $\underline{G}_{-1} = -2 \underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1}) \cdot \underline{C}_{-0}^{-1}$. Then, due to Propositions 5 and 6, one sees that

$$\underline{\Sigma}_{-1}^{\text{VS}} = \underline{C}_{-0}^{-1} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}(\underline{y}_{-1})\right).$$

Now observe that, by definition of $\underline{\dot{e}}_\varepsilon$, we retrieve

$$\text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{\dot{e}}_\varepsilon\right) = \varepsilon^{-1} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}\right) + \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}\right) + \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}(\underline{y}_{-1})\right) + O(\varepsilon) = \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}(\underline{y}_{-1})\right) + O(\varepsilon).$$

As a consequence, from the energy estimation (14), \underline{y}_{-1} satisfies

$$\text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}(\underline{y}_{-1})\right) = 0, \quad (36)$$

that implies $\underline{\Sigma}_{-1}^{\text{VS}} = \underline{0}$. Now, we decompose \underline{y}_{-1} as in Eq. (29): $\underline{y}_{-1} = \underline{y}_{-1}^f + \underline{y}_{-1}^s$. Since $\partial_\tau \underline{y}_0 = 0$, we get

$$0 \stackrel{(36)}{=} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \partial_\tau \underline{e}_{-0}(\underline{y}_{-1}^f)\right) = \partial_\tau \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1}^f)\right).$$

As \underline{y}_{-1}^f has zero mean in τ , it must satisfy

$$\text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1}^f(\tau))\right) = 0. \quad (37)$$

On the other hand, one can show that

$$\underline{\Sigma}_{-1}^{\text{NI}} = (J_0^2 - 1) \underline{G}_{-1} + 2 \mathcal{J}_1 J_0 \underline{C}_{-0}^{-1} = 2 \mathcal{J}_1 \underline{C}_{-0}^{-1} \quad \text{and} \quad \mathcal{J}_1 = J_0 \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1})\right) = \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1})\right).$$

Then, if we take $\underline{w} = \underline{y}_{-1}^s$ in Eq. (35) and we integrate on τ over its period, due to the periodicity assumption on all the terms $\partial_\tau \underline{y}_i$ and \underline{y}_i with respect to τ in $[0, 2\pi/\omega]$, we obtain

$$\frac{1}{2} \int_0^{\frac{2\pi}{\omega}} \int_{\Omega_0} \underline{\Sigma}_{-1}^{\text{NI}} : \underline{e}_{-0}(\underline{y}_{-1}^s) \, d\Omega_0 \, d\tau = \int_0^{\frac{2\pi}{\omega}} \int_{\Omega_0} \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1}^s)\right)^2 \, d\Omega_0 \, d\tau = 0 \implies \text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1}^s)\right) = 0. \quad (38)$$

Therefore, from Eqs. (37) and (38) and by linearity,

$$\text{tr}\left(\underline{C}_{-0}^{-1} \cdot \underline{e}_{-0}(\underline{y}_{-1})\right) = 0 \implies \mathcal{J}_1 = 0 \implies \text{tr}\left(\underline{F}_{-0}^{-1} \cdot \underline{\nabla} \underline{y}_{-1}\right) = 0 \quad (39)$$

and, as a consequence, $\underline{\Sigma}_{-1}^{\text{NI}} = \underline{0}$. Finally, from Eq. (35) and using the periodicity and regularity of \underline{y}_{-1} , we can assert

$$\int_{\Omega_0} \partial_\tau^2 \underline{y}_{-1} \cdot \underline{w} \, d\Omega_0 = 0 \implies \partial_\tau^2 \underline{y}_{-1} = 0 \implies \partial_\tau \underline{y}_{-1} = 0,$$

thus concluding the proof. \square

Hence, from Proposition 7 we can infer that, there exists a vector field $\underline{\Psi}$ such that

$$\underline{y}_{-1}(\underline{\xi}, t) = \underline{F}_{-0}(\underline{\xi}, t) \cdot \underline{\nabla} \times \underline{\Psi}(\underline{\xi}, t).$$

This corresponds to a shear wave generated by the source term, independent of the fast time variable τ . Henceforth, we will show that we actually obtain $\underline{y}_{-1} = \underline{0}$.

5.5. The governing equation of the leading term: the underlying nonlinear problem

The next step of our procedure consists in analysing the equation corresponding to ε^0 in the expansion of the elastodynamic problem (11). Taking into account the results of the propositions above, its expression reads

$$\begin{aligned} \int_{\Omega_0} \rho_0 \left(\partial_{\tau}^2 y_{\underline{2}} + \partial_{\tau}^2 y_{\underline{0}} \right) \cdot \underline{w} \, d\Omega_0 + \frac{\hat{\kappa}}{2} \int_{\Omega_0} \underline{\Sigma}_{\underline{2}}^{\text{NI}} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 + \int_{\Omega_0} \left(\underline{\Sigma}_{\underline{0}}^{\text{TI}} + \underline{\Sigma}^{\text{A}} \right) : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 \\ + \hat{\zeta} \int_{\Omega_0} \underline{\Sigma}_{\underline{2}}^{\text{VS}} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 = \cos(\hat{\omega} \tau) \int_{\Gamma_0} \underline{t}_0 \cdot \underline{w} \, d\Gamma_0. \end{aligned} \quad (40)$$

We highlight that, by definition, $\underline{\Sigma}_{\underline{0}}^{\text{TI}}$ depends only on $y_{\underline{0}}$. Hence, it is independent of τ . We also have

$$\underline{\Sigma}_{\underline{2}}^{\text{NI}} = 2 \mathcal{J}_2 \underline{C}_{\underline{0}}^{-1}$$

where the detailed expression of \mathcal{J}_2 is not needed, and, since, $\partial_{\tau} y_{\underline{1}} = \underline{0}$,

$$\underline{\Sigma}_{\underline{2}}^{\text{VS}} = \underline{C}_{\underline{0}}^{-1} \operatorname{tr} \left(\underline{G}_{\underline{1}} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \right) + \underline{C}_{\underline{0}}^{-1} \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}}(y_{\underline{1}}) \right) + \underline{C}_{\underline{0}}^{-1} \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}}(y_{\underline{2}}) \right).$$

Finally, the expression of $\underline{\Sigma}_{\underline{0}}^{\text{TI}}$ is given in Appendix B.

Proposition 8. *There exists a scalar field p_0 depending on time t such that the couple $(y_{\underline{0}}, p_0)$ is solution of Eq. (19). Moreover,*

$$\frac{\hat{\omega} \hat{\kappa}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \mathcal{J}_2 \, d\tau = p_0, \quad \underline{\Sigma}_{\underline{2}}^{\text{VS}} = \underline{C}_{\underline{0}}^{-1} \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}}(y_{\underline{2}}) \right) \quad \text{and} \quad \int_0^{\frac{2\pi}{\hat{\omega}}} \underline{\Sigma}_{\underline{2}}^{\text{VS}} \, d\tau = \underline{0}. \quad (41)$$

Proof. We first prove the simplified expression for $\underline{\Sigma}_{\underline{2}}^{\text{VS}}$. More precisely, our aim is to prove

$$\operatorname{tr} \left(\underline{G}_{\underline{1}} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \right) + \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}}(y_{\underline{1}}) \right) = 0. \quad (42)$$

First, note that from Eq. (39) we have

$$\partial_{\tau} \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_{\underline{1}}) \right) = 0 \implies \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}}(y_{\underline{1}}) \right) = -\operatorname{tr} \left(\partial_{\tau} \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_{\underline{1}}) \right). \quad (43)$$

Now, by definition,

$$\operatorname{tr} \left(\underline{G}_{\underline{1}} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \right) = \operatorname{tr} \left(-2 \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_{\underline{1}}) \cdot \underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \right) = \operatorname{tr} \left(-2 \underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \cdot \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_{\underline{1}}) \right).$$

Moreover, note that

$$\partial_{\tau} \underline{C}_{\underline{0}}^{-1} = -\underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{C}_{\underline{0}} \cdot \underline{C}_{\underline{0}}^{-1} = -2 \underline{C}_{\underline{0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \cdot \underline{C}_{\underline{0}}^{-1}, \quad (44)$$

therefore, we can state

$$\operatorname{tr} \left(\underline{G}_{\underline{1}} \cdot \partial_{\tau} \underline{e}_{\underline{0}} \right) = \operatorname{tr} \left(\partial_{\tau} \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(y_{\underline{1}}) \right). \quad (45)$$

Combining Eqs. (43) and (45), we obtain Eq. (42), which concludes the first part of the proof. Finally, in order to derive an expression for the governing equation of $y_{\underline{0}}$, we need to discard the terms depending of the fast time variable τ in Eq. (40). Therefore, we integrate Eq. (40) on τ over its period. Since $y_{\underline{2}}$ is smooth and periodic in τ , and \underline{t}_0 has zero mean value on the same period, we derive Eq. (19) using the expression of the stress tensors, as desired. \square

Remark Note that the viscous term does not give any contribution at this stage. Furthermore, (41) relates higher-order corrector terms to p_0 . In what follows, we will show that it is a constraint on the divergence of the (slowly varying) second-order corrector $y_{\underline{2}}$.

5.6. The governing equation of the first-order corrector

We are now interested in the equation associated with ε^1 in problem (11). It reads

$$\begin{aligned} & \int_{\Omega_0} \rho_0 (\partial_{\tau}^2 y_3 + 2 \partial_{\tau}^2 y_2 + \partial_{\tau}^2 y_1) \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} \left((\underline{\Sigma}_{\underline{0}}^{\text{II}} + \underline{\Sigma}^A) : \underline{e}_{\underline{-1}}(\underline{w}) + \underline{\Sigma}_{\underline{-1}}^{\text{II}} : \underline{e}_{\underline{0}}(\underline{w}) \right) d\Omega_0 \\ & + \frac{\hat{\kappa}}{2} \int_{\Omega_0} \left(\underline{\Sigma}_{\underline{-2}}^{\text{NI}} : \underline{e}_{\underline{-1}}(\underline{w}) + \underline{\Sigma}_{\underline{-3}}^{\text{NI}} : \underline{e}_{\underline{0}}(\underline{w}) \right) d\Omega_0 + \hat{\zeta} \int_{\Omega_0} \left(\underline{\Sigma}_{\underline{-2}}^{\text{VS}} : \underline{e}_{\underline{-1}}(\underline{w}) + \underline{\Sigma}_{\underline{-3}}^{\text{VS}} : \underline{e}_{\underline{0}}(\underline{w}) \right) d\Omega_0 = \cos(\hat{\omega} \tau) \int_{\Gamma_0} \underline{t}_1 \cdot \underline{w} \, d\Gamma_0. \end{aligned} \quad (46)$$

After some algebra, the terms $\underline{\Sigma}_{\underline{-3}}^{\text{NI}}$ and $\underline{\Sigma}_{\underline{-3}}^{\text{VS}}$ read:

$$\underline{\Sigma}_{\underline{-3}}^{\text{NI}} = 2 \mathcal{J}_2 \underline{G}_{\underline{-1}} + 2 \mathcal{J}_3 \underline{C}_{\underline{-0}}^{-1}$$

and

$$\underline{\Sigma}_{\underline{-3}}^{\text{VS}} = \underline{C}_{\underline{-0}}^{-1} \text{tr}(\underline{D}_{\underline{-3}}) + \underline{G}_{\underline{-1}} \text{tr}(\underline{C}_{\underline{-0}}^{-1} \cdot \partial_{\tau} \underline{e}_{\underline{-0}}(y_{\underline{-2}})),$$

where $\underline{D}_{\underline{-3}}$ is a tensor depending on $(y_{\underline{0}}, y_{\underline{-1}}, y_{\underline{-2}}, y_{\underline{-3}})$, whose expression is of no practical interest. The detailed expression of the tensor $\underline{\Sigma}_{\underline{-1}}^{\text{II}}$ can be found in Appendix B. For the sake of completeness, we recall here that it corresponds to

$$\underline{\Sigma}_{\underline{-1}}^{\text{II}} = \underline{A}_{\underline{-0}} : \underline{e}_{\underline{-0}}(y_{\underline{-1}}).$$

Therefore, it does not depend on τ . Since we want to derive the governing equation of $y_{\underline{-1}}$ (that does not depend on τ), it is natural to integrate Eq. (46) on τ over $[0, \hat{\omega}/(2\pi)]$. Using similar arguments to the previous section (namely, the periodicity of the quantities depending on τ), one can show that

$$\begin{aligned} & \int_{\Omega_0} \rho_0 \partial_{\tau}^2 y_{\underline{-1}} \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} p_1 \underline{C}_{\underline{-0}}^{-1} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 + \int_{\Omega_0} \underline{\Sigma}_{\underline{-1}}^{\text{II}} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 + \int_{\Omega_0} p_0 \underline{G}_{\underline{-1}} : \underline{e}_{\underline{0}}(\underline{w}) \, d\Omega_0 \\ & + \int_{\Omega_0} \left(\underline{\Sigma}_{\underline{-0}}^{\text{II}} + \underline{\Sigma}^A + p_0 \underline{C}_{\underline{-0}}^{-1} \right) : \underline{e}_{\underline{-1}}(\underline{w}) \, d\Omega_0 = 0, \end{aligned} \quad (47)$$

where the scalar field p_1 satisfies

$$\frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \left(\hat{\kappa} \mathcal{J}_3 + \hat{\zeta} \text{tr}(\underline{D}_{\underline{-3}}) \right) d\tau = p_1. \quad (48)$$

Proposition 9. *The first-order corrector $y_{\underline{-1}}$ satisfies*

$$y_{\underline{-1}} = \underline{0}.$$

Proof. The proof consists in showing that the couple $(y_{\underline{-1}}, p_1)$ satisfies Problem (20) with no source terms. First, by algebraic manipulations and using the definition of the stress tensors, it is easy to see that Eq. (47) corresponds to the first equation of Eq. (20) with $\tilde{y} = y_{\underline{-1}}$, $\tilde{p} = p_1$, and $s(t, \underline{w}) = 0$. Then, since we have shown previously that $J_1 = 0$, we can write

$$\int_{\Omega_0} \underline{C}_{\underline{-0}}^{-1} : \underline{e}_{\underline{-0}}(y_{\underline{-1}}) q \, d\Omega_0 = 0,$$

retrieving the second equation in (20), with $r(t, q)$ equal to zero. Finally, since initial conditions are assumed to be zero, and problem (20) is well-posed, the only solution of our problem is

$$y_{\underline{-1}} = \underline{0}, \quad p_1 = 0,$$

completing the proof. \square

Remark Since $y_{\underline{-1}}$ vanishes, numerous simplifications can be deduced in the expansion of the mechanical quantities in Section 3.2. Among them, one can deduce

$$\mathcal{J}_2 = \text{tr}(\underline{C}_{\underline{-0}}^{-1} \cdot \underline{e}_{\underline{-0}}(y_{\underline{-2}})), \quad \underline{G}_{\underline{-2}} = -2 \underline{C}_{\underline{-0}}^{-1} \cdot \underline{e}_{\underline{-0}}(y_{\underline{-2}}) \cdot \underline{C}_{\underline{-0}}^{-1}, \quad \underline{t}_1 = \underline{0} \quad \text{and} \quad \underline{\varepsilon}_{\underline{-\varepsilon}} = \underline{e}_{\underline{-0}} + \sum_{i=2}^3 \varepsilon^i \underline{e}_{\underline{-0}}(y_{\underline{-i}}) + o(\varepsilon^3).$$

5.7. The governing equation of the fast-oscillating second-order corrector: the ARF pressure field

The second main result of this work concerns the derivation of the governing equation of the pressure wave propagation associated with the ARF. To do so, we consider again the term corresponding to ε^0 in the expansion of problem (11). More precisely, if we inject Eq. (19) in Eq. (40), we obtain

$$\int_{\Omega_0} \rho_0 \partial_{\tau}^2 \underline{y}_{-2} \cdot \underline{w} \, d\Omega_0 + \int_{\Omega_0} \left(\frac{\hat{\kappa}}{2} \underline{\Sigma}_{-2}^{\text{NI}} - \text{p}_0 \underline{C}_{-2}^{-1} \right) : \underline{e}_{-2}(\underline{w}) \, d\Omega_0 + \hat{\zeta} \int_{\Omega_0} \underline{\Sigma}_{-2}^{\text{VS}} : \underline{e}_{-2}(\underline{w}) \, d\Omega_0 = \cos(\hat{\omega} \tau) \int_{\Gamma_0} \underline{t}_0 \cdot \underline{w} \, d\Gamma_0, \quad (49)$$

where $\underline{\Sigma}_{-2}^{\text{VS}}$ is given in Eq. (41) and

$$\frac{\hat{\kappa}}{2} \underline{\Sigma}_{-2}^{\text{NI}} - \text{p}_0 \underline{C}_{-2}^{-1} = \hat{\kappa} \left(\partial_2 - \frac{\hat{\omega}}{2\pi} \int_0^{2\pi} \partial_2 \, d\tau \right) \underline{C}_{-2}^{-1}. \quad (50)$$

Now, to go further in our analysis, we need to decompose \underline{y}_{-2} as in Eq. (29):

$$\underline{y}_{-2} = \underline{y}_{-p} + \underline{y}_{-s} \quad \text{with} \quad \underline{y}_{-p} := \underline{y}_{-2}^f \quad \text{and} \quad \underline{y}_{-s} := \underline{y}_{-2}^s. \quad (51)$$

From Remark 5.6 one can deduce that

$$\frac{\hat{\kappa}}{2} \underline{\Sigma}_{-2}^{\text{NI}} - \text{p}_0 \underline{C}_{-2}^{-1} = \hat{\kappa} \, \text{tr} \left(\underline{C}_{-2}^{-1} \cdot \underline{e}_{-2}(\underline{y}_{-p}) \right) \underline{C}_{-2}^{-1}. \quad (52)$$

The main result of this section is the following proposition.

Proposition 10. *There exist a scalar field p such that the irrotational field \underline{y}_{-p} satisfies Eq. (21).*

Proof. Let us consider Eq. (49). If we use Eq. (52) and the definition of the stress tensor $\underline{\Sigma}_{-2}^{\text{VS}}$, we obtain the following equation :

$$\begin{aligned} \int_{\Omega_0} \rho_0 \partial_{\tau}^2 \underline{y}_{-p} \cdot \underline{w} \, d\Omega_0 + \hat{\kappa} \int_{\Omega_0} \text{tr} \left(\underline{C}_{-2}^{-1} \cdot \underline{e}_{-2}(\underline{y}_{-p}) \right) \text{tr} \left(\underline{C}_{-2}^{-1} \cdot \underline{e}_{-2}(\underline{w}) \right) \, d\Omega_0 + \hat{\zeta} \int_{\Omega_0} \text{tr} \left(\underline{C}_{-2}^{-1} \cdot \partial_{\tau} \underline{e}_{-2}(\underline{y}_{-p}) \right) \text{tr} \left(\underline{C}_{-2}^{-1} \cdot \underline{e}_{-2}(\underline{w}) \right) \, d\Omega_0 \\ = \cos(\hat{\omega} \tau) \int_{\Gamma_0} g(\underline{\xi}, t) \left| \underline{F}_{-2}^{-T} \underline{n}_0 \right| \underline{w} \cdot \underline{F}_{-2}^{-T} \cdot \underline{n}_0 \, d\Gamma_0. \end{aligned}$$

The equation above can be rewritten in the deformed configuration as

$$\int_{\Omega(t)} \rho_0 \partial_{\tau}^2 \underline{y}_{-p} \cdot \underline{w} \, d\Omega + \hat{\kappa} \int_{\Omega(t)} \text{div}_{\underline{x}} \underline{y}_{-p} \cdot \text{div}_{\underline{x}} \underline{w} \, d\Omega + \hat{\zeta} \int_{\Omega(t)} \text{div}_{\underline{x}} \partial_{\tau} \underline{y}_{-p} \cdot \text{div}_{\underline{x}} \underline{w} \, d\Omega = \cos(\hat{\omega} \tau) \int_{\Gamma(t)} g(\underline{\xi}, t) \underline{n} \cdot \underline{w} \, d\Gamma, \quad (53)$$

since $J_0 = 1$. In strong form, Eq. (53) reads

$$\begin{cases} \rho_0 \partial_{\tau}^2 \underline{y}_{-p} - \hat{\kappa} \underline{\nabla}_{\underline{x}} \text{div}_{\underline{x}} \underline{y}_{-p} - \hat{\zeta} \underline{\nabla}_{\underline{x}} \text{div}_{\underline{x}} \partial_{\tau} \underline{y}_{-p} = 0 & \text{in } \Omega(t), \\ (\hat{\kappa} + \hat{\zeta} \partial_{\tau}) \text{div}_{\underline{x}} \underline{y}_{-p} \cdot \underline{n} = g \cos(\hat{\omega} \tau) \underline{n} & \text{on } \Gamma_N(t). \end{cases} \quad (54)$$

This shows that \underline{y}_{-p} is irrotational: indeed, since $(\rho_0, \hat{\kappa}, \hat{\zeta})$ are constant, we have $\underline{\nabla}_{\underline{x}} \times \underline{y}_{-p} = 0$. Finally, observe that Eq. (54) can be recast as the mixed problem (21). \square

5.8. The governing equation of the slow-varying second-order corrector: the shear wave propagation

At last, we give an explicit expression of the governing equation for the second-order corrector y_2^s as defined in Eq. (51). To this end, we need to evaluate the term corresponding to ε^2 in the expansion of the problem (11). Using the results above, namely, $J_0 = 1$, $\partial_\tau y_0 = \underline{0}$, $y_{-1} = \underline{0}$, it reads

$$\begin{aligned} \int_{\Omega_0} \rho_0 (\partial_\tau^2 y_4 + 2 \partial_{\tau\tau}^2 y_3 + \partial_\tau^2 y_2) \cdot \underline{w} \, d\Omega_0 + \frac{\hat{\kappa}}{2} \int_{\Omega_0} (\underline{\Sigma}_2^{\text{NI}} : \underline{e}_2(\underline{w}) + \underline{\Sigma}_4^{\text{NI}} : \underline{e}_0(\underline{w})) \, d\Omega_0 \\ + \int_{\Omega_0} (\underline{\Sigma}_2^{\text{TI}} : \underline{e}_0(\underline{w}) + (\underline{\Sigma}_0^{\text{TI}} + \underline{\Sigma}^A) : \underline{e}_2(\underline{w})) \, d\Omega_0 \\ + \hat{\zeta} \int_{\Omega_0} (\underline{\Sigma}_2^{\text{VS}} : \underline{e}_2(\underline{w}) + \underline{\Sigma}_4^{\text{VS}} : \underline{e}_0(\underline{w})) \, d\Omega_0 = \cos(\hat{\omega} \tau) \int_{\Gamma_0} \underline{t}_2 \cdot \underline{w} \, d\Gamma_0. \end{aligned} \quad (55)$$

Note that, since $y_{-1} = \underline{0}$, the equation above only involves even terms of the expansion of the stress tensors. Moreover, we have $\underline{\Sigma}_2^{\text{TI}} = \underline{\mathbf{A}} : \underline{e}_0(y_{-2})$ and $\underline{\mathbf{A}}$ is given in Appendix B. We just highlight here that it is linear in y_{-2} . We recall that

$$\underline{\Sigma}_2^{\text{NI}} = 2 \mathcal{J}_2 \underline{C}_0^{-1} = 2 \operatorname{tr} \left(\underline{C}_0^{-1} \cdot \underline{e}_0(y_{-2}) \right) \underline{C}_0^{-1}, \quad \underline{\Sigma}_2^{\text{VS}} = \underline{C}_0^{-1} \operatorname{tr} \left(\underline{C}_0^{-1} \cdot \partial_\tau \underline{e}_0(y_{-2}) \right).$$

Finally, since $y_{-1} = 0$, the terms $\underline{\Sigma}_4^{\text{NI}}$ and $\underline{\Sigma}_4^{\text{VS}}$ reduce to

$$\underline{\Sigma}_4^{\text{NI}} = 2 \mathcal{J}_2 \underline{G} + (\mathcal{J}_2^2 + 2 \mathcal{J}_4) \underline{C}_0^{-1}, \quad \underline{\Sigma}_4^{\text{VS}} = \underline{C}_0^{-1} \operatorname{tr} \left(\underline{D}_4 \right) + \underline{G} \operatorname{tr} \left(\underline{C}_0^{-1} \cdot \partial_\tau \underline{e}_0(y_{-2}) \right),$$

where \underline{D}_4 is a tensor field and \mathcal{J}_4 a scalar field whose exact expressions are not needed, and where the expression of $\underline{G} = -2 \underline{C}_0^{-1} \cdot \underline{e}_0(y_{-2}) \cdot \underline{C}_0^{-1}$ has already been given in Remark 5.6. We want to derive the governing equation of the slowly varying vector field y_2^s given in the decomposition Eq. (51), therefore we integrate Eq. (55) on τ over its period. Since this last step is fundamental, we now detail each term. First, we use the decomposition

$$y_2 = y_{-p} + y_s$$

to write

$$\frac{\hat{\omega}}{2\pi} \int_0^{2\pi} \rho_0 (\partial_\tau^2 y_4 + 2 \partial_{\tau\tau}^2 y_3 + \partial_\tau^2 y_2) \cdot \underline{w} \, d\tau = \rho_0 \partial_\tau^2 y_s. \quad (56)$$

In addition,

$$\begin{aligned} \frac{\hat{\omega}}{2\pi} \int_0^{2\pi} \left(\frac{\hat{\kappa}}{2} \underline{\Sigma}_2^{\text{NI}} + \hat{\zeta} \underline{\Sigma}_2^{\text{VS}} \right) : \underline{e}_2(\underline{w}) \, d\tau = \mathbf{p}_0 \underline{C}_0^{-1} : \underline{e}_s(\underline{w}) \\ + \frac{\hat{\omega}}{2\pi} \int_0^{2\pi} \operatorname{tr} \left(\hat{\kappa} \underline{C}_0^{-1} \cdot \underline{e}_0(y_{-p}) + \hat{\zeta} \underline{C}_0^{-1} \cdot \underline{e}_0(\partial_\tau y_{-p}) \right) \underline{C}_0^{-1} : \underline{e}_{-p}(\underline{w}) \, d\tau, \end{aligned} \quad (57)$$

where we have defined

$$\underline{e}_s(\underline{w}) := \frac{1}{2} (\underline{\nabla}^T y_s \cdot \underline{\nabla} \underline{w} + \underline{\nabla}^T \underline{w} \cdot \underline{\nabla} y_s), \quad \underline{e}_{-p}(\underline{w}) := \frac{1}{2} (\underline{\nabla}^T y_{-p} \cdot \underline{\nabla} \underline{w} + \underline{\nabla}^T \underline{w} \cdot \underline{\nabla} y_{-p}).$$

Similarly, one can show that there exists a scalar field \mathbf{p} independent of τ such that

$$\begin{aligned} \frac{\hat{\omega}}{2\pi} \int_0^{2\pi} \left(\frac{\hat{\kappa}}{2} \underline{\Sigma}_4^{\text{NI}} + \hat{\zeta} \underline{\Sigma}_4^{\text{VS}} \right) : \underline{e}_0(\underline{w}) \, d\tau = \mathbf{p} \underline{C}_0^{-1} : \underline{e}_0(\underline{w}) \\ + \frac{\hat{\omega}}{\pi} \int_0^{2\pi} \operatorname{tr} \left(\hat{\kappa} \underline{C}_0^{-1} \cdot \underline{e}_0(y_{-p}) + \hat{\zeta} \underline{C}_0^{-1} \cdot \underline{e}_0(\partial_\tau y_{-p}) \right) \left(\underline{C}_0^{-1} \cdot \underline{e}_0(y_{-p}) \cdot \underline{C}_0^{-1} \right) : \underline{e}_0(\underline{w}) \, d\tau. \end{aligned} \quad (58)$$

Finally, the transversely isotropic tensors and the activation term satisfy

$$\frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \left(\underline{\Sigma}^{\text{TI}} : \underline{e}_{\underline{0}}(\underline{w}) + \left(\underline{\Sigma}^{\text{TI}} + \underline{\Sigma}^{\text{A}} \right) : \underline{e}_{\underline{2}}(\underline{w}) \right) d\tau = \underline{e}_{\underline{0}}(\underline{y}_{\underline{s}}) : \underline{\mathbf{A}}_{\underline{0}} : \underline{e}_{\underline{0}}(\underline{w}) + \left(\underline{\Sigma}^{\text{TI}} + \underline{\Sigma}^{\text{A}} \right) : \underline{e}_{\underline{s}}(\underline{w}), \quad (59)$$

Note that we have introduced the unknown p and we need an equation for it. In order to obtain this equation, we simply observe that from Eq. (41) and the expression of \mathcal{J}_2 given in Remark 5.6, we have

$$p_0 = \frac{\hat{\omega} \hat{\kappa}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \mathcal{J}_2 d\tau = \hat{\kappa} \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{y}_{\underline{s}}) \right). \quad (60)$$

We can now state the main result of this section.

Proposition 11. *The couple $(\underline{y}_{\underline{s}}, p)$ is solution of Eq. (20), with*

$$s(t, \underline{w}) = \frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \int_{\Omega(t)} \operatorname{div}_{\underline{x}} \left((\hat{\kappa} + \hat{\zeta} \partial_{\tau}) \underline{y}_{\underline{p}} \right) \left(\underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \right)^T : \underline{\nabla}_{\underline{x}} \underline{w} d\Omega d\tau, \quad r(t, q) = \frac{1}{\hat{\kappa}} \int_{\Omega_0} p_0 q d\Omega_0. \quad (61)$$

Proof. Considering Eqs. (56), (57), (58) and (60), one can see that the couple $(\underline{y}_{\underline{s}}, p)$ satisfies the first equation of (20) with source term $s(t, \underline{w})$ given by

$$s(t, \underline{w}) = -\frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \int_{\Omega_0} \operatorname{tr} \left(\hat{\kappa} \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{y}_{\underline{p}}) + \hat{\zeta} \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\partial_{\tau} \underline{y}_{\underline{p}}) \right) \left(\underline{C}_{\underline{0}}^{-1} : \underline{e}_{\underline{p}}(\underline{w}) - 2 \left(\underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{y}_{\underline{p}}) \cdot \underline{C}_{\underline{0}}^{-1} \right) : \underline{e}_{\underline{0}}(\underline{w}) \right) d\Omega_0 d\tau.$$

Moreover, from Eq. (60) one can see that the second equation of (20) is satisfied with $r(t, q)$ given by

$$r(t, q) = \frac{1}{\hat{\kappa}} \int_{\Omega_0} p_0 q d\Omega_0.$$

In order to conclude the proof, we only need to check that $s(t, \underline{w})$ can be rewritten as in Eq. (61). First, a simplified expression of $s(t, \underline{w})$ can be obtained by observing that

$$\begin{aligned} \underline{C}_{\underline{0}}^{-1} : \underline{e}_{\underline{p}}(\underline{w}) - 2 \left(\underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{y}_{\underline{p}}) \cdot \underline{C}_{\underline{0}}^{-1} \right) : \underline{e}_{\underline{0}}(\underline{w}) &= \operatorname{tr} \left(\underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{p}}(\underline{w}) - 2 \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{y}_{\underline{p}}) \cdot \underline{C}_{\underline{0}}^{-1} \cdot \underline{e}_{\underline{0}}(\underline{w}) \right) \\ &= -\operatorname{tr} \left(\underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \cdot \underline{F}_{\underline{0}}^{-1} \cdot \underline{\nabla}_{\underline{x}} \underline{w} \cdot \underline{F}_{\underline{0}}^{-1} \right). \end{aligned}$$

As a consequence, we have the following expression for $s(t, \underline{w})$:

$$s(t, \underline{w}) = \frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \int_{\Omega_0} \operatorname{tr} \left(\hat{\kappa} \underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \cdot \underline{F}_{\underline{0}}^{-1} + \hat{\zeta} \partial_{\tau} \underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \cdot \underline{F}_{\underline{0}}^{-1} \right) \operatorname{tr} \left(\underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \cdot \underline{F}_{\underline{0}}^{-1} \cdot \underline{\nabla}_{\underline{x}} \underline{w} \cdot \underline{F}_{\underline{0}}^{-1} \right) d\Omega_0 d\tau. \quad (62)$$

We can see that this is a nonlinear (quadratic) contribution only depending on $\underline{y}_{\underline{p}}$. Thus, it is a source term (accounting for the ARF contribution) and, although $\underline{y}_{\underline{p}}$ has zero mean in τ , it gives a non-zero contribution in the equation, due to nonlinearity. We can rewrite Eq. (62) in the deformed configuration. It becomes

$$s(t, \underline{w}) = \frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \int_{\Omega(t)} \operatorname{tr} \left(\hat{\kappa} \underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} + \hat{\zeta} \partial_{\tau} \underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \right) \operatorname{tr} \left(\underline{\nabla}_{\underline{x}} \underline{y}_{\underline{p}} \cdot \underline{\nabla}_{\underline{x}} \underline{w} \right) d\Omega d\tau, \quad (63)$$

which concludes the proof. \square

Remark Note that p represents a Lagrange multiplier for the solution $\underline{y}_{\underline{s}}$. We also emphasise that, if the solution of the nonlinear problem $\underline{y}_{\underline{0}}$ is zero, then the Lagrangian multiplier p_0 is also zero. This implies that, in this case, the term $\underline{y}_{\underline{s}}$ is divergence-free and $\underline{y}_{\underline{s}} = \underline{\nabla} \times \underline{\Psi}$ for some vector field $\underline{\Psi}$.

5.9. Proof of Corollary 3

In order to study the governing equations of \underline{y}_{-p} and to further simplify the expression of $s(t, \underline{w})$, we need to reduce System (54) to a frequency problem on the pressure variable p . Due to the periodic structure in τ of the surface source term, Eq. (54) can be recast in the frequency domain. The scalar field $p = \text{div}_{\underline{x}} \underline{y}_{-p}$ satisfies

$$p = \text{Re} \left(e^{-i\hat{\omega}\tau} \hat{p} \right), \quad (64)$$

where \hat{p} is solution of

$$\begin{cases} \Delta_{\underline{x}} \hat{p} + \alpha \hat{p} = 0 & \text{in } \Omega(t), \\ (\hat{k} - i \hat{\omega} \hat{\zeta}) \hat{p}(\underline{x}, t) = g(\underline{x}, t) & \text{on } \Gamma(t), \end{cases} \quad (65)$$

with

$$\alpha := \frac{\rho_0 \hat{\omega}^2}{\hat{k} - i \hat{\omega} \hat{\zeta}} \in \mathbb{C}. \quad (66)$$

Eq. (65) represents for each time t a Helmholtz equation governing the propagation of the pushing pressure wave corresponding to the ARF. Note that t plays the role of a parameter in this equation, due to Eq. (64). Problem (65) is well-posed, for all time t , because $\text{Im}(\alpha) \neq 0$. To continue in the simplification process we introduce the function \hat{y}_{-p} such that $\hat{y}_{-p} = -\alpha^{-1} \nabla_{\underline{x}} \hat{p}$. By definition of \hat{p} , we retrieve that $\text{div}_{\underline{x}} \hat{y}_{-p} = \hat{p}$. In addition, the function \underline{y}_{-p} satisfies

$$\underline{y}_{-p} = \text{Re} \left(\hat{y}_{-p} e^{-i\hat{\omega}\tau} \right) \quad \text{and} \quad (\hat{k} + \hat{\zeta} \partial_{\tau}) \underline{y}_{-p} = \text{Re} \left((\hat{k} - i \hat{\omega} \hat{\zeta}) \hat{y}_{-p} e^{-i\hat{\omega}\tau} \right) = \rho_0 \hat{\omega}^2 \text{Re} \left(\alpha^{-1} \hat{y}_{-p} e^{-i\hat{\omega}\tau} \right). \quad (67)$$

As an intermediate step towards the proof of the corollary, we state the following proposition that enables to deduce an alternative expression of the source term $s(t, \underline{w})$ of Proposition 11.

Corollary 12. *The following equalities hold:*

$$\frac{\hat{\omega}}{2\pi} \int_0^{2\pi/\hat{\omega}} \text{div}_{\underline{x}} \left((\hat{k} + \hat{\zeta} \partial_{\tau}) \underline{y}_{-p} \right) (\nabla_{\underline{x}} \underline{y}_{-p})^T d\tau = -\frac{\rho_0 \hat{\omega}^2}{2|\alpha|^2} \text{Re} \left(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}}) \right)$$

and

$$\underline{\text{div}}_{\underline{x}} \text{Re} \left(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}}) \right) = \frac{1}{2} \nabla_{\underline{x}} |\nabla_{\underline{x}} \hat{p}|^2 - \frac{1}{2} \frac{\kappa |\alpha|^2}{\rho_0 \hat{\omega}^2} \nabla_{\underline{x}} |\hat{p}|^2 - \frac{\hat{\zeta} |\alpha|^2}{\rho_0 \hat{\omega}} \text{Im} \left(\hat{p} \nabla_{\underline{x}}(\bar{\hat{p}}) \right).$$

Proof. Observe that, using (67), we can write

$$S := \frac{\hat{\omega}}{2\pi} \int_0^{2\pi/\hat{\omega}} \text{div}_{\underline{x}} \left((\hat{k} + \hat{\zeta} \partial_{\tau}) \underline{y}_{-p} \right) (\nabla_{\underline{x}} \underline{y}_{-p})^T d\tau = \rho_0 \hat{\omega}^2 \frac{\hat{\omega}}{2\pi} \int_0^{2\pi/\hat{\omega}} \text{Re} \left(\alpha^{-1} \text{div}_{\underline{x}} \hat{y}_{-p} e^{-i\hat{\omega}\tau} \right) \text{Re} \left(\nabla_{\underline{x}} \hat{y}_{-p} e^{-i\hat{\omega}\tau} \right)^T d\tau.$$

However, since for any $(a, b) \in \mathbb{C}^2$, we have

$$\frac{\hat{\omega}}{2\pi} \int_0^{2\pi/\hat{\omega}} \text{Re}(ae^{-i\hat{\omega}\tau}) \text{Re}(be^{-i\hat{\omega}\tau}) d\tau = \frac{1}{2} \text{Re}(a\bar{b}),$$

by extension of this property and using the definition of \hat{p} , one can show that

$$S = \frac{\rho_0 \hat{\omega}^2}{2} \text{Re} \left(\alpha^{-1} \text{div}_{\underline{x}} \hat{y}_{-p} (\nabla_{\underline{x}} \bar{\hat{y}}_{-p})^T \right) = -\frac{\rho_0 \hat{\omega}^2 |\alpha|^{-2}}{2} \text{Re} \left(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}}) \right), \quad (68)$$

that corresponds to the first equality to be proved. In order to prove the second equality, observe that

$$\underline{\text{div}}_{\underline{x}} \left(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}}) \right) = \underline{H}_{\underline{x}}(\bar{\hat{p}}) \cdot \nabla_{\underline{x}} \hat{p} + \hat{p} \nabla_{\underline{x}} \Delta_{\underline{x}} \bar{\hat{p}} = \underline{V} + \underline{W},$$

with

$$\underline{V} := \underline{H}_{\underline{x}}(\bar{\hat{p}}) \cdot \underline{\nabla}_{\underline{x}}(\hat{p}), \quad \underline{W} := \hat{p} \underline{\nabla}_{\underline{x}}(\Delta_{\underline{x}} \bar{\hat{p}}) = -\bar{\alpha} \hat{p} \underline{\nabla}_{\underline{x}} \bar{\hat{p}}.$$

Only the real part of V and W needs to be specified. We have

$$\operatorname{Re}(\underline{V}) = \underline{H}_{\underline{x}}(\operatorname{Re}(\hat{p})) \cdot \underline{\nabla}_{\underline{x}} \operatorname{Re}(\hat{p}) + \underline{H}_{\underline{x}}(\operatorname{Im}(\hat{p})) \cdot \underline{\nabla}_{\underline{x}} \operatorname{Im}(\hat{p}).$$

Now, since

$$\underline{H}_{\underline{x}}(\operatorname{Re}(\hat{p})) \cdot \underline{\nabla}_{\underline{x}} \operatorname{Re}(\hat{p}) = \sum_j \frac{\partial^2 \operatorname{Re}(\hat{p})}{\partial x_i \partial x_j} \frac{\partial \operatorname{Re}(\hat{p})}{\partial x_j} = \frac{1}{2} \sum_j \underline{\nabla}_{\underline{x}} \left(\frac{\partial \operatorname{Re}(\hat{p})}{\partial x_j} \right)^2,$$

and analogously for $\underline{H}_{\underline{x}}(\operatorname{Im}(\hat{p})) \cdot \underline{\nabla}_{\underline{x}} \operatorname{Im}(\hat{p})$, we can simplify \underline{V} as

$$\operatorname{Re}(\underline{V}) = \frac{1}{2} \sum_j \underline{\nabla}_{\underline{x}} |\partial_{x_j} \hat{p}|^2 = \frac{1}{2} \underline{\nabla}_{\underline{x}} |\underline{\nabla}_{\underline{x}} \hat{p}|^2. \quad (69)$$

On the other hand, we can express

$$\operatorname{Re}(\underline{W}) = -\frac{\kappa |\alpha|^2}{\rho_0 \hat{\omega}^2} \operatorname{Re}(\hat{p} \underline{\nabla}_{\underline{x}}(\bar{\hat{p}})) - \frac{\hat{\zeta} |\alpha|^2}{\rho_0 \hat{\omega}} \operatorname{Im}(\hat{p} \underline{\nabla}_{\underline{x}}(\bar{\hat{p}})). \quad (70)$$

Since $2 \operatorname{Re}(\hat{p} \underline{\nabla}_{\underline{x}} \bar{\hat{p}}) = \underline{\nabla}_{\underline{x}} |\hat{p}|^2$ we finally have

$$\operatorname{Re}(\underline{W}) = -\frac{1}{2} \frac{\kappa |\alpha|^2}{\rho_0 \hat{\omega}^2} \underline{\nabla}_{\underline{x}} |\hat{p}|^2 - \frac{\hat{\zeta} |\alpha|^2}{\rho_0 \hat{\omega}} \operatorname{Im}(\hat{p} \underline{\nabla}_{\underline{x}}(\bar{\hat{p}})).$$

Collecting the two results above, we obtain the second equality of the statement. \square

Proposition 13. *There exists another scalar field \bar{p} such that the couple (y, \bar{p}) satisfies Eq. (20) with*

$$s(t, \underline{w}) = \int_{\Omega(t)} \underline{E}_{ARF} \cdot \underline{w} \, d\Omega + \int_{\Gamma_N(t)} \underline{w} \cdot \underline{\sigma}_{bndy} \cdot \underline{n} \, d\Gamma, \quad r(t, q) = \frac{1}{\hat{\kappa}} \int_{\Omega_0} p_0 q \, d\Omega_0,$$

where \underline{E}_{ARF} is defined as

$$\underline{E}_{ARF} = -\frac{\hat{\zeta} \hat{\omega}}{2} \operatorname{Im}(\hat{p} \underline{\nabla}_{\underline{x}} \bar{\hat{p}}), \quad (71)$$

and $\underline{\sigma}_{bndy}$ is given by

$$\underline{\sigma}_{bndy} = \frac{1}{4|\alpha|^2} (\rho_0 \hat{\omega}^2 |\underline{\nabla}_{\underline{x}} \hat{p}|^2 - \hat{\kappa} |\alpha|^2 |\hat{p}|^2) \underline{\mathbb{1}} - \frac{\rho_0 \hat{\omega}^2}{2|\alpha|^2} \operatorname{Re}(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}})).$$

Proof. From the first equality of Corollary (12) we have that the source term $s(t, \underline{w})$ given in Proposition 11 satisfies, for all test function \underline{w} real

$$S := \frac{\hat{\omega}}{2\pi} \int_0^{\frac{2\pi}{\hat{\omega}}} \int_{\Omega(t)} \operatorname{div}_{\underline{x}}((\hat{\kappa} + \hat{\zeta} \partial_\tau) \underline{y}_{-p}) (\underline{\nabla}_{\underline{x}} \underline{y}_{-p})^T : \underline{\nabla}_{\underline{x}} \underline{w} \, d\Omega \, d\tau = -\frac{\rho_0 \hat{\omega}^2 |\alpha|^{-2}}{2} \operatorname{Re} \int_{\Omega(t)} \hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}}) : \underline{\nabla}_{\underline{x}} \underline{w} \, d\Omega \, d\tau.$$

Using Green's formula we obtain

$$\frac{2|\alpha|^2}{\rho_0 \hat{\omega}^2} S = \operatorname{Re} \int_{\Omega(t)} \operatorname{div}_{\underline{x}}(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}})) \cdot \underline{w} \, d\Omega - \operatorname{Re} \int_{\Gamma(t)} \hat{p} \underline{w} \cdot \underline{H}_{\underline{x}}(\bar{\hat{p}}) \cdot \underline{n} \, d\Gamma.$$

Thanks to the second equality of Corollary (12) we have

$$\begin{aligned} \operatorname{Re} \int_{\Omega(t)} \underline{\operatorname{div}}_{\underline{x}}(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}})) \cdot \underline{w} \, d\Omega &= \frac{1}{2} \int_{\Omega(t)} (\underline{\nabla}_{\underline{x}} |\underline{\nabla}_{\underline{x}} \hat{p}|^2) \cdot \underline{w} \, d\Omega - \frac{\kappa |\alpha|^2}{2 \rho_0 \hat{\omega}^2} \int_{\Omega(t)} (\underline{\nabla}_{\underline{x}} |\hat{p}|^2) \cdot \underline{w} \, d\Omega \\ &\quad - \frac{\hat{\zeta} |\alpha|^2}{\rho_0 \hat{\omega}} \int_{\Omega(t)} \operatorname{Im}(\hat{p} \underline{\nabla}_{\underline{x}}(\bar{\hat{p}})) \cdot \underline{w} \, d\Omega. \end{aligned}$$

Using Green's formula again, we find

$$\begin{aligned} \operatorname{Re} \int_{\Omega(t)} \underline{\operatorname{div}}_{\underline{x}}(\hat{p} \underline{H}_{\underline{x}}(\bar{\hat{p}})) \cdot \underline{w} \, d\Omega &= \frac{1}{2} \int_{\Gamma(t)} |\underline{\nabla}_{\underline{x}} \hat{p}|^2 \underline{w} \cdot \underline{n} \, d\Gamma - \frac{\kappa |\alpha|^2}{2 \rho_0 \hat{\omega}^2} \int_{\Gamma(t)} |\hat{p}|^2 \underline{w} \cdot \underline{n} \, d\Gamma \\ &\quad - \frac{\hat{\zeta} |\alpha|^2}{\rho_0 \hat{\omega}} \int_{\Omega(t)} \operatorname{Im}(\hat{p} \underline{\nabla}_{\underline{x}}(\bar{\hat{p}})) \cdot \underline{w} \, d\Omega - \int_{\Omega(t)} \check{p} \operatorname{div}_{\underline{x}} \underline{w} \, d\Omega. \end{aligned}$$

where \check{p} is some scalar field that depends on \hat{p} . Collecting all the results above, we obtain

$$S = \int_{\Omega(t)} \underline{F}_{ARF} \cdot \underline{w} \, d\Omega + \int_{\Gamma_N(t)} \underline{w} \cdot \underline{\sigma}_{\underline{bndy}} \cdot \underline{n} \, d\Gamma - \int_{\Omega_0} \check{p} \underline{C}_{=0}^{-1} : \underline{e}_{=0}(\underline{w}) \, d\Omega_0$$

for some scalar field \check{p} . This means that there exists another scalar field $\check{p} = p + \check{p}$ such that the couple $(\underline{y}_s, \check{p})$ satisfies Eq. (20) with $s(t, \underline{w})$ and $r(t, q)$ as in the statement of the Proposition. \square

Appendix A. Proof of Proposition 1

As preliminary results, we provide the following propositions.

Proposition 14. *For any $\kappa_* > 2$, there exists $c \in \mathbb{R}$ such that for all $\kappa > \kappa_*$ we have*

$$W^{NI} \geq J^2 + c.$$

Proof. For the sake of simplicity, we set $k := \kappa/2$ in Eq. (4) and define $I := J^2$. Our objective is to prove the following inequality:

$$k(I - 1) - k \log I \geq I + c, \quad c \in \mathbb{R}. \quad (\text{A.1})$$

We define $\mathcal{L}(I) := (k - 1)I - k \log I$. Then, Eq. (A.1) reads

$$\mathcal{L}(I) \geq I + c. \quad (\text{A.2})$$

Note that, since $k > 1$, we have

$$\lim_{I \rightarrow 0} \mathcal{L}(I) = +\infty, \quad \lim_{I \rightarrow +\infty} \mathcal{L}(I) = +\infty,$$

and therefore, since $\mathcal{L}(I)$ is differentiable, the minimum I^* of the function $\mathcal{L}(I)$ can be found in the interval $(0, +\infty)$ by imposing

$$\mathcal{L}'(I^*) = 0 \implies (k - 1) - \frac{k}{I^*} = 0 \implies I^* = \frac{k}{k - 1}.$$

Now

$$\min_{I \in (0, +\infty)} \mathcal{L}(I) = \mathcal{L}(I^*) = k - k \log\left(\frac{k}{k - 1}\right),$$

thus Eq. (A.2) is satisfied if and only if

$$k \log\left(\frac{k}{k-1}\right) \leq -c. \quad (\text{A.3})$$

In addition, since $k > \kappa^*/2 > 1$, then we can show that there exists $c^* > 1$ such that

$$\lim_{k \rightarrow \kappa^*/2} k \log\left(\frac{k}{k-1}\right) = c^*.$$

Moreover, the left-hand side of Eq. (A.3) is monotone decreasing, and

$$\lim_{k \rightarrow +\infty} k \log\left(\frac{k}{k-1}\right) = 1.$$

Consequently, Eq. (A.3) is verified with $-c = c^*$, thus concluding the proof. \square

Proposition 15. *For any $\kappa_* > 2$, there exists $c \in \mathbb{R}$, depending on $\{\kappa_*, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ only, such that for all $\kappa > \kappa_*$,*

$$W^{\text{TI}} + \kappa_5 W^{\text{NI}} \geq \kappa_5 I_2 + c, \quad W^{\text{TI}} + \kappa_5 W^{\text{NI}} \geq \frac{\kappa_3 \kappa_4}{2} I_4^2 + c \quad \text{and} \quad W^{\text{TI}} + \kappa_5 W^{\text{NI}} \geq \kappa_3 \sqrt{\kappa_4} I_4 + c.$$

Proof. We first recall the definition of the potential W^{TI} . It reads

$$W^{\text{TI}} = \kappa_1 e^{\kappa_2(I_1-3)^2} + \kappa_3 e^{\kappa_4(I_4-1)^2} + \kappa_5(I_2 - 2 \log(J^2) - 3).$$

Now, let us prove the first inequality of the Proposition. We have, since $\log x \leq x/e$ and thanks to Proposition 14,

$$c + \kappa_5 I_2 \leq W^{\text{TI}} + 2 \kappa_5 \log(J^2) \leq W^{\text{TI}} + \frac{2}{e} \kappa_5 J^2 \Rightarrow c + \kappa_5 I_2 \leq W^{\text{TI}} + \kappa_5 W^{\text{NI}}.$$

Now, observing that $e^x \geq x$, and using the fact that $I_2 \geq 0$, we get

$$W^{\text{TI}} + \kappa_5 W^{\text{NI}} \geq \kappa_3 \kappa_4 (I_4^2 - 2I_4 + 1) - 3\kappa_5.$$

Using the inequality $2I_4 \leq I_4^2/2 + 2$, we obtain the second inequality of the proposition. Finally, using the property that $e^{c^* x^2} \geq x$ with $c^* = e^{-1}/2 < 1$, and using this inequality with $x = \sqrt{\kappa_3} |I_4 - 1| / \sqrt{c^*}$, we retrieve

$$W^{\text{TI}} + \kappa_5 W^{\text{NI}} \geq \kappa_3 \frac{\sqrt{\kappa_4}}{\sqrt{c^*}} |I_4 - 1| - 3\kappa_5.$$

The third inequality then follows straightforwardly. Note that I_1 could be estimated in a similar way, but such estimate is not used in what follows. \square

Note that, if we consider the stress tensors associated with the parametric family of solutions $\underline{y}_\varepsilon$:

$$W_\varepsilon^{\text{NI}} = \frac{\hat{\kappa} \varepsilon^{-2}}{2} \left((J_\varepsilon^2 - 1) - \log(J_\varepsilon^2) \right), \quad W_\varepsilon^{\text{TI}} = \kappa_1 e^{\kappa_2(I_{1,\varepsilon}-3)^2} + \kappa_3 e^{\kappa_4(I_{4,\varepsilon}-1)^2} + \kappa_5(I_{2,\varepsilon} - 2 \log(J_\varepsilon^2) - 3), \quad W_\varepsilon^{\text{VS}} = \frac{\zeta}{2} \text{tr} \left(\underline{\underline{C}}_\varepsilon^{-1} \cdot \underline{\underline{\dot{e}}}_\varepsilon \right)^2,$$

then we can state that, assuming $\varepsilon > 0$ small enough, that there exists $c \in \mathbb{R}$ depending on $\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ such that we have

$$W_\varepsilon^{\text{NI}} \geq J_\varepsilon^2 + c.$$

and

$$W_\varepsilon^{\text{TI}} + \kappa_5 W_\varepsilon^{\text{NI}} \geq \kappa_5 I_{2,\varepsilon} + c, \quad W_\varepsilon^{\text{TI}} + \kappa_5 W_\varepsilon^{\text{NI}} \geq \frac{\kappa_3 \kappa_4}{2} I_{4,\varepsilon}^2 + c \quad \text{and} \quad W_\varepsilon^{\text{TI}} + \kappa_5 W_\varepsilon^{\text{NI}} \geq \kappa_3 \sqrt{\kappa_4} I_{4,\varepsilon} + c.$$

Now observe that, because of the regularity assumption on the domain Ω_0 and g , one can assume that there exists $\ell \in C^0([0, T]; W^{1,\infty}(\Omega_0))$ such that

$$\forall t \in [0, T] \quad \ell(\cdot, t)|_{\Gamma_0} = g(\cdot, t). \quad (\text{A.4})$$

Then, as a preliminary estimation, we derive an estimate on $\mathcal{P}_\varepsilon^{ext}$, that we rewrite as a volume integral. First, let us take into account Eq. (A.4), and rewrite $\mathcal{P}_\varepsilon^{ext}$ as

$$\mathcal{P}_\varepsilon^{ext} = \int_{\Gamma_0} \cos(\hat{\omega} t/\varepsilon) \ell J_\varepsilon \left| \underline{F}_\varepsilon^{-T} \cdot \underline{n}_0 \right| \partial_{t\underline{y}} \cdot \underline{F}_\varepsilon^{-T} \cdot \underline{n}_0 \, d\Gamma_0. \quad (\text{A.5})$$

We now consider Eq. (A.5) in the deformed domain $\Omega(t)$, with the usual convention of implicit composition with the deformation map $\underline{\phi} : \underline{\Omega}_0 \rightarrow \Omega_\varepsilon(t) \subset \mathbb{R}^3$ with $\underline{\phi}(\underline{\xi}, t) = \underline{y}_\varepsilon(\underline{\xi}, t) + \underline{\xi}$. Applying the divergence theorem we have

$$\int_{\partial\Omega_\varepsilon} \ell \partial_{t\underline{y}} \cdot \underline{n} \, d\Gamma = \int_{\Omega_\varepsilon(t)} \underline{\nabla}_x \cdot \partial_{t\underline{y}} \ell \, d\Omega + \int_{\Omega_\varepsilon(t)} \partial_{t\underline{y}} \cdot \underline{\nabla}_x \ell \, d\Omega. \quad (\text{A.6})$$

Eq. (A.6) can be rewritten in Lagrangian formulation, leading to

$$\int_{\Gamma_0} \ell J_\varepsilon \left| \underline{F}_\varepsilon^{-T} \cdot \underline{n}_0 \right| \underline{F}_\varepsilon^{-T} \cdot \underline{n}_0 \cdot \partial_{t\underline{y}} \, d\Gamma_0 = \int_{\Omega_0} J_\varepsilon \operatorname{tr} \left(\underline{F}_\varepsilon^{-1} \cdot \underline{\nabla} \partial_{t\underline{y}} \right) \ell \, d\Omega_0 + \int_{\Omega_0} J_\varepsilon \underline{F}_\varepsilon^{-T} \cdot \underline{\nabla} \ell \cdot \partial_{t\underline{y}} \, d\Omega_0. \quad (\text{A.7})$$

If we multiply Eq. (A.7) by $\cos(\hat{\omega} t/\varepsilon)$, we obtain

$$\mathcal{P}_\varepsilon^{ext} = \int_{\Omega_0} \cos(\hat{\omega} t/\varepsilon) J_\varepsilon \operatorname{tr} \left(\underline{F}_\varepsilon^{-1} \cdot \underline{\nabla} \partial_{t\underline{y}} \right) \ell \, d\Omega_0 + \int_{\Omega_0} \cos(\hat{\omega} t/\varepsilon) J_\varepsilon \underline{F}_\varepsilon^{-T} \cdot \underline{\nabla} \ell \cdot \partial_{t\underline{y}} \, d\Omega_0. \quad (\text{A.8})$$

We also need another expression of the source term $\mathcal{P}_\varepsilon^A$. We can rewrite, by definition of the tensor $\underline{\dot{\varepsilon}}_\varepsilon$,

$$\mathcal{P}_\varepsilon^A = - \int_{\Omega_0} \sigma_a \underline{\tau}_1 \cdot \underline{\dot{\varepsilon}}_\varepsilon \cdot \underline{\tau}_1 \, d\Omega_0 = - \frac{1}{2} \int_{\Omega_0} \sigma_a \underline{\tau}_1 \cdot \partial_t \left(\underline{F}_\varepsilon^T \underline{F}_\varepsilon \right) \cdot \underline{\tau}_1 \, d\Omega_0.$$

Hence, by integrating over time, we find

$$\int_0^t \mathcal{P}_\varepsilon^A(s) \, ds = \frac{1}{2} \int_0^t \int_{\Omega_0} \partial_t \sigma_a(\underline{\xi}, s) I_{4,\varepsilon}(\underline{\xi}, s) \, d\Omega_0 \, ds - \frac{1}{2} \int_{\Omega_0} \sigma_a(\underline{\xi}, t) I_{4,\varepsilon}(\underline{\xi}, t) \, d\Omega_0. \quad (\text{A.9})$$

where $I_{4,\varepsilon} = \underline{\tau}_1 \cdot \left(\underline{F}_\varepsilon^T \cdot \underline{F}_\varepsilon \right) \cdot \underline{\tau}_1$. We are now able to prove Proposition 1.

Proof. We want to prove that the total internal energy $\mathcal{E}_\varepsilon^{tot}$ is bounded at every time $t \in [0, T]$, namely there exists a finite constant $C > 0$ independent of ε such that

$$\mathcal{E}_\varepsilon^{tot}(t) \leq C \quad \forall t \in [0, T].$$

Note that the value of C will change along every line of the proof, for the sake of simplicity. In order to retrieve this estimate, we need to control the source term contributions. First, let us integrate with respect to time equation (12).

We obtain

$$\tilde{\mathcal{E}}_\varepsilon^{tot}(t) = \int_0^t \mathcal{P}_\varepsilon^{ext}(s) \, ds + \int_0^t \mathcal{P}_\varepsilon^A(s) \, ds, \quad \tilde{\mathcal{E}}_\varepsilon^{tot}(t) := \mathcal{E}_\varepsilon^{tot}(t) + \int_0^t \mathcal{E}_\varepsilon^{VS}(s) \, ds. \quad (\text{A.10})$$

We now find an estimate for the term $\mathcal{P}_\varepsilon^{ext}$. Applying the Cauchy-Schwarz inequality on Eq. (A.8), we obtain

$$|\mathcal{P}_\varepsilon^{ext}| \leq C \|J_\varepsilon\|_{L^2(\Omega_0)} \left\| \operatorname{tr} \left(\underline{F}_\varepsilon^{-1} \cdot \underline{\nabla} \partial_{t\underline{y}} \right) \right\|_{L^2(\Omega_0)} + C \left\| \operatorname{adj} \underline{F}_\varepsilon \right\|_{L^2(\Omega_0)} \left\| \partial_{t\underline{y}} \right\|_{L^2(\Omega_0)},$$

since $\operatorname{adj} \underline{F}_\varepsilon = J_\varepsilon \underline{F}_\varepsilon^{-1}$, $|\cos(\hat{\omega} t/\varepsilon)| \leq 1$ and where C depends on ℓ only here. Furthermore, from the Young inequality,

$$|\mathcal{P}_\varepsilon^{ext}| \leq C \left(\|J_\varepsilon\|_{L^2(\Omega_0)}^2 + \left\| \operatorname{tr} \left(\underline{F}_\varepsilon^{-1} \cdot \underline{\nabla} \partial_{t\underline{y}} \right) \right\|_{L^2(\Omega_0)}^2 + \left\| \operatorname{adj} \underline{F}_\varepsilon \right\|_{L^2(\Omega_0)}^2 + \left\| \partial_{t\underline{y}} \right\|_{L^2(\Omega_0)}^2 \right).$$

We now estimate each term using the energy. From Proposition 14, we can state that

$$\|J_\varepsilon\|_{L^2(\Omega_0)}^2 - C \leq \mathcal{E}_\varepsilon^{\text{NI}} \leq \mathcal{E}_\varepsilon^{\text{tot}}(t).$$

and from Proposition 15 (first inequality) we have

$$\|\text{adj } \underline{\underline{F}}_\varepsilon\|_{L^2(\Omega_0)}^2 - C = \|I_{2,\varepsilon}\|_{L^1(\Omega_0)} - C \leq \frac{1}{\kappa_5} \mathcal{E}_\varepsilon^{\text{PI}} + \mathcal{E}_\varepsilon^{\text{NI}} \leq \left(\frac{1}{\kappa_5} + 1\right) \mathcal{E}_\varepsilon^{\text{tot}}(t), \quad \text{and} \quad \rho_0 \|\partial_t \underline{y}_\varepsilon\|_{L^2(\Omega_0)}^2 \leq \mathcal{K}_\varepsilon \leq \mathcal{E}_\varepsilon^{\text{tot}}(t).$$

Finally, we have

$$\int_0^t \left\| \text{tr} \left(\underline{\underline{F}}_\varepsilon^{-1} \cdot \underline{\underline{\nabla}} \partial_t \underline{y}_\varepsilon \right) \right\|_{L^2(\Omega_0)}^2 ds = \frac{\varepsilon}{\tilde{\zeta}} \int_0^t \mathcal{E}_\varepsilon^{\text{VS}}(s) ds.$$

Collecting the results above, we find

$$\int_0^t \mathcal{P}_\varepsilon^{\text{ext}}(s) ds \leq C \left(1 + \varepsilon \int_0^t \mathcal{E}_\varepsilon^{\text{VS}}(s) ds + \int_0^t \mathcal{E}_\varepsilon^{\text{tot}}(s) ds \right). \quad (\text{A.11})$$

We now find an estimate for the term $\mathcal{P}_\varepsilon^{\text{A}}$. From Eq. (A.9) we have

$$\int_0^t \mathcal{P}_\varepsilon^{\text{A}}(s) ds \leq \frac{1}{2} \|\partial_t \sigma_a\|_{L^\infty((0,T) \times \Omega)} \int_0^t \|I_{4,\varepsilon}(\cdot, s)\|_{L^1(\Omega_0)} ds + \frac{1}{2} \|\sigma_a\|_{L^\infty((0,T); L^2(\Omega_0))} \|I_{4,\varepsilon}(\cdot, t)\|_{L^2(\Omega_0)}$$

Therefore, using Proposition 15 (second and third inequalities), we get

$$\int_0^t \mathcal{P}_\varepsilon^{\text{A}}(s) ds \leq C \left(\sqrt{\mathcal{E}_\varepsilon^{\text{tot}}(t)} + \int_0^t \mathcal{E}_\varepsilon^{\text{tot}}(s) ds \right) \quad (\text{A.12})$$

Injecting Eq. (A.11) and Eq. (A.12) into (A.10) we obtain the energy estimate

$$\mathcal{E}_\varepsilon^{\text{tot}}(t) + (1 - C\varepsilon) \int_0^t \mathcal{E}_\varepsilon^{\text{VS}}(s) ds \leq C \left(1 + \sqrt{\mathcal{E}_\varepsilon^{\text{tot}}(t)} + \int_0^t \mathcal{E}_\varepsilon^{\text{tot}}(s) ds \right).$$

Then, using the Young inequality and assuming ε small enough, one can show that

$$\tilde{\mathcal{E}}_\varepsilon^{\text{tot}}(t) = \mathcal{E}_\varepsilon^{\text{tot}}(t) + \int_0^t \mathcal{E}_\varepsilon^{\text{VS}}(s) ds \leq C \left(1 + \int_0^t \tilde{\mathcal{E}}_\varepsilon^{\text{tot}}(s) ds \right),$$

thus, by the Grönwall inequality in integral form, we can state that

$$\tilde{\mathcal{E}}_\varepsilon^{\text{tot}}(t) \leq C \exp \left(\int_0^t C ds \right).$$

As a consequence, $\mathcal{E}_\varepsilon^{\text{tot}}(t) \leq \tilde{\mathcal{E}}_\varepsilon^{\text{tot}}(t) \leq C$, thus concluding the proof. \square

Appendix B. Full expression of a linearised stress tensor term

We define the transversely isotropic potential associated to the displacement \underline{y}_0 as

$$W_0^{\text{PI}} = \kappa_1 e^{\kappa_2 (I_{1,0} - 3)^2} + \kappa_3 e^{\kappa_4 (I_{4,0} - 1)^2} + \kappa_5 (I_{2,0} - 2 \log(J_0^2) - 3),$$

with

$$I_{1,0} = 3 + 2 \text{tr} \left(\underline{\underline{e}}_{\underline{\underline{0}}} \right), \quad I_{4,0} = 1 + 2 \underline{\underline{\tau}}_1 \cdot \underline{\underline{e}}_{\underline{\underline{0}}} \cdot \underline{\underline{\tau}}_1, \quad I_{2,0} = |\text{adj } \underline{\underline{F}}_{\underline{\underline{0}}}|^2 = 3 + 4 \text{tr} \left(\underline{\underline{e}}_{\underline{\underline{0}}} \right) + 2 \text{tr}^2 \left(\underline{\underline{e}}_{\underline{\underline{0}}} \right) - 2 \text{tr} \left(\underline{\underline{e}}_{\underline{\underline{0}}}^2 \right).$$

The corresponding stress tensor is given by

$$\underline{\underline{\Sigma}}_0^{\text{TI}} = \frac{\partial W_0^{\text{TI}}}{\partial \underline{\underline{e}}_0} = K_{1,0} (I_{1,0} - 3) \underline{\underline{1}} + K_{4,0} (I_{4,0} - 1) \underline{\underline{\tau}}_1 \otimes \underline{\underline{\tau}}_1 + 2 \kappa_5 (I_{1,0} \underline{\underline{1}} - \underline{\underline{C}}_0^{-1} - 2 \underline{\underline{C}}_0^{-1}),$$

with

$$K_{1,0} := 4\kappa_1\kappa_2 e^{\kappa_2(I_{1,0}-3)^2}, \quad K_{4,0} := 4\kappa_3\kappa_4 e^{\kappa_4(I_{4,0}-1)^2}.$$

One can observe that $\underline{\underline{\Sigma}}_0^{\text{TI}}$ vanishes if $\underline{\underline{e}}_0 = \underline{\underline{0}}$. Furthermore, the full definition of the tensor $\underline{\underline{\mathbf{A}}}_0$ is given, for any test function $\underline{\underline{w}}$, by

$$\underline{\underline{\mathbf{A}}}_0 : \underline{\underline{e}}_0(\underline{\underline{w}}) = \frac{\partial \underline{\underline{\Sigma}}_0^{\text{TI}}}{\partial \underline{\underline{e}}_0} : \underline{\underline{e}}_0(\underline{\underline{w}}) = Q_{1,0} \text{tr}(\underline{\underline{e}}_0(\underline{\underline{w}})) \underline{\underline{1}} + Q_{4,0} (\underline{\underline{\tau}}_1 \cdot \underline{\underline{e}}_0(\underline{\underline{w}}) \cdot \underline{\underline{\tau}}_1) \underline{\underline{\tau}}_1 \otimes \underline{\underline{\tau}}_1 + 4 \kappa_5 (2 \underline{\underline{C}}_0^{-1} \underline{\underline{e}}_0(\underline{\underline{w}}) \underline{\underline{C}}_0^{-1} - \underline{\underline{e}}_0(\underline{\underline{w}}))$$

with

$$Q_{1,0} = 2 K_{1,0} (1 + 2 \kappa_2 (I_{1,0} - 3)^2) + 4 \kappa_5, \quad Q_{4,0} = 2 K_{4,0} (1 + 2 \kappa_2 (I_{4,0} - 1)^2).$$

If $\underline{\underline{e}}_0 = \underline{\underline{0}}$, then

$$\underline{\underline{\mathbf{A}}}_0 : \underline{\underline{e}}_0(\underline{\underline{w}}) = \underline{\underline{\mathbf{A}}}_0 : \underline{\underline{\varepsilon}}(\underline{\underline{w}}) = (8 \kappa_1 \kappa_2 + 4 \kappa_5) \text{tr}(\underline{\underline{\varepsilon}}(\underline{\underline{w}})) \underline{\underline{1}} + 8 \kappa_3 \kappa_4 (\underline{\underline{\tau}}_1 \cdot \underline{\underline{\varepsilon}}(\underline{\underline{w}}) \cdot \underline{\underline{\tau}}_1) \underline{\underline{\tau}}_1 \otimes \underline{\underline{\tau}}_1 + 4 \kappa_5 \underline{\underline{\varepsilon}}(\underline{\underline{w}}),$$

with $\underline{\underline{\varepsilon}}(\underline{\underline{w}}) = \underline{\underline{\nabla}} \underline{\underline{w}} + (\underline{\underline{\nabla}} \underline{\underline{w}})^T$. The expression above shows that $\underline{\underline{\mathbf{A}}}_0$ is a positive fourth-order tensor for small values of $\underline{\underline{e}}_0$. Indeed, for all $\underline{\underline{\varepsilon}} \in \mathcal{L}_s(\mathbb{R}^d)$, we can show that there exists a positive constant α such that

$$\left(\underline{\underline{\mathbf{A}}}_0 : \underline{\underline{\varepsilon}} \right) : \underline{\underline{\varepsilon}} = (8 \kappa_1 \kappa_2 + 4 \kappa_5) \text{tr}(\underline{\underline{\varepsilon}})^2 + 8 \kappa_3 \kappa_4 (\underline{\underline{\tau}}_1 \cdot \underline{\underline{\varepsilon}} \cdot \underline{\underline{\tau}}_1)^2 + 4 \kappa_5 \underline{\underline{\varepsilon}} : \underline{\underline{\varepsilon}} > \alpha |\underline{\underline{\varepsilon}}|^2.$$

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