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Duploid situations in concurrent games

Pierre Clairambault¹ Guillaume Munch-Maccagnoni²

¹: Univ Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP, Lyon, France

²: Team Gallinette, Inria Bretagne, Univ Nantes, LINA, Nantes, France

Impossibility results in denotational semantics

- Classical logic ($\text{CCC} + 0^{0^-} \cong \text{Id}$)
- Untyped λ -calculus with sums ($\text{Bi-CCC} + U^U \cong U$)

\Rightarrow Preorders

Adjunction models (Call-by-push-value)

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Deductive systems (\mathcal{D})

- $A, B \dots \in |\mathcal{D}|, \quad f, g \dots \in \mathcal{D}(A, B)$
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Associativity of composition vs. evaluation order

- $\exists N, \exists f, f \circ^N \perp \neq \perp$ (lazy evaluation)
- $\exists P, \forall g, g \circ^P \perp = \perp$ (strict evaluation)
- $f \circ^N (g \circ^P \perp) \neq \perp$
- $(f \circ^N g) \circ^P \perp = \perp$

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Deductive systems from adjunctions

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Duploids: a characterisation

In \mathcal{D} a deductive system

- f is linear $\iff \forall g, h : (f \circ g) \circ h = f \circ (g \circ h)$
- h is thinkable $\iff \forall f, g : (f \circ g) \circ h = f \circ (g \circ h)$
- A is positive $\iff \forall B, \forall f \in \mathcal{D}(A, B), f$ is linear
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Definition

A **duploid** is a deductive system where:

1. every object has a polarity;
2. for every object, there is a negative object to which it is linearly isomorphic; and
3. for every object, there is a positive object to which it is thinkably isomorphic.

Duploids: a characterisation

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$$\begin{cases} \delta_P \in \mathcal{D}(P, \uparrow P), \delta_P^* \in \mathcal{D}(\uparrow P, P) \\ \delta_P^* \circ (\delta_P \bullet f) = f \\ \delta_P \bullet \delta_P^* = \text{id}_{\uparrow P} \end{cases}$$

$$\begin{cases} \omega_N \in \mathcal{D}(N, \Downarrow N), \omega_N^* \in \mathcal{D}(\Downarrow N, N) \\ (f \circ \omega_N^*) \bullet \omega_N = f \\ \omega_N \circ \omega_N^* = \text{id}_{\Downarrow N} \end{cases}$$

Duploid Situations

Theorem

$$\text{Dupl} \sim \text{Adj}_{eq} \triangleleft \text{Adj}$$

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How can we spot duploids *in vivo*?

1. $\Downarrow \vdash \Uparrow : \mathcal{P} \rightarrow \mathcal{N}$
2. η is pointwise a split mono: $\rho_N \circ \eta_N = \text{id}_N$
3. ε is pointwise a split epi: $\varepsilon_P \bullet \theta_P = \text{id}_P$
4. ρ (*run*) and θ (*think*), as non-natural families, must satisfy:
 - 4.1 $\rho_{\Uparrow} = \Uparrow \varepsilon$ and $\theta_{\Downarrow} = \Downarrow \eta$
 - 4.2 $\rho \circ \rho_{\Uparrow \Downarrow} = \rho \circ \Uparrow \Downarrow \rho$ and $\theta_{\Downarrow \Uparrow} \bullet \theta = \Downarrow \Uparrow \theta \bullet \theta$

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Definition

- $f \in \mathcal{N}(N, M)$ is *linear* if $\rho_M \circ \Uparrow \Downarrow f = f \circ \rho_N$
- $f \in \mathcal{P}(P, Q)$ is *thunkable* if $\theta_Q \bullet f = \Downarrow \Uparrow f \bullet \theta_P$

Proposition: $\Uparrow \vdash_{(\theta, \rho)} \Downarrow : \mathcal{N}_l \rightarrow \mathcal{P}_t$

Duploid Situations

Theorem

Any duploid situation conservatively extends into a duploid satisfying:

$$\begin{array}{lll}
 \Downarrow f = \omega \circ f \circ \omega^* & , & \eta = \delta \bullet \omega & , & \rho = \omega^* \bullet \delta^* \\
 \Uparrow f = \delta \bullet f \bullet \delta^* & , & \varepsilon = \delta^* \circ \omega^* & , & \theta = \omega \circ \delta
 \end{array}$$

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Consider the collage of the adjunction $\Downarrow \vdash \Uparrow : \mathcal{P} \rightarrow \mathcal{N}$:

$$\begin{aligned} \mathcal{D}(P, Q) &\stackrel{\text{def}}{=} \mathcal{P}(P, Q) \\ \mathcal{D}(N, M) &\stackrel{\text{def}}{=} \mathcal{N}(N, M) \\ \mathcal{D}(N, P) &\stackrel{\text{def}}{=} \mathcal{P}(\Downarrow N, P) \cong \mathcal{N}(N, \Uparrow P) \end{aligned}$$

and extend it as follows

$$\mathcal{D}(P, N) \stackrel{\text{def}}{=} \mathcal{P}_t(P, \Downarrow N) \cong \mathcal{N}_l(\Uparrow P, N)$$

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Conjecture: $\mathcal{Dupl} \sim \mathcal{DS} \simeq \mathcal{Adj}_{\text{eq}} \triangleleft \mathcal{Adj}$

CBN: alternating negative strategies between negative arenas

A **term**:

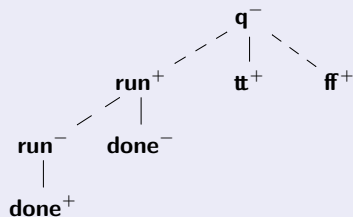
$$\lambda f^{\mathbf{com} \rightarrow \mathbf{com}}. \mathbf{newref} \ r \ \mathbf{in} \ f \ (r := \mathbf{tt}); !r \quad : \quad (\mathbf{com} \rightarrow \mathbf{com}) \rightarrow \mathbb{B}$$

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An **arena**:

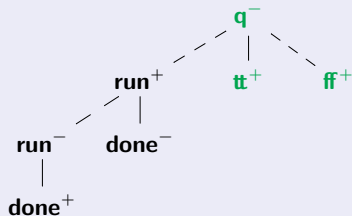


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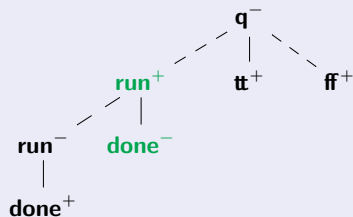


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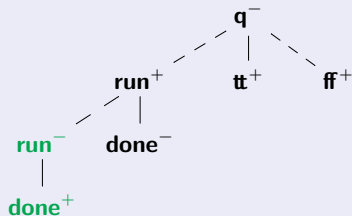


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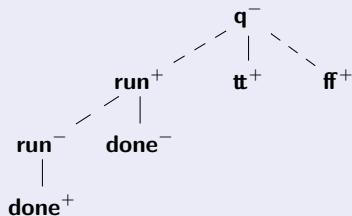


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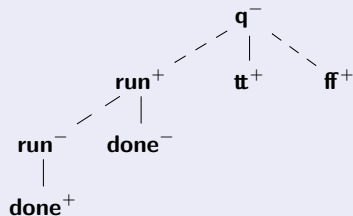
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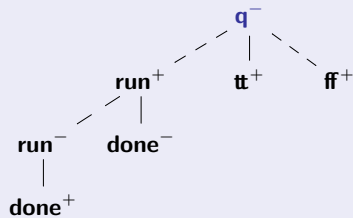
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A **play**:

$$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

q^-

An **arena**:



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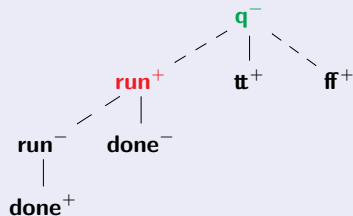
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$$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

q^-

run^+

An **arena**:



CBN: alternating negative strategies between negative arenas

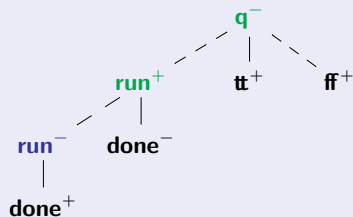
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 run^-

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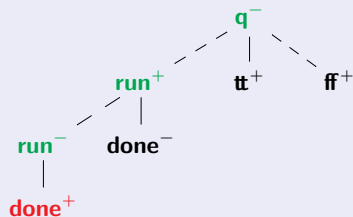
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 run^+
 run^-
 done^+

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CBN: alternating negative strategies between negative arenas

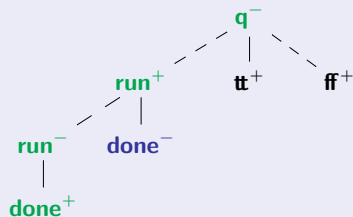
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 q^-
 run^+
 run^-
 done^+
 done^-

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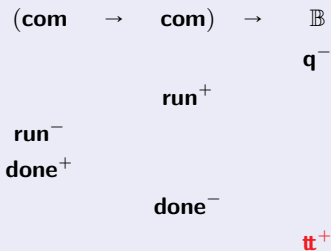


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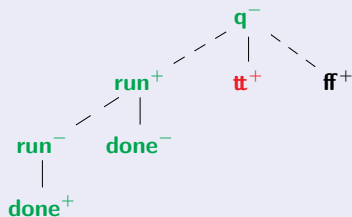
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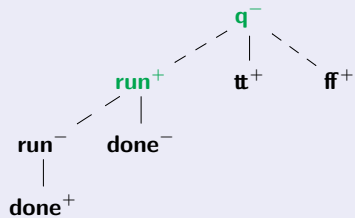
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A **play**:

$$\begin{array}{ccc}
 (\text{com} & \rightarrow & \text{com}) & \rightarrow & \mathbb{B} \\
 & & & & \mathbf{q}^- \\
 & & & & \text{run}^+
 \end{array}$$

An **arena**:



CBN: alternating negative strategies between negative arenas

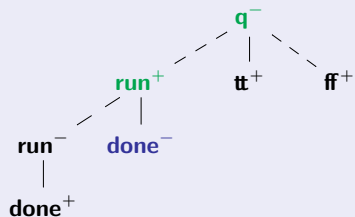
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$
 q^-
 run^+
 done^-

An **arena**:



CBN: alternating negative strategies between negative arenas

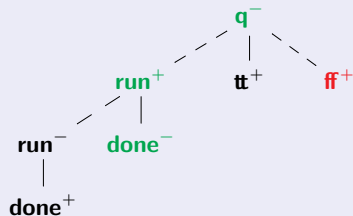
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$
 q^-
 run^+
 done^-
 ff^+

An **arena**:



CBN: alternating negative strategies between negative arenas

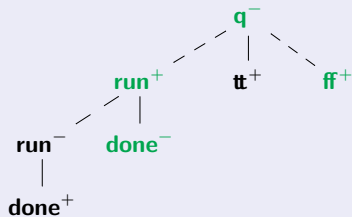
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

(com → com) →	→	ℬ
		q ⁻
run ⁺		
done ⁻		
		ff ⁺

An **arena**:



CBN: alternating negative strategies between negative arenas

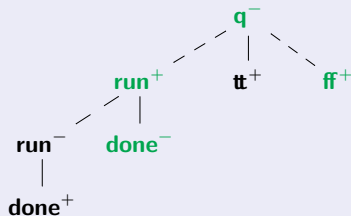
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

(com → com) →	→	ℬ
		q ⁻
run ⁺		
done ⁻		
		ff ⁺

An **arena**:



Alternating plays: alternating, linear orderings of a prefix of the arena, compatible with the order of the arena.

CBN: alternating negative strategies between negative arenas

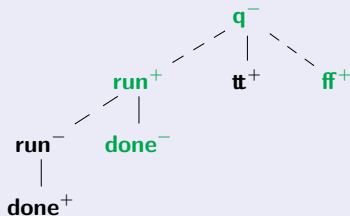
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$\begin{array}{ccc}
 (\text{com} \rightarrow \text{com}) & \rightarrow & \mathbb{B} \\
 & & \mathbf{q}^- \\
 & & \text{run}^+ \\
 & & \text{done}^- \\
 & & \mathbf{ff}^+
 \end{array}$$

An **arena**:



Alternating plays: alternating, linear orderings of a prefix of the arena, compatible with the order of the arena.

Negative, alternating strategy: certain sets of alternating plays.

CBV: alternating negative strategies between positive arenas

A **term**:

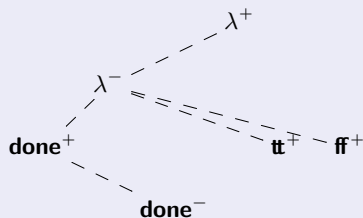
$$\lambda f^{\mathbf{com} \rightarrow \mathbf{com}}. \mathbf{newref} \ r \ \mathbf{in} \ f \ (r := \mathbf{tt}); !r \quad : \quad (\mathbf{com} \rightarrow \mathbf{com}) \rightarrow \mathbb{B}$$

CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

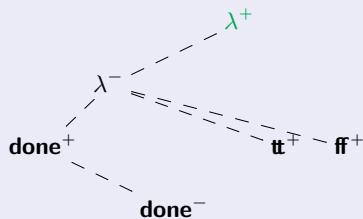


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

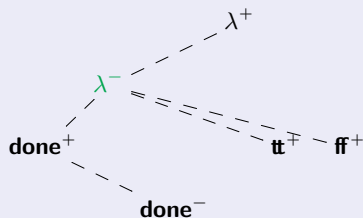


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

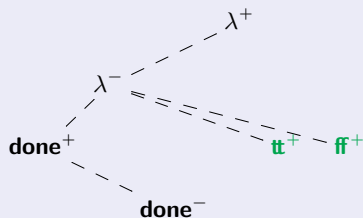


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

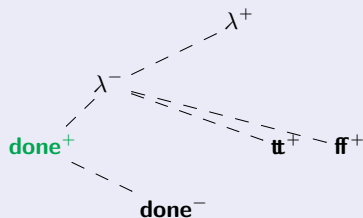


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

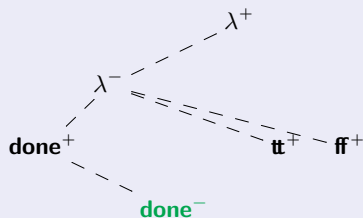


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:



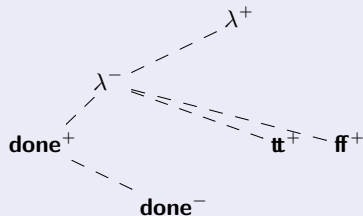
CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$\vdash (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

CBV: alternating negative strategies between positive arenas

A **term**:

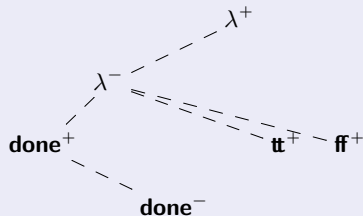
$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$\vdash (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

•⁻

An **arena**:



CBV: alternating negative strategies between positive arenas

A **term**:

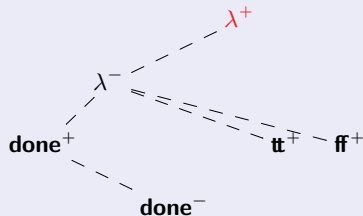
$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$\vdash (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

•⁻ λ⁺

An **arena**:

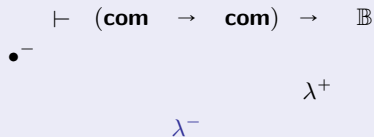


CBV: alternating negative strategies between positive arenas

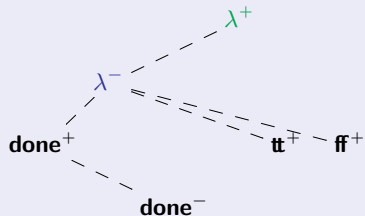
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

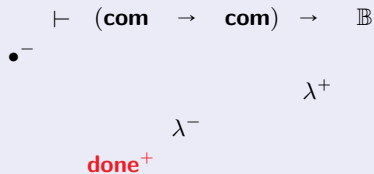


CBV: alternating negative strategies between positive arenas

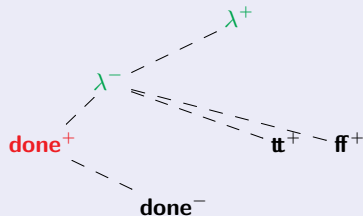
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

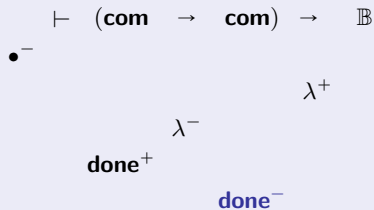


CBV: alternating negative strategies between positive arenas

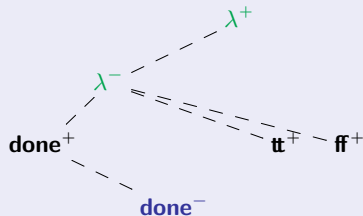
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

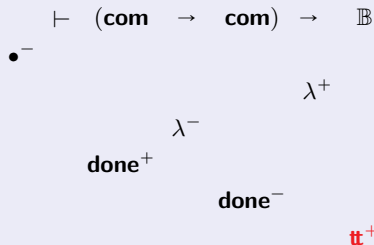


CBV: alternating negative strategies between positive arenas

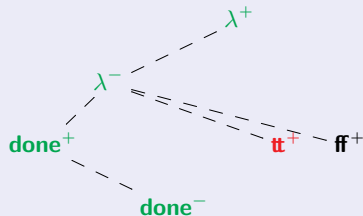
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

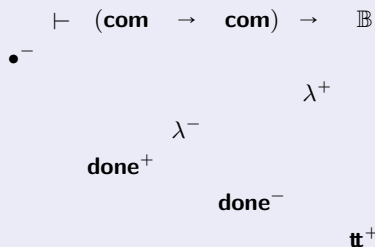


CBV: alternating negative strategies between positive arenas

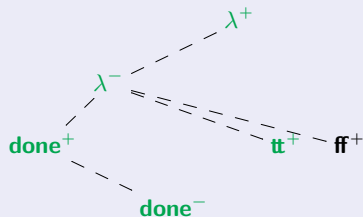
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

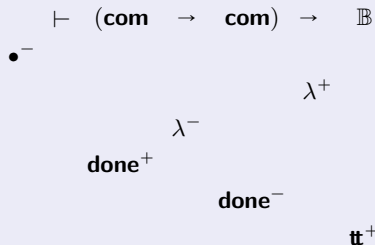


CBV: alternating negative strategies between positive arenas

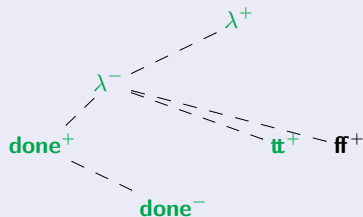
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:



Negative, alternating strategies between **positive arenas**.

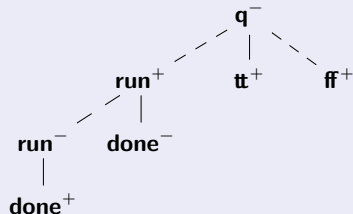
CBN, non-alternating

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

An **arena**:

CBN, non-alternating

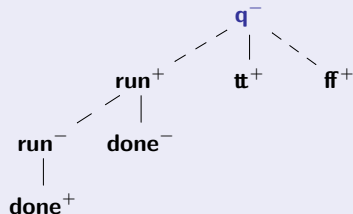
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

\mathbf{q}^-

An **arena**:

CBN, non-alternating

A **term**:

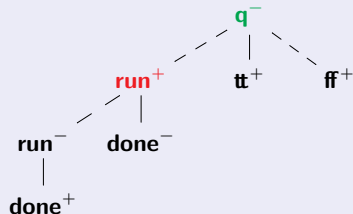
$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:

$$(\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

run^+

\mathbf{q}^-

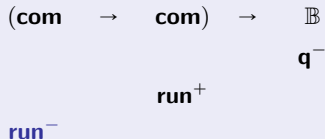
An **arena**:

CBN, non-alternating

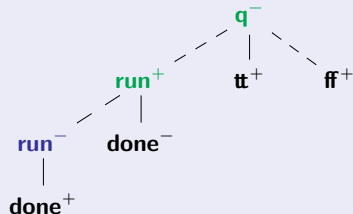
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

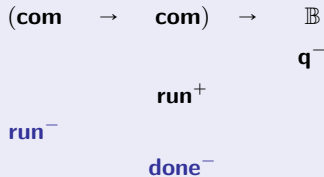


CBN, non-alternating

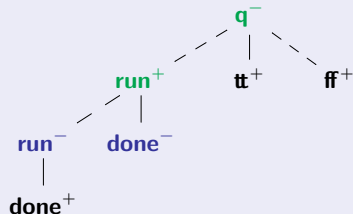
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



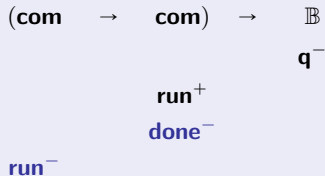
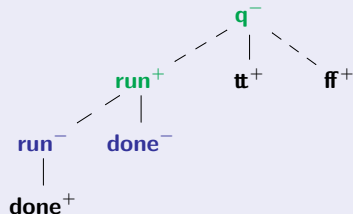
An **arena**:



CBN, non-alternating

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

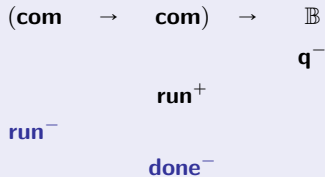
A **play**:An **arena**:

CBN, non-alternating

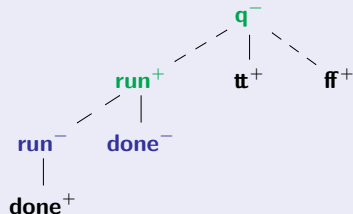
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

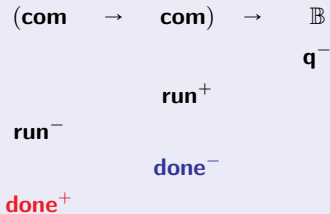


CBN, non-alternating

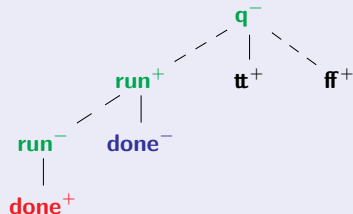
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

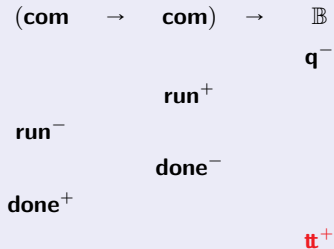


CBN, non-alternating

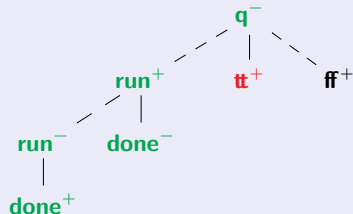
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

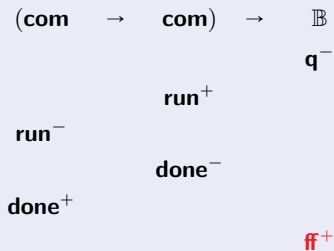


CBN, non-alternating

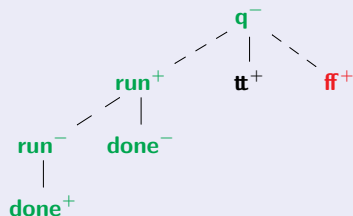
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

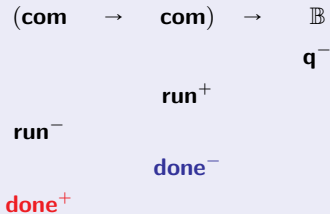


CBN, non-alternating

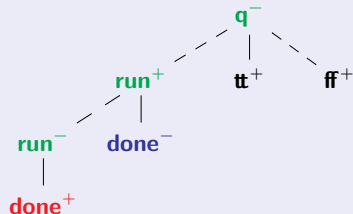
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



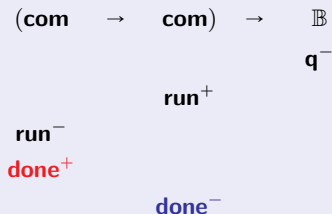
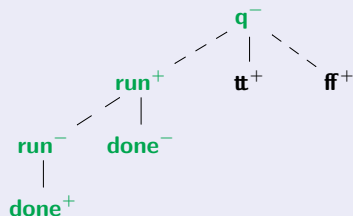
An **arena**:



CBN, non-alternating

A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \text{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

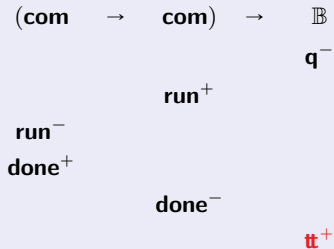
A **play**:An **arena**:

CBN, non-alternating

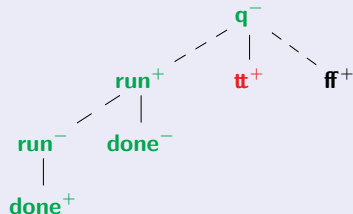
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:

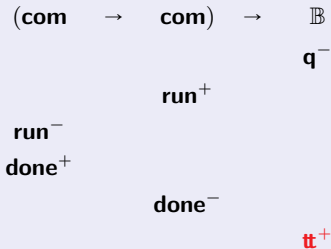


CBN, non-alternating

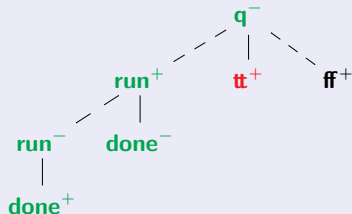
A **term**:

$$\lambda f^{\text{com} \rightarrow \text{com}}. \text{newref } r \text{ in } f(r := \mathbf{tt}); !r \quad : \quad (\text{com} \rightarrow \text{com}) \rightarrow \mathbb{B}$$

A **play**:



An **arena**:



Negative, non-alternating strategies: certain sets of non-alternating plays.

Duploids situations in toy categories of games

The **alternating world**:

 $\mathcal{G}_{\text{alt}}^-$

neg. arenas

neg. strat

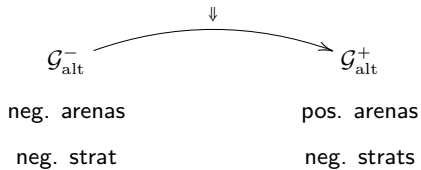
 $\mathcal{G}_{\text{alt}}^+$

pos. arenas

neg. strats

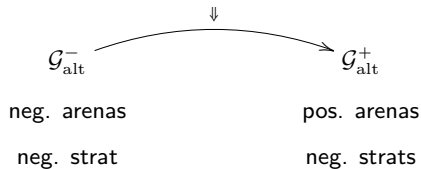
Duploids situations in toy categories of games

The **alternating world**:

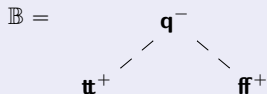


Duploids situations in toy categories of games

The **alternating world**:

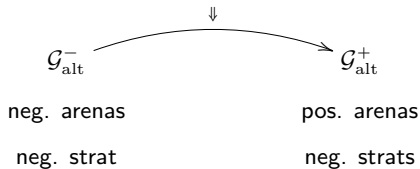


Down-shift on arenas

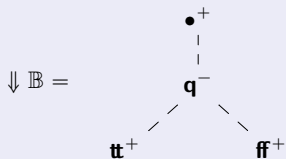


Duploids situations in toy categories of games

The **alternating world**:

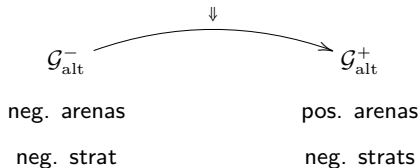


Down-shift on arenas

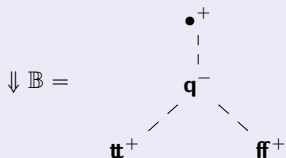


Duploids situations in toy categories of games

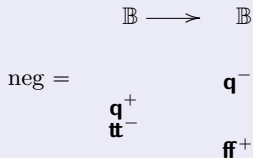
The **alternating world**:



Down-shift on arenas

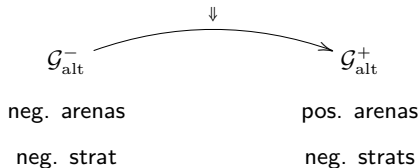


Down-shift on strategies

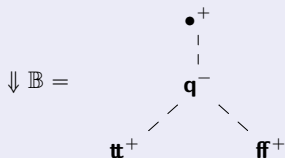


Duploids situations in toy categories of games

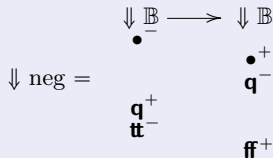
The **alternating world**:



Down-shift on arenas

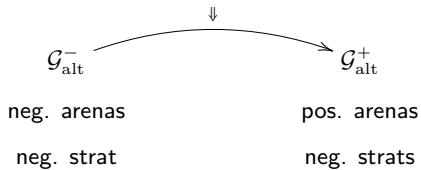


Down-shift on strategies



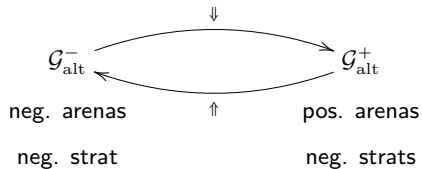
Duploids situations in toy categories of games

The **alternating world**:



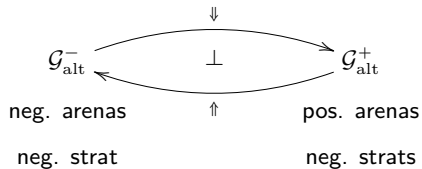
Duploids situations in toy categories of games

The **alternating world**:



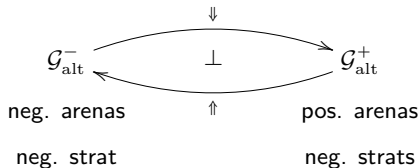
Duploids situations in toy categories of games

The **alternating world**:



Duploids situations in toy categories of games

The **alternating world**:



Unit in $\mathcal{G}_{\text{alt}}^-$ (natural)

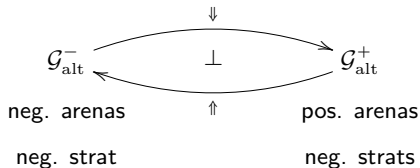
$$\begin{array}{ccc}
 \mathbb{B} & \longrightarrow & \Uparrow \Downarrow \mathbb{B} \\
 & & \bullet^- \\
 & & \bullet^+ \\
 \eta_{\mathbb{B}} = & & \mathbf{q}^- \\
 & & \mathbf{q}^+ \\
 & & \mathbf{tt}^- \\
 & & \mathbf{tt}^+
 \end{array}$$

Co-unit in $\mathcal{G}_{\text{alt}}^+$ (natural)

$$\begin{array}{ccc}
 \Downarrow \Uparrow \mathbb{B}^\perp & \longrightarrow & \mathbb{B}^\perp \\
 & & \bullet^- \\
 & & \bullet^+ \\
 \epsilon_{\mathbb{B}^\perp} = & & \mathbf{q}^- \\
 & & \mathbf{q}^+ \\
 & & \mathbf{tt}^- \\
 & & \mathbf{tt}^+
 \end{array}$$

Duploids situations in toy categories of games

The **alternating world**:



Unit in $\mathcal{G}_{\text{alt}}^-$ (natural)

$$\begin{array}{ccc}
 \mathbb{B} & \longrightarrow & \uparrow \downarrow \mathbb{B} \\
 & & \bullet^- \\
 & & \bullet^+ \\
 \eta_{\mathbb{B}} = & & \mathbf{q}^- \\
 & & \mathbf{q}^+ \\
 & & \mathbf{tt}^- \\
 & & \mathbf{tt}^+
 \end{array}$$

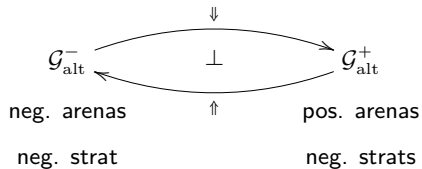
Co-unit in $\mathcal{G}_{\text{alt}}^+$ (natural)

$$\begin{array}{ccc}
 \downarrow \uparrow \mathbb{B}^\perp & \longrightarrow & \mathbb{B}^\perp \\
 \bullet^- \\
 \bullet^+ \\
 \epsilon_{\mathbb{B}^\perp} = & & \mathbf{q}^- \\
 & & \mathbf{q}^+ \\
 & & \mathbf{tt}^- \\
 \mathbf{tt}^+ & &
 \end{array}$$

Variations at the heart of: work by Laurent (LLP), Levy (CBPV), Mellies (analysis of the Blass phenomenon).

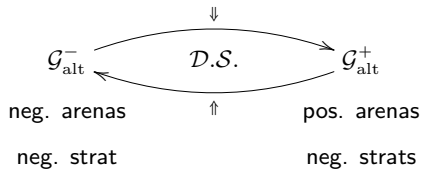
Duploids situations in toy categories of games

The **alternating world**:



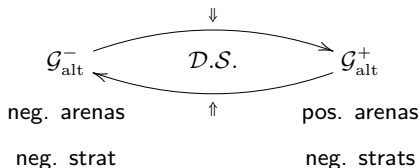
Duploids situations in toy categories of games

The **alternating world**:



Duploids situations in toy categories of games

The **alternating world**:



Execution in $\mathcal{G}_{\text{alt}}^-$ (not natural)

$$\rho_{\mathbb{B}} = \begin{array}{ccc} \uparrow \downarrow \mathbb{B} & \longrightarrow & \mathbb{B} \\ & & \mathbf{q}^- \\ \bullet^+ & & \\ \bullet^- & & \\ \mathbf{q}^+ & & \\ \mathbf{tt}^- & & \\ & & \mathbf{tt}^+ \end{array}$$

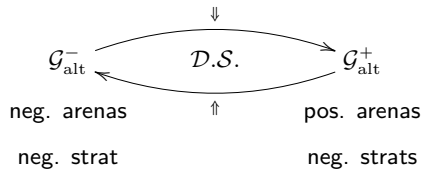
Thinking in $\mathcal{G}_{\text{alt}}^+$ (not natural)

$$\theta_{\mathbb{B}^\perp} = \begin{array}{ccc} \mathbb{B}^\perp & \longrightarrow & \downarrow \uparrow \mathbb{B}^\perp \\ & & \mathbf{q}^- \\ & & \bullet^+ \\ & & \bullet^- \\ & & \mathbf{q}^+ \\ & & \mathbf{tt}^- \\ \mathbf{tt}^+ & & \end{array}$$

Naturality: no, except in the subcategories of **linear** and **thinkable** strategies.

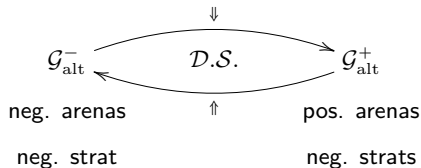
Duploids situations in toy categories of games

The **alternating world**:



Duploids situations in toy categories of games

The **alternating world**:



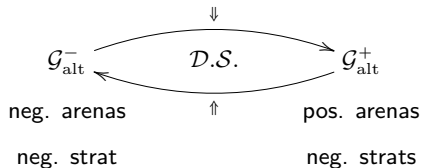
The **non-alternating world**:

\mathcal{G}^-
 neg. arenas
 neg. strats

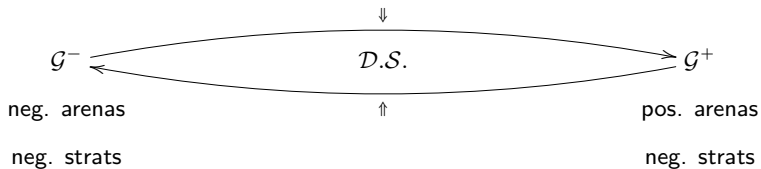
\mathcal{G}^+
 pos. arenas
 neg. strats

Duploids situations in toy categories of games

The **alternating world**:

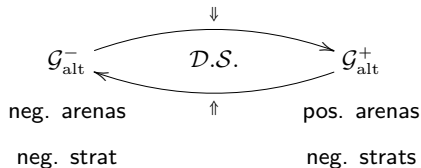


The **non-alternating world**:



Duploids situations in toy categories of games

The **alternating world**:



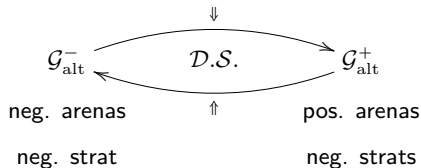
The **non-alternating world**:

\mathcal{G}^-
 neg. arenas
 neg. strats

\mathcal{G}^+
 pos. arenas
 neg. strats

Duploids situations in toy categories of games

The **alternating world**:

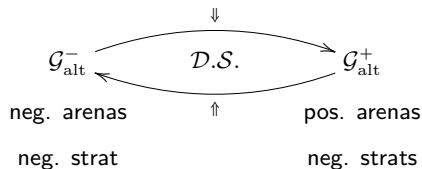


The **non-alternating world**:

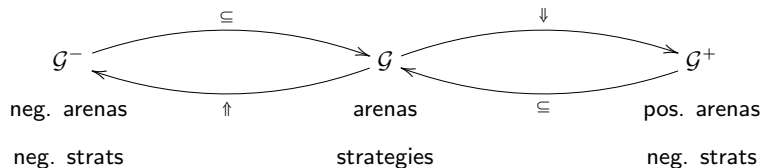


Duploids situations in toy categories of games

The **alternating world**:

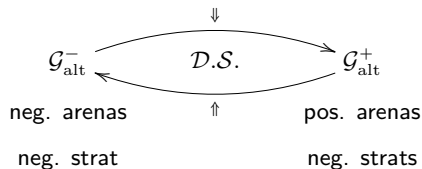


The **non-alternating world**:

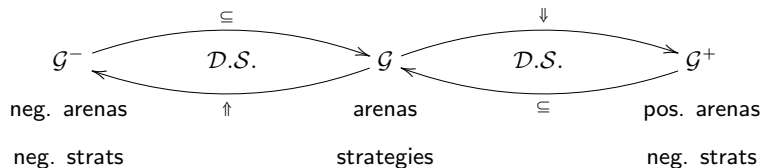


Duploids situations in toy categories of games

The **alternating world**:

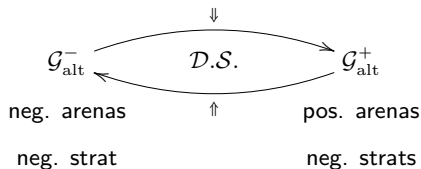


The **non-alternating world**:

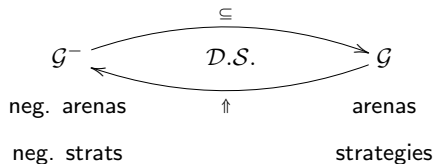


Duploids situations in toy categories of games

The **alternating world**:

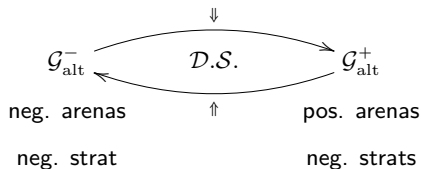


The **non-alternating world**:

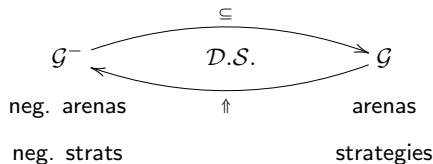


Duploids situations in toy categories of games

The **alternating world**:



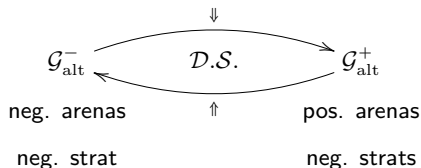
The **non-alternating world**:



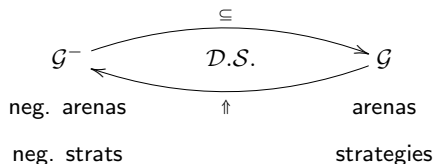
Note: This works in **non-alternating play-based games**, but also some **event-structure** based games (including edcs).

Duploids situations in toy categories of games

The **alternating world**:

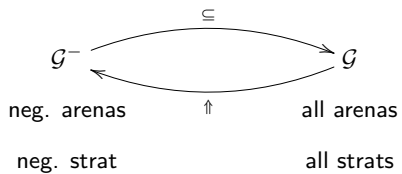


The **non-alternating world**:

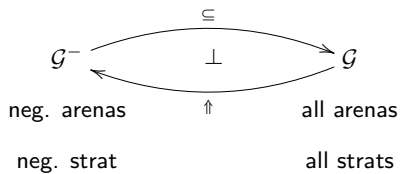


Note: This works in **non-alternating play-based games**, but also some **event-structure** based games (including edcs). **Not** in Conway games.

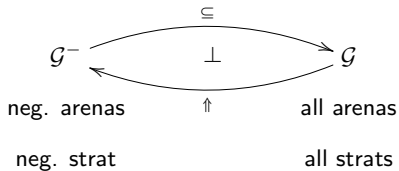
Duploids in non-alternating/concurrent games



Duploids in non-alternating/concurrent games



Duploids in non-alternating/concurrent games

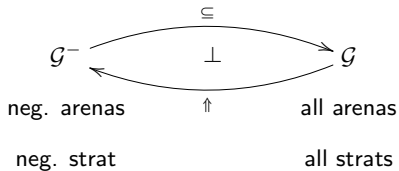


Unit in \mathcal{G}^- (natural)

$$N \longrightarrow \uparrow N$$

$$\eta_N =$$

Duploids in non-alternating/concurrent games



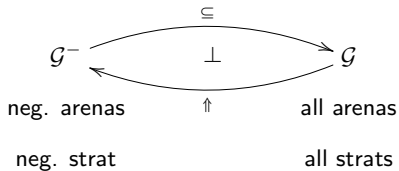
Unit in \mathcal{G}^- (natural)

$$N \longrightarrow \uparrow N$$

•⁻

$$\eta_N =$$

Duploids in non-alternating/concurrent games



Unit in \mathcal{G}^- (natural)

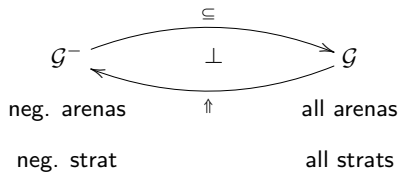
$$N \longrightarrow \uparrow N$$

$$\bullet^-$$

$$n_1^-$$

$$\eta_N =$$

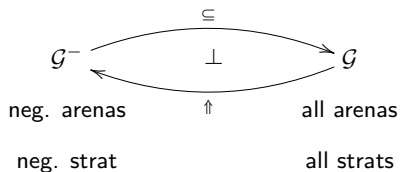
Duploids in non-alternating/concurrent games



Unit in \mathcal{G}^- (natural)

$$\eta_N = n_1^+ \longrightarrow \begin{matrix} \uparrow N \\ \bullet^- \\ n_1^- \end{matrix}$$

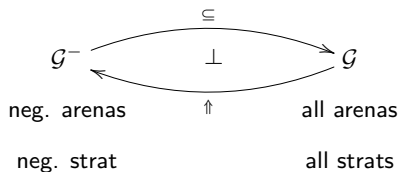
Duploids in non-alternating/concurrent games



Unit in \mathcal{G}^- (natural)

$$\eta_N = n_1^+ \longrightarrow \begin{array}{c} \uparrow N \\ \bullet^- \\ n_1^- \\ n_2^- \\ n_3^- \end{array}$$

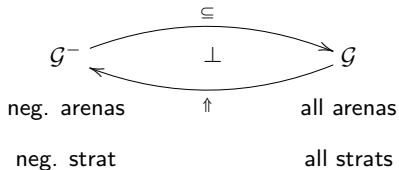
Duploids in non-alternating/concurrent games



Unit in \mathcal{G}^- (natural)

$$\eta_N = \begin{array}{ccc}
 N & \longrightarrow & \uparrow N \\
 & & \bullet^- \\
 & & n_1^- \\
 n_1^+ & & n_2^- \\
 & & n_3^- \\
 n_3^+ & & \\
 n_2^+ & &
 \end{array}$$

Duploids in non-alternating/concurrent games



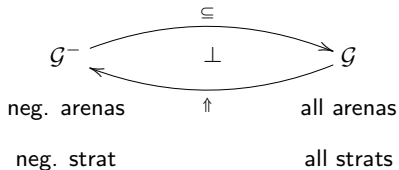
Unit in \mathcal{G}^- (natural)

$$\eta_N = \begin{array}{ccc}
 N & \longrightarrow & \uparrow N \\
 & & \bullet^- \\
 & & n_1^- \\
 n_1^+ & & \\
 & & n_2^- \\
 & & n_3^- \\
 n_3^+ & & \\
 n_2^+ & &
 \end{array}$$

Co-unit in \mathcal{G} (natural)

$$\epsilon_A = \begin{array}{ccc}
 & & \uparrow A \\
 & & \longrightarrow A \\
 & & \\
 & & \\
 & & \\
 & & \\
 & & \\
 & &
 \end{array}$$

Duploids in non-alternating/concurrent games



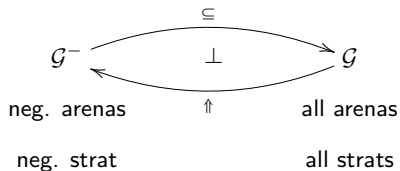
Unit in \mathcal{G}^- (natural)

$$\begin{array}{ccc}
 N & \longrightarrow & \uparrow N \\
 & & \bullet^- \\
 & & n_1^- \\
 \eta_N = & n_1^+ & \\
 & & n_2^- \\
 & & n_3^- \\
 & n_3^+ & \\
 & n_2^+ &
 \end{array}$$

Co-unit in \mathcal{G} (natural)

$$\begin{array}{ccc}
 \uparrow A & \longrightarrow & A \\
 & & a_1^- \\
 \epsilon_A = & &
 \end{array}$$

Duploids in non-alternating/concurrent games



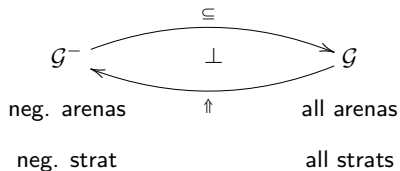
Unit in \mathcal{G}^- (natural)

$$\begin{array}{ccc}
 N & \longrightarrow & \uparrow N \\
 & & \bullet^- \\
 & & n_1^- \\
 \eta_N = & n_1^+ & \\
 & & n_2^- \\
 & & n_3^- \\
 & n_3^+ & \\
 & n_2^+ &
 \end{array}$$

Co-unit in \mathcal{G} (natural)

$$\begin{array}{ccc}
 \uparrow A & \longrightarrow & A \\
 & & a_1^- \\
 \bullet^+ & & \\
 \epsilon_A = & &
 \end{array}$$

Duploids in non-alternating/concurrent games



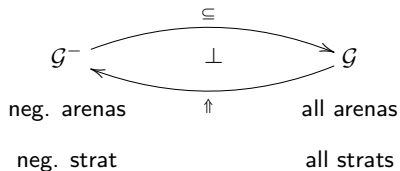
Unit in \mathcal{G}^- (natural)

$$\eta_N = \begin{array}{ccc} N & \longrightarrow & \uparrow N \\ & & \bullet^- \\ & & n_1^- \\ n_1^+ & & \\ & & n_2^- \\ & & n_3^- \\ n_3^+ & & \\ n_2^+ & & \end{array}$$

Co-unit in \mathcal{G} (natural)

$$\epsilon_A = \begin{array}{ccc} \uparrow A & \longrightarrow & A \\ & & a_1^- \\ \bullet^+ & & \\ a_1^+ & & \end{array}$$

Duploids in non-alternating/concurrent games



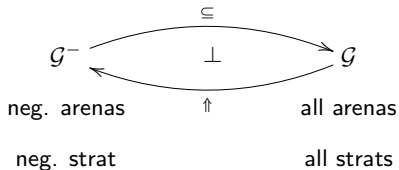
Unit in \mathcal{G}^- (natural)

$$\eta_N = \begin{array}{ccc}
 N & \longrightarrow & \uparrow N \\
 & & \bullet^- \\
 & & n_1^- \\
 n_1^+ & & \\
 & & n_2^- \\
 & & n_3^- \\
 n_3^+ & & \\
 n_2^+ & &
 \end{array}$$

Co-unit in \mathcal{G} (natural)

$$\epsilon_A = \begin{array}{ccc}
 \uparrow A & \longrightarrow & A \\
 & & a_1^- \\
 \bullet^+ & & \\
 a_1^+ & & \\
 a_2^- & & \\
 & & a_3^-
 \end{array}$$

Duploids in non-alternating/concurrent games



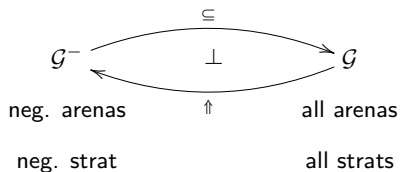
Unit in \mathcal{G}^- (natural)

$$\eta_N = \begin{array}{ccc} N & \longrightarrow & \uparrow N \\ & & \bullet^- \\ & & n_1^- \\ n_1^+ & & \\ & & n_2^- \\ & & n_3^- \\ n_3^+ & & \\ n_2^+ & & \end{array}$$

Co-unit in \mathcal{G} (natural)

$$\epsilon_A = \begin{array}{ccc} \uparrow A & \longrightarrow & A \\ & & a_1^- \\ & & \bullet^+ \\ & & a_1^+ \\ \epsilon_A = & & a_2^- \\ & & a_3^- \\ & & \\ & & a_3^+ \end{array}$$

Duploids in non-alternating/concurrent games



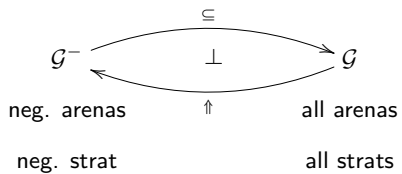
Unit in \mathcal{G}^- (natural)

$$\eta_N = \begin{array}{ccc}
 N & \longrightarrow & \uparrow N \\
 & & \bullet^- \\
 & & n_1^- \\
 n_1^+ & & \\
 & & n_2^- \\
 & & n_3^- \\
 n_3^+ & & \\
 n_2^+ & &
 \end{array}$$

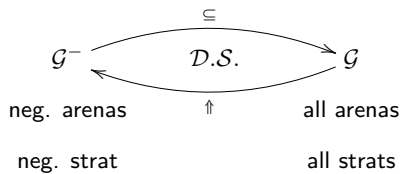
Co-unit in \mathcal{G} (natural)

$$\epsilon_A = \begin{array}{ccc}
 \uparrow A & \longrightarrow & A \\
 & & a_1^- \\
 & & \bullet^+ \\
 & & a_1^+ \\
 \epsilon_A = & & a_2^- \\
 & & a_3^- \\
 & & a_3^+ \\
 & & a_2^+
 \end{array}$$

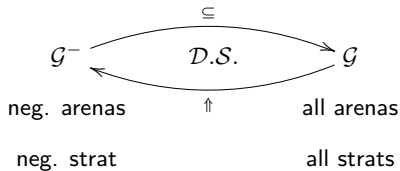
Duploids in non-alternating/concurrent games



Duploids in non-alternating/concurrent games



Duploids in non-alternating/concurrent games

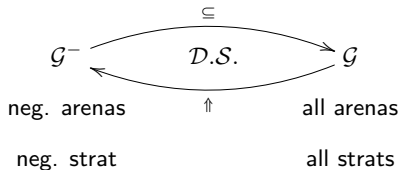


Execution in \mathcal{G}^- (not natural)

$$\uparrow N \longrightarrow N$$

$$\rho_N =$$

Duploids in non-alternating/concurrent games



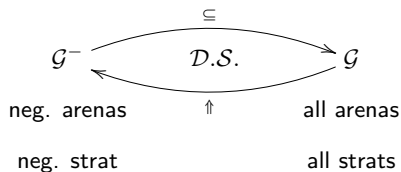
Execution in \mathcal{G}^- (not natural)

$$\uparrow N \longrightarrow N$$

$$n_1^-$$

$$\rho_N =$$

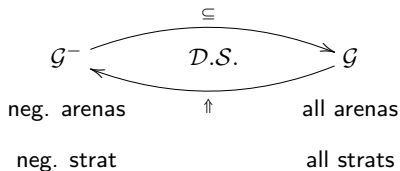
Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

$$\begin{array}{l}
 \uparrow N \longrightarrow N \\
 \bullet^+ \quad n_1^- \\
 n_1^+ \\
 \rho_N =
 \end{array}$$

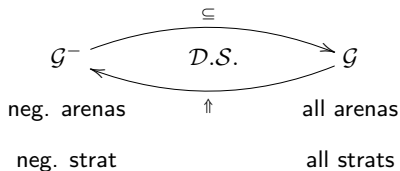
Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

$$\rho_N = \begin{array}{ccc} \uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & \\ & & n_2^- \\ n_3^- & & \end{array}$$

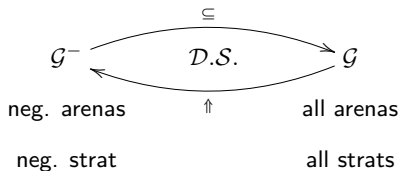
Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

$$\begin{array}{l}
 \uparrow N \longrightarrow N \\
 \bullet^+ \quad n_1^- \\
 \rho_N = \quad n_1^+ \quad n_2^- \\
 \quad \quad n_3^- \\
 \quad \quad n_2^+ \quad n_3^+
 \end{array}$$

Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

$$\begin{array}{l}
 \uparrow N \longrightarrow N \\
 \bullet^+ \qquad \qquad n_1^- \\
 n_1^+ \qquad \qquad n_2^- \\
 n_3^- \qquad \qquad n_3^+ \\
 n_2^+ \qquad \qquad n_2^+
 \end{array}$$

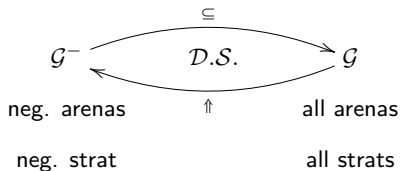
$\rho_N =$

Thinking in \mathcal{G} (not natural)

$$A \longrightarrow \uparrow A$$

$\theta_A =$

Duploids in non-alternating/concurrent games



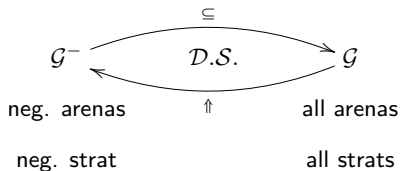
Execution in \mathcal{G}^- (not natural)

$$\rho_N = \begin{array}{ccc} \uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & n_2^- \\ & & \\ n_3^- & & \\ n_2^+ & & n_3^+ \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\theta_A = \begin{array}{ccc} A & \longrightarrow & \uparrow A \\ a_1^- & & \end{array}$$

Duploids in non-alternating/concurrent games



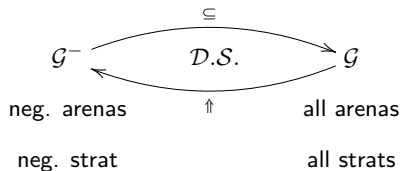
Execution in \mathcal{G}^- (not natural)

$$\begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 \bullet^+ & & n_1^- \\
 n_1^+ & & \\
 \rho_N = & & n_2^- \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\begin{array}{ccc}
 A & \longrightarrow & \uparrow A \\
 a_1^- & & \bullet^- \\
 \theta_A = & &
 \end{array}$$

Duploids in non-alternating/concurrent games



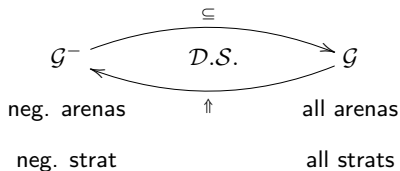
Execution in \mathcal{G}^- (not natural)

$$\rho_N = \begin{array}{ccc} \uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & \\ & & n_2^- \\ n_3^- & & \\ n_2^+ & & \\ & & n_3^+ \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\theta_A = \begin{array}{ccc} A & \longrightarrow & \uparrow A \\ a_1^- & & \\ & & \bullet^- \\ & & a_1^+ \end{array}$$

Duploids in non-alternating/concurrent games



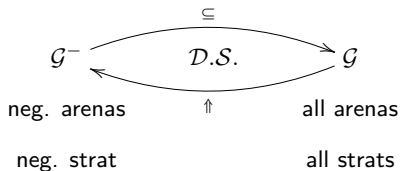
Execution in \mathcal{G}^- (not natural)

$$\rho_N = \begin{array}{ccc} \uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & n_2^- \\ & & \\ n_3^- & & \\ n_2^+ & & \\ & & n_3^+ \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\theta_A = \begin{array}{ccc} A & \longrightarrow & \uparrow A \\ a_1^- & & \bullet^- \\ & & a_1^+ \\ a_2^- & & \end{array}$$

Duploids in non-alternating/concurrent games



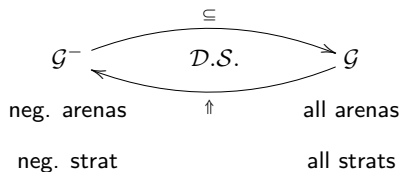
Execution in \mathcal{G}^- (not natural)

$$\rho_N = \begin{array}{ccc} \uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & n_2^- \\ & & \\ n_3^- & & \\ n_2^+ & & n_3^+ \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\theta_A = \begin{array}{ccc} A & \longrightarrow & \uparrow A \\ a_1^- & & \bullet^- \\ & & a_1^+ \\ a_2^- & & a_2^+ \end{array}$$

Duploids in non-alternating/concurrent games



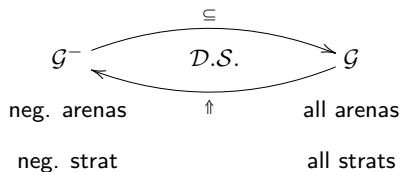
Execution in \mathcal{G}^- (not natural)

$$\begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & n_2^- \\
 \rho_N = & & \\
 n_3^- & & \\
 n_2^+ & & n_3^+
 \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\begin{array}{ccc}
 A & \longrightarrow & \uparrow A \\
 a_1^- & & \bullet^- \\
 & & a_1^+ \\
 \theta_A = & a_2^- & a_2^+ \\
 & & a_3^-
 \end{array}$$

Duploids in non-alternating/concurrent games



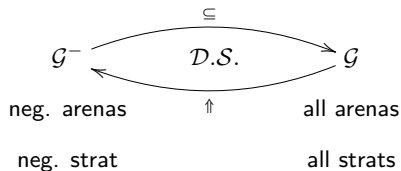
Execution in \mathcal{G}^- (not natural)

$$\begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & n_2^- \\
 \rho_N = & & \\
 n_3^- & & \\
 n_2^+ & & n_3^+
 \end{array}$$

Thinking in \mathcal{G} (not natural)

$$\begin{array}{ccc}
 A & \longrightarrow & \uparrow A \\
 a_1^- & & \bullet^- \\
 \theta_A = & & a_1^+ \\
 a_2^- & & a_2^+ \\
 & & a_3^- \\
 a_3^+ & &
 \end{array}$$

Duploids in non-alternating/concurrent games



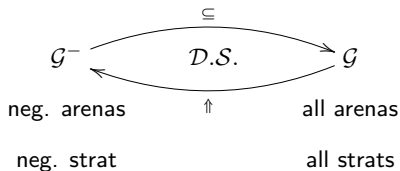
Execution in \mathcal{G}^- (not natural)

$$\rho_N = \begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & n_2^- \\
 & & \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

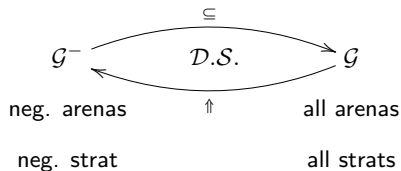
$$\rho_N = \begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & n_2^- \\
 & & \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

\Leftrightarrow

Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

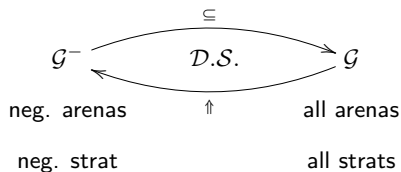
$$\begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & \\
 \rho_N = & & n_2^- \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$$M \xleftrightarrow{\quad} N$$

Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

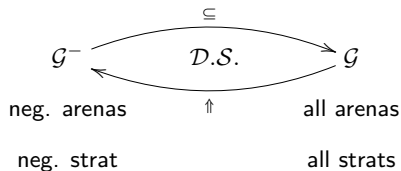
$$\begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & \\
 \rho_N = & & n_2^- \\
 & & \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$$\begin{array}{ccc}
 & \Leftrightarrow & \\
 M & \longrightarrow & N \\
 & & n_1
 \end{array}$$

Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

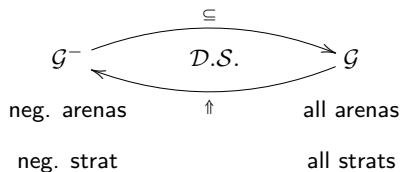
$$\rho_N = \begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & \\
 & & n_2^- \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$$\begin{array}{ccc}
 & \Leftrightarrow & \\
 M & \longrightarrow & N \\
 & & n_1 \\
 & & \dots \\
 & & n_k
 \end{array}$$

Duploids in non-alternating/concurrent games



Execution in \mathcal{G}^- (not natural)

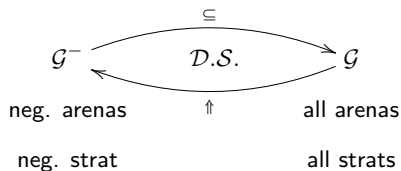
$$\begin{array}{ccc}
 \uparrow N & \longrightarrow & N \\
 & & n_1^- \\
 \bullet^+ & & \\
 n_1^+ & & \\
 \rho_N = & & n_2^- \\
 & & \\
 n_3^- & & \\
 n_2^+ & & \\
 & & n_3^+
 \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$$\begin{array}{ccc}
 & \Leftrightarrow & \\
 M & \longrightarrow & N \\
 & & n_1 \\
 & & \dots \\
 m^+ & & n_k
 \end{array}$$

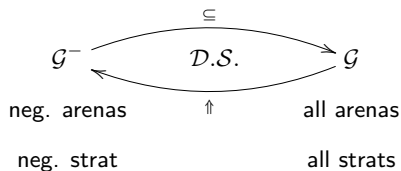
Duploids in non-alternating/concurrent games



Thinking in \mathcal{G} (not natural)

$$\theta_A = \begin{array}{ccc} A & \longrightarrow & \uparrow A \\ a_1^- & & \bullet^- \\ & & a_1^+ \\ & & a_2^+ \\ & & a_3^- \\ a_3^+ & & \end{array}$$

Duploids in non-alternating/concurrent games



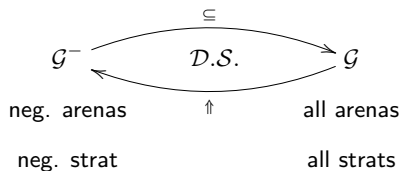
Proposition

A strategy $\sigma : A \rightarrow B$ is **thinkable** if it commutes with θ

Thinking in \mathcal{G} (not natural)

$$\begin{array}{ccc}
 A & \longrightarrow & \uparrow A \\
 a_1^- & & \bullet^- \\
 & & a_1^+ \\
 \theta_A = & a_2^- & a_2^+ \\
 & & a_3^- \\
 & a_3^+ &
 \end{array}$$

Duploids in non-alternating/concurrent games



Proposition

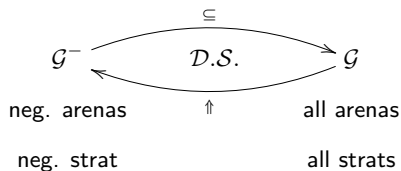
A strategy $\sigma : A \rightarrow B$ is **thinkable** if it commutes with θ

\Leftrightarrow

Thinking in \mathcal{G} (not natural)

$$\theta_A = \begin{array}{ccc} A & \longrightarrow & \uparrow A \\ a_1^- & & \bullet^- \\ & & a_1^+ \\ & & a_2^+ \\ & & a_3^- \\ a_3^+ & & \end{array}$$

Duploids in non-alternating/concurrent games



Proposition

A strategy $\sigma : A \rightarrow B$ is **thinkable** if it commutes with θ

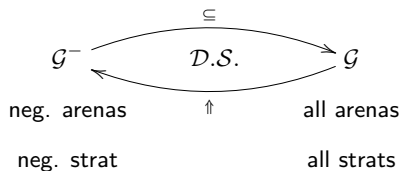
$$A \xrightarrow{\quad} B$$

\Leftrightarrow

Thinking in \mathcal{G} (not natural)

$$\begin{array}{ccc}
 A & \longrightarrow & \uparrow A \\
 a_1^- & & \bullet^- \\
 & & a_1^+ \\
 \theta_A = & a_2^- & a_2^+ \\
 & & a_3^- \\
 & a_3^+ &
 \end{array}$$

Duploids in non-alternating/concurrent games



Proposition

A strategy $\sigma : A \rightarrow B$ is **thinkable** if it commutes with θ

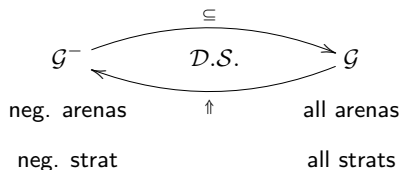
$$A \xrightarrow{\Leftrightarrow} B$$

a^+

Thinking in \mathcal{G} (not natural)

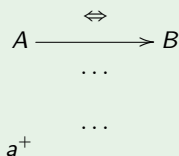
$$\begin{array}{ccc}
 A & \longrightarrow & \uparrow A \\
 a_1^- & & \bullet^- \\
 & & a_1^+ \\
 \theta_A = & a_2^- & a_2^+ \\
 & & a_3^- \\
 & a_3^+ &
 \end{array}$$

Duploids in non-alternating/concurrent games

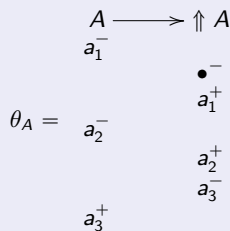


Proposition

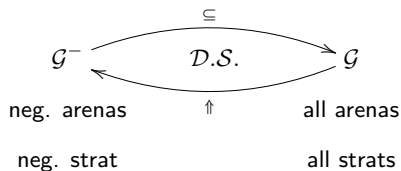
A strategy $\sigma : A \rightarrow B$ is **thinkable** if it commutes with θ



Thinking in \mathcal{G} (not natural)

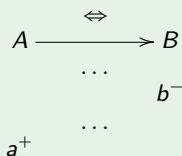


Duploids in non-alternating/concurrent games

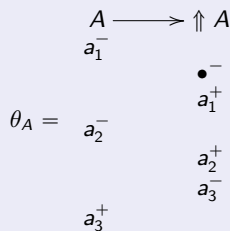


Proposition

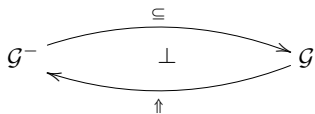
A strategy $\sigma : A \rightarrow B$ is **thunkable** if it commutes with θ



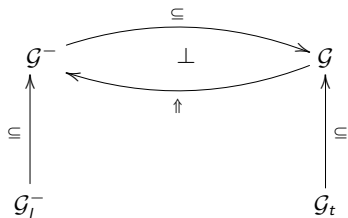
Thinking in \mathcal{G} (not natural)



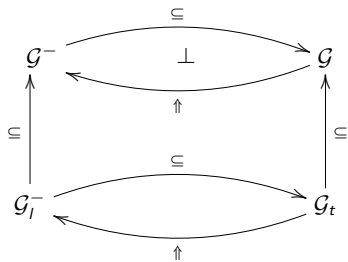
Duploids in non-alternating/concurrent games



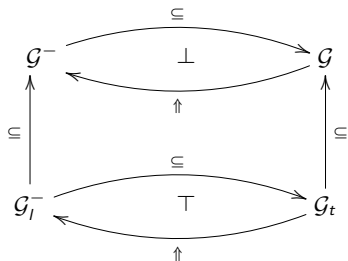
Duploids in non-alternating/concurrent games



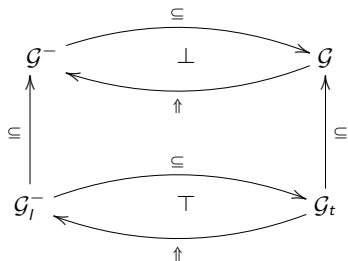
Duploids in non-alternating/concurrent games



Duploids in non-alternating/concurrent games

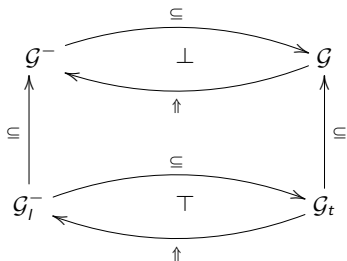


Duploids in non-alternating/concurrent games



$$\begin{aligned}
 |\mathcal{D}| &= |\mathcal{G}^-| + |\mathcal{G}| \\
 \mathcal{D}(A, B) &= \mathcal{G}(A, B) \\
 \mathcal{D}(N, M) &= \mathcal{G}(N, M) \\
 \mathcal{D}(N, P) &= \mathcal{G}(N, P) \cong \mathcal{G}^-(N, \uparrow P) \\
 \mathcal{D}(P, N) &= \mathcal{G}_t(P, N) \cong \mathcal{G}_l^-(\uparrow P, N)
 \end{aligned}$$

Duploids in non-alternating/concurrent games



$$\begin{aligned}
 |\mathcal{D}| &= |\mathcal{G}^-| + |\mathcal{G}| \\
 \mathcal{D}(A, B) &= \mathcal{G}(A, B) \\
 \mathcal{D}(N, M) &= \mathcal{G}(N, M) \\
 \mathcal{D}(N, P) &= \mathcal{G}(N, P) \cong \mathcal{G}^-(N, \uparrow P) \\
 \mathcal{D}(P, N) &= \mathcal{G}_t(P, N) \cong \mathcal{G}_l^-(\uparrow P, N)
 \end{aligned}$$

Duality/composition: three duploids for the price of one!