Tropical cellular automata: why urban fires propagate according to polyhedral balls

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Abstract. In order to analyse the propagation of fire in urban areas, we study a deterministic percolation model on a regular grid in which fire propagates from a point to a bounded neighbourhood of this point, with time constants depending on the jump. Using discrete geometry methods, we obtain an explicit formula for the propagation speed. In particular, we show that for a large time horizon, the wave front is close to the boundary of a ball with respect to a polyhedral weak-Minkowski semi-norm, which can be determined analytically from the time constants. We illustrate the model by simulations on data from the Kobe fire following the 1995 Southern Hyōgo Prefecture Earthquake, indicating that this deterministic model gives an accurate account of actual urban fires.

Keywords: Cellular automata, first passage percolation, tropical geometry, Hamilton-Jacobi PDE, Wulff shape, Shapley-Folkman theorem

1 Introduction

1.1 Motivation

We study fire propagation - a process which is different in wildland and cities. In the latter, fire spread is inherently discrete and so Cellular Automata (CA) seems an appropriate framework for studying this phenomenon and that has been the approach of our work. There have been many great and often well documented fires in urban cities - all serving as good frames of reference for our research. Some considered examples are: Roma (64 AD), London (1666), Paris (1871), San Francisco (1906) and Kobe (1995).

Several works have developed stochastic cellular automata models to represent the propagation of fires, using carefully chosen parameters inferred from physical considerations. These include the work of Zhao [25,26], and of Ohgai, Gohnai and Watabe [20]. In these works, every cell (lattice point) typically represents a building. Propagation occurs from a cell to its neighbour cells with associated probabilities determined by factors like distance and wind. Moreover, the model associates to each cell a state variable representing the degree of progress of the fire. This degree takes values in a finite set (from "ignition" to "structure collapse") and it influences the propagation to neighbouring cells. The validity of these models was confirmed in [25,26,20] by simulations which accurately reproduce historical data of the Kobe fire (1995).

However, the complexity inherent to such stochastic models makes it difficult to obtain analytical results, according to existing theories. For example, we may wish to find analytical formulae for propagation speeds to points at large distances from the source of the fire, or predict how the shape of the ignited region varies as a function of parameters like the speed and direction of the wind.

1.2 Contribution

In this paper, we study a discrete model of fire propagation in urban areas, which we call "tropical cellular automaton", owing to connections with tropical algebra. This model is essentially a multi-state deterministic version of first passage percolation, for which there are times (not necessarily rational) associated with the neighbourhood elements. In its simplest version, presented in Section 2, it consists of a finite subset $E \subset \mathbb{Z}^d$, equipped with a valuation $\tau : E \mapsto \mathbb{R}_{\geqslant 0}$. Every lattice point has only two possible states, "original stage", and "fire". If a cell at position x becomes ignited at time t, then fire propagates to point $x + y$ with $y \in E$ at time $t + \tau(y)$. Assuming only the cell 0 is ignited at instant 0, we denote by $v(x)$ the first propagation time at point $x \in \mathbb{Z}^d$. In a more refined version, described in Section 3, the propagation time from a cell to its neighbours also depends on a discrete state representing the age of the fire in the the cell, in accordance with earlier work [25,26,20].

We denote by $v(\alpha)$ the time at which the fire reaches point $\alpha \in \mathbb{Z}^d$. The following is the main result of this paper.

Theorem 1. *Given a tropical cellular automaton model on $\mathbb{Z}^d$, defined as in Section 2, there is a polyhedral weak Minkowski semi-norm ρ such that, for all $\alpha \in \mathbb{Z}^d$, $\lim_{k\to\infty,\, k\alpha\in\mathbb{Z}^d} k^{-1}v(k\alpha) = \rho(\alpha)$.*

By *polyhedral weak Minkowski semi-norm*, we mean a polyhedral, convex, and positively homogeneous map, not necessarily symmetric - meaning that $\rho(\alpha)$ and $\rho(-\alpha)$ may differ. Our terminology arises from metric geometry, where weak norms are familiar objects [21]. Indeed, owing to the influence of wind, we cannot hope for symmetry in general. We also don't necessarily assume that $\rho(\alpha) = 0$ implies $\alpha = 0$, hence the term "semi-norm". The simplest example of polyhedral weak Minkowski semi-norm is the following deformation of the L_1 norm in $\mathbb{Z}^2$,

$$\rho(x) = \max(p_1^+ x_1, -p_1^- x_1) + \max(p_2^+ x_2, -p_2^- x_2) \tag{1}$$

where $p_1^\pm, p_2^\pm > 0$. If the axis of x_1 is oriented from West to East for increasing values of x_1, then the numbers p_1^+ and p_1^- represent respectively the inverses of the propagation speed in the West and East directions. A similar interpretation applies to $p_2^\pm$. If $p_i^\pm \equiv 1$, ρ is nothing but the L_1 norm. Theorem 1 and its proof show that for a time t large enough, the region reached by the fire will be approximately the ball of radius t in the semi-norm ρ, see Proposition 1 below.

In cities with a Manhattan type structure, with propagation to a von Neumann neighbourhood of a cell, we will actually get a deformed L_1 ball like (1). An examination of the celebrated map by John Leake of the extension of London's great 1666 fire shows that even in less regular cities, with only an approximately quadrangular street organisation, the "deformed L_1 ball" is still visible (Figure 1). We also present in Section 4 simulations regarding the Kobe fire following the earthquake of 1995. The earthquake broke the fire hydrants and, so, there were few resources available to limit the propagation of the fire during the first hours, making of Kobe data a suitable case study to validate our model. We shall see that the deterministic model allows us to reproduce in an accurate way historical data. Surprisingly, the results obtained from the deterministic model are sometimes more accurate than the ones obtained by a stochastic CA model (stochastic models tend to "round" the deformed L_1 ball which does appear on historical data, see Figure 3). This suggests that in an urban area fire propagation has some deterministic features. Whereas certain conditions, like the location and time of the initial ignition sources, are inherently random, deterministic propagation to nearest neighbours is a dominant factor.

Fig. 1. Extract from "An exact survey of the streets, lanes and churches contained within the ruins of the city of London", map of John Leake (1667), source www.bl.uk/onlinegallery. Theorem 1 shows that in a city with a regular quadrangular map, the fire propagates according to a deformed L_1 (weak-)ball. Even in a city like 17th century London, which is only roughly quadrangular, the propagation is approximately polyhedral (a deformed L_1 ball is superposed in yellow, to be compared with the final extension of the fire). Note that the approximation is incorrect at the North-West and West of London, owing to the presence of the wall of London and to other factors.

1.3 Related work

The present work took inspiration from two classes of models: first passage percolation, and Hamilton-Jacobi PDE of eikonal type.

We refer the reader to [16,2] for general overviews of results in first passage percolation. Theorem 1 is a deterministic version of the Cox-Durett theorem [5], showing the existence of a limit shape. Moreover, results of Cox-Kesten [6], and Kesten [16] show that the propagation speed is continuous with respect to weak convergence of i.i.d. distributions of propagation times to nearest neighbours. In this way a basic version of our deterministic model can be recovered as a limit of the classical stochastic model. In the stochastic setting, limit shapes can be arbitrarily complex and they are generally hard to determine, see [7]. In contrast, in the deterministic setting, the limiting shape is a polyhedron with an explicit combinatorial characterisation. Results of this kind go back to Willson [24], see also related work by Gravner and Griffeath [13]. Willson studied discrete time cellular automata in which propagation over one time unit is determined by an "ordered transition rule". Willson showed that the limit shape, called the "Wulff shape" as it originates from crystallography, is a rational polytope. In contrast, the limit shape in our basic model may be an irrational polytope. Moreover, in the extended model of Section 3, our representation in Theorem 4 implies that the limit shape is, in general, only a finite union of polytopes. So our results seems to differ in generality with the Wulff shape limit result in [24]. We also note that the present proof is different, as it relies on the Shapley-Falkman theorem, leading to tight metric estimates, see Theorem 3. It would be of interest to unify the previously mentioned results in a common setting.

First passage percolation is of course one of the most "purified" cellular automata models, we refer the reader to the survey [9] for an overview of cellular automata models. Here, we draw especially inspiration from the stochastic models of fire propagation by Zhao [25,26] and of Ohgai et al. [20].

Other models of fire propagation, in the wildland, rely on PDE, see for instance [17]. The present discrete model is akin to the discretisation of a Hamilton-Jacobi PDE of eikonal type by a semi-Lagrangean scheme, the set E representing a neighbourhood or "stencil". The present results reveal some discrete geometry properties of such discrete schemes. We note that some discrete geometry aspects have appeared in the study of fast marching methods for Hamilton-Jacobi PDE in Finsler structures [19].

A last source of inspiration arises from tropical algebra. The latter deals with semi-fields in which the structure laws are the maximum (or minimum) and the addition. In the basic version of our model, propagation is represented by an idempotent element in the group algebra of $\mathbb{Z}^d$ over the min-plus model of the tropical semi-field, therefore providing an application to the recently studied group algebras in characteristic one [4]. In fact, our results may be thought of as an extension of tropical spectral theory [1,3] to tropical group algebras. We finally note that tropical geometry has previously appeared in the study of more sophisticated cellular automata models, including the tropical analogue of the KdV equation [14], or sandpile models [15].

The paper is organised as follows. Our main results concerning the simplified model are in section 2. The more refined model is presented in section 3. We report experimental results in Section 4.

2 The basic tropical cellular automaton model

This section is on the purified tropical cellular automaton model of a binary nature, where a cell is either ignited or is not. It appears to be a deterministic first passage percolation model.

2.1 The model

We work in the integer lattice $\mathbb{Z}^d$ and assume without loss of generality that the source of fire is at the origin 0. Let $E \subset \mathbb{Z}^d$ be a finite set corresponding to the set of points in $\mathbb{Z}^d$ to which fire may propagate directly from the source. For all $x \in E$ denote by $\tau(x) \in \mathbb{R}_{\geqslant 0} := \{t \in \mathbb{R} \mid t \geqslant 0\}$ the corresponding time of propagation. Due to the assumptions of homogeneity of the system, it follows for all $x \in \mathbb{Z}^d$ that $x + E$ is the set of points to which fire may propagate directly from x.

Given $k \in \mathbb{N} := \{0, 1, \dots\}$ such that $k \geqslant 1$ and $x \in \mathbb{Z}^d$ we define

$$\tau^k(x) = \inf \tau(y_1) + \cdots + \tau(y_k),\ x = y_1 + \cdots + y_k,\ y_1, \dots, y_k \in E \ , \tag{2}$$

with the convention that an infimum over an empty set is $+\infty$. We also adopt the convention that $\tau^0(0) = 0$ and $\tau^0(x) = +\infty$ if $x \neq 0$. Thus, $\tau^k(x)$ represents the propagation time in exactly k steps from the origin to point x, defined as the infimum of the times of a sequence of k elementary jumps allowing the fire to propagate from 0 to x. We will also be interested in the propagation time to x in *at most* k steps, $\tau^{[k]}(x) := \inf_{0 \leqslant m \leqslant k} \tau^m(x)$. Finally, the propagation time to x is defined as

$$\tau^*(x) := \inf_{0 \leqslant k} \tau^k(x) = \lim_{k \to \infty} \tau^{[k]}(x) \ . \tag{3}$$

We shall always assume in the sequel that $\mathbb{N}E = \mathbb{Z}^d$. Here, $\mathbb{N}E := \cup_{k \in \mathbb{N}} (k \cdot E)$, where $k \cdot E := E + \cdots + E$ (k times), and $+$ denotes the Minkowski sum of sets, meaning that for $X, Y \subset \mathbb{R}^d$, $X + Y := \{x + y \mid x \in X, y \in Y\}$. The assumption that $\mathbb{N}E = \mathbb{Z}^d$ entails that every point in $\mathbb{Z}^d$ is reachable from the initial source in k steps for some $k \in \mathbb{N}$. It is equivalent to τ^* taking only finite values on $\mathbb{Z}^d$. We assume $0 \in E$ with $\tau(0) = 0$, it follows $\tau^{[k]} = \tau^k$.

2.2 Inf-convolution algebra

Given two maps $f, g : \mathbb{Z}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, the inf-convolution $f \,\square\, g : \mathbb{Z}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, is defined by $f \,\square\, g(x) = \inf_{y \in \mathbb{Z}^d} (f(x - y) + g(y))$, see [22]. We extend τ to a function defined on $\mathbb{Z}^d$, by setting $\tau(x) = +\infty$ if $x \in \mathbb{Z}^d \setminus E$. We shall equip the space of functions $\mathbb{Z}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ with the pointwise order $\leqslant$.

The *convex hull* of a function $f : \mathbb{Z}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, denoted by $\operatorname{co} f$, is the function $\mathbb{R}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ defined as the supremum of the affine functions $g : \mathbb{R}^d \to \mathbb{R}$ such that $g(x) \leqslant f(x)$ for all $x \in \mathbb{Z}^d$. Extending f to a function

$\mathbb{R}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ by setting $f(x) = +\infty$ if $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$, we see that this is a special case of the general notion of convex hull of a function [22, §5].

The *domain* of f is $\operatorname{dom} f := \{x \in \mathbb{Z}^d \mid f(x) < \infty\}$. We note that the domain of $\operatorname{co} f$ is precisely the convex hull of the set $\operatorname{dom} f$ (the smallest convex set containing $\operatorname{dom} f$).

A basic result of convex analysis [22, Th. 5.4] shows that convex functions are stable by inf-convolutions. We deduce that for all $f : \mathbb{Z}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, $(\operatorname{co} f)^k = \operatorname{co}(f^k)$, where $\cdot^k$ refers to the kth power for the inf-convolution product.

2.3 Main points of the proof of Theorem 1

We next present the main steps of the proof of Theorem 1. We shall need several lemmas from convex analysis. We refer the reader to [22] for the main results which we use, we will give more specific references when appropriate.

Lemma 1. *Let $x \in \mathbb{R}^d$, then, for all $k \geqslant 1$,*

$$(\operatorname{co} \tau)^k(x) = k \operatorname{co} \tau\,(x/k)\,.$$

If $f : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ is a convex function containing 0 as an interior point to its domain, it is known that, for all $u \in \mathbb{R}^d$,

$$\lim_{s \to 0^+} s^{-1}(f(su) - f(0)) = \inf_{s>0} s^{-1}(f(su) - f(0)) \in \mathbb{R}\ . \tag{4}$$

This quantity is denoted by $f'(0;u)$, it is called *the directional derivative* of f at point 0 in the direction u. It satisfies

$$f'(0;u) = \sup_{p \in \partial f(0)} \langle p, u\rangle \tag{5}$$

where $\partial f(0)$ denotes the subdifferential of f at point 0, i.e., the set of vectors p such that the inequality $f(x) - f(0) \geqslant \langle p, x\rangle$ holds for all $x \in \mathbb{R}^d$, see [22, § 23].

Lemma 2.

$$\inf_{k \geqslant 1} k \operatorname{co} \tau(x/k) = (\operatorname{co} \tau)'(0;x)\ .$$

We shall use the notation $f^* := \inf_{k \geqslant 0} f^k$ for $f : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, recalling that the kth power refers to the inf-convolution product on $\mathbb{R}^d$.

Lemma 3.

$$(\operatorname{co} \tau)^*(x) = (\operatorname{co} \tau)'(0;x).$$

The proof of our main result relies on the Shapley-Folkman theorem. The latter is a fundamental theorem of discrete convexity, which implies that the rescaled Minkowski sum $k^{-1}(X_1 + \cdots + X_k)$ of a sequence $X_1, \ldots, X_k$ of uniformly bounded subsets of $\mathbb{R}^d$, converges to a convex set. We state here the special case of this theorem in which $X_1 = \cdots = X_k$.

Theorem 2 (Shapley-Folkman, see e.g. [23, Th.3.1.2]). *Let X be a finite set in $\mathbb{Z}^d$ and $k \geqslant 1$. Let $\operatorname{co} X$ denote the convex hull of X in $\mathbb{R}^d$ and consider a point $x \in k \operatorname{co} X$, the kth Minkowski sum of $\operatorname{co} X$. Then we can find a family of vectors $(x_i)_{1 \leqslant i \leqslant k}$, and a subset $I \subset \{1, \ldots, k\}$ of cardinality at most d, such that*

$$x = \sum_{i \in I} x_i + \sum_{j \in \{1,\ldots,k\} \setminus I} x_j ,$$

where $(\forall i \in I) x_i \in \operatorname{co} X$ and $(\forall j \in \{1, \ldots, k\} \setminus I) x_j \in X$.

Let us recall that a convex map is polyhedral if its epigraph is a polyhedron [22, § 19]. The following result, derived from Theorem 2, shows that the propagation time τ^* is at finite distance, in the sup-norm, from a polyhedral weak Minkowski semi-norm.

Theorem 3. *The map*

$$\rho(x) := (\operatorname{co} \tau)' (0; x)$$

is polyhedral, convex and positively homogeneous of degree one. Moreover,

$$\rho(x) \leqslant \tau^*(x) \leqslant M + \rho(x) \tag{6}$$

where $M := \sup_{z \in (d+1) \operatorname{co} E} \tau^*(z)$.

Theorem 1 stated in the introduction follows readily from this result. Define now $B(s) := \{x : \tau^*(x) \leqslant s\}$, the set of cells reached in time less than or equal to s. Also, consider the ball of radius s in the weak Minkowski semi-norm ρ: $B_\rho(s) := \{x : \rho(x) \leqslant s\}$. The following property is an immediate consequence of Theorem 3:

Proposition 1. *For all $s \geqslant M$, we have $B_\rho(s - M) \subseteq B(s) \subseteq B_\rho(s)$.*

Computing the polyhedral weak Minkowski semi-norm ρ boils down to computing the facets of a d-dimensional polytope defined by its vertices. It follows from McMullen's upper bound theorem [18] that the double description algorithm of [11] runs in polynomial time if the dimension is fixed. Therefore:

Proposition 2. *If the dimension d is fixed, the polyhedral weak Minkowski semi-norm ρ arising in Theorem 3 can be computed in a number of arithmetic operations that is polynomial in the cardinality of E.*

Remark 1. Theorem 1 can be interpreted in terms of tropical group algebras [4]. Group algebra arise classically in representation theory. In the present setting, they represent invariant metrics on groups. Let $\mathbb{R}^+_{\min}$ denote the semiring $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$, equipped with the addition "$a + b$" $:= \min(a, b)$ and the multiplication "$a \times b$" $:= a + b$. The tropical group algebra $\mathbb{R}^+_{\min}[\mathbb{Z}^d]$ consists of the set of functions from $\mathbb{Z}^d$ to $\mathbb{R}^+_{\min}$ that take infinite values except at a finite number of points. These functions are equipped with the operations of infimum and of inf-convolution. The *completed group algebra* $\mathbb{R}^+_{\min}[[\mathbb{Z}^d]]$ consists of all functions from $\mathbb{Z}^d$ to $\mathbb{R}^+_{\min}$. The propagation time is given by $\tau^* = \inf_{k \geqslant 0} \tau^k$, the unary operation

$\tau \mapsto \tau^*$ being known as the *Kleene star* in language theory. The element τ^* is idempotent in the completed group group algebra, meaning that $(\tau^*)^2 = \tau^*$. This entails the triangular inequality $\tau^*(x+y) \leqslant \tau^*(x) + \tau^*(y)$. Hence, $(x, y) \mapsto \tau(x-y)$ defines a weak metric [21] on $\mathbb{Z}^d$, invariant by translation.

3 Fire spread model with an age structured tropical cellular automaton

We now present the refinement of the previous model which we implemented to validate our approach on Kobe's fire example. Every cell will be equipped with a discrete state variable representing the degree of progression or "age" of the fire.

3.1 Age structured model

Cells. Cells form the state space for the CA model by dividing the area of interest into a lattice grid with elements from $\mathbb{Z}^2$. As time moves forward in discrete steps the state of each cell will change according to the rules that we next describe.

The size of the cells should correspond to a real life surface area in the city of interest. As such, a cell size should not be too large since then a single cell may correspond to multiple buildings and cannot be expected to describe the state of each building simultaneously at any given time. On the other hand, the cell sizes should not be prohibitively small so as to incur an unnecessary computational penalty. Based on these considerations and on the work in [25,26] we have used grid cells of area approximately 0.2m$\times$0.2m. A plausible concern may be that a single building described by multiple grid cells may experience different states of fire at different points in the building at some fixed time in the simulation. We will see that this is not an issue in practice. Indeed, it is logical that in a very large building, all parts would not ignite simultaneously and fire may propagate through the building in a way similar to if the building had been made of several smaller buildings.

Cell states. Each cell is either of building type or empty space. The cells of building type may take states $\{0\}$ to $\{5\}$. State $\{0\}$ defines a cell which is not yet ignited but has the capacity to become ignited. States $\{1\}$ through $\{4\}$ represent the different stages of fire development, namely: ignition, flashover, full development and structure fire. Finally, state $\{5\}$ signifies collapse. Colour codes were assigned to each cell state for intuitive interpretation. White is used for empty space and blue for building cells which are not yet ignited. A red spectrum is used for states 1 through 4. State 5 is represented in black.

Cell neighbourhood. By utilising a regular lattice state space, it is possible to describe the cell neighbourhood in simple terms. Based on [25,26] and the natural assumption that the L_1 norm is most suitable for modelling fire spread radius, we use a cell neighbourhood of L_1 distance at most 30. In our model, the neighbourhood of a cell is the set of cells which lie within this distance.

Propagation rules Once a cell is ignited (state $\{1\}$) we employ a deterministic transition through subsequent states. This transition is independent of neighbouring cells. If we denote by t_i the transition time from state $\{i\}$ to state $\{i+1\}$ for $1 \leqslant i \leqslant 4$, then (based on [25,26]) we take the following values, measured in minutes: $t_1 = 4, t_2 = 6.5, t_3 = 15, t_4 = 25$. Thus, once a cell x transitions to state $\{1\}$ (becomes ignited) a timer is started for cell x, and this cell will pass to state $\{2\}$ after time t_1. The transitions to more advanced fire ages occur in a similar manner.

In addition, fires propagates to neighbours. For simplicity, we assume that the propagation of fire to a neighbour cell only occurs when the fire reaches the flashover state in the current cell. That is to say, if a cell x reaches state $\{2\}$ at time t, then all cells within the previously defined neighbourhood of x change to state $\{1\}$, immediately, at time t (unless previously ignited).

3.2 Analysis of the age structured model

The cellular automaton which we just defined is an extension of the basic model of Section 2, which can be formalised in the following general setting. Every cell is now equipped with an age taking values in a finite set S (above, $S = \{0, \dots, 5\}$). If cell $x \in \mathbb{Z}^d$ is at age $i \in S$, the fire propagates by a direct jump to cell $x + y \in \mathbb{Z}^d$ at age $j \in S$, after a time denoted by $\tau(i, j, y) \in \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$. We assume that $\tau(i, j, y) = +\infty$ except when y belongs to a non-empty finite set $E_{ij} \subset \mathbb{Z}^d$. The minimal propagation time from cell 0 at age i to cell x at age j is denoted by $\tau^*(i, j, x)$. It is defined as in (3), mutatis mutandis, i.e., $\tau^*(i, j, x) = \inf_{k \geqslant 0} \tau^k(i, j, x)$, where $\tau^k(i, j, x)$ is the infimum of $\tau(i_0, i_1, y_1) + \dots + \tau(i_{k-1}, i_k, y_k)$ over all sequences $i_0, \dots, i_k \in S$ and $y_1, \dots, y_k \in \mathbb{Z}^d$ such that $y_1 + \dots + y_k = x$, with $i_0 = i$ and $i_k = j$.

We next sketch the extension of our main result to this age structured model, leaving details to a further work. The completed group algebra $\mathbb{R}^+_{\min}[[\mathbb{Z}^d]]$ is equipped with the binary operations of infimum, inf-convolution, and with the unary Kleene star operation, sending $f \in \mathbb{R}^+_{\min}[[\mathbb{Z}^d]]$ to f^*, as defined in Remark 1. An element of $\mathbb{R}^+_{\min}[\mathbb{Z}^d]$ is said to be *rational* if it can be written as a finite well formed expression involving elements of the (uncompleted) group algebra $\mathbb{R}^+_{\min}[\mathbb{Z}^d]$ and the operations inf, inf-convolution, and Kleene star.

The following result shows that the propagation times are given by "star height one" rational expressions in the completed group algebra.

Theorem 4. *For all $i, j \in S$, there exists a finite family of functions $u_\ell, w_\ell \in \mathbb{R}^+_{\min}[\mathbb{Z}^d]$, $1 \leqslant \ell \leqslant L$ (depending on i, j), such that*

$$\tau^*(i, j, \cdot) = \inf_{1 \leqslant \ell \leqslant L} u_\ell \,\square\, w_\ell^* \ . \tag{7}$$

This follows from Theorem 14 of [12], building on results by Eilenberg and Schützenberger [8] characterising the rational subsets of commutative monoids.

The essential terms in (7) are the idempotent functions w_ℓ^*, which were effectively approximated in Theorem 3. Hence, (7) allows us to reduce the analysis of the age structured model to the analysis of the basic model. Note that for this generalised model, the limit shape is generally an union of *distinct* polyhedral balls, arising from the different idempotents w_ℓ^* in (7).

4 Simulation and validation

In order to see the theory in action a basic model was created using the MATLAB software and applied to a real site case study. The case study is a small area of Kobe, Japan and records the effects of a fire provoked by the earthquake of 1995. The earthquake took place at 5h47. The fire lasted for approximately 15 hours. Wind speeds remained relatively small for the duration of the fire and we feel justified in neglecting the effects of wind in our simple model.

Cells are given states from $\{0\}$ to $\{5\}$ where: "$\{0\}$" means "not ignited yet but has the capacity to become ignited"; states $\{1\}$ through $\{4\}$ signify the different stages of fire from "ignited" to "structure fire" and finally; $\{5\}$ represents collapse. Colours in Figure 2 and Figure 3 are associated with their respective states in an intuitive way with a red spectrum representing the stages of fire and black representing collapse, as said above. It is interesting that the purely deterministic model is providing such a high degree of accuracy.

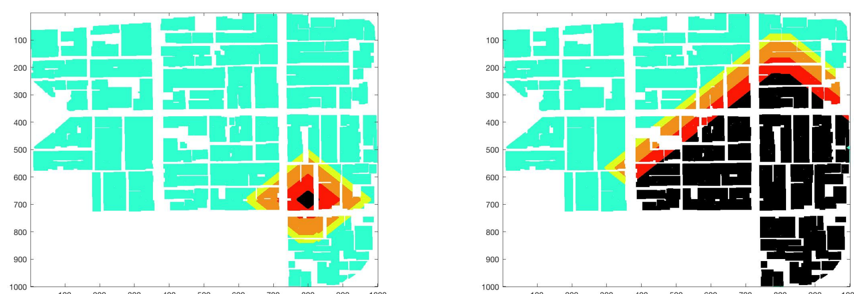

Fig. 2. Results obtained by simulating the tropical CA model. Propagation on January 17, 1995, at 7h00 (left) and at 10h00 (right). The ignited region has a L_1 shape.

5 Concluding remarks

We studied a deterministic cellular automaton model representing propagation of fire in a urban area, and showed that fire propagates according to a polyhedral ball. This assumes that the area is homogeneous. In application to many real urban settings, different areas of the city will possess different geometric features. As such, it makes sense that a cell at one point in the city will have a different

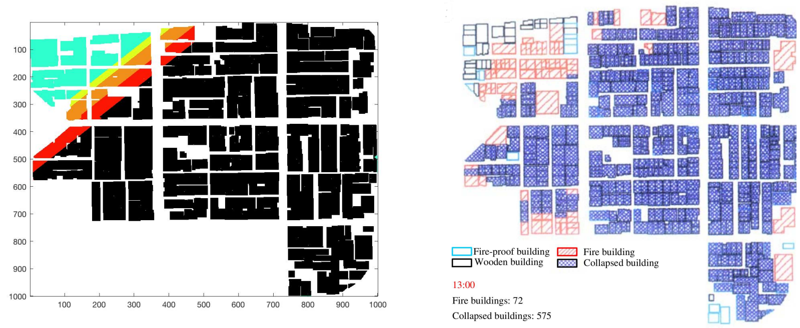

Fig. 3. Results obtained by simulating the tropical CA model (left), at 13h00, versus the local investigated results for the Kobe fire [10] at the same time, reproduced from [25, Fig. 9a] (right). The polyhedral geometry of the destructed region and of the firefront are visible (especially on the top left block), both in the simulation and on historical data.

propagation neighbourhood to another cell at a different point in the city. This should correspond to a notion of discrete Finsler structure, the study of which is left to a further work.

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