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Submerging CSIDH

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Outline

- 1 CSIDH
- 2 Hidden Shift Algorithms
- 3 Computing a group action
- 4 Ordinary curves
- 5 Conclusion

Group action

A **group** G acts on a **set** X .

$$h * (g * x) = (h \cdot g) * x$$

Easy

- Operations in G ;
- Action $g * x$, $g \in G$, $x \in X$.

Hard

- Find g from x and $x' = g * x$.

CSIDH

In the case of CSIDH [CLM⁺]

Set

Montgomery curves on \mathbb{F}_p :

$$E_A : y^2 = x^3 + Ax^2 + x .$$

Endomorphism Ring

- $End_{p^2}(E_A)$: Order of a quaternion algebra
- $End_p(E_A) = \mathbb{Z}[\pi] = \mathbb{Z}[\sqrt{-p}]$

Group

Isogenies between those curves, which correspond exactly to $Cl\mathcal{O}$ where $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$.

Parameters rationale

The base field is \mathbb{F}_p for $p = 4\ell_1 \cdots \ell_u - 1$, with ℓ_1, \dots, ℓ_u small primes.

It turns out that each ℓ_i gives an isogeny $[l_i]$ of **small** degree ℓ_i , very easy to compute (as $[l_i]^{-1}$).

$\mathcal{Cl}\mathcal{O}$ is spanned by products of the form:

$$\prod_{i=1}^u [l_i]^{e_i}$$

for $e_i \in \{-m \dots m\}$ and $2m + 1 \simeq p^{1/(2u)}$ ($\mathcal{Cl}\mathcal{O}$ has $O(\sqrt{p})$ elements).

The one-way commutative group action!

Computing the action of $[b] = \prod_{i=1}^u [l_i]^{e_i}$:

Apply successively um isogenies of degree $\leq \ell_u$.

Find $[b]$ such that $[b] \cdot E = E'$:

Compute an isogeny between two curves E and E' .

Commutative group action

$$[b] \cdot E = E' \Rightarrow \forall [a] \in \mathcal{Cl}\mathcal{O}, [ab] \cdot E = [a] \cdot E'$$

CSIDH parameters for NIST security levels

| Level | $\log_2 p$ | # primes | Isogeny range | Estimated quantum query cost |
|--------|------------|----------|---------------|------------------------------|
| NIST 1 | 512 | 74 | 5 | 2^{62} |
| NIST 3 | 1024 | 132 | 7 | 2^{94} |
| NIST 5 | 1792 | 209 | 10 | 2^{129} |

Hidden Shift Algorithms

Hidden Shift

$$f(x) = g(x + s), x \in \mathbb{G}. \quad \text{Find } s$$

Quantum Algorithms

- $\mathcal{O}\left(8^{\sqrt{n}}\right)$ in $\mathbb{Z}/(2^n\mathbb{Z})$ [Kup05]
- $\mathcal{O}\left(8^{\sqrt{\log_2(N)}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$ [Kup05]
- $\tilde{\mathcal{O}}\left(3^{\sqrt{2\log_3(N)}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$, N smooth [Kup05]
- $\tilde{\mathcal{O}}\left(2^{\sqrt{2n\log_2(n)}}\right)$ in $\mathbb{Z}/(2^n\mathbb{Z})$, polynomial memory [Reg04]
- $\tilde{\mathcal{O}}\left(2^{\sqrt{2\log_2(N)}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$, with QRAM [Kup13]
- $\tilde{\mathcal{O}}\left(2^{\sqrt{2\log_2(N)\log_2(\log_2(N))}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$, polynomial memory [CJS14]
- $2^{\sqrt{2\log_2(3)n}}$ in $\mathbb{Z}/(2^n\mathbb{Z})$ [BNP18]

What we have

Hidden shift algorithm for $\mathbb{Z}/(2^n\mathbb{Z})$ that costs $2^{\sqrt{2\log_2(3)n}}$

What we need

Precise cost for a hidden shift algorithm for $\mathbb{Z}/(N\mathbb{Z})$

Hidden Shift in $\mathbb{Z}/(2^n\mathbb{Z})$

Oracle

$$O : \begin{array}{l} |0\rangle |x\rangle |0\rangle \mapsto |0\rangle |x\rangle |f(x)\rangle \\ |1\rangle |x\rangle |0\rangle \mapsto |1\rangle |x\rangle |g(x)\rangle \end{array}$$

Sampling

$$O \left(\frac{1}{2^{(n+1)/2}} \sum_{i=0}^{2^n} (|0\rangle + |1\rangle) |i\rangle |0\rangle \right) = \frac{1}{2^{(n+1)/2}} \sum_{f(x)} (|0\rangle |x\rangle + |1\rangle |x+s\rangle) |f(x)\rangle$$

Quantum Fourier Transform

$$|\psi_\ell\rangle = |0\rangle + \exp\left(2i\pi s \frac{\ell}{2^n}\right) |1\rangle, \ell$$

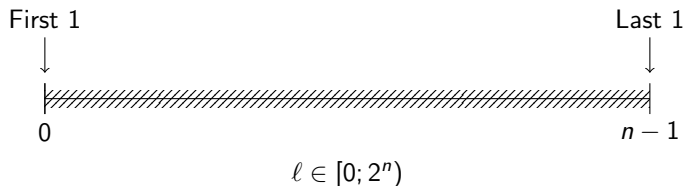
Combining the qubits

Targets

$$\begin{aligned} |\psi_{2^n-1}\rangle &= |0\rangle + (-1)^s |1\rangle \\ |\psi_{2^n-2}\rangle &= |0\rangle + (-1)^{\lfloor s/2 \rfloor} \exp\left(2i\pi \frac{s \bmod 2}{4}\right) |1\rangle \\ &\dots \end{aligned}$$

Combination

$$(l_1, l_2) \mapsto l_1 \pm l_2 \pmod{2^n}$$



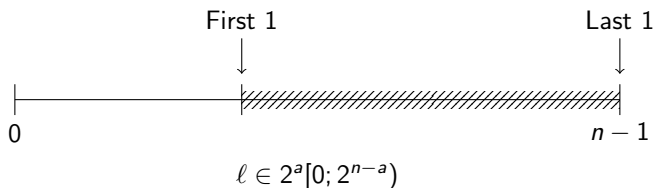
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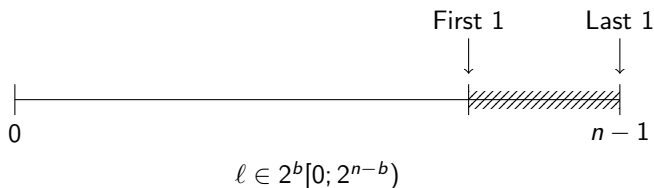
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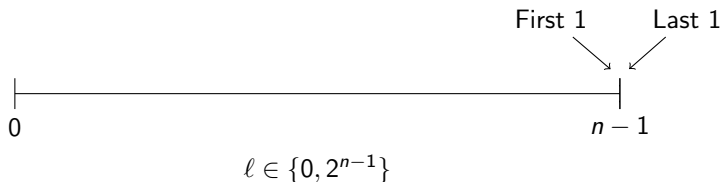
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Combination

$$(l_1, l_2) \mapsto l_1 \pm l_2 \pmod{2^n}$$



Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

Situation

Elements $|\psi_\ell\rangle = |0\rangle + \exp(2i\pi s \frac{\ell}{N}) |1\rangle$

Targets $\bigotimes_{i=0}^n |\psi_{2^i}\rangle \simeq QFT |t\rangle, \frac{t}{2^n} \simeq \frac{s}{N}$

Combination $(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2 \pmod N$

Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

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Combination $(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2$ in \mathbb{Z}

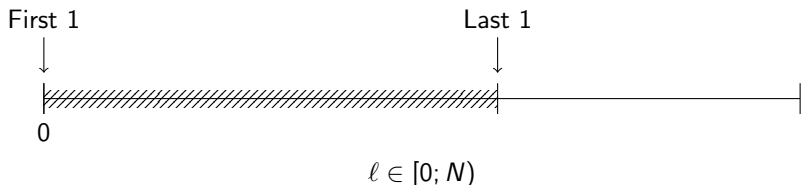
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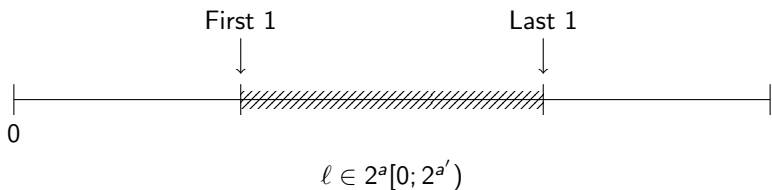
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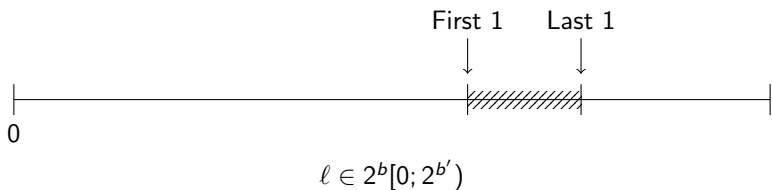
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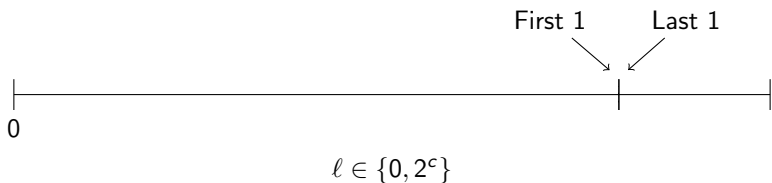
Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

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Combination $(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2$ in \mathbb{Z}



Cost for CSIDH

Final complexity

- Around $5 \times 2^{1.8\sqrt{\log_2(N)}}$ (simulated!) queries to f and g and quantum memory
- Log. overhead for classical time and memory

Costs for CSIDH (\log_2)

| $\log_2(p)$ | n | Our Hidden Shift query cost | Query cost estimation from [CLM ⁺] |
|-------------|-----|-----------------------------|--|
| 512 | 256 | 32.5 | 62 |
| 1024 | 512 | 44.5 | 94 |
| 1792 | 896 | 57.5 | 129 |

Computing a group action

Objectives

Target

Find an **efficient** procedure to compute:

$$[g] \cdot E$$

where E is a CSIDH curve and $[g] \in \mathcal{ClO}$, in superposition over the whole group \mathcal{ClO} .

General situation

Direct computation of $[g] \cdot E$ is expensive

In CSIDH

Computing the $[l_i] \cdot E$ is cheap

Cost reduction

Strategy

- Decompose $[g] = \prod [l_i]^{e_i}$
- Ensure (e_1, \dots, e_k) is small
- Compute $\prod [l_i]^{e_i}$

In Practice

- Precompute a short basis B of $\{(e_1, \dots, e_k) \mid \prod [l_i]^{e_i} = 1\}$ (BKZ-20)
- Quantumly decompose $[g]$ over $[l_i]$ (Shor)
- Reduce the size of the exponents using B (Babai)
- Compute the isogeny

Overhead between 2^5 and 2^8 theoretically, heuristically between 2 and 5.

Ordinary curves

The Couveignes–Rostovtsev–Stolbunov scheme

- In general, in the ordinary case, one can find ideal classes to span $\mathcal{Cl}\mathcal{O}$, but they cost much more.
- Taking $u = \frac{1}{2} \frac{\log p}{\log 3}$:

$$\mathcal{Cl}\mathcal{O} \simeq \{[l_1]^{e_1} \cdots [l_u]^{e_u}, e_i \in \{-1, 0, 1\}\} .$$

Two choices

- Keep this basis: the dimension increases. The approximation factor remains good in practice: 2^3 for $\log_2 p = 512$ to 2^4 for $\log_2 p = 1024$. could increase up to 2^{15} (in practice better).
- Take a smaller dimension and bigger exponents (asymptotically better) [BFJ16, BJI18].

De Feo–Kieffer–Smith's scheme [FKS18]

Intermediate situation. Products are of the form:

$$[l_1]^{e_1} \cdots [l_u]^{e_u} \cdots [l_{u+v}]^{e_{u+v}} .$$

The e_i have different ranges $-m_i \dots m_i$ and some **must be** positive.

We can adapt!

- Take the weights m_i into account in \mathcal{L} .
- Adapt the CVP instance to force some coordinates to be positive.

Overhead 2^5 w.r.t a classical group action (better than taking a naïve decomposition).

$\Rightarrow 2^{38}$ equivalent classical group actions for 56-bit parameters proposed in [FKS18].

Conclusion

Conclusion

We have estimated the cost of Kuperberg's algorithm.

- To reach the NIST security levels in **queries**, parameters should be multiplied by 4.





We have estimated the time to attack CSIDH.

- To reach the NIST security levels in **time**, parameters should be doubled to tripled.

| Level | Original $\log_2 p$ | Corrected $\log_2 p$ |
|--------|------------------------|-------------------------|
| NIST 1 | 512 | 900 |
| NIST 3 | 1024 | 2500 |
| NIST 5 | 1792 | 5000 |

Thank you!

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