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# Beyond Vehicle Routing: a General Purpose Branch-Cut-and-Price Code for Applications with Resource Constrained Shortest Path (RCSP) Pricing

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# Modern Branch-Cut-and-Price for Vehicle Routing

- ▶ Bucket graph-based labelling algorithm for the RCSP pricing [Righini and Salani, 2006] [Sadykov et al., 2017]
- ▶ (Dynamic) partially elementary path (*ng*-path) relaxation [Baldacci et al., 2011b] [Roberti and Mingozzi, 2014] [Bulhoes et al., 2018b]
- ▶ Automatic dual price smoothing stabilization [Wentges, 1997] [Pessoa et al., 2017]
- ▶ Reduced cost fixing of (bucket) arcs in the pricing problem [Ibaraki and Nakamura, 1994] [Irnich et al., 2010] [Sadykov et al., 2017]
- ▶ Rounded Capacity Cuts [Laporte and Nobert, 1983] [Lysgaard et al., 2004]
- ▶ Limited-Memory Rank-1 Cuts [Jepsen et al., 2008] [Pecin et al., 2017b] [Pecin et al., 2017c] [Pecin et al., 2017a]
- ▶ Enumeration of elementary routes [Baldacci et al., 2008]
- ▶ Multi-phase strong branching [Pecin et al., 2017b]
- ▶ Generic (strong) diving heuristic [Sadykov et al., 2018]

# Motivation

An expert team needs **several months of work** to  
implement a state-of-the-art  
Branch-Cut-and-Price algorithm

## **Our objective**

A framework and a modelling tool which can be used to  
develop modern Branch-Cut-and-Price algorithms for specific  
problems faster and more easily

## Set partitioning formulation over constrained path variables, with additional constraints and variables

- ▶  $\lambda_p^k = 1$  iff resource constrained path  $p \in \mathcal{P}^k$  in directed graph  $G^k = (V^k, A^k)$ ,  $k \in K$ , participates in the solution.
- ▶  $x^{p,k} \in \{0, 1\}^{|A^k|}$  — characteristic vector of path  $p \in \mathcal{P}^k$

$$\min \sum_{k \in K} \sum_{p \in \mathcal{P}^k} (c^k x^{p,k}) \lambda_p^k + f y$$

$$\sum_{k \in K} \sum_{p \in \mathcal{P}^k} \sum_{a \in E_j} x_a^{p,k} \lambda_p^k = (\leq) 1, \quad j \in J$$

$$\sum_{k \in K} \sum_{p \in \mathcal{P}^k} (D^k x^{p,k}) \lambda_p^k + D^0 y \geq d$$

$$L^k \leq \sum_{p \in \mathcal{P}^k} \lambda_p^k \leq U^k \quad \forall k \in K$$

$$y \in \mathbb{N}^{n_I} \times \mathbb{R}^{n_F}$$

$$\lambda_p^k \in \mathbb{N} \quad \forall k \in K, p \in \mathcal{P}^k$$

## Elementarity (or packing) sets

$$E_j \subseteq \bigcup_{k \in K} A^k, \quad \forall j \in J$$

For each  $j \in J$ , at most one arc  $a \in E_j$  can appear in the global solution of the problem.

Knowledge about elementarity sets allows us to apply important techniques,

**critical for obtaining the state-of-the-art performance:**

- ▶ Partially elementary paths (*ng*-paths) relaxation
- ▶ Limited-memory rank-1 packing cuts
- ▶ Enumeration of elementary routes

# Information about the graphs in the model

For each graph  $G^k$ ,  $k \in K$

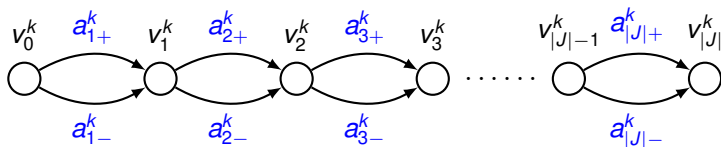
- ▶ Sets of vertices, arcs, resources  $V^k$ ,  $A^k$ ,  $R^k$
- ▶ Source and the sink:  $v_{\text{source}}^k$ ,  $v_{\text{sink}}^k$
- ▶ Non-negative resource consumption of main resources  $M^k \subseteq R^k$  on arcs:  $q_{a,r}^k \in \mathbb{R}_+$ ,  $a \in A^k$ ,  $r \in M^k$
- ▶ Cycles of zero consumption of all main resources should not exist
- ▶ Unrestricted resource consumption of other resources on arcs:  $q_{a,r}^k \in \mathbb{R}$ ,  $a \in A^k$ ,  $r \in R^k \setminus M^k$
- ▶ Resource consumption bounds on vertices:  $[l_{v,r}^k, u_{v,r}^k]$ ,  $v \in V^k$ ,  $r \in R^k$

# Example 1: Generalized Assignment Problem

## The problem data

- ▶ set  $J$  of tasks, set  $K$  of machines of capacity  $Q^k$ ,  $k \in K$
- ▶ assignment cost  $c_j^k$  and size  $w_j^k$ ,  $j \in J$ ,  $k \in K$

## Graph $G^k$



- ▶ One resource with consumption:  $q_{a_{j+}^k}^k = w_j^k$ ,  $q_{a_{j-}^k}^k = 0$ ,  $j \in J$
- ▶ Consumption bounds:  $[l_{v_j^k}^k, u_{v_j^k}^k] = [0, Q^k]$ ,  $j \in J \cup \{0\}$ .



# Generalized Assignment: formulation

$$\begin{aligned} \min \quad & \sum_{k \in K} \sum_{p \in \mathcal{P}^k} (c^k x^{p,k}) \lambda_p^k \\ & \sum_{k \in K} \sum_{p \in \mathcal{P}^k} x_{a_{j+}^k}^{p,k} \lambda_p^k = 1, \quad j \in J \\ & \sum_{p \in \mathcal{P}^k} \lambda_p^k \leq 1 \quad \forall k \in K \\ & \lambda_p^k \in \mathbb{N} \quad \forall k \in K, p \in \mathcal{P}^k \end{aligned}$$

## Elementarity sets

$$E_j = \left\{ a_{j+}^k \right\}_{k \in K}, \quad j \in J$$

## Example 2: Bin Packing and Vector Packing

### The problem data

- ▶ set  $J$  of items, bin capacity  $Q_1$  (and  $Q_2$ )
- ▶ item weights  $w_{j,1}$  (and  $w_{j,2}$ )

### Graph $G$

- ▶  $|J| + 2$  nodes :  $v_{\text{source}} = v_0, v_1, \dots, v_{|J|}, v_{\text{sink}} = v_{|J|+1}$
- ▶  $O(|J|^2)$  arcs :  $a_{j'j} = (v_{j'}, v_j), j', j \in J \cup \{0, |J| + 1\}, j' < j$ .
- ▶ One (two) resources with consumption  $q_{a_{j'j},1} = w_{j1}$   
( $q_{a_{j'j},2} = w_{j2}$ )
- ▶ Consumption bounds:

$$[l_{v_j,1}, u_{v_j,1}] = [0, Q_1], [l_{v_j,2}, u_{v_j,2}] = [0, Q_2], j \in J \cup \{0, |J| + 1\}$$

# Bin Packing and Vector Packing: formulation

$$\begin{aligned} \min \quad & \sum_{p \in \mathcal{P}} \sum_{j \in \mathcal{J}} x_{a_{0j}}^p \lambda_p \\ & \sum_{p \in \mathcal{P}} \sum_{\substack{j' \in \mathcal{J}: \\ j' < j}} x_{a_{j'j}}^p \lambda_p = 1, \quad j \in \mathcal{J} \\ & \lambda_p \in \mathbb{N}, \quad \forall p \in \mathcal{P} \end{aligned}$$

Elementarity sets

$$E_j = \{a_{j'j}\}_{j' \in \mathcal{J}, j' < j}, \quad j \in \mathcal{J}$$

## Example 3: Team Orienteering Problem

### The problem data

- ▶ Set  $J$  of locations, start and end points,  $m$  team members
- ▶ Profits  $c_j$ ,  $j \in J$ , maximum tour length  $Q$ .

### Graph $G$

- ▶  $|J| + 2$  nodes :  $v_{\text{source}} = v_0, v_1, \dots, v_{|J|}, v_{\text{sink}} = v_{|J|+1}$ .
- ▶  $O(|J|^2)$  arcs :  $a_{j'j} = (v_{j'}, v_j), j' \in J \cup \{0\}, j \in J \cup \{|J| + 1\}$ .
- ▶ One resource:  $q_{a_{j'j}} = \text{Euclidean distance between } j' \text{ and } j$
- ▶ Consumption bounds:  $[l_{v_j}, u_{v_j}] = [0, Q], j \in J \cup \{0, |J| + 1\}$ .

### Variants

- ▶ Capacitated Team Orienteering Problem (CTOP)
- ▶ Capacitated Profitable Tour Problem (CPTP)

# Team Orienteering Problem: formulation

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j y_j \\ \sum_{p \in \mathcal{P}} \sum_{j' \in J \cup \{0\}} x_{a_{j'j}}^p \lambda_p + y_j &= 1, \quad j \in J \\ \sum_{p \in \mathcal{P}} \lambda_p &= m, \\ y_j &\in \{0, 1\} \quad \forall j \in J \\ \lambda_p &\in \mathbb{N} \quad \forall p \in \mathcal{P} \end{aligned}$$

## Elementarity sets

$$E_j = \{a_{j'j}\}_{j' \in J \cup \{0\}}, \quad j \in J$$

## Example 4: Pickup and Delivery with Time Windows

### The problem data

- ▶ Set  $J$  of requests, set  $I$  of pickup ( $I^p$ ), delivery ( $I^d$ ) and depot ( $j = 0, |I| - 1$ ) points, distances  $d_{j'j}, j', i \in I$
- ▶  $m$  vehicles with capacity  $Q$  and fixed cost  $f$
- ▶ Request sizes  $w_j, j \in J$ .
- ▶ Time windows  $[b_i, e_i]$  and service times  $s_i, i \in I$ .

### Graph $G$

- ▶  $|I|$  nodes :  $v_{\text{source}} = v_0, v_1, \dots, v_{\text{sink}} = v_{|I|-1}$ .
- ▶  $O(|I|^2)$  arcs :  $a_{j'j} = (v_{j'}, v_j), j' \in I \setminus \{|I| - 1\}, j \in J \setminus \{0\}$ .
- ▶  $2 + |J|$  resources :
  - ▶ Time (main) resource:  $q_{a_{j'i},1} = s_{j'} + d_{j'i}$
  - ▶ Capacity resource:  $q_{a_{j'i},2} = \begin{cases} w_i, & i \in I^p, \\ -w_{i-|J|}, & i \in I^d. \end{cases}$
  - ▶ One binary resource for every request

# Pickup and Delivery with Time Windows: formulation

$$\begin{aligned} \min \sum_{p \in \mathcal{P}} \left( \sum_{i \in I \setminus \{0\}} f \cdot x_{a_{0i}}^p + \sum_{i', i \in I} d_{i'i} x_{a_{i'i}} \right) \lambda_p \\ \sum_{p \in \mathcal{P}} \sum_{i' \in I \setminus \{i\}} x_{a_{i'i}}^p \lambda_p = 1, \quad i \in I^p \\ \sum_{p \in \mathcal{P}} \lambda_p \leq m, \\ \lambda_p \in \mathbb{N} \quad \forall p \in \mathcal{P} \end{aligned}$$

## Elementarity sets

$$E_i = \{a_{i'i} \mid i' \in I \setminus \{i\}\}, \quad i \in I^p$$

## Non-robust rank-1 cuts [Jepsen et al., 2008] [Pecin et al., 2017c]

Each cut  $\eta \in \mathcal{N}$  is obtained by a **Chvátal-Gomory rounding** of a set  $\mathcal{C}_\eta \subseteq \mathcal{J}$  of **set packing constraints** using a vector of multipliers  $\rho^\eta$  ( $0 < \rho_j^\eta < 1, j \in \mathcal{C}_\eta$ )

$$\sum_{k \in K} \sum_{p \in \mathcal{P}^k} \left[ \sum_{j \in \mathcal{C}_\eta} \rho_j^\eta \sum_{a \in E_j} x_a^p \right] \lambda_p \leq \left[ \sum_{j \in \mathcal{C}_\eta} \rho_j^\eta \right]$$

- ▶ Each active cut  $\eta \in \mathcal{N}$  **adds one resource** in the RCSP pricing
- ▶ **Limited-memory technique** [Pecin et al., 2017b] is critical to reduce the impact on the pricing problem difficulty: for each  $E_j, j \in \mathcal{C}_\eta$ , a memory (on vertices or on arcs) is defined at the separation, **making the resource local**



## Enumeration of elementary paths [Baldacci et al., 2008]

- ▶ We try to enumerate all elementary **paths with reduced cost smaller than the current primal-dual gap** in each graph  $G^k$
- ▶ A labelling algorithm is used for enumeration
- ▶ If  $G^k$  is enumerated, the **pricing can be done by inspection**
- ▶ If all graphs are enumerated and the total number of paths is “small”, the problem can be **finished by a MIP solver**

### Sufficient condition to apply

- ▶ Arcs in the **same elementarity set** should have the **same coefficients** in the “core” master constraints (excluding cuts and branching constraints)
- ▶ Arcs not in an elementarity set should not participate in the “core” master constraints

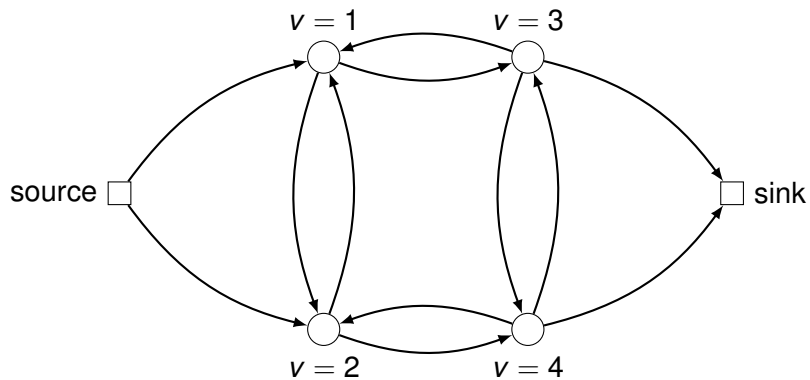
## Pricing: structure of RCSP problem instances

- ▶ A directed **graph**  $G = (V, A)$ .
- ▶ **Unrestricted in sign** reduced **costs**  $\bar{c}_a$  on arcs  $a \in A$
- ▶ Main resources with **non-negative resource consumption**  $d_{a,r} \in \mathbb{R}_+$ ,  $a \in A$ ,  $r \in M$
- ▶ Possibly other resources with unrestricted resource consumption  $d_{a,r} \in \mathbb{R}$ ,  $a \in A$ ,  $r \in R \setminus M$ .
- ▶ Up to  $\approx 500 - 1000$  of (more or less) **local** binary or (small) integer **resources**

### We want to

Find a walk from the source to the sink minimizing the total reduced cost respecting the resource constraints, as well as many other (up to 1000) different near-optimal feasible walks

## Pricing: original graph



# Pricing : the bucket graph [Sadykov et al., 2017]

Example with two main resources

$v = 1$



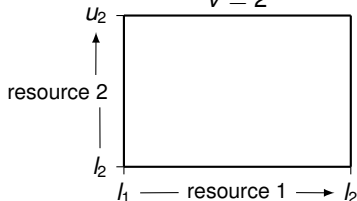
$v = 3$



source

sink

$v = 2$

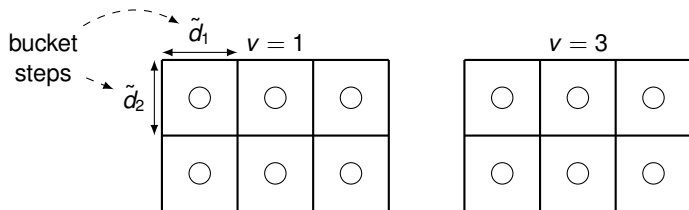


$v = 4$



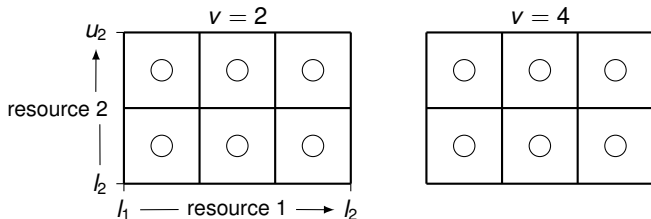
# Pricing : the bucket graph [Sadykov et al., 2017]

Example with two main resources



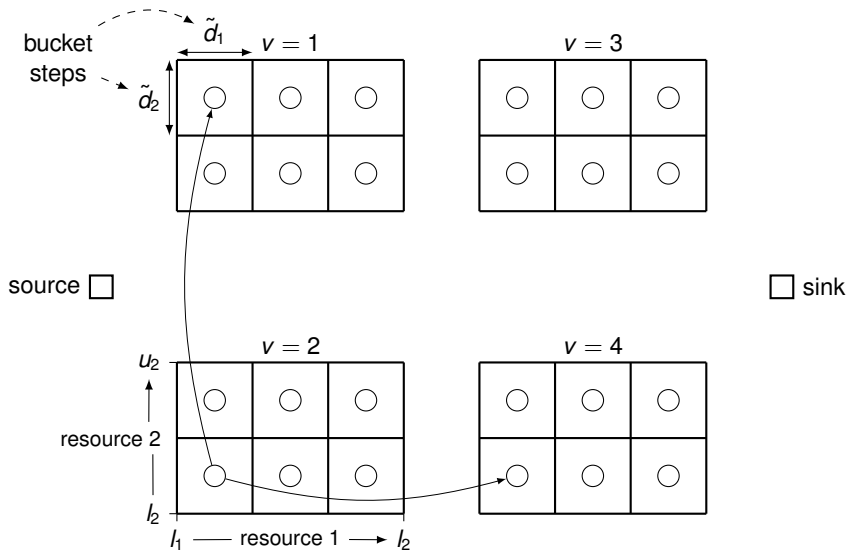
source

sink



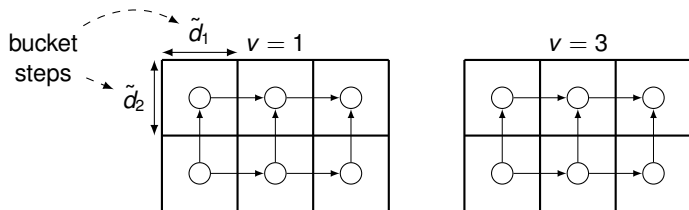
# Pricing : the bucket graph [Sadykov et al., 2017]

Example with two main resources



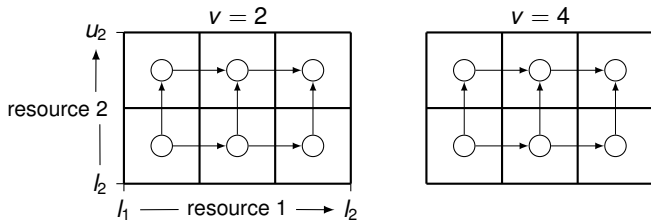
# Pricing : the bucket graph [Sadykov et al., 2017]

Example with two main resources



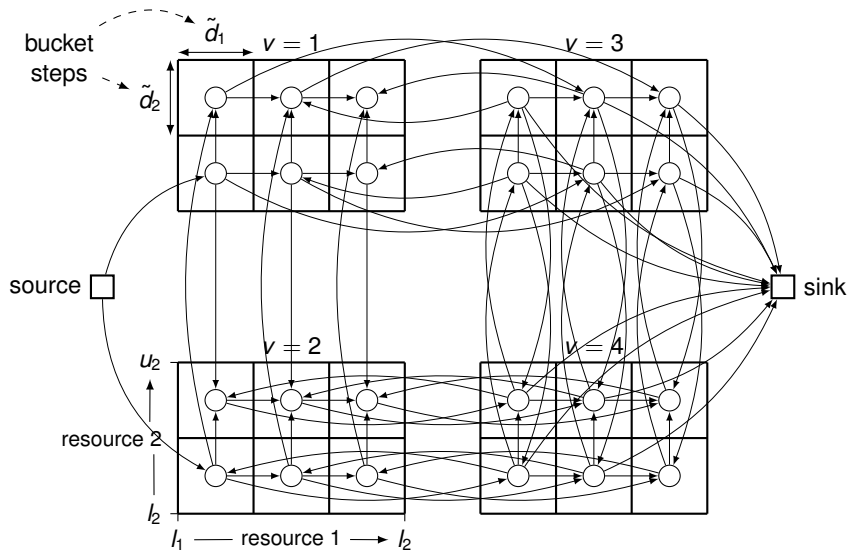
source

sink



# Pricing : the bucket graph [Sadykov et al., 2017]

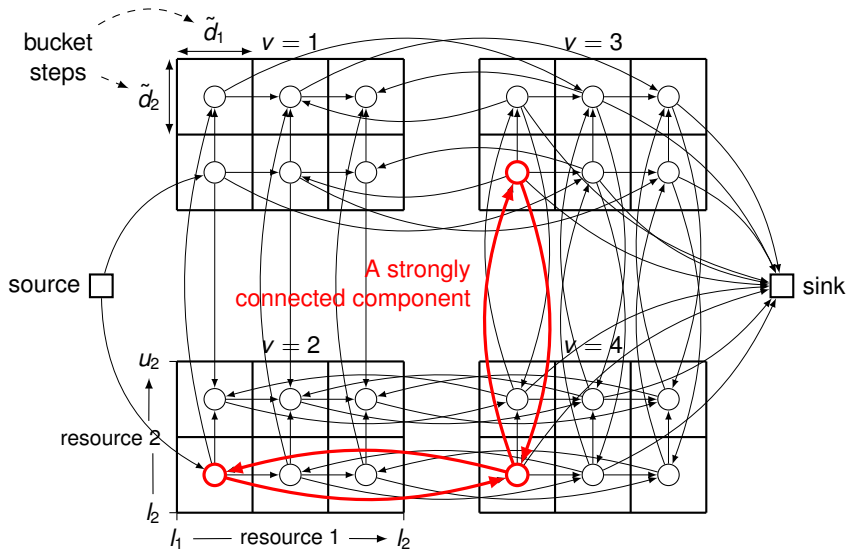
Example with two main resources





# Pricing : the bucket graph [Sadykov et al., 2017]

Example with two main resources



# Generalized Assignment: computational results

## Comparison with on classic instances in class D

$ K $	$ J $	[Posta et al., 2012]	Our BCP
5	100	6s	58s
10	100	55s	1m19s
20	100	1h47m	12m48s
5	200	4s	4m30s
10	200	26m02s	1h40m

## Comparison on random instances in class D

$ K $	$ J $	Solved within 3000 seconds	
		[Nauss, 2003]	Our BCP
25	90	0/10	10/10
25	100	0/10	6/10
30	100	0/10	9/10

Gurobi 7.5 can solve only 1 instance in 3 hours.

# Bin Packing: computational results

Comparison with the best BCP [Belov and Scheithauer, 2006]  
(results from [Delorme et al., 2016])

$N$  — number of instances solved within 1 hour

$T$  — average time

Instance class	$ J $	Best BCP		Our BCP	
		$N$	$T$	$N$	$T$
Falkenauer T	60-501	80/80	>24s	80/80	35s
Hard28	160-200	28/28	7s	28/28	54s
ANI200	201	50/50	2m24s	50/50	56s
ANI400	402	1/50	>1h	50/50	14m17s

## State-of-the-art

Iterative aggregation-disaggregation approach by  
[Clautiaux et al., 2017]

# Vertex Packing: computational results

Comparison with the state-of-the-art on the **harderst**  
**2-resources 200-items instances** by [Caprara and Toth, 2001]

Algorithm	Class 1		Class 4	
	<i>N</i>	<i>T</i>	<i>N</i>	<i>T</i>
[Brandão and Pedroso, 2016]	10/10	2h07m	0/10	>2h
[Hu et al., 2017]	0/10	>10m	0/10	>10m
[Hessler et al., 2017]	3/10	>47m	0/10	>1h
Our BCP	10/10	<b>3m53s</b>	<b>10/10</b>	7m40s
	Class 5		Class 9	
[Brandão and Pedroso, 2016]	0/10	>2h	0/10	>2h
[Hu et al., 2017]	7/10	>6m	0/10	>10m
[Hessler et al., 2017]	7/10	>41m	0/10	>1h
Our BCP	7/10	>27m	<b>8/10</b>	>19m

# Team Orienteering: computational results

## Standard Team Orienteering problem

Comparison with the state-of-the-art on the most difficult class 4 of classic instances with 100 locations by

[Chao et al., 1996]

Algorithm	$N$	$T$
[Bianchessi et al., 2018]	39/60	>15m
Our BCP	55/60	>8m

## Capacitated Team Orienteering problem

Comparison with the state-of-the-art on the basic and most difficult instances with 51-200 locations by [Archetti et al., 2009]

Algorithm	$N$	$T$
[Archetti et al., 2013]	6/14	>35m
Our BCP	13/14	>7m

## Other computational results

### Capacitated Profitable Tour Problem

Comparison with the state-of-the-art on the **open instances with 51-200** locations by [Archetti et al., 2009]

Algorithm	$N$	$T$
[Bulhoes et al., 2018a]	0/28	>1h
Our BCP	<b>24/28</b>	>9m

### Pickup and Delivery Problem With Time Windows

Comparison with the state-of-the-art on 40 **classic instances with 30-75 requests** by [Ropke and Cordeau, 2009]

Algorithm	$N$	$T$
[Baldacci et al., 2011a]	32/40	>13m44s
[Gschwind et al., 2018]	33/40	>12m31s
Our BCP	<b>40/40</b>	5m06s

Our BCP is not competitive on instances by [Li and Lim, 2003].

# Conclusions

- ▶ Our generic BCP algorithm showed good performance **approaching or outperforming the state-of-the-art** for apparently different problems
- ▶ Relatively short development time

## Issues to address

- ▶ Dependence on initial primal bounds in many cases
- ▶ Involved parameterisation
- ▶ The code needs much more testing before it can be used externally

# Perspectives

July 2018 A state-of-the-art **web-based solver** for some **Vehicle Routing** problems for ISMP conference

- ▶ Capacitated VRP
- ▶ VRP with time windows
- ▶ Heterogeneous fleet VRP
- ▶ Multi-depot VRP
- ▶ Site-dependent VRP

≈ 2019 A **Julia-JuMP based modelling tool** with the precompiled library implementing the modern Branch-Cut-and-Price



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