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## A tractable Multi-Partitions Clustering

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## Outline

- 1 Extension of the variable selection to variable clustering
  - Variable selection in clustering
  - Multiple Gaussian Mixture
  - Proposed Multiple Partitions Mixture
  - Properties of the model
- 2 Parameters estimation and model selection
  - Maximum likelihood inference
  - Penalized observed-data likelihood
  - Integrated complete-data likelihood
- 3 Numerical experiments
  - NBA team data
  - Wine data
- 4 Conclusion and perspectives

## Data

$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  composed of  $n$  independent observations  $\mathbf{x}_i = (x_{i1}, \dots, x_{id})$  defined on  $\mathbb{R}^d$ .

## Goal

Cluster the data in  $G$  clusters.

## What variables use in clustering?

- Well-posed problem in the supervised classification setting with objective criteria: error rate, AUC, ...
- Ill-posed problem in clustering since the class variable is not known by advance. Thus, what are the most relevant variables with respect to this unknown variable?
- Pragmatic solution 1: Prior choice of the practitioner among available variables (according to some focus)
- Pragmatic solution 2: Posterior analysis of the correlation between the predicted cluster (based on all the variables) and each variable

## The model based clustering solution

- Mixture models allow to perform clustering by modelling the distribution of the data as a mixture of  $G$  components each one corresponding to a cluster.
- Thus possibility to suppose that some variables do not depend (directly) on the cluster in the probabilistic model.

## Some references

- Raftery & Dean (2006): some classifying, and some redundant variables. Redundant variables independent of the cluster given the classifying variables.
- Maugis & *al.* (2009): refinement of Raftery & Dean (2006) by specifying the role of each variable (classifying, redundant or independant).

## Advantages of these approaches

- Improve the accuracy of the clustering by decreasing the variance of the estimators.
- Allow some specific interpretation of the classifying variables.

## Limitations of these approaches

- Combinatorial problem to select the best model with these refined approaches
- Search too hard to perform when the number of variables is large

## Solution: use simpler models for a better search (Marbac & Serdki 2016,2017)

- Assumption of conditional independence of the classifying variables given the cluster
- Non-classifying variables are independent
- Optimisation of the integrated classification likelihood (ICL)
- Better results than previous approaches on large number of variables with moderated sample size
- The independence assumption allows to easily consider the heterogeneous data setting

## Several clustering variables

- The variables in the data can convey several clustering view points with respect to different groups of variables
- Allow to find some clustering which could be hidden by other variables

## Some references

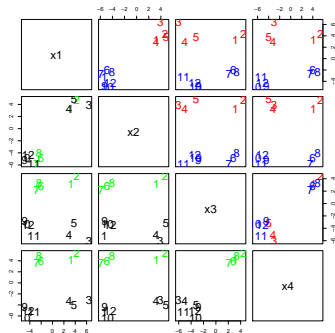
- Galimberti & *al.* (2007): First proposition of a multiple Gaussian mixture model
- Galimberti & *al.* (2017): Refinement of the previous model with ideas similar to Raftery & Dean (2006) et Maugis & *al.* (2009)

## Remarks

- Smart modelling of the role of each variable
- Search hard to perform when the number of variables is large
- Specific to the Gaussian setting

## Illustration of a multiple partition

id	$x_1$	$x_2$	$x_3$	$x_4$	$z_1$	$z_2$
1	3.23	3.28	3.14	4.08	1	1
2	4.26	4.7	4.41	5.36	1	1
3	6.43	3.94	-6.09	-3.03	1	2
4	2.93	3.22	-5.05	-3.2	1	2
5	3.77	4.88	-3.21	-4	1	2
6	-2.03	-4.9	2.81	4	2	1
7	-2.78	-5.87	2.11	3.57	2	1
8	-2.38	-4.06	3.35	4.91	2	1
9	-4.88	-5.26	-2.86	-4.42	2	2
10	-5.01	-4.83	-3.3	-6.42	2	2
11	-3.25	-5.84	-5.23	-5.35	2	2
12	-4.28	-4.36	-3.38	-5.14	2	2





## Main assumptions

- The variables can be decomposed in  $B$  independent blocks.
- The block  $b$  follows a mixture with  $G_b$  components (for  $b = 1, \dots, B$ ), with the assumption of class conditional independence of the variables.

## Notations

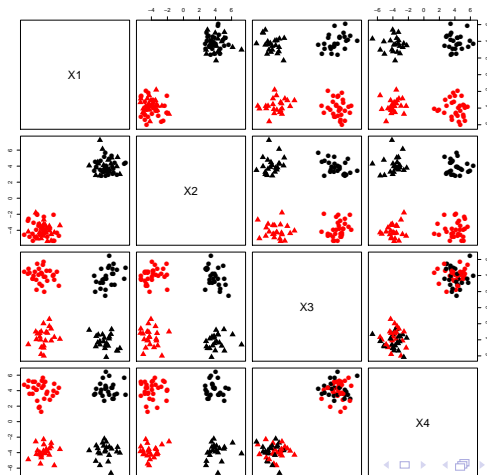
- $\omega = (\omega_j; j = 1, \dots, d)$  the repartition of the variables in blocks;  $\omega_j = b$  if variable  $j$  belongs to block  $b$ .
- $\mathbf{m} = (G_1, \dots, G_B, \omega)$  defines the model
- $\Omega_b = \{j : \omega_j = b\}$  the subset of variables belonging to block  $b$
- $\theta = (\pi, \alpha)$  model parameters
- $p(\cdot | \alpha_{jg})$  the pdf of the distribution of parameters  $\alpha_{jg}$ .

## Probability distribution function of $\mathbf{x}_i$

$$p(\mathbf{x}_i | \mathbf{m}, \theta) = \prod_{b=1}^B p(\mathbf{x}_{i\{b\}} | \mathbf{m}, \theta) \text{ with } p(\mathbf{x}_{i\{b\}} | \mathbf{m}, \theta) = \sum_{g=1}^{G_b} \pi_{bg} \prod_{j \in \Omega_b} p(x_{ij} | \alpha_{jg}),$$

Illustration :

- $n = 100$  from a MPM with  $B = 2$  blocks of two variables.
- Variable 1 and 2 belong to block 1 and variables 3 and 4 in block 2
- Each block follows a bi-component Gaussian mixture (i.e.,  $G_b = 2$ ) with equal proportions (i.e.,  $\pi_{bg} = 1/2$ ) and  $\mu_{j1} = 4$ ,  $\mu_{j2} = -4$  and  $\sigma_{jg} = 1$ .



## Remarks

- Different partitions explained by subsets of variables.
- Generalizes approaches used for variable selection in model-based clustering (if  $B = 2$  and  $G_1 = 1$  then variables belonging to block 1 are not relevant for the clustering, while variables belonging to block 2 are relevant)
- MGMM permits **variable selection** and **multiple partitions** explained by **subsets of variables** (variables classification).
- Sparse model: number of parameters  $\nu_m = \sum_{b=1}^B (G_b - 1) + G_b \sum_{j \in \Omega_b} \nu_j$
- Better model search expected than in the model of Galimberti & al. (2017)
- Natural extension to the heterogeneous data setting

## Identifiability

Model identifiability is directly obtained from the identifiability of Gaussian mixture with local independence (Teicher, 1963, 1967).

## Observed-data likelihood for sample $\mathbf{x}$ and model $m$

$$\ell(\boldsymbol{\theta}|\mathbf{m}, \mathbf{x}) = \sum_{b=1}^B \sum_{i=1}^n \ln \left( \sum_{g=1}^{G_b} \pi_{bg} \prod_{j \in \Omega_b} p(x_{ij}|\boldsymbol{\alpha}_{jg}) \right).$$

## Latent partitions

- $B$  independent mixtures
- $\mathbf{z} = (\mathbf{z}_{ib}; i = 1, \dots, n; b = 1, \dots, B)$  vectors of the component memberships
- $\mathbf{z}_{ib} = (z_{ib1}, \dots, z_{ibG_b})$  where  $z_{ibg} = 1$  if observation  $i$  arose from component  $g$  for block  $b$ , and  $z_{ibg} = 0$  otherwise

## Completed-data likelihood for sample $\mathbf{x}$ and model $m$

$$\ell(\boldsymbol{\theta}|\mathbf{m}, \mathbf{x}, \mathbf{z}) = \sum_{b=1}^B \ln p(\mathbf{z}_b|\boldsymbol{\pi}_b) + \sum_{j=1}^d \ln p(\mathbf{x}_j|\mathbf{z}_{\omega_j}, \boldsymbol{\alpha}_j),$$

where  $\ln p(\mathbf{z}_b|\boldsymbol{\pi}_b) = \sum_{i=1}^n \sum_{g=1}^{G_b} z_{ibg} \ln \pi_{bg}$  and

$\ln p(\mathbf{x}_j|\mathbf{z}_b, \boldsymbol{\alpha}_j) = \sum_{i=1}^n \sum_{g=1}^{G_b} z_{ibg} \ln p(x_{ij}|\boldsymbol{\alpha}_{jg})$ .

## EM algorithm

Starting from the initial value  $\theta^{[0]}$ , iteration  $[r]$  is composed of two steps:  
**E-step** Computation of the fuzzy partitions  $t_{ibg}^{[r]} := \mathbb{E}[Z_{ibg} | \mathbf{x}_{i\{b\}}, \mathbf{m}, \theta^{[r-1]}]$ ,  
 hence for  $b = 1, \dots, B$ , for  $g = 1, \dots, G_b$ , for  $i = 1, \dots, n$

$$t_{ibg}^{[r]} = \frac{\pi_{bg}^{[r-1]} \prod_{j \in \Omega_b} p(x_{ij} | \alpha_{jg}^{[r-1]})}{\sum_{k=1}^{G_b} \pi_{bk}^{[r-1]} \prod_{j \in \Omega_b} p(x_{ij} | \alpha_{jk}^{[r-1]})},$$

**M-step** Maximization of the expected value of the complete-data log-likelihood over the parameters,

$$\pi_{bg}^{[r]} = \frac{n_{bg}^{[r]}}{n} \text{ and } \alpha_{jg}^{[r]} = \arg \max_{\alpha_{jg} \in \Theta_j} Q(\alpha_{jg} | \mathbf{x}_j, \mathbf{t}_{\omega_{jg}}^{[r]}),$$

where  $Q(\alpha_{jg} | \mathbf{x}_j, \mathbf{t}_b) = \sum_{i=1}^n t_{ibg} \ln p(x_{ij} | \alpha_{jg})$ .

## Remarks

- Independence between the  $B$  blocks of variables permits to maximize the observed-data log-likelihood on each block separately.
- Possible modification to perform the block estimation and the parameter inference simultaneously.
- In practice the number of blocks, the repartition of variables into blocks, and the number of classes per block are unknown.

## Model collection $\mathcal{M}$

$$\mathcal{M} = \{\mathbf{m} : \omega_j \leq B_{\max} \text{ and } G_b \leq G_{\max}; j = 1, \dots, d; b = 1, \dots, B_{\max}\},$$

where  $B_{\max}$  is the maximum number of blocks and  $G_{\max}$  is the maximum number of components within block.

## Model selection

Model selection often achieved by searching the model  $\mathbf{m}^*$  maximizing the BIC criterion which is defined by

$$\text{BIC}(\mathbf{m}) = \max_{\boldsymbol{\theta}_{\mathbf{m}}} \ell_{\text{pen}}(\boldsymbol{\theta}_{\mathbf{m}} | \mathbf{m}, \mathbf{x})$$

where

$$\ell_{\text{pen}}(\boldsymbol{\theta}_{\mathbf{m}} | \mathbf{m}, \mathbf{x}) = \ell(\boldsymbol{\theta}_{\mathbf{m}} | \mathbf{m}, \mathbf{x}) - \frac{\nu_{\mathbf{m}}}{2} \ln n.$$

## Remark

$$(\mathbf{m}^*, \hat{\boldsymbol{\theta}}_{\mathbf{m}^*}) = \arg \max_{(\mathbf{m}, \boldsymbol{\theta}_{\mathbf{m}})} \ell_{\text{pen}}(\boldsymbol{\theta}_{\mathbf{m}} | \mathbf{m}, \mathbf{x})$$

## Penalised completed likelihood

$$\begin{aligned}\ell_{pen}(\boldsymbol{\theta}_m | \mathbf{m}, \mathbf{x}, \mathbf{z}) &= \ell(\boldsymbol{\theta}_m | \mathbf{m}, \mathbf{x}, \mathbf{z}) - \frac{\nu_m}{2} \log n \\ &= \sum_{b=1}^B \ln p(\mathbf{z}_b | \boldsymbol{\pi}_b) - \frac{G_b - 1}{2} \ln n + \sum_{j=1}^d \ln p(\mathbf{x}_j | \mathbf{z}_{\omega_j}, \boldsymbol{\alpha}_j) - \frac{\nu_j G_{\omega_j}}{2} \ln n,\end{aligned}$$

## Consequence

- For  $\mathbf{z}_b$  fixed: possibility to re-affect each variable **individually** to the most accurate block.
- Thus computationnaly attractive



## Combinatorial model selection through a modified the EM algorithm for $B$ and $(G_1, \dots, G_B)$ fixed : choice of $\mathbf{m} \Leftrightarrow$ choice of $\omega$

The EM algorithm to achieve  $\arg \max_{(\omega, \theta)} \ell_{pen}(\theta_{\mathbf{m}} | \mathbf{m}, \mathbf{x})$ , starting from  $(\omega^{[0]}, \theta^{[0]})$  is at iteration  $[r]$ :

**E-step** Computation of the fuzzy partitions  $t_{ibg}^{[r]} := \mathbb{E}[Z_{ibg} | \mathbf{x}_i, \mathbf{m}, \theta^{[r-1]}]$ , hence for  $b = 1, \dots, B$ , for  $g = 1, \dots, G_b$ , for  $i = 1, \dots, n$

$$t_{ibg}^{[r]} = \frac{\pi_{bg}^{[r-1]} \prod_{j \in \Omega_b^{[r-1]}} p(x_{ij} | \alpha_{jg}^{[r-1]})}{\sum_{k=1}^{G_b} \pi_{bk}^{[r-1]} \prod_{j \in \Omega_b^{[r-1]}} p(x_{ij} | \alpha_{jk}^{[r-1]})},$$

**M-step1** Updating the affectation of the variables to blocks

$$\omega_j^{[r]} = \arg \max_{\omega_j \in \{1, \dots, B\}} \left( \sum_{g=1}^{G_{\omega_j}} \max_{\alpha_{jg} \in \Theta_j} Q(\alpha_{jg} | \mathbf{x}_j, \mathbf{t}_{\omega_j g}^{[r]}) - \frac{\nu_j G_{\omega_j}}{2} \ln n \right),$$

**M-step2** Updating the model parameters

$$\pi_{bg}^{[r]} = \frac{n_{bg}^{[r]}}{n} \text{ and } \alpha_{jg}^{[r]} = \arg \max_{\alpha_{jg} \in \Theta_j} Q(\alpha_{jg} | \mathbf{x}_j, \mathbf{t}_{\omega_j g}^{[r]}).$$

## Integrated complete-data likelihood

$$p(\mathbf{x}, \mathbf{z} | \mathbf{m}) = \int p(\mathbf{x}, \mathbf{z} | \mathbf{m}, \boldsymbol{\theta}) p(\boldsymbol{\theta} | \mathbf{m}) d\boldsymbol{\theta}.$$

## Assumptions

- Independence between the prior distributions
- Standard conjugate priors
- Closed form of the complete-data integrated likelihood

## MICL (maximum integrated complete-data likelihood) criterion

$$\text{MICL}(\mathbf{m}) = \ln p(\mathbf{x}, \mathbf{z}_m^* | \mathbf{m}) \text{ with } \mathbf{z}_m^* = \arg \max_{\mathbf{z}_m} \ln p(\mathbf{x}, \mathbf{z} | \mathbf{m}).$$

Thus

$$(\mathbf{m}^*, \mathbf{z}_{m^*}^*) = \arg \max_{(\mathbf{m}, \mathbf{z}_m)} \ln p(\mathbf{x}, \mathbf{z} | \mathbf{m}).$$

## Motivations

- Criteria based on the integrated complete-data likelihood are popular for model-based clustering
- Take into account the clustering purpose : model the data distribution and provide well-separated components

## Optimisation of MICL over $(\mathbf{z}, \omega)$ for $B$ and $G_1, \dots, G_B$ fixed

Starting at the initial value  $\omega^{[0]}$ , each  $\omega_j$  is uniformly sampled among  $\{1, \dots, B\}$ , the algorithm at iteration  $[r]$  is

**Partition step:** find  $\mathbf{z}_b^{[r]}$  such that for all  $b = 1, \dots, B$

$$p(\mathbf{x}_{\{b\}}^{[r-1]}, \mathbf{z}_b^{[r]}) \geq p(\mathbf{x}_{\{b\}}^{[r-1]}, \mathbf{z}_b^{[r-1]}),$$

where  $\mathbf{x}_{\{b\}}^{[r-1]} = (\mathbf{x}_j; \omega^{[r-1]} = b)$ .

**Model step:** find  $\omega^{[r]}$  such that for  $j = 1, \dots, d$

$$\omega_j^{[r]} = \arg \max_{b \in \{1, \dots, B\}} p(\mathbf{x}_j | \mathbf{z}_b^{[r]}).$$

## Data description

NBA teams for the season 2016/2017 described by 16 numerical variables<sup>1</sup>

- total minutes played (min)
- field goals made rate (fgmr)
- field goals attempted (fga)
- three-pointers made rate (3pmr)
- three-pointer attempted (3pa)
- free throws made (ftm)
- free throw attempted (fta)
- offensive rebounds (or)
- total rebounds (tr)
- assists (as)
- steals (st)
- turnovers (to)
- blocks (bk)
- personal fouls (pf)
- technical fouls (tc)
- points (pts)

Model selection with BIC

$B$	BIC	Time (s)	Block	G	variables
1	-1932	7	1	2	all the variables
2	-1915	47	1	3	fgmr, fga, 3pmr, 3pa, tr, as, to, pts
			2	1	min, ftmr, fta, orr, st, bk, pf, tc
3	-1909	170	1	2	fgmr, 3pmr, pf
			2	2	fga, 3pa, tr, as, st, to, pts
			3	1	min, ftmr, fta, orr, bk, tc

<sup>1</sup><http://www.dougstats.com/16-17RD.Team.Opp.txt>

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## Model parameters

### First block: offensive vs defensive teams

Three features: field goals made rate, three points made rate and personal fouls

		fmgr	3pmr	pf
offensive teams: ( $\pi_{11} = 0.57$ ) high shooting ability low personal fouls	mean	0.468	0.371	1628.042
	sd	0.009	0.010	76.138
defensive teams: ( $\pi_{12} = 0.43$ ) low shooting ability high personal fouls	mean	0.446	0.342	1635.503
	sd	0.005	0.010	173.701

### Second block: two better statistics teams for general performances vs others

Seven features: field goal attempted, 3 points attempted, total rebounds, assists, steals, turnovers and points

		fga	3pa	tr	as	st	to	pts
GS Warriors: ( $\pi_{11} = 0.07$ ) Houston Rockets	mean	7144	2934	3642	2281	728	1186	9481
	sd	3	372	3	211	59	3	22
Other: ( $\pi_{11} = 0.93$ ) teams	mean	6992	2162	3564	1825	626	1091	8600
	sd	183	262	141	131	45	105	259

### Third block: Six features detected as irrelevant for clustering.

## Data description

### Data<sup>1</sup>

- 27 chemical and physical properties of three types of Italian wines: Barolo, Grignolino, Barbera
- Data collected during the time period of 1970–1979

### Models selected by BIC

$B$	BIC	Time	Block	G	ARI
1	-6025.00	30	1	4	0.78
2	-5947.88	280	1	3	0.87
			2	4	0.16
3	-5921.42	1590	1	4	0.74
			2	4	0.20
			3	2	0.02
4	-5918.06	6065	1	4	0.75
			2	2	0.21
			3	3	0.02
			4	2	0.00

<sup>1</sup>available in the package pgmm

## Model interpretation

### Block 1: the type of wines (ARI=0.75)

19 variables: Alcohol, Sugar-free Extract, Tartaric Acid, Uronic Acids, Alcalinity of Ash, Calcium, Magnesium, Phosphate, Total Phenols, Flavanoids, Non-flavanoid Phenols, Proanthocyanins, Color Intensity, Hue, OD280/OD315 of Diluted Wines, OD280/OD315 of Flavanoids, Glycerol, 2-3-Butanediol, Proline

	Barolo	Grignolino	Barbera
Class 1	0	45	0
Class 2	0	5	48
Class 3	58	1	0
Class 4	1	20	0

### Block 2: year of production

4 variables: Fixed Acidity, Malic Acid, pH, Total Nitrogen

	Year								
	1970	1971	1972	1973	1974	1975	1976	1978	1979
Class 1	8	25	4	27	31	3	1	0	0
Class 2	1	3	3	2	14	6	16	29	5



## Conclusion

- Proposition of model-based clustering with several class variables, each one explaining the heterogeneity of a block of variables:
  - Find groups of variables producing the same clustering of the individuals
  - Interpret the clustering produced for each group of variables
- Model search performed simultaneously with parameters estimation
- Proposed model can be used in the heterogeneous data settings

## Perspectives

- Consider the semi-supervised setting in the multi-partition framework for new partitions discovery: *i. e.*  $\mathbf{z}_1$  known and  $\mathbf{z}_2$  unknown.
- Extension to heterogeneous co-clustering by adding one level of latent variable to summarize the multi-partition by a single partition, while keeping the partition of the variables.
- Derive some k-means type multi-partition similar to Witten & Tibshirani (2010) in the variable selection framework to deal with the very high dimensional setting.