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Anne Canteaut

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# Exploiting algebraic properties of block ciphers

**Anne Canteaut**

`Anne.Canteaut@inria.fr`

`https://www.paris.inria.fr/secret/Anne.Canteaut/`

COST Training School, Torremolinos, February 2018

# Random behaviour of cryptographic primitives

Cryptographic primitives should behave like random functions.

Security proofs of many constructions assume random building blocks

- modes of operation;
- sponge construction...

A distinguishing property may lead to some attacks

e.g., finding the plaintext among a few possibilities.

For iterated constructions

a distinguisher on a round-reduced version may be exploited for recovering the key (e.g., last-round attacks).

## Algebraic Normal Form

**Algebraic Normal Form of  $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ :**

unique polynomial representation in  $\mathbb{F}_2[x_1, \dots, x_n] / (x_1^2 - x_1, \dots, x_n^2 - x_n)$ .

**Monomials of  $n$  variables**

$$\prod_{i=1}^n x_i^{u_i} = x^u \text{ with } u \in \mathbb{F}_2^n$$

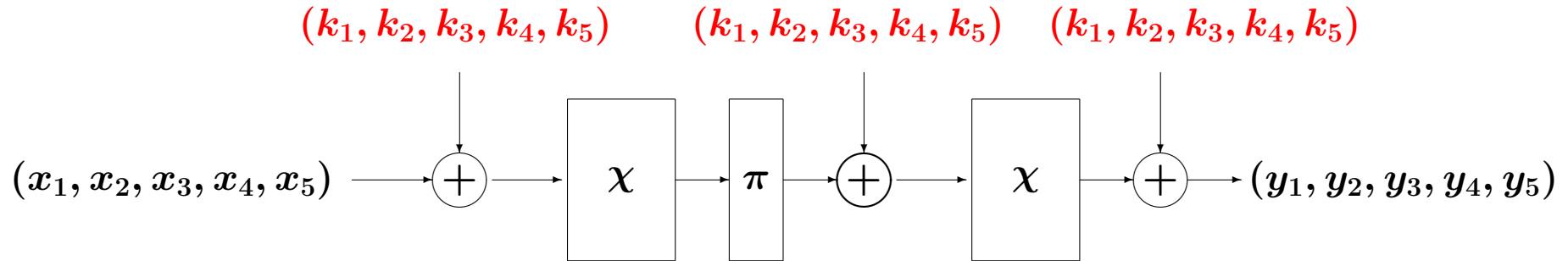
**Example:**

For  $u = (0101)$ ,

$$x^u = x_4^0 x_3^1 x_2^0 x_1^1 = x_3 x_1$$

$$f(x_1, \dots, x_n) = \sum_{u \in \mathbb{F}_2^n} c_u x^u \text{ with } c_u \in \mathbb{F}_2$$

## Example



$$\begin{aligned}
 y_1 = & k_2k_3k_4k_5 + k_2k_3k_4x_5 + k_2k_3k_5x_4 + k_2k_3x_4x_5 + k_2k_4k_5x_3 + k_2k_4x_3x_5 + k_2k_5x_3x_4 \\
 & + k_2x_3x_4x_5 + k_3k_4k_5x_2 + k_3k_4x_2x_5 + k_3k_5x_2x_4 + k_3x_2x_4x_5 + k_4k_5x_2x_3 + k_4x_2x_3x_5 \\
 & + k_5x_2x_3x_4 + x_2x_3x_4x_5 + k_2k_3k_4 + k_2k_3k_5 + k_2k_3x_4 + k_2k_3x_5 + k_2k_4k_5 + k_2k_4x_5 \\
 & + k_2k_5x_3 + k_2k_5x_4 + k_2x_3x_5 + k_2x_4x_5 + k_3k_4k_5 + k_3k_4x_2 + k_3k_4x_5 + k_3k_5x_4 + k_3x_2x_4 \\
 & + k_3x_4x_5 + k_4k_5x_2 + k_4k_5x_3 + k_4x_2x_5 + k_4x_3x_5 + k_5x_2x_4 + k_5x_3x_4 + x_2x_4x_5 + x_3x_4x_5 \\
 & + k_1k_3 + k_1x_3 + k_2k_3 + k_2k_5 + k_2x_5 + k_3k_5 + k_3x_1 + k_3x_3 + k_3x_5 + k_4k_5 \\
 & + k_4x_2 + k_4x_3 + k_4x_5 + k_5x_3 + k_5x_4 + x_1x_3 + x_2x_4 + x_3x_4 + x_3x_5 + x_4x_5 \\
 & + k_2 + k_3 + k_5 + x_2 + x_3 + x_5
 \end{aligned}$$

## ANF of a random function

### Uniform distribution over all functions:

equivalent to the uniform distribution over all ANFs.

→ each monomial appears with probability  $\frac{1}{2}$ .

### Uniform distribution over all permutations:

open problem.

- all coordinates of a permutation of  $\mathbb{F}_2^n$  have degree at most  $(n - 1)$ .
- almost all permutations of  $\mathbb{F}_2^n$  have degree  $(n - 1)$  [Wells 69], [Das 02], [Konyagin-Pappalardi 02]

# Higher-order differential attacks

## Higher-order differentials [Lai 94]

Differential (derivative) of  $f$  w.r.t.  $\alpha \in \mathbb{F}_2^n$ :

$$D_\alpha f : x \mapsto f(x \oplus \alpha) \oplus f(x)$$

Higher-order differential.

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_d} f(x) = \bigoplus_{v \in \langle \alpha_1, \dots, \alpha_d \rangle} f(x \oplus v) := D_{\langle \alpha_1, \dots, \alpha_d \rangle} f(x)$$

$$\deg D_V f \leq \deg f - \dim V$$

Then, for any subspace  $V$  with  $\dim V > \deg f$

$$\forall a \in \mathbb{F}_2^n, D_V f(a) = \bigoplus_{v \in V} f(a \oplus v) = 0$$



## Higher-order differential attacks [Knudsen 94]

for all keys,  $f_k$  has a low degree in  $x$

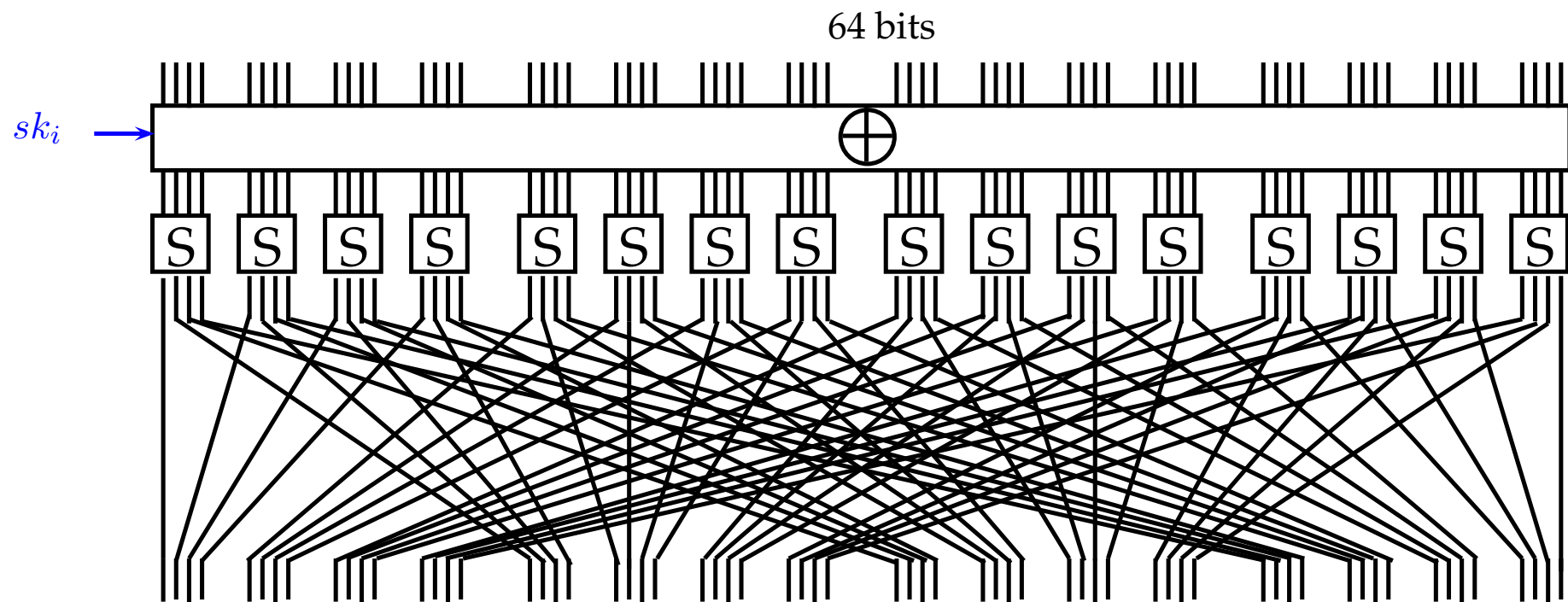
For any affine subspace  $a + V$  with  $\dim V > \deg_x f$

$$\bigoplus_{x \in a + V} f_k(x) = 0$$

Distinguisher with data complexity proportionnal to  $2^{\deg_x f}$

# Degree of an iterated block cipher

# PRESENT [Bogdanov et al. 07]



31 rounds (+ a key addition)

## Algebraic degree of PRESENT with a fixed key

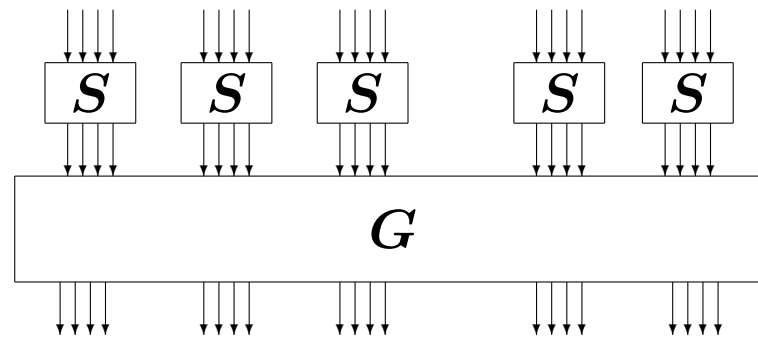
$$\deg S = 3$$

After  $r$  rounds,

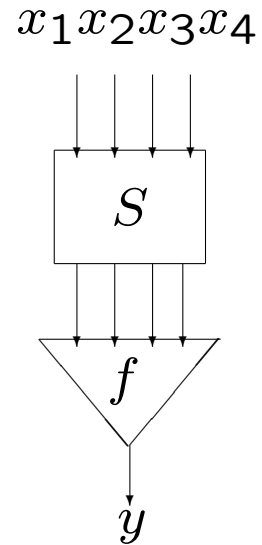
$$\deg_x E^r \leq 3^r$$

$r$	1	2	3	4	5	6	7
deg	3	9	27	63	63	63	63

## Using the particular form of the Sbox layer



## Exercise: find the maximal degree of the following 4-bit function

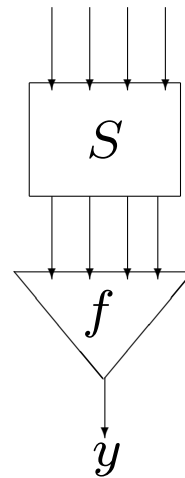


with  $f(y_1, y_2, y_3, y_4) = y_1y_2$ .

$x$	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5
$S_1(x)$	1	0	1	0	0	1	1	0	0	1	1	0	0	0	1	1
$S_2(x)$	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0	0
$S_3(x)$	1	1	0	1	1	1	1	0	0	0	0	0	1	0	0	1
$S_4(x)$	1	1	1	1	0	1	0	1	0	0	1	1	0	0	0	0

## Exercise: find the maximal degree of the following 4-bit function

$x_1x_2x_3x_4$



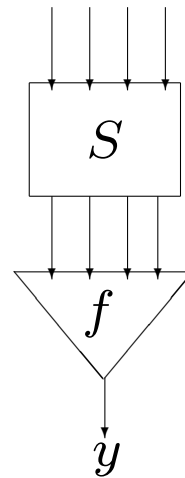
with  $f(y_1, y_2, y_3, y_4) = y_1y_2$ .

$x$	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5
$S_1(x)$	1	0	1	0	0	1	1	0	0	1	1	0	0	0	1	1
$S_2(x)$	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0	0
$S_3(x)$	1	1	0	1	1	1	1	0	0	0	0	0	1	0	0	1
$S_4(x)$	1	1	1	1	0	1	0	1	0	0	1	1	0	0	0	0

$$\deg S_1 S_2 \leq 3$$

## Exercise: find the maximal degree of the following 4-bit function

$x_1x_2x_3x_4$



with  $f(y_1, y_2, y_3, y_4) = y_1y_2y_3$ .

$x$	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5
$S_1(x)$	1	0	1	0	0	1	1	0	0	1	1	0	0	0	1	1
$S_2(x)$	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0	0
$S_3(x)$	1	1	0	1	1	1	1	0	0	0	0	0	1	0	0	1
$S_4(x)$	1	1	1	1	0	1	0	1	0	0	1	1	0	0	0	0

$$\deg S_1S_2S_3 \leq 3$$



## Product of some coordinates of a permutation

$\delta_k$  = maximal degree of the product of  $k$  coordinates of  $S$

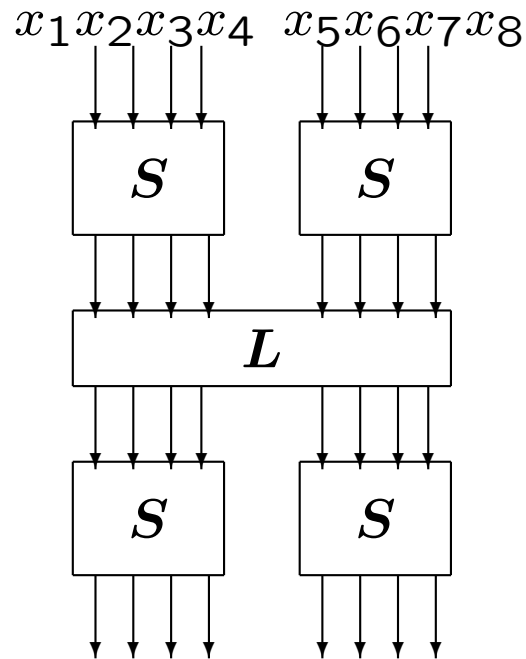
In our example:

$k$	1	2	3	4
$\delta_k$	3	3	3	4

**Proposition.** If  $S$  is a permutation of  $\mathbb{F}_2^n$ ,

$$\delta_k = n \text{ if and only if } k = n$$

Exercise: find the maximal degree of the following 2-round cipher

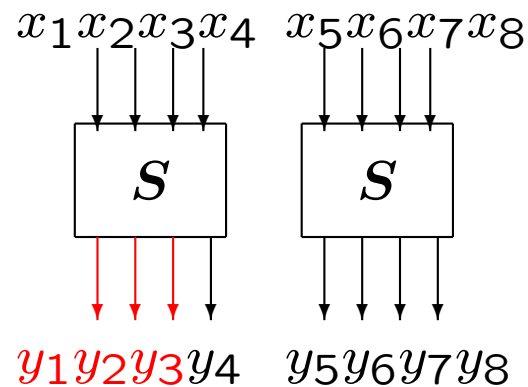


where the 4-bit Sbox  $S$  is such that the maximal degree  $\delta_k$  of the product of  $k$  of its coordinates is given by

$k$	1	2	3	4
$\delta_k$	3	3	3	4

**Exercise: find the maximal degree of the following 2-round cipher**

$f$  is a function of degree  $\leq 3$  of the output of the first Sbox layer



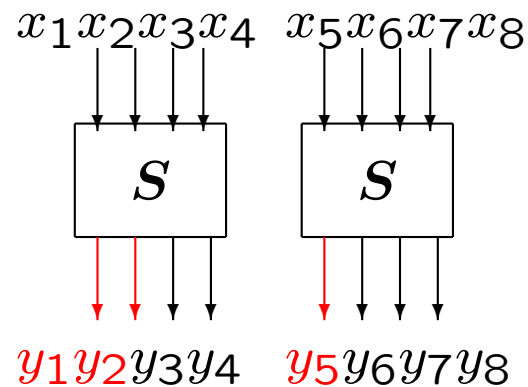
**First case:**

$$f(x_1, \dots, x_8) = y_1 y_2 y_3 = S_1(x_1, \dots, x_4) S_2(x_1, \dots, x_4) S_3(x_1, \dots, x_4)$$

$$\Rightarrow \deg f \leq 3$$

**Exercise: find the maximal degree of the following 2-round cipher**

$f$  is a function of degree  $\leq 3$  of the output of the first Sbox layer



**Second case:**

$$f(x_1, \dots, x_8) = y_1 y_2 y_5 = S_1(x_1, \dots, x_4) S_2(x_1, \dots, x_4) S_1(x_5, \dots, x_8)$$

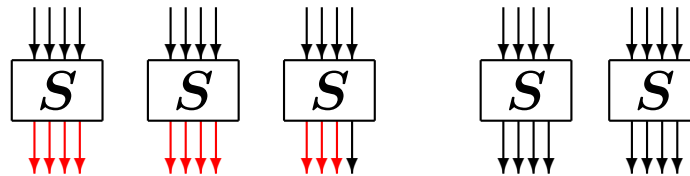
$$\Rightarrow \deg f \leq 3 + 3 = 6$$

## Degree of the product $f$ of $d$ output coordinates

### A fundamental parameter:

$\delta_k =$  maximal degree of the product of  $k$  coordinates of  $S$

**Example:**  $d = 11$



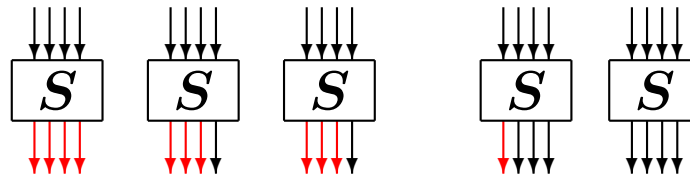
$$\deg f \leq 2\delta_4 + \delta_3$$

## Degree of the product $f$ of $d$ output coordinates

### A fundamental parameter:

$\delta_k =$  maximal degree of the product of  $k$  coordinates of  $S$

**Example:**  $d = 11$



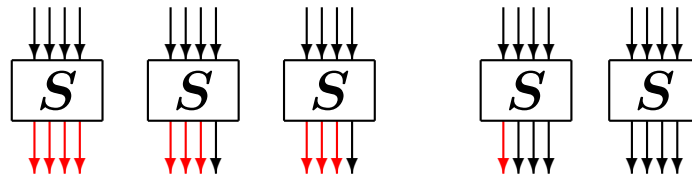
$$\deg f \leq \delta_4 + 2\delta_3 + \delta_1$$

## Degree of the product $f$ of $d$ output coordinates

### A fundamental parameter:

$\delta_k =$  maximal degree of the product of  $k$  coordinates of  $S$

**Example:**  $d = 11$



$$\deg f \leq \max_{(x_1, \dots, x_4)} (x_1 \delta_1 + \dots + x_4 \delta_4)$$

$$\text{with } x_1 + 2x_2 + 3x_3 + 4x_4 = d .$$

## A new bound [Boura, C., De Cannière 09]

**Theorem.** Let  $F = (S, \dots, S)$  where  $S$  is a permutation of  $\mathbb{F}_2^m$ .  
Then,

$$\deg(G \circ F) \leq n - \frac{n - \deg G}{\gamma(S)}$$

where

$$\gamma(S) = \max_{1 \leq k \leq m-1} \frac{m - k}{m - \delta_k(S)}.$$

Most notably,

$$\gamma(S) \leq m - 1 \text{ with equality iff } \deg S = m - 1.$$



## Our example

$$\gamma(S) = \max_{1 \leq k < 4} \frac{4 - k}{4 - \delta_k(S)}$$

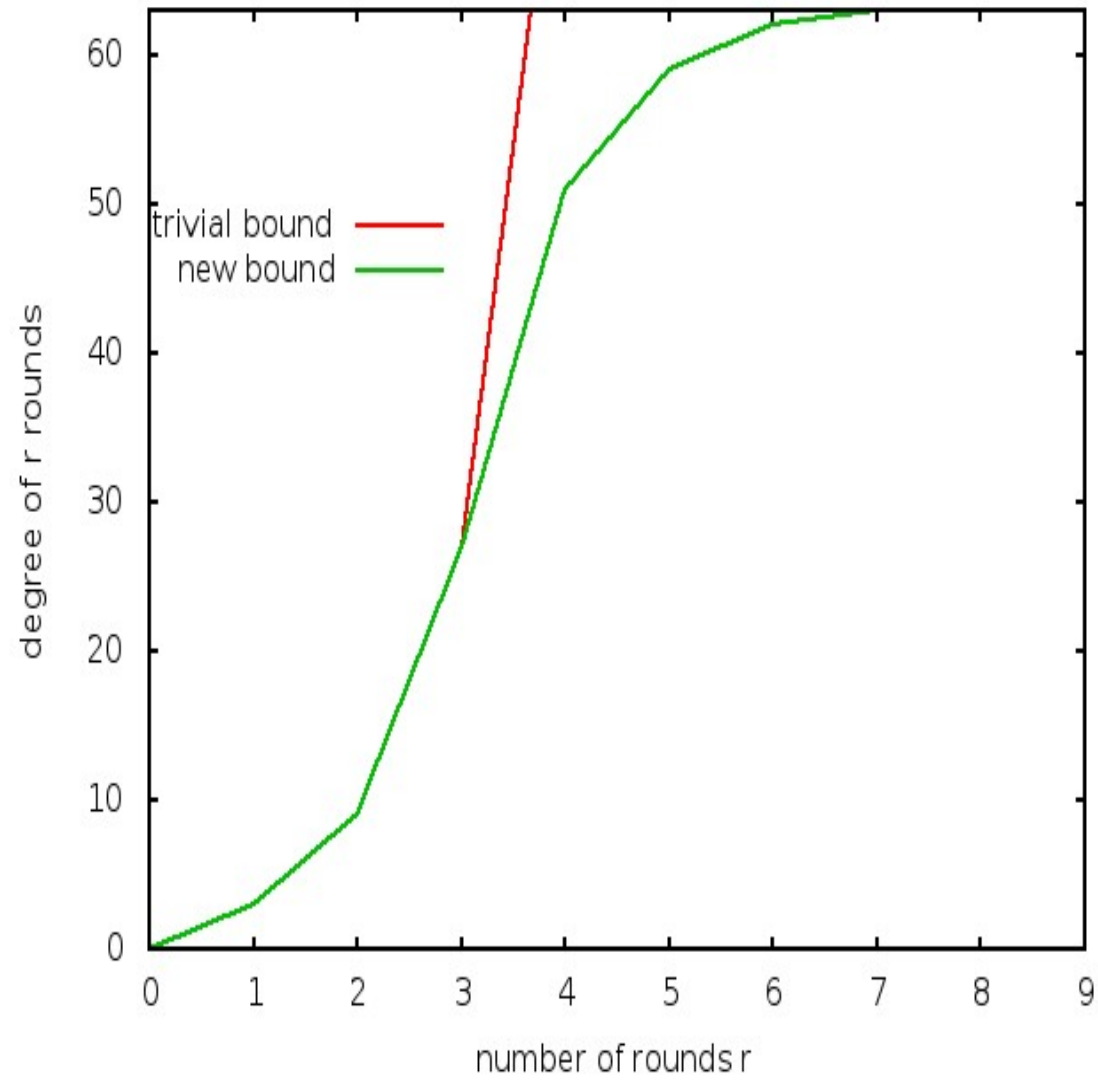
$k$	1	2	3	4
$\delta_k(S)$	3	3	3	4

$$\gamma(S) \leq \max \left( \frac{3}{1}, \frac{2}{1}, \frac{2}{1} \right) = 3$$

We deduce

$$\deg(G \circ F) \leq n - \frac{n - \deg G}{3}$$

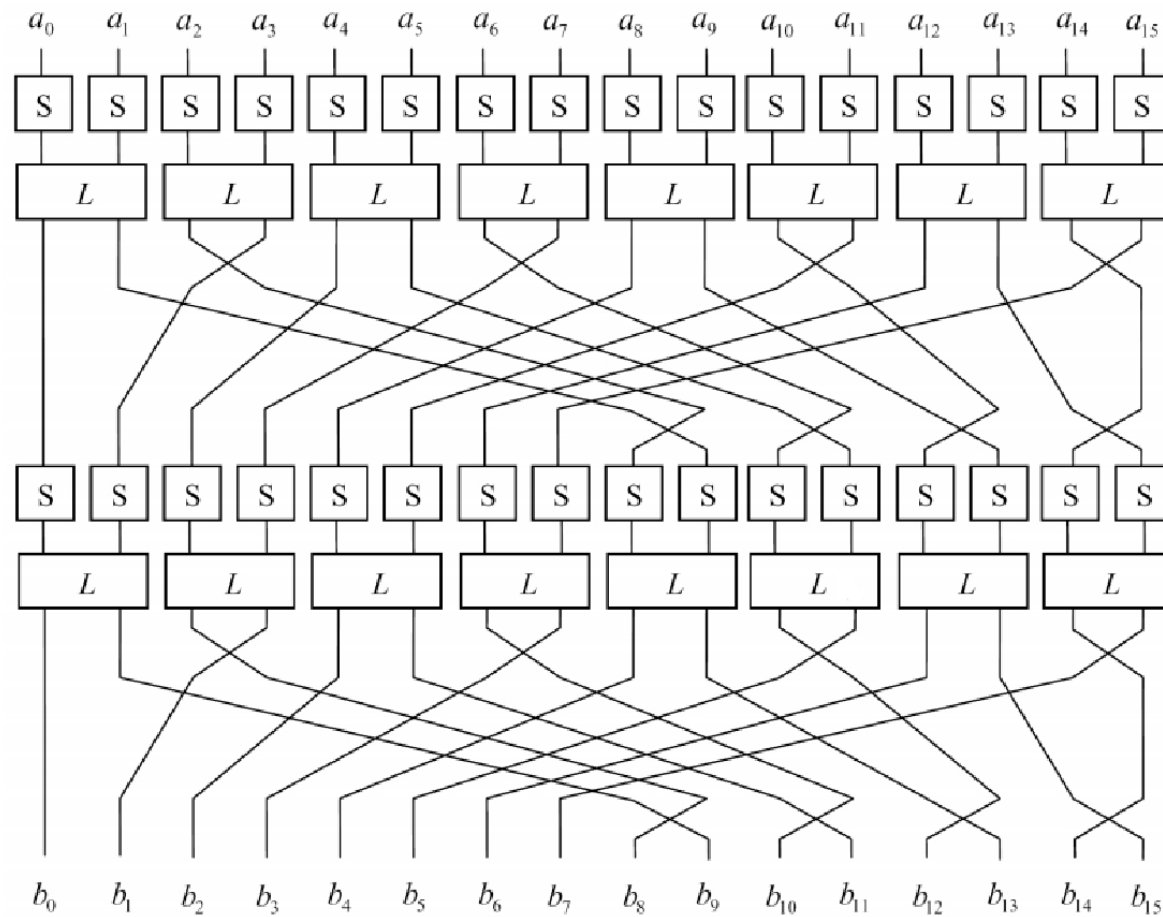
## Our example



# How does the degree increase with the number of rounds?

Inspired by JH [Wu 2007]

block size = 64 bits with a 4-bit Sbox



How does the degree increase with the number of rounds?

**SuperBox view:**

$r$  rounds without the last linear layer = each output depends on  $2^{r+1}$  input bits.

**For  $r = 3$ :**

$$n = 16, \gamma(S) = 3, \deg G \leq 6$$

$$\Rightarrow \deg(G \circ F) \leq 16 - \frac{16 - 6}{3} = 12.66$$

## How does the degree increase with the number of rounds?

### SuperBox view:

$r$  rounds without the last linear layer = each output depends on  $2^{r+1}$  input bits.

### For $r = 3$ :

$$n = 16, \gamma(S) = 3, \deg G \leq 6$$

$$\Rightarrow \deg(G \circ F) \leq 16 - \frac{16 - 6}{3} = 12.66$$

# rounds	block size	bound
2	8	6
3	16	12
4	32	25
5	64	51
6	64	59
7	64	62
8	64	63

## Key recovery on 6 rounds

Let  $G_k$  be the first 5 rounds of the cipher

$$\deg G_k \leq 51$$

$$\Rightarrow \bigoplus_{x \in a+V} G_k(x) = 0 \text{ for any } (a+V) \text{ with } \dim V = 52$$

## Key recovery on 6 rounds

**Input:**  $2^{52}$  pc pairs  $(m_i, c_i)$  with  $m_i$  in an affine subspace of dim 52

**For recovering the  $j$ -th key byte  $k_j$ :**

For each possible value for  $k_j$

For  $i$  from 0 to  $(2^{52} - 1)$

$$y_{i,j} \leftarrow \mathcal{R}^{-1}(c_{i,j} \oplus k_j)$$

If  $\bigoplus_i y_{i,j} = 0$

Return  $k_j$

$$D = 2^{52} \text{ and } T = 8 \times 2^{60}$$

## Are there better choices for $V$ ?

For  $V = a + \langle e_0, \dots, e_{27} \rangle$ ,

$$V = \boxed{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline c & c & c & c & c & c & c & c & c & A & A & A & A & A & A & A \\ \hline \end{array}}$$

- **invariant** under the key addition and the first Sbox layer
- Let  $H_k =$  first linear layer + rounds 2 to 5.

Since  $\deg H_k \leq 25$  and  $\dim V = 28$ ,

$$\bigoplus_{x \in a+V} H_k(x) = \bigoplus_{x \in b+V} G_k(x) = D_V G_k(b) = 0$$

$$\Rightarrow D = 2^{28} \text{ and } T = 8 \times 2^{36}$$



## What does it mean?

$$G_k(x) = \sum_{u \in \mathbb{F}_2^n} c_u x^u \text{ with } c_u = \bigoplus_{x \preceq u} G_k(x)$$

For  $V = \langle e_0, \dots, e_{27} \rangle = \{x : x \preceq 0 \dots 0 \underbrace{1 \dots 1}_{28}\}$

$$\Rightarrow \bigoplus_{x \in a+V} G_k(x) = 0 \text{ for all } a \in \mathbb{F}_2^n$$

means that the ANF of  $G_k$  does not contain any **monomial multiple of  $x_0 \dots x_{27}$**

$\Rightarrow$  **Cube distinguisher** with data complexity  $2^{wt(u)}$  [Aumasson, Dinur, Meier, Shamir 09]

## Many refinements

- zero-sum distinguishers on SHA-3 [Boura, C., De Cannière 11]
- interpolation attack on Low-MC [Dinur, Liu, Meier, Wang 15]
- attack on the full Misty-1 using the division property [Todo 15]