Secure building-blocks against differential and linear attacks
Anne Canteaut

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Secure building-blocks
against differential and linear attacks

Anne Canteaut

Anne.Canteaut@inria.fr
https://www.paris.inria.fr/secret/Anne.Canteaut/

COST Training School, Torremolinos, February 2018
Key-alternating block ciphers

where each $F^{(i)}$ is a permutation of $F^m_2$. 

$K$ master key

key schedule

plaintext $x$ $+$ $F(1)$ $+$ $F(2)$ $+$ $\ldots$ $+$ $F(r)$ $+$ ciphertext $y$

where each $F^{(i)}$ is a permutation of $F^m_2$. 

Round function in a substitution-permutation network

\[ x^{(i)} \]

\[ S S S S S S S S S S S \]

linear diffusion

\[ k_i \]

\[ x^{(i+1)} \]
Outline

- Representations of Sboxes
- Linear approximations of a Boolean function and Walsh transform
- Resistance to differential attacks
- Finding good Sboxes
- Security criteria for the linear layer
Representations of Sboxes
**Boolean functions**

**Definition.** A Boolean function of $n$ variables is a function from $F_2^n$ into $F_2$.

**Truth table of a Boolean function.**

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$f(x_1, x_2, x_3)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Value vector of $f$:** word of $2^n$ bits corresponding to all $f(x), x \in F_2^n$. 
**Definition.** A vectorial Boolean function with $n$ inputs and $m$ outputs is a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^m$:

$$S : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$$

$$(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_m)$$

Each function $S_i : (x_1, \ldots, x_n) \mapsto y_i$ is called a coordinate of $S$.

**Example.**

<table>
<thead>
<tr>
<th>$x$</th>
<th>0 1 2 3 4 5 6 7 8 9 a b c d e f</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(x)$</td>
<td>f e b c 6 d 7 8 0 3 9 a 4 2 1 5</td>
</tr>
<tr>
<td>$S_1(x)$</td>
<td>1 0 1 0 0 1 1 0 0 1 1 0 0 0 1 1</td>
</tr>
<tr>
<td>$S_2(x)$</td>
<td>1 1 1 0 1 0 1 0 0 1 0 1 0 1 0 0</td>
</tr>
<tr>
<td>$S_3(x)$</td>
<td>1 1 0 1 1 1 1 0 0 0 0 0 1 0 0 1</td>
</tr>
<tr>
<td>$S_4(x)$</td>
<td>1 1 1 1 0 1 0 1 0 0 1 1 0 0 0 0</td>
</tr>
</tbody>
</table>
Hamming weight of a Boolean function.

The Hamming weight of a Boolean function \( f \), \( wt(f) \), is the Hamming weight of its value vector.

A function of \( n \) variables is balanced if and only if \( wt(f) = 2^{n-1} \).

Proposition. A vectorial function \( S \) with \( n \) inputs and \( n \) outputs is a permutation if and only if any nonzero linear combination of its coordinates

\[
x \mapsto \bigoplus_{i=1}^{n} \lambda_i S_i(x), \quad \lambda = (\lambda_1, \ldots, \lambda_n) \neq 0
\]

is a balanced Boolean function.
Algebraic normal form (ANF)

Monomials in $\mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 + x_1, \ldots, x_n^2 + x_n)$:

$$\{x^u, \ u \in \mathbb{F}_2^n\} \text{ where } x^u = \prod_{i=1}^{n} x_i^{u_i}.$$ 

Example: $x^{1011} = x_1^1x_2^0x_3^1x_4^1 = x_1x_3x_4$.

Proposition.

Any Boolean function of $n$ variables has a unique polynomial representation in $\mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 + x_1, \ldots, x_n^2 + x_n)$:

$$f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{F}_2^n} a_u x^u, \ a_u \in \mathbb{F}_2.$$ 

Moreover, the coefficients of the ANF and the values of $f$ satisfy:

$$a_u = \bigoplus_{x \preceq u} f(x) \text{ and } f(u) = \bigoplus_{x \preceq u} a_x,$$

where $x \preceq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$. 
Example

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$f(x_1, x_2, x_3)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$a_{000} = f(000) = 0$

$a_{100} = f(100) \oplus f(000) = 1$

$a_{010} = f(010) \oplus f(000) = 0$

$a_{110} = f(110) \oplus f(010) \oplus f(100) \oplus f(000) = 1$

$a_{001} = f(001) \oplus f(000) = 0$

$a_{101} = f(101) \oplus f(001) \oplus f(100) \oplus f(000) = 0$

$a_{011} = f(011) \oplus f(001) \oplus f(010) \oplus f(000) = 1$

$a_{111} = \bigoplus_{x \in \mathbb{F}_2^3} f(x) = wt(f) \mod 2 = 0$

\[ f = x_1 \oplus x_1x_2 \oplus x_2x_3. \]
Computing the ANF

\( n = 3: \)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(0) )</td>
<td>( f(0) )</td>
<td>( f(0) \oplus f(1) )</td>
<td>( f(0) \oplus f(1) )</td>
<td>( f(0) \oplus f(1) )</td>
<td>( f(0) \oplus f(1) )</td>
<td>( f(0) \oplus f(1) )</td>
<td>( f(0) \oplus f(1) )</td>
<td></td>
</tr>
<tr>
<td>( f(1) )</td>
<td>( f(1) )</td>
<td>( f(0) \oplus f(2) )</td>
<td>( f(0) \oplus f(2) )</td>
<td>( f(0) \oplus f(2) )</td>
<td>( f(0) \oplus f(2) )</td>
<td>( f(0) \oplus f(2) )</td>
<td>( f(0) \oplus f(2) )</td>
<td></td>
</tr>
<tr>
<td>( f(2) )</td>
<td>( f(2) )</td>
<td>( f(0) \oplus f(3) )</td>
<td>( f(0) \oplus f(3) )</td>
<td>( f(0) \oplus f(3) )</td>
<td>( f(0) \oplus f(3) )</td>
<td>( f(0) \oplus f(3) )</td>
<td>( f(0) \oplus f(3) )</td>
<td></td>
</tr>
<tr>
<td>( f(3) )</td>
<td>( f(3) )</td>
<td>( f(0) \oplus f(4) )</td>
<td>( f(0) \oplus f(4) )</td>
<td>( f(0) \oplus f(4) )</td>
<td>( f(0) \oplus f(4) )</td>
<td>( f(0) \oplus f(4) )</td>
<td>( f(0) \oplus f(4) )</td>
<td></td>
</tr>
<tr>
<td>( f(4) )</td>
<td>( f(4) )</td>
<td>( f(0) \oplus f(5) )</td>
<td>( f(0) \oplus f(5) )</td>
<td>( f(0) \oplus f(5) )</td>
<td>( f(0) \oplus f(5) )</td>
<td>( f(0) \oplus f(5) )</td>
<td>( f(0) \oplus f(5) )</td>
<td></td>
</tr>
<tr>
<td>( f(5) )</td>
<td>( f(5) )</td>
<td>( f(0) \oplus f(6) )</td>
<td>( f(0) \oplus f(6) )</td>
<td>( f(0) \oplus f(6) )</td>
<td>( f(0) \oplus f(6) )</td>
<td>( f(0) \oplus f(6) )</td>
<td>( f(0) \oplus f(6) )</td>
<td></td>
</tr>
<tr>
<td>( f(6) )</td>
<td>( f(6) )</td>
<td>( f(0) \oplus f(7) )</td>
<td>( f(0) \oplus f(7) )</td>
<td>( f(0) \oplus f(7) )</td>
<td>( f(0) \oplus f(7) )</td>
<td>( f(0) \oplus f(7) )</td>
<td>( f(0) \oplus f(7) )</td>
<td></td>
</tr>
<tr>
<td>( f(7) )</td>
<td>( f(7) )</td>
<td>( f(0) \oplus f(8) )</td>
<td>( f(0) \oplus f(8) )</td>
<td>( f(0) \oplus f(8) )</td>
<td>( f(0) \oplus f(8) )</td>
<td>( f(0) \oplus f(8) )</td>
<td>( f(0) \oplus f(8) )</td>
<td></td>
</tr>
</tbody>
</table>

first step:

\[ f(2i + 1) \leftarrow f(2i + 1) \oplus f(2i) \]

second step:

\[ f(4i + j + 2) \leftarrow f(4i + j + 2) \oplus f(4i + j), \ \forall 0 \leq j < 2 \]

third step:

\[ f(8i + j + 4) \leftarrow f(8i + j + 4) \oplus f(8i + j), \ \forall 0 \leq j < 4 \]
Computing the ANF

When the value vector is stored as a 32-bit integer $x$:

\[
\begin{align*}
x &\ ^= (x \ & 0x55555555) \ll 1; \\
x &\ ^= (x \ & 0x33333333) \ll 2; \\
x &\ ^= (x \ & 0x0f0f0f0f) \ll 4; \\
x &\ ^= (x \ & 0x00ff00ff) \ll 8; \\
x &\ ^= x \ll 16;
\end{align*}
\]
Definition.
The degree of a Boolean function is the degree of the largest monomial in its ANF.

Proposition.
The weight of an $n$-variable function $f$ is odd if and only if $\deg f = n$.

Definition.
The degree of a vectorial function $S$ with $n$ inputs and $m$ outputs is the maximal degree of its coordinates.
### Example

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(x)$</td>
<td>f</td>
<td>e</td>
<td>b</td>
<td>c</td>
<td>6</td>
<td>d</td>
<td>7</td>
<td>8</td>
<td>0</td>
<td>3</td>
<td>9</td>
<td>a</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$S_1(x)$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$S_2(x)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_3(x)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$S_4(x)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
S_1 = 1 + x_1 + x_3 + x_2x_3 + x_4 + x_2x_4 + x_3x_4 + x_1x_3x_4 + x_2x_3x_4
\]

\[
S_2 = 1 + x_1x_2 + x_1x_3 + x_1x_2x_3 + x_4 + x_1x_4 + x_1x_2x_4 + x_1x_3x_4
\]

\[
S_3 = 1 + x_2 + x_1x_2 + x_2x_3 + x_4 + x_2x_4 + x_1x_2x_4 + x_3x_4 + x_1x_3x_4
\]

\[
S_4 = 1 + x_3 + x_1x_3 + x_4 + x_2x_4 + x_3x_4 + x_1x_3x_4 + x_2x_3x_4
\]
Identifying $\mathbb{F}_2^n$ with a finite field

$\mathbb{F}_2^n$ is identified with the finite field with $2^n$ elements.

$$F_{2^n} = \{0\} \cup \{\alpha^i, \ 0 \leq i \leq 2^n - 2\}$$

where $\alpha$ is a root of a primitive polynomial of degree $n$.

⇒ for any $i$, $\alpha^i = \sum_{j=0}^{n-1} \lambda_j \alpha^j$

Example for $n = 4$:

primitive polynomial: $1 + x + x^4$, $\alpha$ a root of this polynomial.

<table>
<thead>
<tr>
<th>$\mathbb{F}_{24}$</th>
<th>0</th>
<th>1</th>
<th>$\alpha$</th>
<th>$\alpha^2$</th>
<th>$\alpha^3$</th>
<th>$\alpha^4$</th>
<th>$\alpha^5$</th>
<th>$\alpha^6$</th>
<th>$\alpha^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\alpha$</td>
<td>$\alpha^2$</td>
<td>$\alpha^3$</td>
<td>$\alpha + 1$</td>
<td>$\alpha^2 + \alpha$</td>
<td>$\alpha^3 + \alpha^2$</td>
<td>$\alpha^3 + \alpha + 1$</td>
</tr>
<tr>
<td>$\mathbb{F}_2^4$</td>
<td>0000</td>
<td>0001</td>
<td>0010</td>
<td>0100</td>
<td>1000</td>
<td>0011</td>
<td>0110</td>
<td>1100</td>
<td>1011</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha^8$</th>
<th>$\alpha^9$</th>
<th>$\alpha^{10}$</th>
<th>$\alpha^{11}$</th>
<th>$\alpha^{12}$</th>
<th>$\alpha^{13}$</th>
<th>$\alpha^{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha^2 + 1$</td>
<td>$\alpha^3 + \alpha$</td>
<td>$\alpha^2 + \alpha + 1$</td>
<td>$\alpha^3 + \alpha^2 + \alpha$</td>
<td>$\alpha^3 + \alpha^2 + \alpha + 1$</td>
<td>$\alpha^3 + \alpha^2 + 1$</td>
<td>$\alpha^3 + 1$</td>
</tr>
<tr>
<td>0101</td>
<td>1010</td>
<td>0111</td>
<td>1110</td>
<td>1111</td>
<td>1101</td>
<td>1001</td>
</tr>
</tbody>
</table>
The univariate representation of Sboxes

Any vectorial function with \( n \) inputs and \( n \) outputs can be seen as

\[
S : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}
\]

Then,

\[
S(X) = \sum_{i=0}^{2^n-1} c_i X^i, \ c_i \in \mathbb{F}_{2^n}.
\]

**Example:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(x) )</td>
<td>f</td>
<td>e</td>
<td>b</td>
<td>c</td>
<td>6</td>
<td>d</td>
<td>7</td>
<td>8</td>
<td>0</td>
<td>3</td>
<td>9</td>
<td>a</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

\[
S(X) = \alpha^{12} + \alpha^2 X + \alpha^{13} X^2 + \alpha^6 X^3 + \alpha^{10} X^4 + \alpha^5 X^5 + \alpha^{10} X^6 + \alpha^2 X^7 + \alpha^9 X^8 + \alpha^4 X^9 + \alpha^7 X^{10} + \alpha^7 X^{11} + \alpha^5 X^{12} + X^{13} + \alpha^6 X^{14}
\]
Linear approximations of a function
and Walsh transform
Linear attacks [Matsui 93]

Idea.
Use linear relations between the input and output bits of the cipher which hold with probability significantly greater or significantly less than $1/2$.

$a \in \mathbb{F}_2^n$: input mask

$b \in \mathbb{F}_2^n$: output mask

\[ \left| \Pr_x [a \cdot x \oplus b \cdot E_k(x) = 0] - \frac{1}{2} \right| \]

For our 4-bit Sbox.

\[ x_1 \oplus x_4 \oplus S_2(x) = 0x9 \cdot x \oplus 0x2 \cdot S(x) \]

equals 0 with probability $\frac{1}{8}$. 
Computing the probabilities of all linear relations

Bias of a Boolean function.

For any Boolean function $f$ of $n$ variables

$$\mathcal{E}(f) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)} = 2^n - 2\text{wt}(f).$$

Equivalently,

$$\Pr[f(x) = 0] = \frac{\text{wt}(f)}{2^n} = \frac{1}{2} \left(1 - \frac{\mathcal{E}(f)}{2^n}\right).$$

→ we need to compute the biases of all Boolean functions

$$x \mapsto b \cdot S(x) + a \cdot x.$$
Linear approximations of an Sbox

\begin{array}{|c|cccccccccccc|}
\hline
a \backslash b & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & a & b & c & d & e & f \\
\hline
\hline
\hline
\end{array}

\[
\Pr[a \cdot x + b \cdot S(x) = 0] = \frac{1}{2} \left( 1 + \frac{\mathcal{E}(a, b)}{2^n} \right)
\]

For instance, for \( a = 0x9 \) and \( b = 0x2 \), we have \( p = \frac{1}{2}(1 - \frac{12}{16}) = \frac{1}{8} \).
Walsh transform of a Boolean function $f$ of $n$ variables

\[ \mathbb{F}_2^n \longrightarrow \mathbb{Z} \]

\[ a \longmapsto E(f + \ell_a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + a \cdot x} \]

where \( \ell_a : x \longmapsto a \cdot x \)

Walsh transform of a vectorial function $S$:

\[ \mathbb{F}_2^n \times \mathbb{F}_2^m \longrightarrow \mathbb{Z} \]

\[ (a, b) \longmapsto E(b \cdot S + \ell_a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{b \cdot S(x) + a \cdot x} \]
Computing the Walsh transform

\[
\begin{align*}
T(x) &= (-1)^f(x) \\
\begin{array}{c|cccccccc}
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
\end{array}
\end{align*}
\]

| step 1 | 0 & 2 & 2 & 0 & 0 & 2 & -2 & 0 |
| step 2 | 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 |
| \(E(f + \ell_a)\) | 0 & 4 & 0 & 4 & 4 & 0 & -4 & 0 |

first step: \[T(2i) \leftarrow T(2i) + T(2i + 1)\]
\[T(2i + 1) \leftarrow T(2i) - T(2i + 1)\]

second step: \[T(4i + j) \leftarrow T(4i + j) + T(4i + j + 2), \ \forall 0 \leq j < 2\]
\[T(4i + j + 2) \leftarrow T(4i + j) - T(4i + j + 2), \ \forall 0 \leq j < 2\]

third step: \[T(8i + j) \leftarrow T(8i + j) + T(8i + j + 4), \ \forall 0 \leq j < 4\]
\[T(8i + j + 4) \leftarrow T(8i + j) - T(8i + j + 4), \ \forall 0 \leq j < 4\]

Complexity: \(n2^n\) operations.
Some basic properties of the Walsh transform

Lemma:
\[ \mathcal{E}(\ell_a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{a \cdot x} = \begin{cases} 2^n & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases} \]

Proposition. The Walsh transform is an involution (up to a multiplicative constant): for any \( x \in \mathbb{F}_2^n \),
\[
\sum_{a \in \mathbb{F}_2^n} \mathcal{E}(f + \ell_a)(-1)^{a \cdot x} = \sum_{a \in \mathbb{F}_2^n} \sum_{u \in \mathbb{F}_2^n} (-1)^{f(u) + a \cdot u + a \cdot x} \\
= \sum_{u \in \mathbb{F}_2^n} (-1)^{f(u)} \sum_{a \in \mathbb{F}_2^n} (-1)^{a \cdot (x + u)} \\
= 2^n (-1)^{f(x)}
\]
Some basic properties of the Walsh transform

Parseval equality.

$$\sum_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) = 2^{2n}.$$

Proof.

$$\sum_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) = \sum_{a \in \mathbb{F}_2^n} \left( \sum_{x \in \mathbb{F}_2^n} (-1)f(x) + a \cdot x \right) \left( \sum_{y \in \mathbb{F}_2^n} (-1)f(y) + a \cdot y \right)$$

$$= \sum_{x \in \mathbb{F}_2^n} \sum_{y \in \mathbb{F}_2^n} (-1)f(x) + f(y) \sum_{a \in \mathbb{F}_2^n} (-1)^{a \cdot (x+y)}$$

$$= 2^n \sum_{x \in \mathbb{F}_2^n} (-1)f(x) + f(x)$$

$$= 2^{2n}.$$

[Check it on each column of the table on Slide 19]
Definition. For any Boolean function $f$ of $n$ variables,

$$\mathcal{L}(f) = \max_a |\mathcal{E}(f + \ell_a)|$$

is called the linearity of $f$ (highest bias for an affine approximation).

$$\mathcal{N}\mathcal{L}(f) = 2^{n-1} - \frac{1}{2} \mathcal{L}(f)$$

is called the nonlinearity of $f$ (distance of $f$ to the affine functions).
Can we say something about $\mathcal{L}(f)$?

$$\mathcal{L}(f) = \max_a |\mathcal{E}(f + \ell_a)|$$

**Theorem.** [Rothaus 76] For any Boolean function with $n$ variables,

$$\mathcal{L}(f) \geq 2^{\frac{n}{2}},$$

with equality for even $n$ only. The functions achieving this bound are called bent functions. They are not balanced.

**Proof.** From Parseval equality:

$$2^{2n} = \sum_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) \leq \max_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) \times 2^n = 2^n \mathcal{L}^2(f)$$

with equality if and only if all $\mathcal{E}^2(f + \ell_a)$ are equal.

Then, $\mathcal{L}(f) \geq 2^{\frac{n}{2}}$ with equality if and only if

$$\mathcal{E}(f + \ell_a) = \pm 2^{\frac{n}{2}}, \forall a \in \mathbb{F}_2^n.$$
Can we say something about $\mathcal{L}(f)$?

What is the lowest possible value for $\mathcal{L}(f)$ when $n$ is odd? When $f$ is balanced?

**Functions of degree 2.**

For $n$ odd, $n = 2t + 1$

$$x_1x_2 \oplus x_3x_4 \oplus \cdots \oplus x_{2t-1}x_{2t} \oplus x_{2t+1}$$

satisfies $\mathcal{L}(f) = 2^{n+1/2}$. Moreover, $f$ is balanced and

$$\forall a \in \mathbb{F}_2^n, \ E(f + \ell_a) \in \{0, \pm 2^{n+1/2}\}.$$

**Theorem.**

$$2^{n/2} \leq \min_{f \in \text{Bool}_n} \mathcal{L}(f) \leq 2^{n+1/2}$$
Boolean functions with a low linearity

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<tr>
<th>$n$</th>
<th>$\min_{f \in \text{Bool}_n} \mathcal{L}(f)$</th>
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<td>[Berlekamp-Welch 72]</td>
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Open problem. Find the lowest possible linearity for a Boolean function of $n$ variables, where $n$ is odd and $n \geq 9.$
Balanced Boolean functions with a low linearity

<table>
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Open problem. Find the highest possible nonlinearity for a balanced Boolean function of $n$ variables, where $n$ is even and $n \geq 8$.

Proposition. [Katz 71] If $f$ is balanced, all values $E(f + l_a)$ are divisible by $2^\left\lceil \frac{n-1}{\deg f} \right\rceil + 1$, i.e., at least by 4 (and by 8 if $\deg f < n - 1$).
Linearity of an Sbox

Criterion on the Sbox.
All linear approximations of $S$ should have a small bias, i.e.,

$$\mathcal{L}(S) = \max_{a \in \mathbb{F}_2^n, b \in \mathbb{F}_2^n, b \neq 0} |\mathcal{E}(b \cdot S + \ell_a)| = \max_{b \neq 0} \mathcal{L}(b \cdot S)$$

must be as small as possible.

$$\mathcal{N} \mathcal{L}(S) = 2^{n-1} - \frac{1}{2} \mathcal{L}(S)$$

is called the nonlinearity of $S$. 
Sboxes with a low linearity

What is the lowest possible value for $\mathcal{L}(S)$ when $S$ is a vectorial function with $n$ inputs and $n$ outputs?

**Theorem.** [Chabaud-Vaudenay 94] For any function $S$ with $n$ inputs and $n$ outputs,

$$\mathcal{L}(S) \geq 2^{\frac{n+1}{2}},$$

with equality for odd $n$ only. The functions achieving this bound are called *almost bent functions*.

**For $n$ even.**

There exist Sboxes with

$$\mathcal{L}(S) = 2^{\frac{n+2}{2}}$$

but we do not known if this value is minimal.
Resistance to differential attacks
## Difference distribution table of an Sbox

$$\delta_S(a, b) = \# \{ X \in F_2^n, \ S(X \oplus a) \oplus S(X) = b \}$$

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</table>
Resistance to differential attacks

Criterion on the Sbox. [Nyberg-Knudsen 92]
All entries in the difference table of $S$ should be small.

$$
\delta(S) = \max_{a,b \neq 0} \# \{X \in \mathbb{F}_2^n, \ S(X \oplus a) \oplus S(X) = b \}
$$

must be as small as possible.

$\delta(S)$ is called the differential uniformity of $S$ (always even).

Theorem. For any Sbox $S$ with $n$ inputs and $n$ outputs,

$$\delta(S) \geq 2 .$$

The functions achieving this bound are called almost perfect nonlinear functions (APN).
Link between the difference and square correlation tables

**Theorem.** [Chabaud Vaudenay 94][Blondeau Nyberg 13]

There is a one-to-one correspondence between the DDT

\[ \delta(a, b), \ a, b \in \mathbb{F}_2^n \]

and the squared LAT

\[ \mathcal{E}^2(a, b), \ a, b \in \mathbb{F}_2^n \]

\[ \mathcal{E}^2(u, v) = \sum_{a, b \in \mathbb{F}_2^n} (-1)^{a \cdot u + b \cdot v} \delta(a, b) \]

\[ \delta(a, b) = 2^{-2n} \sum_{u, v \in \mathbb{F}_2^n} (-1)^{a \cdot u + b \cdot v} \mathcal{E}^2(u, v) \]

There is a one-to-one correspondence between the Sbox and the LAT.

But several Sboxes may have the same squared LAT.
Finding good Sboxes
Affine equivalence between Sboxes

$S_1$ and $S_2$ are **affinely equivalent** if there exist two affine permutations of $F_2^n$, $A_1$ and $A_2$, such that

$$S_2 = A_2 \circ S_1 \circ A_1$$

Then,

$$\delta(S_2) = \delta(S_1) \text{ and } \mathcal{L}(S_2) = \mathcal{L}(S_1)$$
Permutations of $\mathbb{F}_2^n$, $n$ odd

\[ \mathcal{L}(S) \geq 2^{\frac{n+1}{2}} \text{ and } \delta(S) \geq 2 \]

- Any AB Sbox (i.e., with $\mathcal{L}(S) = 2^{\frac{n+1}{2}}$) is APN [Chabaud-Vaudenay 94].

- The converse holds for some specific cases only, e.g for quadratic APN Sboxes [Carlet-Charpin-Zinoviev 98]

- AB functions over $\mathbb{F}_2^n$ have degree at most $\frac{n+1}{2}$. 
Known AB permutations of $\mathbb{F}_2^n$, $n$ odd

Monomials permutations $S(x) = x^s$ over $\mathbb{F}_{2^n}$.

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<td>quadratic</td>
<td>$2^i + 1$ with $\gcd(i, n) = 1$, $1 \leq i \leq t$</td>
<td>[Gold 68],[Nyberg 93]</td>
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<td>Kasami</td>
<td>$2^{2i} - 2^i + 1$ with $\gcd(i, n) = 1$</td>
<td>[Kasami 71]</td>
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<td>$2^t + 3$</td>
<td>[Dobbertin 98]</td>
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<td>[C.-Charpin-Dobbertin 00]</td>
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<td>Niho</td>
<td>$2^t + 2^{\frac{t}{2}} - 1$ if $t$ is even</td>
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<td>$2^t + 2^{\frac{3t+1}{2}} - 1$ if $t$ is odd</td>
<td>[Xiang-Hollmann 01]</td>
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</table>

Non-monomial permutations. [Budaghyan-Carlet-Leander08]

For $n$ odd, divisible by 3 and not by 9.

$$S(x) = x^{2^i+1} + ux^{2^j\frac{n}{3}+2^{(3-j)\frac{n}{3}+i}} \text{ with } \gcd(i, n) = 1 \text{ and } j = i\frac{n}{3} \mod 3$$
Permutations of $F_{2}^{4}$

$\delta(S) \geq 4$ and $\mathcal{L}(S) \geq 8$

16 classes of optimal Sboxes [Leander-Poschmann 07]

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</tbody>
</table>
Permutations of $F_2^6$

$\delta(S) \geq 2$ and $L(S) \geq 12$

$$S = \{0, 54, 48, 13, 15, 18, 53, 35, 25, 63, 45, 52, 3, 20, 41, 33, 59, 36, 2, 34, 10, 8, 57, 37, 60, 19, 42, 14, 50, 26, 58, 24, 39, 27, 21, 17, 16, 29, 1, 62, 47, 40, 51, 56, 7, 43, 44, 38, 31, 11, 4, 28, 61, 46, 5, 49, 9, 6, 23, 32, 30, 12, 55, 22\};$$

satisfies

$$\delta(S) = 2, \ deg\ S = 4 \ and \ L(S) = 16 \ [Dillon\ 09]$$

The corresponding univariate polynomial over $F_{26}$ contains 52 nonzero monomials (out of 56 possible monomials of degree at most 4).

This is the only known APN permutation with an even number of variables.
Good permutations of $F_{2}^{n}$, $n$ even

Usually, we search for permutations $S$ with

$$\delta(S) = 4 \text{ and } L(S) = 2^{\frac{n+2}{2}}.$$

Monomials permutations $S(x) = x^{s}$ over $F_{2^{n}}$.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Condition</th>
<th>Reference</th>
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<tbody>
<tr>
<td>$2^{i} + 1, \gcd(i, n) = 2$</td>
<td>$n \equiv 2 \mod 4$</td>
<td>[Gold 68]</td>
</tr>
<tr>
<td>$2^{2i} - 2^{i} + 1, \gcd(i, n) = 2$</td>
<td>$n \equiv 2 \mod 4$</td>
<td>[Kasami 71]</td>
</tr>
<tr>
<td>$2^{\frac{n}{2}} + 2^{\frac{n}{4}} + 1$</td>
<td>$n \equiv 4 \mod 8$</td>
<td>[Bracken-Leander 10]</td>
</tr>
<tr>
<td>$2^{n} - 2$</td>
<td></td>
<td>[Lachaud-Wolfmann 90]</td>
</tr>
</tbody>
</table>

The last one is affinely equivalent to the AES Sbox.
Security criteria for the linear layer
A two-round characteristic

![Diagram of a two-round characteristic]

\[
\text{EDP}((a, c, L(c), b)) = \prod_{i=1}^{t} \frac{\delta_S(a_i, c_i)}{2^m} \prod_{j=1}^{t} \frac{\delta_S(L(c)_j, b_j)}{2^m}
\]
A two-round characteristic

\[
\begin{align*}
\text{EDP}((a, c, L(c), b)) & \leq \left(2^{-m} \delta(S) \right)^{\text{wt}(c)+\text{wt}(L(c))} \\
& \leq \left(2^{-m} \delta(S) \right)^d
\end{align*}
\]
Differential branch number of $L$ over $\mathbb{F}_2^m$

minimal number of active Sboxes within a 2-round characteristic

$$d = d_{\text{min}}(\mathcal{C}_L) \text{ where } \mathcal{C}_L = \{(x, L(x)), x \in (\mathbb{F}_2^m)^t\}$$

$\mathcal{C}_L$ is a code over $\mathbb{F}_2^m$ of length $2t$ and size $(2^m)^t$.

Maximizing the differential branch number.

From Singleton’s bound,

$$d_{\text{min}}(\mathcal{C}_L) \leq t + 1$$

with equality for MDS codes.
A differential may aggregate many differential characteristics.

**Bound on the 2-round MEDP** [Hong et al.00][Daemen-Rijmen02]:

\[
\text{MEDP}_2 \leq \left(2^{-m}\delta(S)\right)^{d-1}
\]

where \(d\) is the differential branch number of \(L\) over \(\mathbb{F}_2^m\).
In the AES:

• \( S(x) = A(x^{254}) \) over \( \mathbb{F}_{2^8} \) where \( A \) is an affine permutation of \( \mathbb{F}_{2^8} \). Then, \( \delta(S) = 4 \).

• \( L = \text{MixColumns} \) is such that \( C_L \) is an \([8,4,5]\) MDS code over \( \mathbb{F}_{2^8} \).

For any 2-round characteristic \( \Omega \),

\[
\text{EDP}(\Omega) \leq \left( \frac{\delta(S)}{2^m} \right)^d = 2^{-30}
\]

For any 2-round differential \((a,b)\),

\[
\text{EDP}(a,b) \leq \left( \frac{\delta(S)}{2^m} \right)^{d-1} = 2^{-24}
\]
For linear cryptanalysis

Expected squared correlation (linear potential) of a mask \((u, v)\):

\[
\text{ELP}(u, v) = 2^{-2n-\kappa} \sum_{k \in \mathbb{F}_2^\kappa} \left( \sum_{x \in \mathbb{F}_2^n} (-1)^{u \cdot x + v \cdot E_k(x)} \right)^2
\]

Expected squared correlation of a 2-round linear trail \((a, L^T(c), c, b)\):

\[
\text{ELP}((a, L^T(c), c, b)) \leq \left( 2^{-2m\mathcal{L}(S)^2} \right)^{\text{wt}(L^T(c)) + \text{wt}(c)} \leq \left( 2^{-m\mathcal{L}(S)} \right)^{2d'}
\]
Linear branch number of $L$ over $\mathbb{F}_2^m$

$$d' = d_{\text{min}}(C'_L) \text{ where } C'_L = \{(L^T(x), x), x \in (\mathbb{F}_2^m)^t\}$$

$C'_L$ is the dual (orthogonal) of $C_L$: 

$$\forall x, y : (L^T(x), x) \cdot (y, L(y)) = L^T(x) \cdot y \oplus x \cdot L(y) = 0$$

Then, $C'_L$ is MDS if and only if $C_L$ is MDS.
In the AES:

- $S(x) = A(x^{254})$ over $\mathbb{F}_{2^8}$ where $A$ is an affine permutation of $\mathbb{F}_{2^8}$. Then, $\mathcal{L}(S) = 2^5$.

- $L = \text{MixColumns}$ is such that $\mathcal{C}_L$ is an $[8, 4, 5]$ MDS code over $\mathbb{F}_{2^8}$

For any 2-round linear trail $\Omega$,

$$\text{ELP}(\Omega) \leq \left( \frac{\mathcal{L}(S)}{2^m} \right)^{2d'} = 2^{-30}$$

For any 2-round linear approximation $(a, b)$,

$$\text{ELP}(a, b) \leq \left( \frac{\mathcal{L}(S)}{2^m} \right)^{2(d'-1)} = 2^{-24}$$
More detailed lecture notes