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# On nonlinear approximations and the linear hull effect

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ASK 2018, Kolkata

# Linear approximations

## Linear approximations

$$\Pr[\alpha \cdot x + \beta \cdot F(x) = 0] \text{ far from } \frac{1}{2}$$

quantified by:

$$\text{cor}_F(\alpha, \beta) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + \beta \cdot F(x)}$$

since

$$\Pr[\alpha \cdot x + \beta \cdot F(x) = 0] = \frac{1}{2} (1 + \text{cor}_F(\alpha, \beta))$$

## Linear approximations with correlation $\pm 1$

$F$  has a linear approximation with correlation  $\pm 1$   
iff it has a component of degree 1.

$\Rightarrow$  This never occurs for one-round SPN (except for trivial Sboxes)

### An alternative formulation:

$$\text{cor}_F(\alpha, \beta) = -1 + 2^{-n+2} \#\{x \in \langle \alpha \rangle^\perp \text{ such that } F(x) \in \langle \beta \rangle^\perp\}$$

$$\Rightarrow \text{cor}_F(\alpha, \beta) = \pm 1 \text{ iff } F(\langle \alpha \rangle^\perp) = \langle \beta \rangle^\perp \text{ or } \mathbb{F}_2^n \setminus \langle \beta \rangle^\perp.$$

## Linear approximations over several rounds [Daemen 95][Nyberg 01]

$$\text{cor}_{G \circ F}(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_2^n} \text{cor}_F(\alpha, \gamma) \text{cor}_G(\gamma, \beta) .$$

If one dominant trail  $(\alpha, \gamma_0, \beta)$ :

$$\text{cor}_{G \circ F}(\alpha, \beta) \simeq \text{cor}_F(\alpha, \gamma_0) \text{cor}_G(\gamma_0, \beta) .$$

Otherwise, linear hull effect.

## Two-round approximations with correlation $\pm 1$

For a two-round SPN

$$\text{cor}_{L \circ S}(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_2^n} \text{cor}_S(\alpha, \gamma) \text{cor}_L(\gamma, \beta) = \text{cor}_S(\alpha, L^T(\beta)) .$$

$$\text{cor}_{\mathcal{R} \circ \text{Add}_k \circ \mathcal{R}}(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_2^n} (-1)^{k \cdot \gamma} \text{cor}_S(\alpha, L^T(\gamma)) \text{cor}_S(\gamma, L^T(\beta)) .$$

**Question:** can we get a correlation  $\pm 1$  for a two-round approximation for some fixed  $k$ ?

# **Nonlinear approximations and invariants**



## Nonlinear approximations

Let  $g$  and  $h$  be two balanced Boolean functions of  $n$  variables.

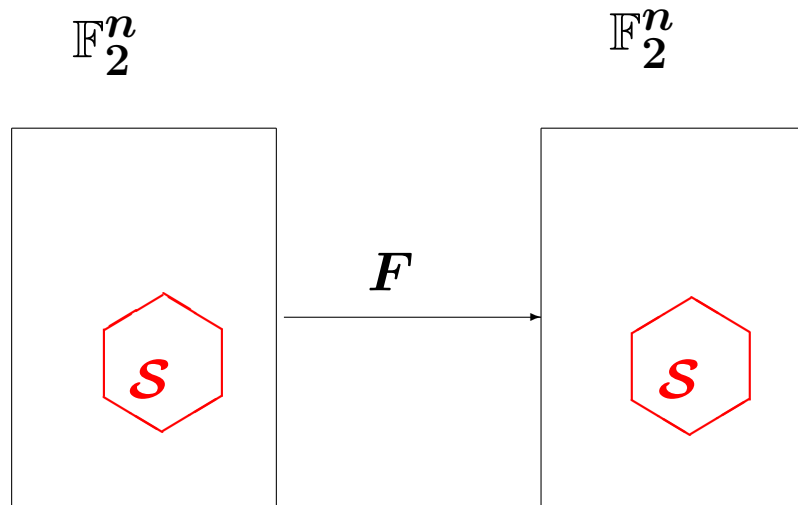
$$\Pr[g(x) + h(F(x)) = 0] \text{ far from } \frac{1}{2}.$$

quantified by:

$$\text{cor}_F(g, h) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{g(x) + h(F(x))}$$

## Nonlinear invariants [Todo-Leander-Sasaki 16]

Non-trivial partition of  $\mathbb{F}_2^n$  invariant under  $F$ :



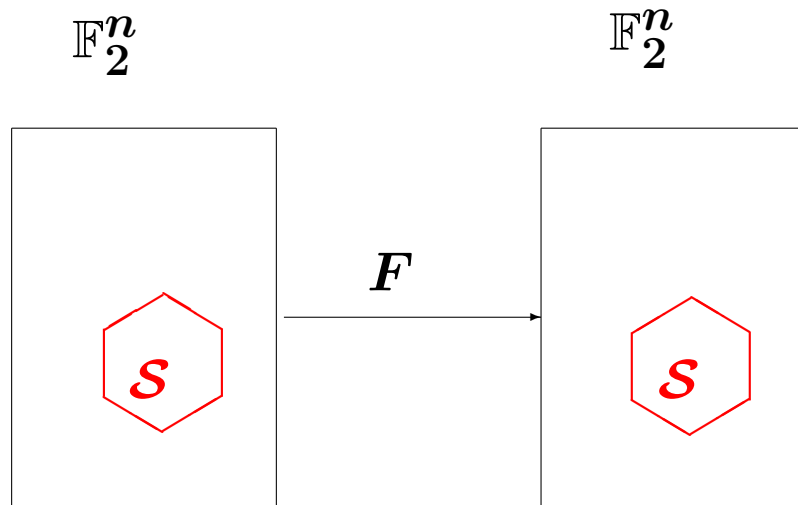
$S$ : any subset of  $\mathbb{F}_2^n$

$$F(S) = S$$

$$\text{or } F(S) = \mathbb{F}_2^n \setminus S$$

# The nonlinear invariant attack [Todo-Leander-Sasaki 16]

Non-trivial partition of  $\mathbb{F}_2^n$  invariant under  $F$ :



$\mathcal{S}$ : any subset of  $\mathbb{F}_2^n$

$$F(\mathcal{S}) = \mathcal{S}$$

$$\text{or } F(\mathcal{S}) = \mathbb{F}_2^n \setminus \mathcal{S}$$

**Equivalently:**

Let  $g$  be the Boolean function defined by  $g(x) := 1$  iff  $x \in \mathcal{S}$

$$\forall x \in \mathbb{F}_2^n, g(F(x)) = g(x) \text{ or } \forall x \in \mathbb{F}_2^n, g(F(x)) = g(x) + 1$$

Such a  $g$  is called an **invariant** for  $F$ .

## Nonlinear approximations with correlation $\pm 1$

$g$  is an invariant for  $F$  if and only if

$$\text{cor}_F(g, g) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{g(x) + g(F(x))} = \pm 1$$

## Nonlinear approximations as a combination of linear approximations

$$\text{cor}_F(g, h) = \sum_{\gamma, \gamma' \in \mathbb{F}_2^n} \text{cor}_g(\gamma) \text{cor}_F(\gamma, \gamma') \text{cor}_h(\gamma') .$$

If  $g = \ell_\alpha$  and  $h = \ell_\beta$ , then

$$\text{cor}_F(g, h) = \text{cor}_F(\alpha, \beta) .$$

Otherwise, we gather together several linear approximations.

# **Nonlinear approximations and the linear hull effect**

## Transforming nonlinear invariants into linear approximations

Let  $g$  be a **balanced** nonlinear invariant for  $F$ .

We can always define a permutation  $\mathcal{G}$  such that  $\alpha \cdot \mathcal{G}(x) = g(x)$ .

Then,

$$\begin{aligned}g(x) + g(F(x)) &= \alpha \cdot \mathcal{G}(x) + \alpha \cdot (\mathcal{G} \circ F)(x) \\ &= \alpha \cdot y + \alpha \cdot (\mathcal{G} \circ F \circ \mathcal{G}^{-1})(y)\end{aligned}$$

The **nonlinear** approximation of  $F$  defined by  $(g, g)$  corresponds to the **linear** approximation  $(\alpha, \alpha)$  of  $F^{\mathcal{G}, \mathcal{G}^{-1}} = \mathcal{G} \circ F \circ \mathcal{G}^{-1}$ .

$$\text{cor}_{F^{\mathcal{G}, \mathcal{G}^{-1}}}(\alpha, \alpha) = \sum_{\gamma_1, \gamma_2 \in \mathbb{F}_2^n} \text{cor}_{\mathcal{G}_\alpha}(\gamma_1) \text{cor}_F(\gamma_1, \gamma_2) \text{cor}_{\mathcal{G}_\alpha}(\gamma_2)$$

The other components of  $\mathcal{G}$  do not matter!

## $\mathcal{G}$ -shifted trails

$$\begin{aligned} E_{(k_0, \dots, k_t)}^{\mathcal{G}, \mathcal{G}^{-1}} &= \mathcal{G} \circ \mathcal{R}_{k_t} \circ \mathcal{R}_{k_{t-1}} \circ \dots \circ \mathcal{R}_{k_0} \circ \mathcal{G}^{-1} \\ &= \mathcal{R}_{k_t}^{\mathcal{G}, \mathcal{G}^{-1}} \circ \mathcal{R}_{k_{t-1}}^{\mathcal{G}, \mathcal{G}^{-1}} \circ \dots \circ \mathcal{R}_{k_0}^{\mathcal{G}, \mathcal{G}^{-1}}. \end{aligned}$$

$$\text{cor}_{E_{(k_0, \dots, k_t)}^{\mathcal{G}, \mathcal{G}^{-1}}}(\alpha, \beta) = \sum_{\gamma_1, \dots, \gamma_{t-1} \in \mathbb{F}_2^n} \prod_{i=0}^{t-1} \text{cor}_{\mathcal{R}_{k_i}^{\mathcal{G}, \mathcal{G}^{-1}}}(\gamma_i, \gamma_{i+1}).$$



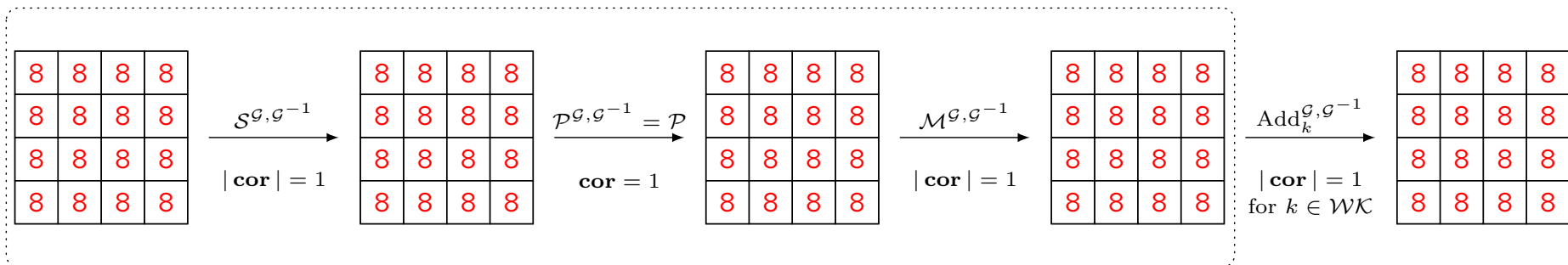
## A one-round $\mathcal{G}$ -shifted trail on Midori-64

$\mathcal{G} = (G, \dots, G)$  where  $G$  is a bijection on 4 bits such that  $\langle 8, G(x) \rangle = g(x)$  with  $g(x) = x_3x_2 + x_2 + x_1 + x_0$  invariant for the Sbox, i.e.

$$|\text{cor}_{\mathcal{S}G, G^{-1}}(8, 8)| = 1 .$$

$$|\text{cor}_{\mathcal{M}\mathcal{G}, \mathcal{G}^{-1}}((8, \dots, 8), (8, \dots, 8))| = 1 .$$

$\Rightarrow$  Iterative one-round trail with correlation  $\pm 1$ :



## A two-round shifted trail on Midori-64 [Beyne 18]

For  $g(x) = x_0x_2 + x_0 + x_1 + x_3$  and  $\alpha = 0x5$ , the Sbox satisfies

$$g(S(x)) + \alpha \cdot x = 1 .$$

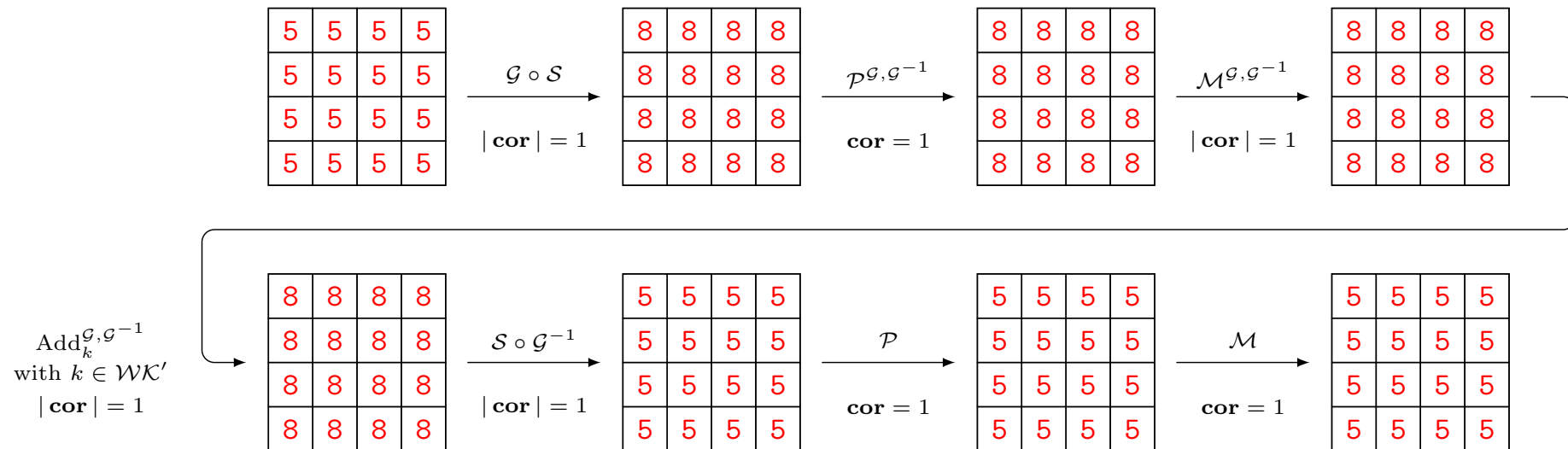
We choose a 4-bit bijection  $G$  such that  $\langle 8, G(x) \rangle = g(x)$ .

Equivalently,

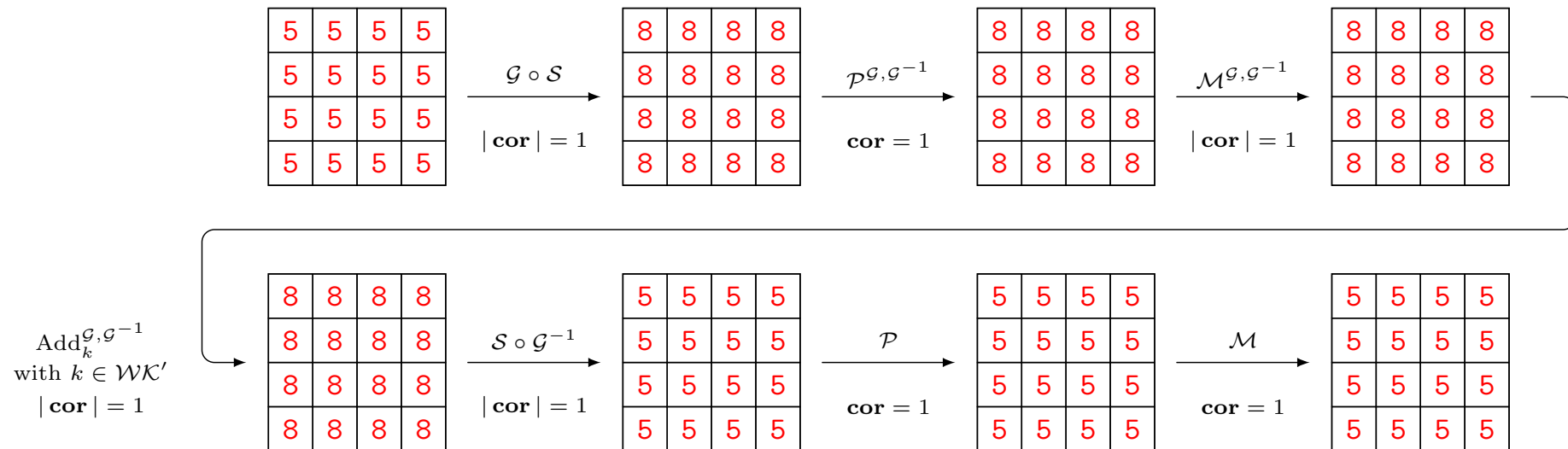
$$\text{cor}_S(\ell_\alpha, g) = \text{cor}_{G \circ S}(\alpha, 8) = -1 .$$

$$|\text{cor}_{\mathcal{M}^{g, g^{-1}}}((8, \dots, 8), (8, \dots, 8))| = 1 .$$

# A two-round shifted trail on Midori-64 [Beyne 18]



# A two-round shifted trail on Midori-64 [Beyne 18]



This is a two-round **linear** approximation with correlation  $\pm 1$ !

## A 4-round $\mathcal{G}$ -shifted trail on Midori-64

$G$  is a bijection on 4 bits such that  $\langle 8, G(x) \rangle = g(x)$   
with  $g(x) = x_3x_2x_1 + x_3x_1 + x_3 + x_2 + x_1 + x_0$  invariant for the Sbox:

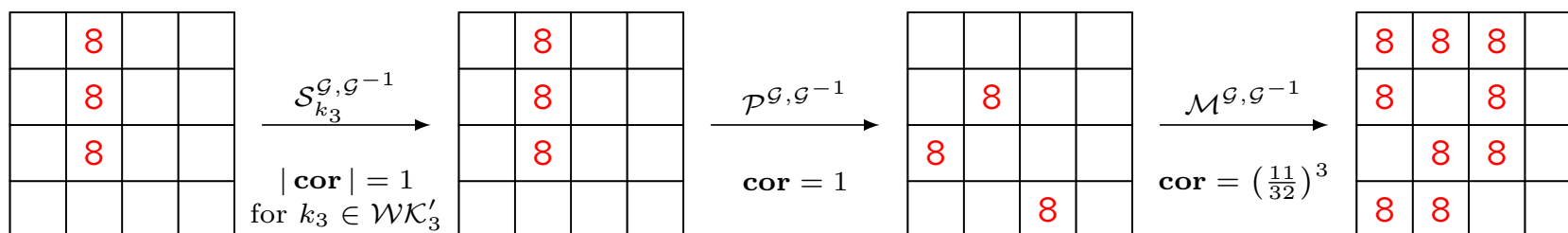
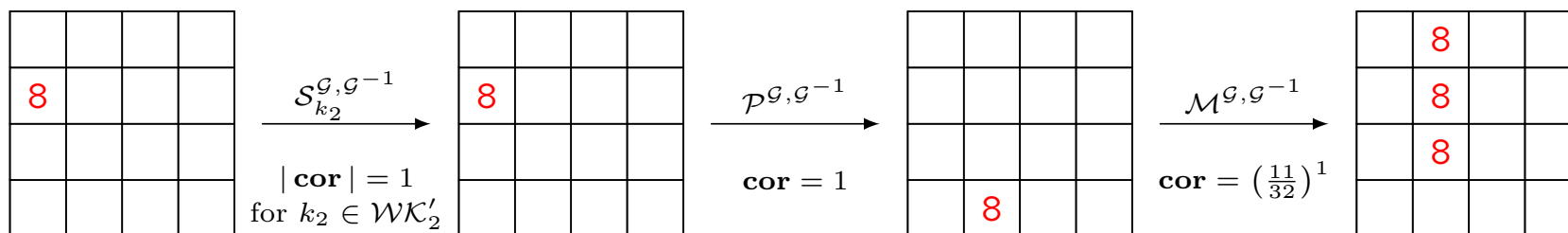
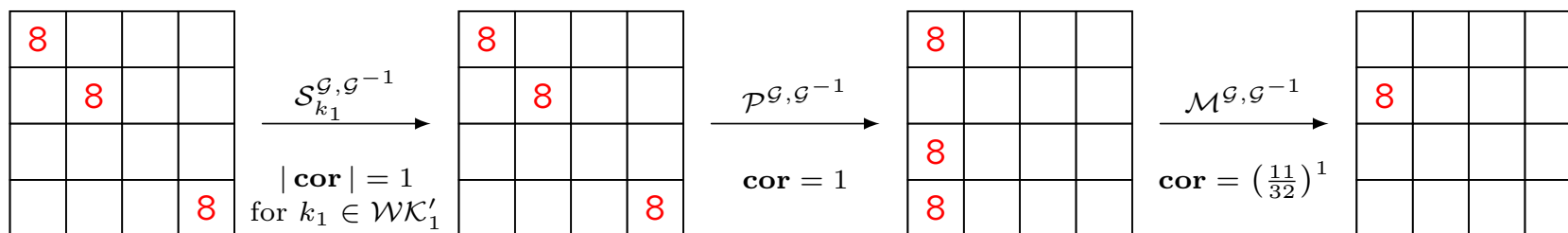
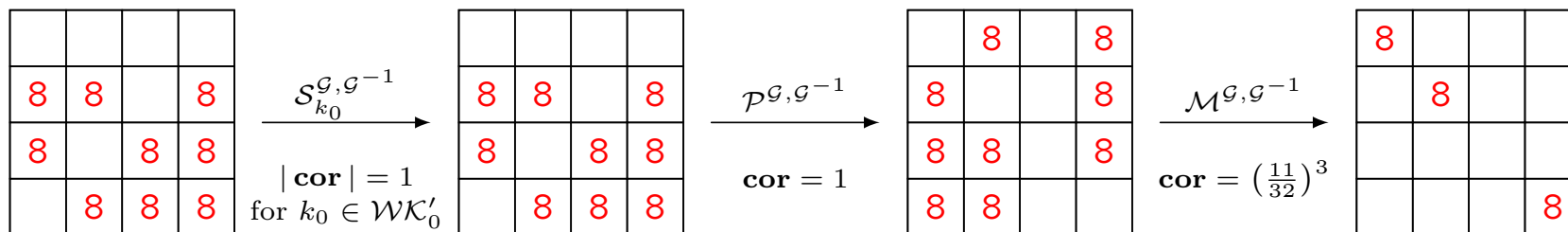
$$|\text{cor}_{SG, G^{-1}}(8, 8)| = 1 .$$

But,

$$|\text{cor}_{MG, G^{-1}}(\alpha, M\alpha)| = \frac{11}{32}$$

if  $\alpha \neq (0, 0, 0, 0)$  and all  $\alpha_i \in \{0, 8\}$ .

## A 4-round $\mathcal{G}$ -shifted trail on Midori-64



## A 4-round $\mathcal{G}$ -shifted trail on Midori-64

The weak keys are those equal to 0 or 1 in all active cells.

**Correlation of the trail:**

$$\left(\frac{11}{32}\right)^8 = 2^{-12.325}$$

**Correlation of the approximation:**

$$\text{cor}_{(\mathcal{R}_{k_3} \circ \dots \circ \mathcal{R}_{k_0})^{\mathcal{G}, \mathcal{G}^{-1}}(\alpha, \alpha)} \simeq 2^{-12.16}$$

**What's about the other trails?** For the first 2 rounds:

- For  $G_1 = [0, 8, c, 4, a, 2, 6, e, 9, 1, d, 5, 3, b, f, 7]$ ,  
35,937  $\mathcal{G}_1$ -shifted linear trails having a nonzero correlation
- For  $G_2 = [0, 9, a, 1, 8, 2, 3, f, c, 4, d, 5, 6, e, b, 7]$ ,  
282,184  $\mathcal{G}_2$ -shifted linear trails having a nonzero correlation

## Another 1-round $\mathcal{G}$ -shifted trail on Midori-64

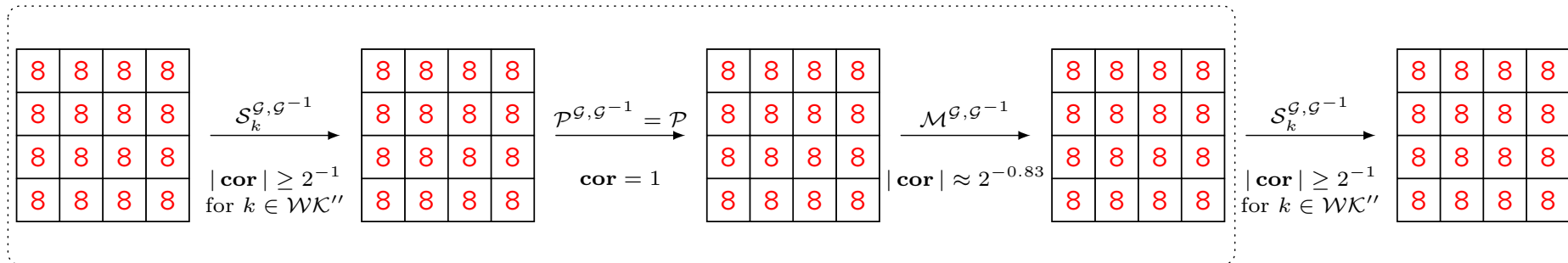
$\mathcal{G} = (G', G, \dots, G)$  where  $G$  is a bijection on 4 bits such that  
 $\langle 8, G(x) \rangle = g(x)$  with  $g(x) = x_3x_2 + x_2 + x_1 + x_0$  invariant for  $S$ ,  
 $\langle 8, G'(x) \rangle = g'(x)$  with  $g'(x) = x_3x_2x_1 + x_3x_1 + x_3 + x_2 + x_1 + x_0$ .

$$|\text{cor}_{S_k^{G', G'^{-1}}}(8, 8)| = \begin{cases} 1 & \text{if } k \in \{0, 1\} \\ 2^{-1} & \text{if } k \notin \{0, 1\} \end{cases} .$$

$$|\text{cor}_{\mathcal{M}^{\mathcal{G}, \mathcal{G}^{-1}}}((8, \dots, 8), (8, \dots, 8))| \simeq 2^{-0.83}$$



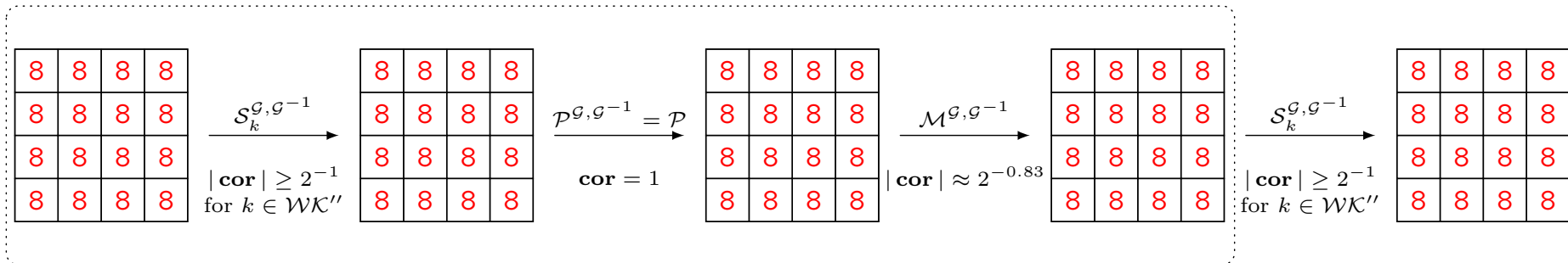
## Another 1-round $\mathcal{G}$ -shifted trail on Midori-64



Correlation of the 16-round trail:

$$\geq \left(2^{-1.83}\right)^{16} = 2^{-29.28}$$

## Another 1-round $\mathcal{G}$ -shifted trail on Midori-64



**Correlation of the 16-round trail:**

$$\geq \left(2^{-1.83}\right)^{16} = 2^{-29.28}$$

**Correlation over 16 rounds:**

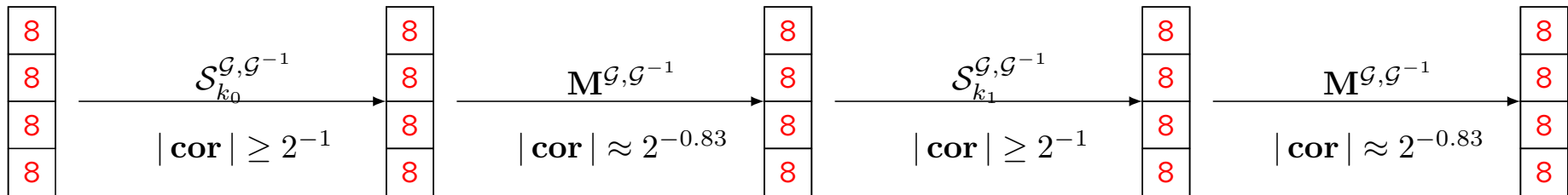
different from the correlation of the trail.

## Focus on a single column

$\mathcal{G} = (G', G, G, G)$  with

$$|\text{cor}_{S_k^{G', G'^{-1}}} (8, 8)| = \begin{cases} 1 & \text{if } k \in \{0, 1\} \\ 2^{-1} & \text{if } k \notin \{0, 1\} \end{cases} .$$

$$|\text{cor}_{M^{\mathcal{G}, \mathcal{G}^{-1}}} ((8, 8, 8, 8), (8, 8, 8, 8))| = 2^{-0.83}$$



If  $k_1 \in \left( \mathbb{F}_2^4 \setminus \{(0, 0, *, *)\} \right) \times \{(0, 0, *, *)\}^3$ ,

$$\text{cor}_{\mathcal{R}_{k_1} \circ \mathcal{R}_{k_0}} ((8, 8, 8, 8), (8, 8, 8, 8)) = 0$$

## Open problems

- When can we approximate the correlation with a single trail?
- Nonlinear approximations as a method for clustering linear approximations to capture the linear hull effect?
  - How general is this?
  - How can we find the appropriate approximation?