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# On Sboxes sharing the same DDT

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## Problem

Find all  $n$ -bit Sboxes having a given difference distribution table.

$\alpha/\beta$	0	1	2	3	4	5	6	7
0	8	.	.	.	.	.	.	.
1	.	2	.	2	.	2	.	2
2	.	.	2	2	.	.	2	2
3	.	2	2	.	.	2	2	.
4	.	.	.	.	2	2	2	2
5	.	2	.	2	2	.	2	.
6	.	.	2	2	2	2	.	.
7	.	2	2	.	2	.	.	2

where  $\delta_F(\alpha, \beta) = \#\{x \in \mathbb{F}_2^n : F(x + \alpha) + F(x) = \beta\}$

## Some trivial properties of the DDT

### Differential uniformity of $F$ :

$$\delta(F) = \max_{\alpha \neq 0, \beta} \delta_F(\alpha, \beta) .$$

$\delta(F) \geq 2$  with equality for **APN functions**.

- All entries in the DDT are even.
- The entries in a row sum to  $2^n$ .

## Related open problems

- Characterization of valid DDTs.
- Characterization of the functions sharing the same DDT.
- **The big APN problem** [Dillon 09]: Does there exist an APN permutation of  $n$  variables with  $n$  even,  $n \geq 8$ ?
- **The crooked conjecture** [Bending, Fon-der-Flaass 98]:  
 $F$  is an APN permutation of degree 2 if and only if the support of every row in the DDT is the complement of a hyperplane.

## Indicator of the DDT [Carlet, Charpin, Zinoviev 98]

### Definition:

For any  $n$ -bit Sbox  $F$ ,  $\gamma_F$  is the Boolean function of  $2n$  variables defined by

$$\gamma_F(\alpha, \beta) = 0 \text{ if and only if } \delta_F(\alpha, \beta) = 0 \text{ or } \alpha = 0 .$$

$\alpha/\beta$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	*	0	*	0	*	0	*
2	0	0	*	*	0	0	*	*
3	0	*	*	0	0	*	*	0
4	0	0	0	0	*	*	*	*
5	0	*	0	*	*	0	*	0
6	0	0	*	*	*	*	0	0
7	0	*	*	0	*	0	0	*

## Two notions of differential equivalence

- DDT-equivalence:

$$F \sim_{\text{DDT}} G \iff \text{DDT}_F = \text{DDT}_G$$

- $\gamma$ -equivalence (aka differential equivalence [Gorodilova 16]):

$$F \sim_{\gamma} G \iff \gamma_F = \gamma_G$$

DDT-equivalence  $\Rightarrow$   $\gamma$ -equivalence





## The two notions are different

For  $n = 4$

$$F = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 14],$$

$$G = [0, 1, 3, 2, 5, 4, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15].$$

$$\text{DDT}_G = \begin{bmatrix} 16 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 16 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 12 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & 4 \end{bmatrix}$$

## $\gamma$ -equivalence and differential uniformity

The following 4-bit Sboxes are  $\gamma$ -equivalent:

$$F_1 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \text{ with } \delta(F_1) = 14,$$

$$F_2 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1] \text{ with } \delta(F_2) = 12,$$

$$F_3 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1] \text{ with } \delta(F_3) = 10,$$

$$F_4 = [1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1] \text{ with } \delta(F_4) = 8.$$

## The two notions coincide in some cases

$\alpha/\beta$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	*	0	*	0	*	0	*
2	0	0	*	*	0	0	*	*
3	0	*	*	0	0	*	*	0
4	0	0	0	0	*	*	*	*
5	0	*	0	*	*	0	*	0
6	0	0	*	*	*	*	0	0
7	0	*	*	0	*	0	0	*

$F \sim_\gamma G$  implies  $F \sim_{\text{DDT}} G$

- when  $F$  is **APN**.
- when  $F$  and  $G$  are **quadratic**.

**Sboxes sharing the same DDT**

## Trivially equivalent Sboxes

### Proposition.

The DDT-equivalence class of an  $n$ -bit Sbox contains all functions of the form

$$x \mapsto F(x + \mathbf{c}) + \mathbf{d}, \text{ for } \mathbf{c}, \mathbf{d} \in \mathbb{F}_2^n.$$

We say that the DDT-equivalence class of  $F$  is **trivial** if it contains trivially equivalent Sboxes only.

### Problems.

- Characterize the Sboxes having a trivial DDT-equivalence class.
- Determine the properties of the Sboxes within a non-trivial DDT-equivalence class.

## An equivalent formulation

$F$  and  $G$  share the same DDT iff they share the same squared LAT.

⇒ Sboxes within the same DDT-equivalence class correspond to LAT with different sign sequences:

$$\mathcal{W}_G(\lambda, \mu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\lambda \cdot G(x) + \mu \cdot x} = (-1)^{s(\lambda, \mu)} \mathcal{W}_F(\lambda, \mu).$$

$F$  and  $G$  are trivially DDT-equivalent Sboxes if and only if

$$s(\lambda, \mu) = d \cdot \lambda + c \cdot \mu.$$

## Algebraic degree of DDT-equivalent Sboxes

**Conjecture** [Gorodilova 16]. If  $F$  is a quadratic APN Sbox, then any  $G$  in the DDT-class of  $F$  satisfies

$$\deg(F + G) \leq 1 .$$

**In general:** For any even  $n$ , all  $n$ -bit Sboxes defined by

$$S(x) = (f(x), c_1, \dots, c_{n-1})$$

where  $f$  is a bent function and  $(c_1, \dots, c_{n-1})$  is a constant, have the same DDT.

All rows are equal to

$$[2^{n-1}, 2^{n-1}, 0, 0, \dots, 0]$$

$\Rightarrow$  There exist Sboxes of any degree between 2 and  $n/2$  in this DDT-equivalence class.

## Extended affine equivalence

### Definition.

Two  $n$ -bit Sboxes  $F$  and  $G$  are extended-affine (EA) equivalent if there exist affine functions  $A_0, A_1, A_2$ , where  $A_1$  and  $A_2$  are bijective such that

$$G = A_1 \circ F \circ A_2 + A_0 .$$

### Proposition (adapted from [Gorodilova 16])

If  $F$  and  $G$  are EA-equivalent, then their DDT-equivalence (resp.  $\gamma$ -equivalence) classes have the same size.

$$\mathcal{C}_{\text{DDT}}(G) = \{A_1 \circ F' \circ A_2 + A_0, \text{ with } F' \in \mathcal{C}_{\text{DDT}}(F)\}$$



## CCZ equivalence [Carlet, Charpin, Zinoviev 98]

### Definition.

Two  $n$ -bit Sboxes  $F$  and  $G$  are said **CCZ equivalent** if

$\{(y, G(y)), y \in \mathbb{F}_2^n\}$  is the image of  $\{(x, F(x)), x \in \mathbb{F}_2^n\}$

by a **linear permutation**  $\mathcal{L}$  of  $\mathbb{F}_2^n \times \mathbb{F}_2^n$ .

In particular, if  $\mathcal{L} : (x, y) \mapsto (L_1(x, y), L_2(x, y))$ , then

$x \mapsto L_1(x, F(x))$  is a **permutation**.

### Open problem [Gorodilova 16]

Does an analogue of the result for **EA** equivalence hold for **CCZ** equivalence?

## CCZ equivalence

### Theorem.

If  $F$  and  $G$  are CCZ-equivalent then

- their DDT-equivalence (resp.  $\gamma$ -equivalence) classes have the same size.
- The DDT-class of  $G$  is obtained by applying the same linear permutation  $\mathcal{L}$  to all functions in the DDT-class of  $F$ .

## Algorithm for determining all Sboxes having a prescribed $\gamma$

**Input** : The indicator of a DDT,

**Output** : All functions having this indicator

**Idea:** Recursive Tree-traversal algorithm

- Tree of depth  $2^n$  : each node at level  $i$  corresponds to one possible value for  $F(i)$ .
- From the constraints of the DDT and the values  $F(0), \dots, F(i-1)$ :
  - find all possible values for  $F(i)$
  - for each of them, move on to the next step  $F(i+1)$ , and backtrack if necessary

**Pruning trick:** We fix

$$F(0) \text{ and } F(1)$$

# Experimental results

# Permutations with optimal differential uniformity

## APN permutations.

The DDT-equivalence classes of all known APN permutations for  $n \leq 9$  are trivial.

## Optimal permutations for $n = 4$ .

The DDT-equivalence classes and the  $\gamma$ -equivalence classes of all permutations  $F$  with  $\delta(F) = 4$  and optimal linearity listed in [Leander, Poschmann 07] are trivial.

## APN non-bijective functions

The DDT-equivalence classes of all known APN functions for  $n \leq 8$  are trivial, except:

- when  $n \equiv 0 \pmod{4}$ : the Gold APN functions with exponents  $2^k + 1$  with  $k = n/2 \pm 1$  [Gorodilova 16]
- for  $n = 6$ : Class 13 in [Brinckmann, Leander 08].

We checked that, for  $n = 6$ , all APN functions of degree  $\leq 3$  are trivial except Class 13.

## Do all permutations have a trivial DDT class?

The following **permutations** share the same DDT.

$x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F(x)$	0	1	2	3	4	5	6	7	8	9	10	11	13	12	15	14	16
$G(x)$	0	1	2	3	4	5	6	7	8	9	10	11	13	12	15	14	16
$x$	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31		
$F(x)$	17	19	18	20	21	23	22	25	24	26	27	28	29	31	30		
$G(x)$	17	19	18	21	20	22	23	24	25	27	26	28	29	31	30		

## Some conclusions and a conjecture

- All Sboxes we found with a non-trivial DDT-equivalence class have **non-distinct rows in their DTT**.
- All rows in the DDT of an APN permutation are distinct.

### **Conjecture.**

The DDT-equivalence class of any APN permutation is trivial.

### **Open problem.**

Find a family of Sboxes for which it can be proved that the DDT-equivalence class is trivial.



## Toy example for $n = 3$

$$\mathcal{R}_i := \{j \mid \mathbf{DDT}(i, j) \neq 0\}.$$

	0	1	2	3	4	5	6	7
0	8	.	.	.	.	.	.	.
1	.	2	.	2	.	2	.	2
2	.	.	2	2	.	.	2	2
3	.	2	2	.	.	2	2	.
4	.	.	.	.	2	2	2	2
5	.	2	.	2	2	.	2	.
6	.	.	2	2	2	2	.	.
7	.	2	2	.	2	.	.	2

- Set  $F(0) = 0$
- $F(0) + F(1) \in \mathcal{R}_1 = \{1, 3, 5, 7\} \Rightarrow$  Set  $F(1) = 1$ .
- $F(0) + F(2) \in \mathcal{R}_2 = \{2, 3, 6, 7\}$  and  $F(1) + F(2) \in \mathcal{R}_3 = \{1, 2, 5, 6\}$   
 $\Rightarrow F(2) \in \{3, 7\}$

# Toy example for $n = 3$

