



The multiple electromagnetic wave scattering by small spheres

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The multiple electromagnetic wave scattering by small spheres

Justine Labat, Victor Péron, Sébastien Tordeux

PhD student in Mathematics

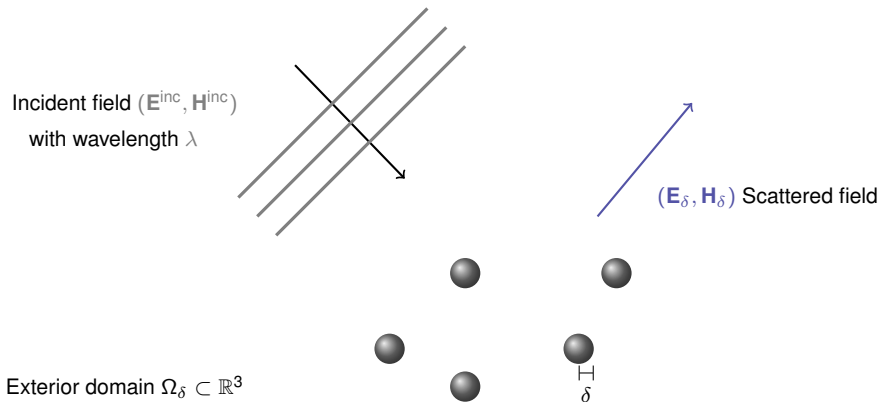
EPC Magique 3D — UPPA-E2S, INRIA Bordeaux Sud-Ouest, LMAP UMR CNRS 5142

Seminar of RTWH Aachen University

Aachen, December 4th 2018

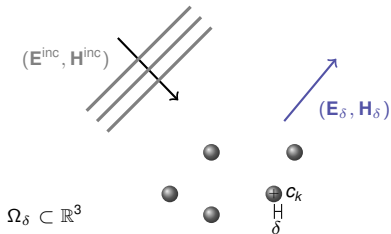


3D Scattering problem by small obstacles



Asymptotic assumption : $\delta \ll \lambda$

Model problem



- Time-harmonic domain
- Homogeneous & isotropic medium Ω_δ
- Perfect conductors $\mathcal{B}(c_k, \delta)$, $k = 1 \dots N_{\text{obs}}$

Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl} \, \mathbf{E}_\delta - i\kappa \mathbf{H}_\delta = 0 & \text{in } \Omega_\delta \\ \mathbf{curl} \, \mathbf{H}_\delta + i\kappa \mathbf{E}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

$$\text{with } \kappa^2 = \omega^2 \mu \left(\epsilon + \frac{i\sigma}{\omega} \right), \Im(\kappa) \geq 0$$

Boundary condition

$$\mathbf{n} \times \mathbf{E}_\delta = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{on } \partial\Omega_\delta$$

Silver-Müller radiation condition

$$r(\mathbf{H}_\delta \times \hat{\mathbf{x}} - \mathbf{E}_\delta) \xrightarrow[r \rightarrow \infty]{} 0 \quad \text{unif. in } \hat{\mathbf{x}} = \frac{\mathbf{x}}{r}$$

Mathematical well-posedness

For any $\mathbf{E}^{\text{inc}} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_\delta)$ there exist a unique solution $(\mathbf{E}_\delta, \mathbf{H}_\delta) \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_\delta)^2$ to the exterior Maxwell problem.

Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Motivations

Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Multiple scattering



Foldy-Lax model

- ✓ Interactions taken into account
- ✓ Low computational cost
- ✓ Meshless method

Superposition
principle



Born approximation

- ✗ No interaction between the obstacles
- ✓ Low computational cost
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Motivations

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Born approximation

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High-order numerical solution

- ✗ High computational cost
 - ✗ volumical methods FEM, DG, ...
 - ~ boundary element methods
- ✗ Mesh refinement
- ✓ High-order spectral method

Outline

1. Single electromagnetic scattering by small spheres

- Asymptotic models
- Numerical results
- Application: the Born approximation

2. Multiple electromagnetic scattering by small spheres

- Foldy-Lax model
- Spectral method
- Numerical results

3. Conclusions and perspectives

Outline

1. Single electromagnetic scattering by small spheres

- Asymptotic models
- Numerical results
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2. Multiple electromagnetic scattering by small spheres

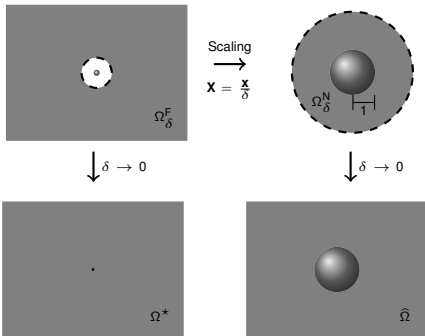
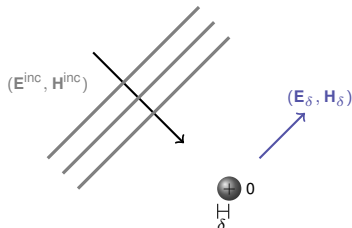
- Foldy-Lax model
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3. Conclusions and perspectives

Approximation of solution to single scattering

Method of **matched asymptotic expansions**

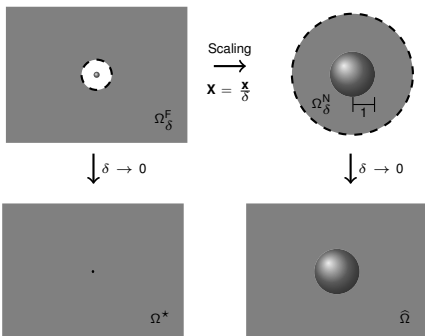
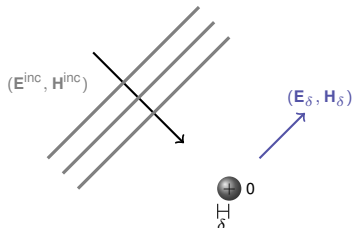
- Domain decomposition
- Local approximations
- Matching procedure



Approximation of solution to single scattering

Method of **matched asymptotic expansions**

- Domain decomposition
- **Local approximations**
- Matching procedure



Asymptotic expansions

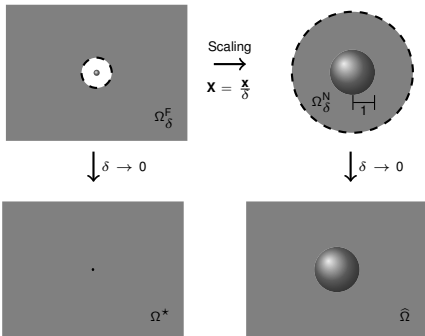
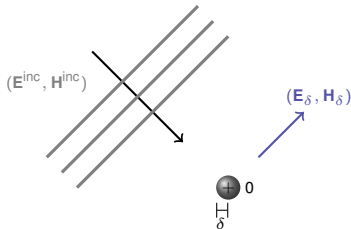
Far field expansion in $\Omega^* = \mathbb{R}^3 \setminus \{0\}$

Near field expansion in $\hat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0, 1)}$

Approximation of solution to single scattering

Method of **matched asymptotic expansions**

- Domain decomposition
- Local approximations
- **Matching procedure**



Asymptotic expansions

Far field expansion in $\Omega^* = \mathbb{R}^3 \setminus \{0\}$

Near field expansion in $\hat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0, 1)}$

Near field



Far field

Asymptotic expansions

Near field expansion

$$\mathbf{E}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{E}}_0(\mathbf{X}) + \delta\widehat{\mathbf{E}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{E}}_2(\mathbf{X}) + \dots$$

$$\mathbf{H}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{H}}_0(\mathbf{X}) + \delta\widehat{\mathbf{H}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{H}}_2(\mathbf{X}) + \dots$$

$\mathbf{X} = \frac{\mathbf{x}}{\delta}$: fast variable

Far field expansion

$$\mathbf{E}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{E}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{E}}_5(\mathbf{x}) + \dots$$

$$\mathbf{H}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{H}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{H}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{H}}_5(\mathbf{x}) + \dots$$

- For an obstacle of arbitrary shape

Numerical solution of elementary problems **independent of δ**

- ▶ Near-field: **stationnary** problems
- ▶ Far-field: **time-harmonic** problems + equivalent multipole distributions

- For a **spherical** obstacle

- ▶ Analytical solutions of elementary problems
- ▶ Equivalent **multipole distributions**

Equivalent multipole distributions

Boundary Value Problem



Problem in the free-space

$$\begin{cases} \operatorname{curl} \hat{\mathbf{E}}_0 = 0 & \text{in } \hat{\Omega} \\ \operatorname{div} \hat{\mathbf{E}}_0 = 0 & \text{in } \hat{\Omega} \\ \mathbf{n} \times \hat{\mathbf{E}}_0 = -\mathbf{n} \times \mathbf{E}^{\text{inc}}(0) & \text{on } \hat{\Gamma} \\ \hat{\mathbf{E}}_0 = O(R^{-1}) & R \rightarrow \infty \end{cases}$$

$$\hat{\mathbf{E}}_0 = -\nabla V_{1,\text{elec}}^{0,\text{stat}} = -\nabla \left(\lim_{\epsilon \rightarrow 0} V_{1,\text{elec}}^{\epsilon,\text{stat}} \right)$$

$$\begin{cases} -\Delta V_{1,\text{elec}}^{\epsilon,\text{stat}} = \varrho_1^\epsilon & \text{in } \mathbb{R}^3 \\ V_{1,\text{elec}}^{\epsilon,\text{stat}} = O(R^{-1}) & R \rightarrow \infty \end{cases}$$

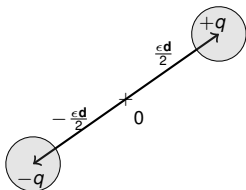


Figure: Electrostatic dipole

Distribution of two electric point-charges

$$\varrho_1^\epsilon = \frac{1}{\epsilon} \left(\delta_{\frac{\epsilon \mathbf{d}}{2}} - \delta_{-\frac{\epsilon \mathbf{d}}{2}} \right)$$

Dipole moment $\mathbf{d}$ related to the BC

$$\mathbf{d} = 4\pi \mathbf{E}^{\text{inc}}(0)$$

Equivalent multipole distributions

Boundary Value Problem



Problem in the free-space

$$\begin{cases} \mathbf{curl} \hat{\mathbf{E}}_0 = 0 & \text{in } \hat{\Omega} \\ \text{div} \hat{\mathbf{E}}_0 = 0 & \text{in } \hat{\Omega} \\ \mathbf{n} \times \hat{\mathbf{E}}_0 = -\mathbf{n} \times \mathbf{E}^{\text{inc}}(0) & \text{on } \hat{\Gamma} \\ \hat{\mathbf{E}}_0 = O(R^{-1}) & R \rightarrow \infty \end{cases}$$

$$\begin{cases} \text{div} \hat{\mathbf{E}}_0 = D\delta_0(\mathbf{d}) & \text{in } \mathbb{R}^3 \\ \mathbf{curl} \hat{\mathbf{E}}_0 = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

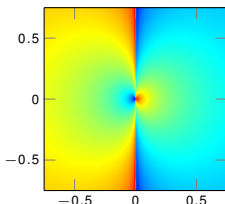


Figure: Electrostatic potential $\epsilon \rightarrow 0$

Distribution of two electric point-charges

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Equivalent multipole distributions

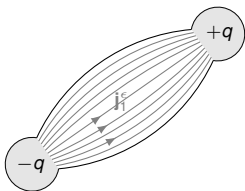
Boundary Value Problem



Problem in the free-space

$$\begin{cases} \operatorname{curl} \tilde{\mathbf{E}}_3 - i\kappa \tilde{\mathbf{H}}_3 = 0 & \text{in } \Omega^* \\ \operatorname{curl} \tilde{\mathbf{H}}_3 + i\kappa \tilde{\mathbf{E}}_3 = 0 & \text{in } \Omega^* \\ \tilde{\mathbf{E}}_3, \tilde{\mathbf{H}}_3 = O(r^{-3}) & r \rightarrow 0 \\ + \text{Silver-Müller condition} & \text{at } \infty \end{cases}$$

$$\begin{aligned} \tilde{\mathbf{E}}_3 &= -\nabla \left(\lim_{\epsilon \rightarrow 0} V_{1,\text{elec}}^\epsilon \right) + i\kappa \lim_{\epsilon \rightarrow 0} \mathbf{A}_{1,\text{elec}}^\epsilon \\ &\quad - \operatorname{curl} \lim_{\epsilon \rightarrow 0} \mathbf{A}_{1,\text{mag}}^\epsilon \\ \tilde{\mathbf{H}}_3 &= -\nabla \left(\lim_{\epsilon \rightarrow 0} V_{1,\text{mag}}^\epsilon \right) + i\kappa \lim_{\epsilon \rightarrow 0} \mathbf{A}_{1,\text{mag}}^\epsilon \\ &\quad + \operatorname{curl} \lim_{\epsilon \rightarrow 0} \mathbf{A}_{1,\text{elec}}^\epsilon \end{aligned}$$



Filiform current distribution

$$\mathbf{j}_1^\epsilon = \frac{i\omega}{\epsilon} \frac{\mathbf{d}_E}{|\mathbf{d}_E|} \delta_{\mathcal{F}_\epsilon}$$

Dipole moment $\mathbf{d}_E$ related to $\hat{\mathbf{E}}_0$

$$\mathbf{d}_E = 4\pi \mathbf{E}^{\text{inc}}(0)$$

Figure: Time-harmonic electric dipole

Equivalent multipole distributions

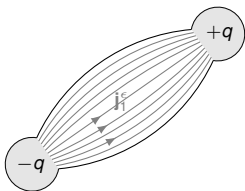
Boundary Value Problem



Problem in the free-space

$$\begin{cases} \mathbf{curl} \, \tilde{\mathbf{E}}_3 - i\kappa \tilde{\mathbf{H}}_3 = 0 & \text{in } \Omega^* \\ \mathbf{curl} \, \tilde{\mathbf{H}}_3 + i\kappa \tilde{\mathbf{E}}_3 = 0 & \text{in } \Omega^* \\ \tilde{\mathbf{E}}_3, \tilde{\mathbf{H}}_3 = O(r^{-3}) & r \rightarrow 0 \\ + \text{Silver-Müller condition} & \text{at } \infty \end{cases}$$

$$\begin{cases} -\Delta V_{1,\text{elec}}^\epsilon - \kappa^2 V_{1,\text{elec}}^\epsilon = \varrho_1^\epsilon & \text{in } \mathbb{R}^3 \\ + \text{Sommerfield condition} & R \rightarrow \infty \\ -\Delta \mathbf{A}_{1,\text{elec}}^\epsilon - \kappa^2 \mathbf{A}_{1,\text{elec}}^\epsilon = \frac{\mathbf{j}_1^\epsilon}{c} & \text{in } \mathbb{R}^3 \\ + \text{Sommerfield condition} & R \rightarrow \infty \end{cases}$$



Filiform current distribution

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$$\begin{cases} \mathbf{curl} \, \tilde{\mathbf{E}}_3 - i\kappa \tilde{\mathbf{H}}_3 = -i\kappa \mathbf{d}_H \delta_0 & \text{in } \mathbb{R}^3 \\ \mathbf{curl} \, \tilde{\mathbf{H}}_3 + i\kappa \tilde{\mathbf{E}}_3 = i\kappa \mathbf{d}_E \delta_0 & \text{in } \mathbb{R}^3 \\ \operatorname{div} \tilde{\mathbf{E}}_3 = D\delta_0(\mathbf{d}_E) & \text{in } \mathbb{R}^3 \\ \operatorname{div} \tilde{\mathbf{H}}_3 = D\delta_0(\mathbf{d}_H) & \text{in } \mathbb{R}^3 \\ + \text{Silver-Müller condition} & \text{at } \infty \end{cases}$$

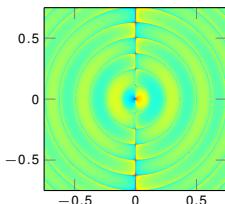


Figure: Electric potential $\epsilon \rightarrow 0$

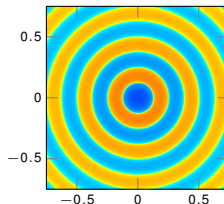


Figure: Magnetic potential $\epsilon \rightarrow 0$

Equivalent multipole distributions

$$\hat{\mathbf{E}}_p = \sum_{\substack{n=1 \\ |n-p| \text{ odd}}}^{p+1} \mathcal{E}_{n,\text{elec}}^{\text{stat}}[\mathbf{u}_{1,n}^{p,E}, \dots, \mathbf{u}_{n,n}^{p,E}] + \sum_{\substack{n=1 \\ |n-p| \text{ even}}}^{p+1} \mathcal{E}_{n,\text{mag}}^{\text{stat}}[\mathbf{u}_{1,n}^{p,H}, \dots, \mathbf{u}_{n,n}^{p,H}]$$

$$\tilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \left\{ \mathcal{E}_{n,\text{elec}}[\mathbf{v}_{1,n}^{p,E}, \dots, \mathbf{v}_{n,n}^{p,E}] + \mathcal{E}_{n,\text{mag}}[\mathbf{v}_{1,n}^{p,H}, \dots, \mathbf{v}_{n,n}^{p,H}] \right\}$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	●	●				
	1						
	2						
Far field	3	● ●	● ●				
	4						
	5						

● Electric field

● Magnetic field

+ Far field **collected dipolar** model



J. Labat, V. Péron, S. Tordeux (Inria Research Report n° 9169, Mars 2018)

Asymptotic Modeling of the Electromagnetic Scattering by Small Spheres Perfectly Conducting.

Equivalent multipole distributions

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	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	●	●				
	1	●	●	●	●		
	2						
Far field	3	● ●	● ●				
	4	● ●	● ●	● ●	● ●		
	5						

● Electric field

● Magnetic field

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$$\tilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \left\{ \mathcal{E}_{n,\text{elec}}[\mathbf{v}_{1,n}^{p,E}, \dots, \mathbf{v}_{n,n}^{p,E}] + \mathcal{E}_{n,\text{mag}}[\mathbf{v}_{1,n}^{p,H}, \dots, \mathbf{v}_{n,n}^{p,H}] \right\}$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	●	●				
	1	●	●	●	●		
	2	●	●	●	●	●	●
Far field	3	● ●	● ●				
	4	● ●	● ●	● ●	● ●		
	5	● ●	● ●	● ●	● ●	● ●	● ●

● Electric field

● Magnetic field

+ Far field **collected dipolar** model



J. Labat, V. Péron, S. Tordeux (Inria Research Report n°9169, Mars 2018)

Asymptotic Modeling of the Electromagnetic Scattering by Small Spheres Perfectly Conducting.

Numerical validation

Small parameter

$$\delta \in \left\{ \frac{1}{10^p}, p = 0.5 : 0.1 : 4 \right\}$$

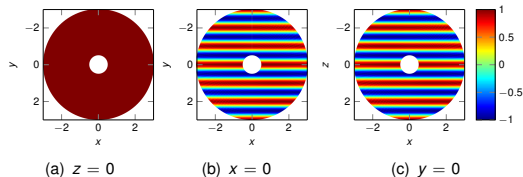


Figure: Incident plane wave (real part)

Physical parameters

- $\varepsilon = 1.0$
- $\mu = 1.0$
- $\sigma = 0$
- $\lambda = 5.0$

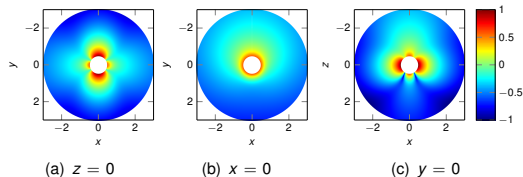


Figure: Reference scattered field (log)

Numerical validation

Small parameter

$$\delta \in \left\{ \frac{1}{10^p}, p = 0.5 : 0.1 : 4 \right\}$$

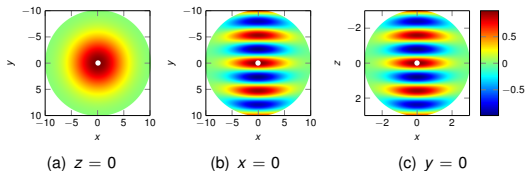


Figure: Incident **gaussian beam** (real part)

Physical parameters

- $\varepsilon = 1.0$
- $\mu = 1.0$
- $\sigma = 0$
- $\lambda = 5.0$

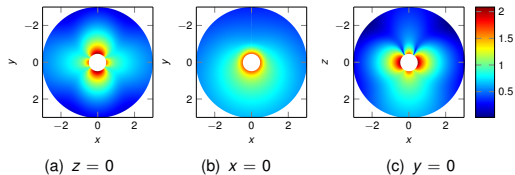


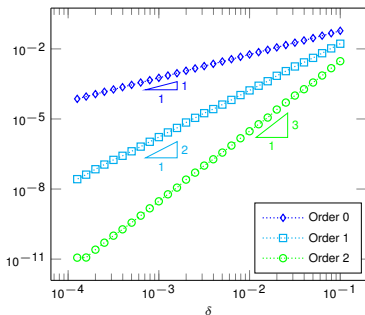
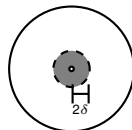
Figure: Reference scattered field (log)

Numerical convergence of near field approximations

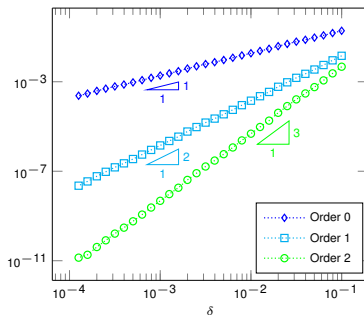
Order 0 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 1 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta \widehat{\mathbf{E}}_1(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 2 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta \widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2 \widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$



(a) Electric field



(b) Magnetic Field

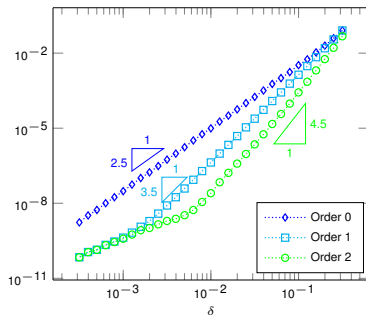
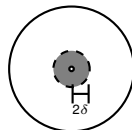
Figure: Relative $\mathbf{L}^2(\Omega_\delta^{2\delta})$ -errors for near field approximations — Incident plane wave

Numerical convergence of near field approximations

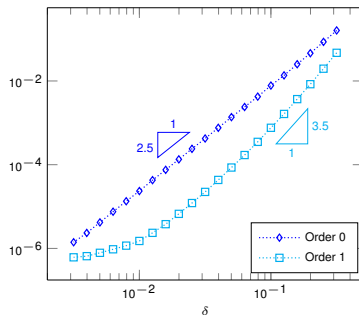
Order 0 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

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Order 2 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta \widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2 \widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$



(a) Electric field



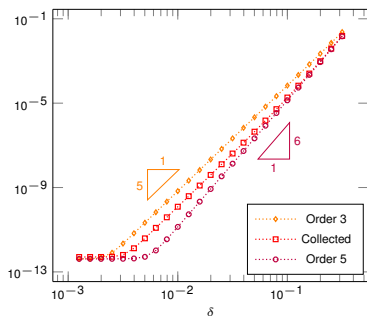
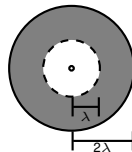
(b) Magnetic field

Figure: Relative $\mathbf{L}^2(\Omega_\delta^{2\delta})$ -errors for near field approximations — Incident **gaussian beam**

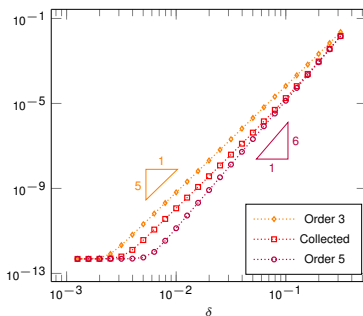
Numerical convergence of far field approximations

Order 3 : $\|\mathbf{E}_\delta - \delta^3 \tilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

Order 5 : $\|\mathbf{E}_\delta - \delta^3 \tilde{\mathbf{E}}_3 - \delta^5 \tilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$



(a) Electric field



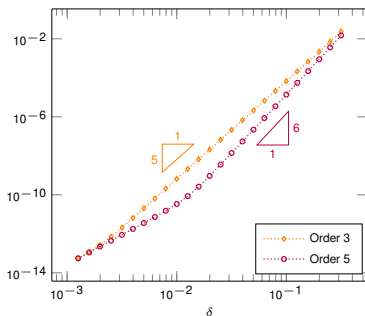
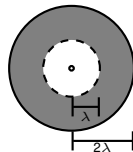
(b) Magnetic field

Figure: Absolute $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$ -errors for far field approximations — Incident plane wave

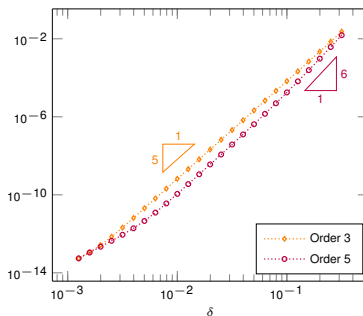
Numerical convergence of far field approximations

Order 3 : $\|\mathbf{E}_\delta - \delta^3 \tilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

Order 5 : $\|\mathbf{E}_\delta - \delta^3 \tilde{\mathbf{E}}_3 - \delta^5 \tilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$



(a) Electric field



(b) Magnetic field

Figure: Absolute $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$ -errors for far field approximations — Incident gaussian beam

Application: the Born approximation

- The electric field is approximated by the **superposition principle**

$$\mathbf{E}_{\delta}(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a **dipolar source** around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \boldsymbol{\varepsilon}_{1,\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \boldsymbol{\varepsilon}_{1,\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k)$$

- The dipole moments $\mathbf{d}_{\delta,k}^{\text{E}}$ and $\mathbf{d}_{\delta,k}^{\text{H}}$ depend on the nature of the obstacles

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- The dipole moments $\mathbf{d}_{\delta,k}^{\text{E}}$ and $\mathbf{d}_{\delta,k}^{\text{H}}$ depend on the nature of the obstacles

For **perfectly conducting** spheres:

- Approximation of order 3

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \mathbf{E}^{\text{inc}}(c_k)$$

$$\mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \mathbf{H}^{\text{inc}}(c_k)$$

- Collected dipolar approximation

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10}\right) \mathbf{E}^{\text{inc}}(c_k) \quad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5}\right) \mathbf{H}^{\text{inc}}(c_k)$$

No interaction are taken into account

Outline

1. Single electromagnetic scattering by small spheres

- Asymptotic models
- Numerical results
- Application: the Born approximation

2. Multiple electromagnetic scattering by small spheres

- Foldy-Lax model
- Spectral method
- Numerical results

3. Conclusions and perspectives

Foldy-Lax model

- The electromagnetic fields are approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \quad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a dipolar source around c_k

$$\begin{aligned} \mathbf{E}_{\delta,k}(\mathbf{x}) &= \boldsymbol{\varepsilon}_{1,\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \boldsymbol{\varepsilon}_{1,\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k) \\ \mathbf{H}_{\delta,k}(\mathbf{x}) &= \boldsymbol{\mathcal{H}}_{1,\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \boldsymbol{\mathcal{H}}_{1,\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k) \end{aligned}$$

For **perfectly conducting** spheres:

- Approximation of order 3

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left(\mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \right) \quad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left(\mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \right)$$

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$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10} \right) \left(\mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \right) \quad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5} \right) \left(\mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \right)$$

Foldy-Lax model

- The electromagnetic fields are approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \quad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a dipolar source around c_k

$$\begin{aligned}\mathbf{E}_{\delta,k}(\mathbf{x}) &= \boldsymbol{\varepsilon}_{1,\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - \mathbf{c}_k) + \boldsymbol{\varepsilon}_{1,\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - \mathbf{c}_k) \\ \mathbf{H}_{\delta,k}(\mathbf{x}) &= \boldsymbol{\mathcal{H}}_{1,\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - \mathbf{c}_k) + \boldsymbol{\mathcal{H}}_{1,\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - \mathbf{c}_k)\end{aligned}$$

- **Vectorial formulation** (3-order approximation)

Find $\mathbf{d} = ((\mathbf{d}_1^E), \dots, (\mathbf{d}_{N_{\text{obs}}}^E), (\mathbf{d}_1^H), \dots, (\mathbf{d}_{N_{\text{obs}}}^H))^\top \in \mathbb{C}^{6N_{\text{obs}}}$ such that

$$\mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

with $\mathbf{f} = \begin{pmatrix} 4\pi \mathbf{E}^{\text{inc}}(c_1) \\ \vdots \\ 4\pi \mathbf{E}^{\text{inc}}(c_{N_{\text{obs}}}) \\ -2\pi \mathbf{H}^{\text{inc}}(c_1) \\ \vdots \\ -2\pi \mathbf{H}^{\text{inc}}(c_{N_{\text{obs}}}) \end{pmatrix}$ and $\mathbb{A}$ the “interaction” matrix

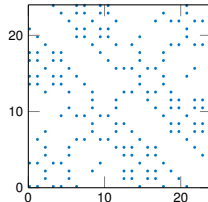


Figure: Skeleton of $\mathbb{A}$ for $N_{\text{obs}} = 4$

Foldy-Lax approximations

- Approximation of order 3

$$\mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

- Collected dipolar approximation

$$\mathbb{D}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\mathbb{D} = \text{diag}(\underbrace{\alpha, \dots, \alpha}_{3N_{\text{obs}}}, \underbrace{\beta, \dots, \beta}_{3N_{\text{obs}}}) \quad \alpha = 1 + \frac{3(\kappa\delta)^2}{10} \quad \beta = 1 - \frac{6(\kappa\delta)^2}{10}$$

- Modified dipolar approximation

$$\tilde{\mathbb{D}}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\tilde{\mathbb{D}} = \text{diag}(\underbrace{\tilde{\alpha}, \dots, \tilde{\alpha}}_{3N_{\text{obs}}}, \underbrace{\tilde{\beta}, \dots, \tilde{\beta}}_{3N_{\text{obs}}}) \quad \tilde{\alpha} = \frac{3i}{2(\kappa\delta)^3} \frac{j_1(\kappa\delta)}{h_1^{(1)}(\kappa\delta)} \quad \tilde{\beta} = -\frac{3i}{(\kappa\delta)^3} \frac{j_1(\kappa\delta) + \kappa\delta j_1'(\kappa\delta)}{h_1^{(1)}(\kappa\delta) + \kappa\delta h_1^{(1)'}(\kappa\delta)}$$

Foldy-Lax approximations

- Approximation of order 3

$$\mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

- Collected dipolar approximation

$$\mathbb{D}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

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Spectral method

- The electromagnetic fields are represented by the **Stratton-Chu formula** for $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) d\mathbf{S}_\mathbf{y}$$

where $\Phi(\mathbf{x}, \mathbf{y}) = \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$

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- Each $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ satisfies the **boundary integral equation**

$$\sum_{j=1}^{N_{\text{obs}}} \mathcal{M}_\Gamma^{kj} \mathbf{p}_j = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{on } \Gamma_\delta^k$$

where $\mathcal{M}_\Gamma^{kj} : \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^j}, \Gamma_\delta^j) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ is the extension of

$$\mathcal{M}_\Gamma^{kj} \lambda(\mathbf{x}_\Gamma) = \mathbf{n}(\mathbf{x}_\Gamma) \times \lim_{\mathbf{x} \rightarrow \mathbf{x}_\Gamma} \mathbf{curl} \int_{\Gamma_\delta^j} \Phi(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) d\mathbf{s}_y \quad \lambda \in \mathcal{C}^\infty(\Gamma_\delta^j) \quad \mathbf{x}_\Gamma \in \Gamma_\delta^k$$

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- Galerkin discretization of the BIE on local spectral basis with N_{mod} modes

$$\mathbf{p}_j(\mathbf{x}) = \sum_{n=1}^{N_{\text{mod}}} \sum_{m=-n}^n p_{n,m}^{j,\perp} \nabla_{\mathbb{S}^2} Y_{n,m}(\hat{\mathbf{x}}_j) + p_{n,m}^{j,\times} \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\hat{\mathbf{x}}_j) \quad \mathbf{x} \in \Gamma_\delta^j$$

with $\hat{\mathbf{x}}_j = \frac{\mathbf{x} - \mathbf{c}_j}{|\mathbf{x} - \mathbf{c}_j|}$ and $\nabla_{\mathbb{S}^2} Y_{n,m}$, $\mathbf{curl}_{\mathbb{S}^2} Y_{n,m}$: complex-valued vector spherical harmonics

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M. Ganesh, S.C. Hawkins (2009)

A high-order algorithm for multiple electromagnetic scattering in three dimensions



H. Barucq, J. Chabassier, H. Pham, S. Tordeux (2017)

Numerical robustness of single-layer method with Fourier basis for multiple obstacle acoustic scattering in homogeneous media

Vectorial formulation

- **Variational** formulation: Find $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ such that

$$\langle \mathcal{M}_\Gamma^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \quad \forall \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- **Vectorial** formulation: Find $\mathbf{p} = ((p_{n,m}^{1,\perp}), \dots, (p_{n,m}^{N_{\text{obs}},\perp}), (p_{n,m}^{1,\times}), \dots, (p_{n,m}^{N_{\text{obs}},\times}))^\top \in \mathbb{C}^{\mathbf{N}}$ s.t.

$$\mathbf{M} \mathbf{p} = \mathbf{f}$$

with $\mathbf{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$ and

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{\perp\perp} & \mathbf{M}_{\perp\times} \\ \mathbf{M}_{\times\perp} & \mathbf{M}_{\times\times} \end{pmatrix} \quad \text{with} \quad \mathbf{M}_{\alpha\beta} = \begin{pmatrix} \mathbf{M}_{\alpha\beta}^{11} & \mathbf{M}_{\alpha\beta}^{12} & \dots & \mathbf{M}_{\alpha\beta}^{1N_{\text{obs}}} \\ \mathbf{M}_{\alpha\beta}^{21} & \mathbf{M}_{\alpha\beta}^{22} & \dots & \mathbf{M}_{\alpha\beta}^{2N_{\text{obs}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{\alpha\beta}^{N_{\text{obs}}1} & \mathbf{M}_{\alpha\beta}^{N_{\text{obs}}2} & \dots & \mathbf{M}_{\alpha\beta}^{N_{\text{obs}}N_{\text{obs}}} \end{pmatrix}$$

$$= \mathbf{M}_{\alpha\beta}(\delta, \mathbf{d}_{jk})$$



Y.-L. Xu (1995)

Electromagnetic scattering by an aggregate of small spheres

Vectorial formulation

- Variational formulation: Find $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ such that

$$\langle \mathcal{M}_{\Gamma}^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \quad \forall \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- Vectorial formulation: Find $\mathbf{p} = ((p_{n,m}^{1,\perp}), \dots, (p_{n,m}^{N_{\text{obs}},\perp}), (p_{n,m}^{1,\times}), \dots, (p_{n,m}^{N_{\text{obs}},\times}))^\top \in \mathbb{C}^N$ s.t.

$$\mathbb{M} \mathbf{p} = \mathbf{f}$$



Computation time
Memory consumption
Ill-conditioned system

- Anti-diagonal coefficients
- Number of unknowns
- Dense matrix



Mex interface (C)
Preconditioning
Smart storage and assembling

- Sub-blocks $\mathbb{M}_{\alpha\beta}$ depend on $\mathbf{d}_{jk}$
- Linear algebra
- Sparse matrix

Linear transformations

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}) \quad \text{with} \quad \mathbf{p} = \mathbf{p}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \mathbb{M} \mathbf{p} = \mathbf{f}$$

- **Step 1:** Change of variable $\mathbb{T}_\delta$

$$\tilde{\mathbf{p}} := \mathbb{T}_\delta \mathbf{p} \quad \implies \quad \mathbf{E}_\delta = \phi(\tilde{\mathbf{p}})$$

- **Step 2:** Permutation matrix $\mathbb{P}$

$$(\text{diag } \mathbb{P} \mathbb{M})_k \neq 0 \quad \forall k = 1, \dots, N$$

- **Step 3:** Diagonal matrix $\mathbb{D}_\delta$

$$\tilde{\mathbb{M}} := \mathbb{D}_\delta^{-1} \mathbb{P} \mathbb{M} \quad \implies \quad (\text{diag } \tilde{\mathbb{M}})_k = 1 \quad \forall k = 1, \dots, N$$

$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}) \quad \text{with} \quad \tilde{\mathbf{p}} = \tilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \tilde{\mathbb{M}} \tilde{\mathbf{p}} = \mathbb{D}_\delta^{-1} \mathbb{P} \mathbf{f}$$

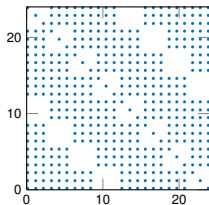
Preconditionning

$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}) \quad \text{with} \quad \tilde{\mathbf{p}} = \tilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \tilde{\mathbf{M}} \tilde{\mathbf{p}} = \mathbf{D}_\delta^{-1} \mathbf{P} \mathbf{f}$$

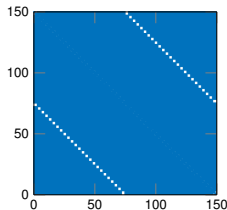


Number of unknowns $\mathbf{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

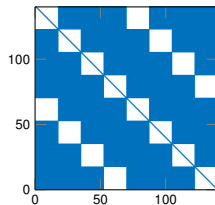
Memory consumption



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$



(b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$



(c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

Figure: Skeleton of $\tilde{\mathbf{M}}$ ▶ Foldy

Preconditionning

$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}) \quad \text{with} \quad \tilde{\mathbf{p}} = \tilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \tilde{\mathbf{M}} \tilde{\mathbf{p}} = \mathbf{D}_\delta^{-1} \mathbf{P} \mathbf{f}$$

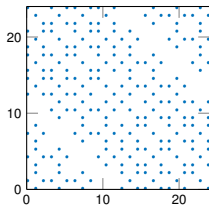


Number of unknowns $\mathbf{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

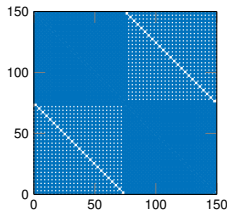
Memory consumption



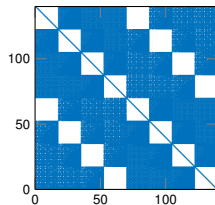
Error of approximation
Analytic preconditionning (dipole)
Algebraic preconditionning (small criterion)



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$



(b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$



(c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

Figure: Skeleton of $\tilde{\mathbf{M}}$

Preconditionning

$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}) \quad \text{with} \quad \tilde{\mathbf{p}} = \tilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \tilde{\mathbf{M}} \tilde{\mathbf{p}} = \mathbf{D}_\delta^{-1} \mathbf{P} \mathbf{f}$$

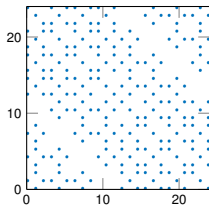


Number of unknowns $\mathbf{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

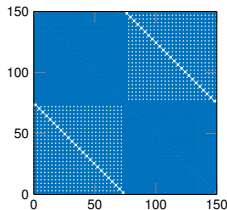
Memory consumption



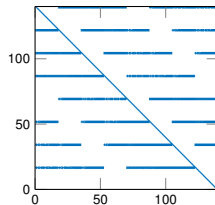
Error of approximation
Analytic preconditionning (dipole)
Algebraic preconditionning (small criterion)



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$



(b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$



(c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

Figure: Skeleton of $\text{Precond}(\tilde{\mathbf{M}})$

Preconditionning

$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}) \quad \text{with} \quad \tilde{\mathbf{p}} = \tilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \tilde{\mathbf{M}} \tilde{\mathbf{p}} = \mathbf{D}_\delta^{-1} \mathbf{P} \mathbf{f}$$



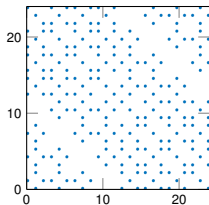
Number of unknowns $\mathbf{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

Memory consumption

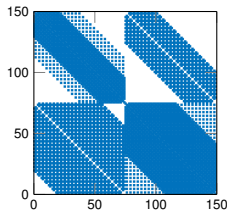


Error of approximation
Analytic preconditionning (dipole)
Algebraic preconditionning (small criterion)

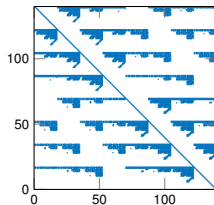
Sparse matrix



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$



(b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$



(c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

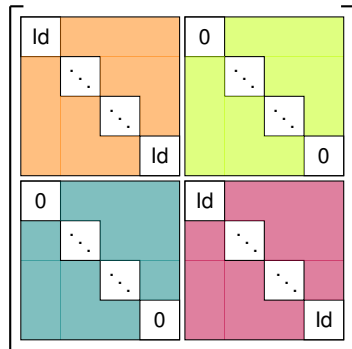
Figure: Skeleton of $\text{Precond}(\tilde{\mathbf{M}})$

Assembling for axisymmetric configurations

- Example: 4 aligned obstacles

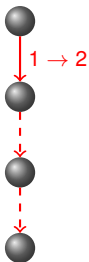


$$\tilde{\mathbf{M}} =$$



The sub-blocks $\mathbf{M}_{\alpha\beta}^{kj}$ depend only on δ and $\mathbf{d}_{jk}$

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} =$$

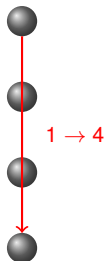
Figure 1 displays four 4x4 grids illustrating the evolution of a 1D Ising spin chain. Each grid shows a sequence of spins (black dots) and their interactions (colored squares). The top-left grid shows the initial state with 'Id' and '1 - 2' labels. The top-right grid shows the first step with '0' and '1' labels. The bottom-left grid shows the second step with '0' and '1' labels. The bottom-right grid shows the third step with 'Id' and '0' labels.

- Storage of sub-blocks

[illegible]

Assembling for axisymmetric configurations

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} =$$

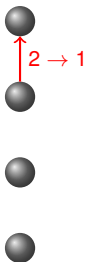
Figure 1 displays four 4x4 grids illustrating the evolution of a 2D Ising spin system. Each grid shows a 4x4 lattice of spins (black dots) and their corresponding energy values (numbers) for each configuration. The grids are labeled 'Id' (Identity) and '0' (Zero). The energy values are: 1, -2, 1, -3, -4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0. The spins are represented by black dots in the top-left, top-right, and bottom-right quadrants of each grid.

- Storage of sub-blocks

[illegible]

Assembling for axisymmetric configurations

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} =$$

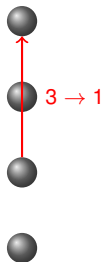
[illegible]

- Storage of sub-blocks

[illegible]

Assembling for axisymmetric configurations

- Example: 4 aligned obstacles



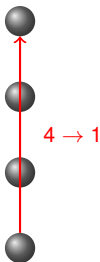
$$\tilde{\mathbf{M}} =$$

Figure 1 displays four 4x4 grids illustrating the evolution of a 2D Ising spin system. The grids are arranged in a 2x2 layout, showing the state of the system at different stages of evolution. The top-left grid shows the initial state with spins 1 (orange), -1 (pink), and 0 (white). The top-right grid shows the state after one iteration, with spins 0 (white), 1 (light green), and -1 (dark green). The bottom-left grid shows the state after two iterations, with spins 0 (white), 1 (light blue), and -1 (dark blue). The bottom-right grid shows the state after three iterations, with spins 0 (white), 1 (light purple), and -1 (dark purple).

- Storage of sub-blocks

[illegible]

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} =$$

Figure 1 illustrates the construction of a 4x4 Latin square. The figure consists of four 4x4 grids arranged in a 2x2 layout, showing the progression of the construction process.

- Top-left grid:** Shows the initial state. The top row contains the labels $1 - 2$, $1 - 3$, and $1 - 4$. The bottom-right cell contains the label Id . The other cells are empty.
- Top-right grid:** Shows the first step. The top row is shifted to the right by one position. The top-right cell now contains 0 . The bottom-right cell still contains Id .
- Bottom-left grid:** Shows the second step. The second row is shifted to the right by one position. The bottom-right cell still contains Id .
- Bottom-right grid:** Shows the final 4x4 Latin square. All rows and columns contain the elements $0, 1, 2, 3$ exactly once. The bottom-right cell contains Id .

- Storage of sub-blocks

[illegible]

Assembling for axisymmetric configurations

- Example: 4 aligned obstacles

 $8(N_{\text{obs}} - 1)$ blocks

instead of $4(N_{\text{obs}})^2$



$$\tilde{\mathbf{M}} =$$

Figure 1 displays four 4x4 grids illustrating the evolution of a 2D Ising spin system. The grids are arranged in a 2x2 layout. The top-left grid shows the initial state with a diagonal of red spins (1) and others white (0). The top-right grid shows the state after one iteration, with red spins spreading. The bottom-left grid shows the state after two iterations, with red spins forming a block. The bottom-right grid shows the state after three iterations, with red spins forming a large block. The labels 'Id' and '0' are placed in the bottom-right and top-left cells respectively, indicating the identity of the spin state.

- Storage of sub-blocks

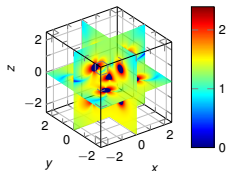
[illegible]

- Iterative solvers: define the **action** of $\tilde{\mathbf{M}}_{\text{Block}}$ on $\mathbf{u}$

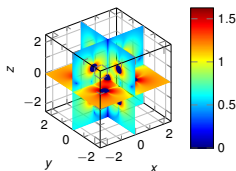
$$\mathcal{A}(\tilde{\mathbb{M}}_{\text{Block}}, \mathbf{u}) := \tilde{\mathbb{M}} \mathbf{u}$$

Numerical validation

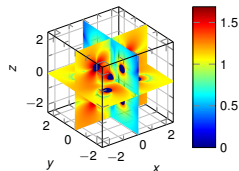
- 13 obstacles shared around the origin — **Arbitrary** configuration



(a) x-coordinates

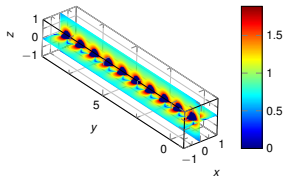


(b) y-coordinates

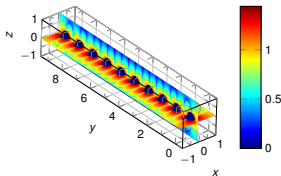


(c) z-coordinates

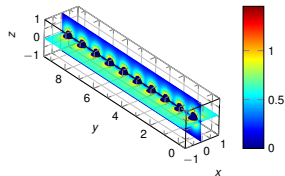
- 10 aligned obstacles — **Axisymmetric** configuration



(d) x-coordinates

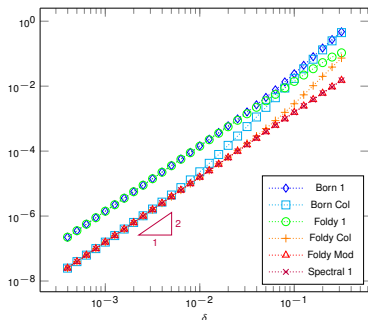


(e) y-coordinates

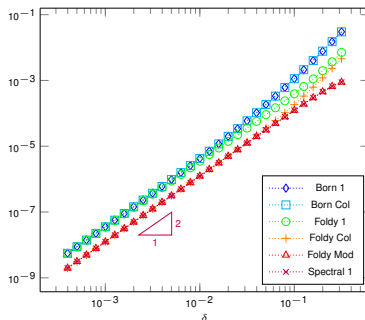


(f) z-coordinates

Numerical convergence of reduced models



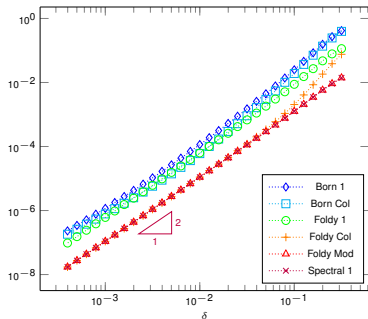
(g) Arbitrary configuration $N_{\text{obs}} = 13$



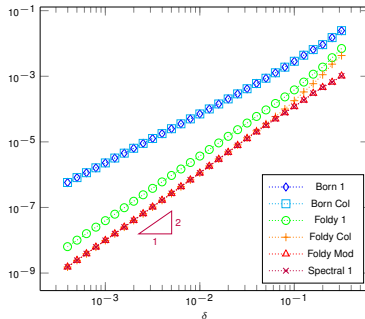
(h) Aligned obstacles $N_{\text{obs}} = 20$

Figure: Relative $L^2(\Omega_\lambda^{2\lambda})$ -errors for electric field with $|\mathbf{d}_{jk}| \approx 1.0$

Numerical convergence of reduced models



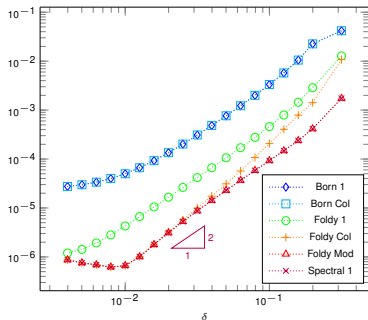
(a) Arbitrary configuration $N_{\text{obs}} = 13$



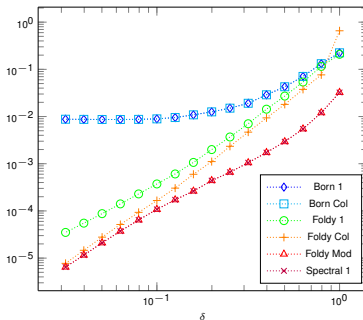
(b) Aligned obstacles $N_{\text{obs}} = 20$

Figure: Relative $L^2(\Omega_\lambda^{2\lambda})$ -errors for electric field with $|\mathbf{d}_{jk}| \approx 2\sqrt{\delta}$

Numerical convergence of reduced models



(a) Arbitrary configuration $N_{\text{obs}} = 13$



(b) Aligned obstacles $N_{\text{obs}} = 20$

Figure: Relative $L^2(\Omega_\lambda^{2\lambda})$ -errors for electric field with $|\mathbf{d}_{jk}| \approx 4\delta$

Time comparison

In terms of accuracy

Modified Foldy $\longleftrightarrow$ Spectral 1

	Modified Foldy			Spectral 1 (arbitrary configuration)		
N_{obs}	100	1000	2000	100	1000	2000
Matrix	0.5588	51.0912	190.6899	10.0850	722.7368	5252
Right hand-side	0.000525	0.000648	0.0232	0.0045	0.0343	0.0722
Inversion	0.0157	7.1581	72.2991	0.0948	10.6212	90.1667
Time linear system	0.5750	58.2499	263.0122	10.1843	733.3923	5342.2
Representation	4.3079	48.4771	105.1629	4.2486	46.5048	89.4991
Total time	4.8829	106.7270	368.1751	14.4329	779.8971	5431.7

Table: Time computation in seconds

Time comparison

In terms of accuracy

Modified Foldy $\longleftrightarrow$ Spectral 1

	Modified Foldy			Spectral 1 (aligned configuration)		
N_{obs}	100	1000	2000	100	1000	2000
Matrix	0.5588	51.0912	190.6899	9.4903	105.2278	208.5052
Right hand-side	0.000525	0.000648	0.0232	0.0042	0.0320	0.0722
Inversion	0.0157	7.1581	72.2991	0.1656	10.8251	88.4188
Time linear system	0.5750	58.2499	263.0122	9.6001	116.0849	298.4607
Representation	4.3079	48.4771	105.1629	4.3087	44.6488	89.8913
Total time	4.8829	106.7270	368.1751	13.9088	160.7337	388.3520

Table: Time computation in seconds

Outline

1. Single electromagnetic scattering by small spheres

- Asymptotic models
- Numerical results
- Application: the Born approximation

2. Multiple electromagnetic scattering by small spheres

- Foldy-Lax model
- Spectral method
- Numerical results

3. Conclusions and perspectives

Conclusions and Perspectives

Conclusion

- ✓ Numerical validation of the matched asymptotic expansions for single scattering
- ✓ Numerical validation of Born and Foldy-Lax models for multiple scattering
- ✓ Derivation and implementation of spectral method at any order

On-going work

- ~ Smart assembling of spectral matrix for axisymmetric configurations
- ~ Numerical validation of spectral method with finite element solutions
- ~ Comparison of preconditionners and iterative solvers

Perspectives

- ✗ Justification of the matched asymptotic expansions
- ✗ High-order Foldy-Lax models for multiple scattering
- ✗ Extension to time-dependent domain
- ✗ Numerical extension to obstacles of arbitrary shape

Thank you for your attention :)