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# On CCZ-Equivalence, Extended-Affine Equivalence and Function Twisting

Anne Canteaut, Léo Perrin

June 18, 2018  
BFA'2018



## Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  and  $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  are *C(arlet)-C(harpin)-Z(inoviev)* equivalent if

$$\Gamma_G = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = L(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}) = L(\Gamma_F),$$

where  $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$  is an affine permutation.

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### Definition (EA-Equivalence; EA-mapping)

$F$  and  $G$  are *E(xtended) A(ffine)* equivalent if  $G(x) = (B \circ F \circ A)(x) + C(x)$ , where  $A, B, C$  are affine and  $A, B$  are permutations; so that

$$\{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = \begin{bmatrix} A^{-1} & 0 \\ CA^{-1} & B \end{bmatrix} (\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}) .$$

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Affine permutations with such linear part are **EA-mappings**; their transposes are **TEA-mappings**

What is the relation between functions that are CCZ- but **not** EA-equivalent?

## Admissible Mapping

For  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ , the affine permutation  $L$  is **admissible for F** if

$$L(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}) = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\}$$

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## Definition (LAT/Walsh Spectrum)

The L(inear) A(pproximation) T(able) of  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  is

$$\mathcal{W}_F(\alpha, \beta) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + \beta \cdot F(x)} .$$



## Structure of this talk

0 - CCZ-Equivalence ; Bijectivity

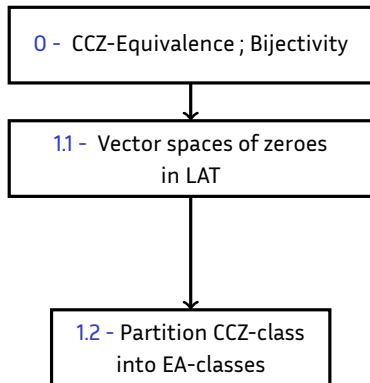
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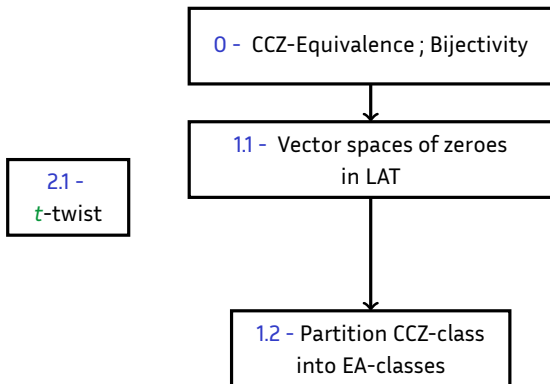


1.1 - Vector spaces of zeroes  
in LAT

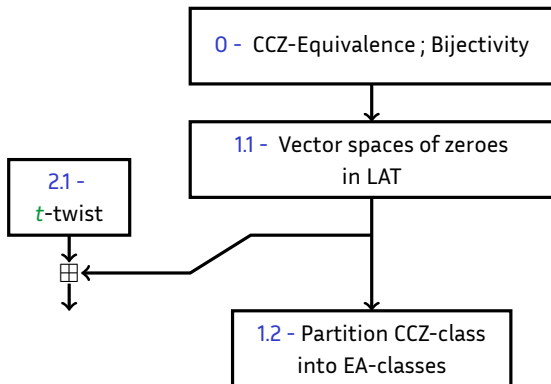
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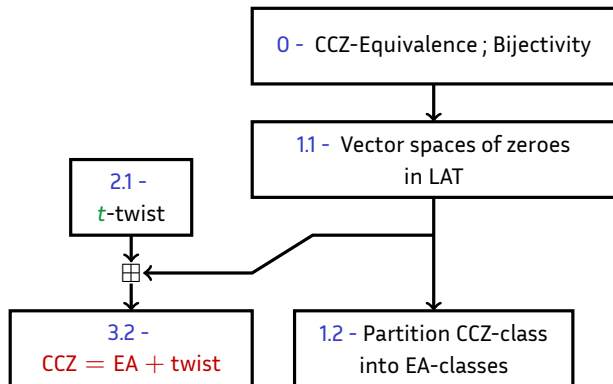
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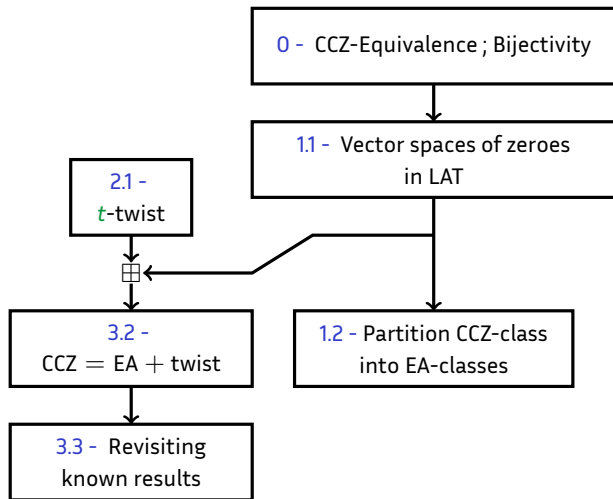
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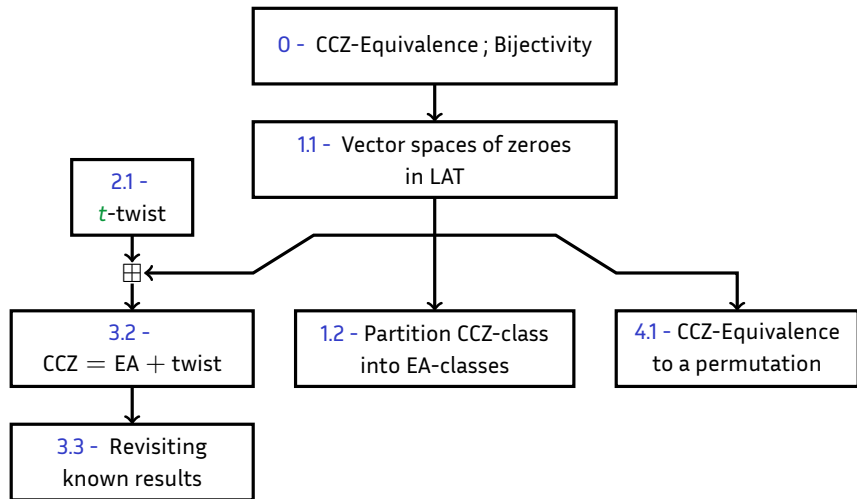
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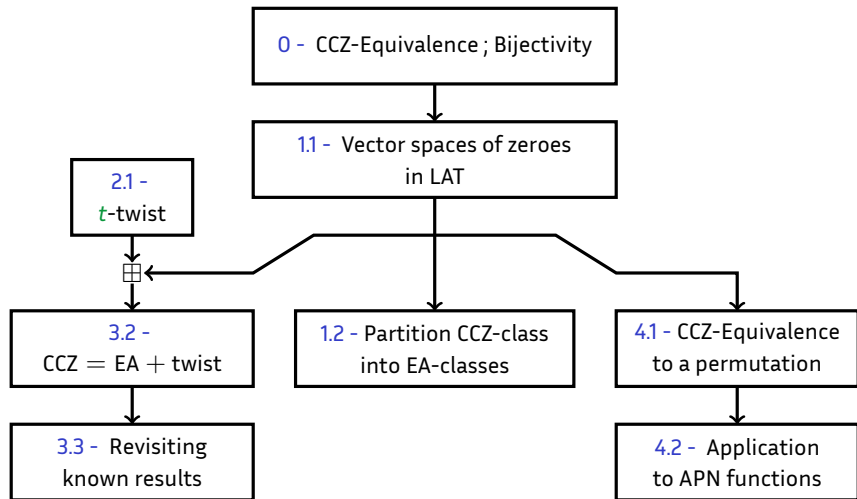


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# Outline

- 1 **CCZ-Equivalence and Vector Spaces of 0**
- 2 Function Twisting
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
- 4 Conclusion

## Plan of this Section

- 1 **CCZ-Equivalence and Vector Spaces of 0**
  - Vector Spaces of Zeroes
  - Partitioning a CCZ-Class into EA-Classes
- 2 Function Twisting
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
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# Walsh Zeros

For all  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ , we have

$$\mathcal{W}_F(\alpha, 0) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + 0 \cdot F(x)} = 0.$$

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### Definition (Walsh Zeroes)

The *Walsh zeroes* of  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  is the set

$$\mathcal{Z}_F = \{u \in \mathbb{F}_2^n \times \mathbb{F}_2^m, \mathcal{W}_F(u) = 0\} \cup \{0\}.$$

With  $\mathcal{V} = \{(x, 0), \forall x \in \mathbb{F}_2^n\} \subset \mathbb{F}_2^{n+m}$ , we have  $\mathcal{V} \subset \mathcal{Z}_F$ .

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Note that if  $\Gamma_G = L(\Gamma_F)$ , then  $\mathcal{Z}_G = (L^T)^{-1}(\mathcal{Z}_F)$ .

# Admissibility for F

## Lemma

Let  $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$  be a linear permutation. It is admissible for  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  if and only if

$$L^T(\mathcal{V}) \subseteq \mathcal{Z}_F$$

## Admissibility of EA-mappings

EA-mappings are admissible for all  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ :

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^T (\mathcal{V}) = \begin{bmatrix} A^T & C^T \\ 0 & B^T \end{bmatrix} \left( \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \forall x \in \mathbb{F}_2^n \right\} \right) = \mathcal{V}.$$



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### Theorem (Budaghyan, Carlet (2011))

*The CCZ-class of a bent function contains only its EA-class.*

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### Proof.

A function is bent

- ⇒ no zeroes outside of  $\mathcal{V}$
- ⇒ no vector spaces of zeroes other than  $\mathcal{V}$
- ⇒ only 1 EA-class



# Permutations

We define

$$\mathcal{V}^\perp = \{(0, y), \forall y \in \mathbb{F}_2^m\} \subset \mathbb{F}_2^{n+m}.$$

## Lemma

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  is a permutation if and only if

$$\mathcal{V}^\perp \subset \mathcal{Z}_F.$$

## EA-classes imply vector spaces

### Lemma

let  $F, G$  and  $G'$  be such that  $\Gamma_G = L(\Gamma_F)$  and  $\Gamma_{G'} = L'(\Gamma_F)$ .  
If  $L(\mathcal{V}) = L'(\mathcal{V})$ , then  $G$  and  $G'$  are EA-equivalent.

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**Can we use this knowledge to partition a CCZ-class into its EA-classes?**



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1 EA-class  $\implies$  1 vector space of zeroes of dimension  $n$  in  $\mathcal{Z}_n$

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1 EA-class  $\implies$  1 vector space of zeroes of dimension  $n$  in  $\mathcal{Z}_n$

Reality takes it back...

The converse of the lemma is wrong.

## Counter-example

Let  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$  be a permutation and let

$$M_n = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

It holds that

$$\begin{aligned} \Gamma_{F^{-1}} &= \{ (x, F(x)), \forall x \in \mathbb{F}_2^n \} \\ &= \{ (F^{-1}(y), (F \circ F^{-1})(y)), \forall y \in \mathbb{F}_2^n \} \\ &= \{ (F^{-1}(y), y), \forall y \in \mathbb{F}_2^n \} \\ &= M_n(\Gamma_F). \end{aligned}$$

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### The contradiction

If  $F$  is an involution then  $\Gamma_F = \Gamma_{F^{-1}} = M_n(\Gamma_F)$

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$$\Rightarrow M_n(\mathcal{V}) = \mathcal{V}^\perp \neq I_n(\mathcal{V})$$

... but  $M_n$  and  $I_n$  send  $\Gamma_F$  in the same EA-class

(namely that of  $F$ ).

## Making the converse work (1/2)

### Definition (CCZ-invariants)

The **CCZ-invariants** of  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$  are the affine permutations  $L$  of  $\mathbb{F}_2^{n+n}$  such that

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### Examples

- For an involution,  $M_n$  is a CCZ-invariant.
- For a quadratic function  $q$ , there are CCZ-invariants with the following linear parts:

$$\begin{bmatrix} I_n & 0 \\ \Delta_{\alpha} q & I_n \end{bmatrix}.$$



## Making the converse work (2/2)

### Theorem (Number of EA-classes)

For  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ , let:

- $s_F$  be the number of vector spaces of dimension  $n$  in  $\mathcal{Z}_F$
- $c_F$  be the number of CCZ-invariants of  $F$
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Then

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### Corollary

If  $c_F = 1$ , then we do have a bijection between EA-classes and vector spaces of 0 of dimension  $n$  in  $\mathcal{Z}_F$ .

# Outline

- 1 CCZ-Equivalence and Vector Spaces of 0
- 2 Function Twisting**
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
- 4 Conclusion

# Plan of this Section

- 1 CCZ-Equivalence and Vector Spaces of 0
- 2 **Function Twisting**
  - The Twist
  - CCZ = EA + Twist
  - Revisiting some Results
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation
- 4 Conclusion

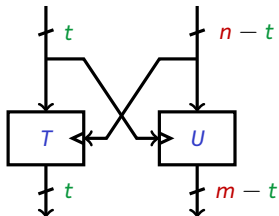
**EA-equivalence is a simple sub-case of CCZ-Equivalence...**

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**What must we add to EA-equivalence to fully describe CCZ-Equivalence?**

## Definition of the Twist

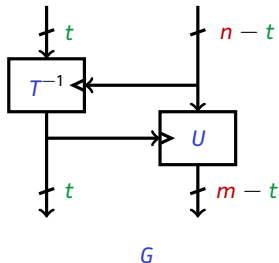
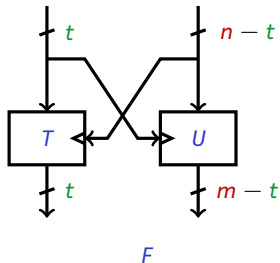
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## Definition of the Twist

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If  $T$  is a permutation for all secondary inputs, then we define the  $t$ -twist equivalent of  $F$  as  $G$ , where

$$G(x, y) = (T_y^{-1}(x), U_{T_y^{-1}(x)}(y))$$

for all  $(x, y) \in \mathbb{F}_2^t \times \mathbb{F}_2^{n-t}$ .

# Examples of Twisting

- Inversion is an  $n$ -twist.

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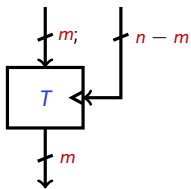
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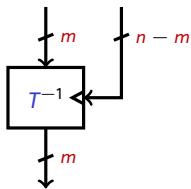
- Inversion is an  $n$ -twist.
- Open and closed butterflies operating on  $n$  bits are obtained from another with an  $(n/2)$ -twist.
- Some degenerate cases exist for  $t = m$  and  $n = n$ .

## Examples of Twisting

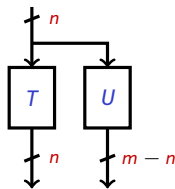
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- Open and closed butterflies operating on  $n$  bits are obtained from another with an  $(n/2)$ -twist.
- Some degenerate cases exist for  $t = m$  and  $n = n$ .



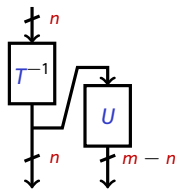
$t = m$  (start)



$t = m$  (end)



$t = n$  (start)



$t = n$  (end)

# Swap Matrices

The **swap matrix** permuting  $\mathbb{F}_2^{n+m}$  is defined for  $t \leq \min(n, m)$  as

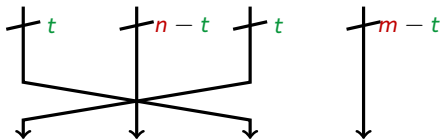
$$M_t = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & I_{n-t} & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-t} \end{bmatrix}.$$

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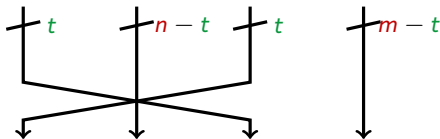


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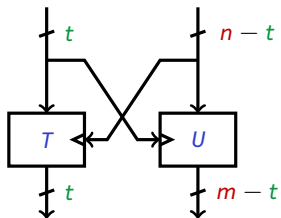


For all  $t \leq \min(n, m)$ ,  $M_t$  is an **orthogonal** and **symmetric involution**.



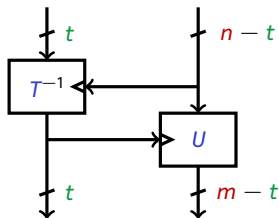
# Swap Matrices and Twisting

$$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$$



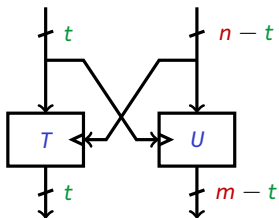
$t$ -twist

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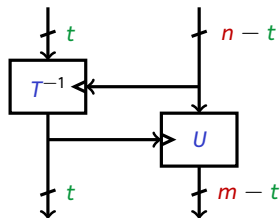
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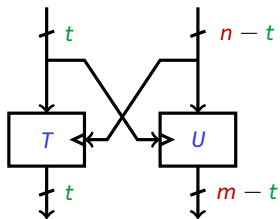


$$\Gamma_G = \{ (x, G(x)), \forall x \in \mathbb{F}_2^n \}$$

$M_t$

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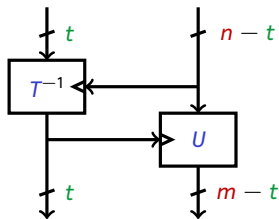
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$M_t$

$$\mathcal{W}_F(u) = \mathcal{W}_G(M_t(u))$$

# Twisting and CCZ-Class

## Lemma

*Twisting preserves the CCZ-equivalence class.*

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## Theorem

If  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  and  $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  are CCZ-equivalent, then

$$\Gamma_G = (B \times M_t \times A)(\Gamma_F),$$

where  $A$  and  $B$  are EA-mappings and where

$$t = \dim(\text{proj}_{\mathcal{V}^\perp}((A^T \times M_t \times B^T)(\mathcal{V}))) .$$

In other words, EA-equivalence and twists are sufficient to fully describe CCZ-equivalence!

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## Corollary

If a function is CCZ-equivalent but not EA-equivalent to another function, then they have to be EA-equivalent to functions for which a  $t$ -twist is possible.

## Proof sketch

1. As  $F$  is CCZ-equivalent to  $G$ , there is a linear permutation  $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$  such that

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1+2+lem. As  $L^T(\mathcal{V}) = (A^T \times M_t)(\mathcal{V})$ , the functions  $G$  and  $G'$  such that  $\Gamma_G = L(\Gamma_F)$  and  $\Gamma_{G'} = (A^T \times M_t)(\Gamma_F)$  are EA-equivalent.

We conclude that

$$\Gamma_G = (B \times M_t \times A)(\Gamma_F).$$

# Usage?

**What can we do with this knowledge?**

# Boolean Functions

## Theorem (Budaghyan, Carlet (2011))

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$\implies F(x||y) = x \oplus f(y), \forall (x, y) \in \mathbb{F}_2 \times \mathbb{F}_2^{n-1}$ ,

$\implies$  1-twisting  $F$  does not change the EA-class

$\implies$  it is impossible to leave the EA-class of  $F$





## Modular Addition (1/2)

### Theorem (Schulte-Geers'13)

Addition modulo  $2^m$  is CCZ-equivalent to

$$q(x, y) = (0, x_0y_0, x_0y_0 + x_1y_1, \dots, x_0y_0 + \dots + x_{n-2}y_{n-2}),$$

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It holds that

$$L^{-1} = \underbrace{\begin{bmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ I_m & 0 & I_m \end{bmatrix}}_{A_1} \times \underbrace{\begin{bmatrix} 0 & 0 & I_m \\ 0 & I_m & 0 \\ I_m & 0 & 0 \end{bmatrix}}_{M_m} \times \underbrace{\begin{bmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ 0 & I_m & I_m \end{bmatrix}}_{A_2}.$$

## Modular Addition (2/2)

### Lemma

Let  $T_z^{\boxplus} : \mathbb{F}_2^m \rightarrow \mathbb{F}_2^m$  be defined by

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Let  $v = T_z^{\boxplus}(x)$ . Then:

$$\begin{cases} v_0 &= x_0 \\ v_{i+1} &= x_i + x_{i+1} + v_i z_i \end{cases} \quad \text{and, convertly,} \quad \begin{cases} x_0 &= v_0 \\ x_{i+1} &= x_i + v_{i+1} + v_i z_i. \end{cases}$$

# Outline

- 1 CCZ-Equivalence and Vector Spaces of 0
- 2 Function Twisting
- 3 Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation**
- 4 Conclusion

## Plan of this Section

- 1 CCZ-Equivalence and Vector Spaces of 0
- 2 Function Twisting
- 3 **Necessary and Efficient Conditions for CCZ-Equivalence to a Permutation**
  - Efficient Criteria
  - Applications to APN Functions
- 4 Conclusion



## Another Problem

**How do we know if a function is CCZ-equivalent to a permutation?**

## Remainder

Recall that  $F$  is a permutation if and only if  $\mathcal{V} \subset \mathcal{Z}_F$  and  $\mathcal{V}^\perp \subset \mathcal{Z}_F$ .

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### Lemma

$G$  is CCZ-equivalent to a permutation if and only if

$$V = L(\mathcal{V}) \subset \mathcal{Z}_G \text{ and } V' = L(\mathcal{V}^\perp) \subset \mathcal{Z}_G$$

for some linear permutation  $L$ . Note that

$$\text{span}(V \cup V') = \mathbb{F}_2^n \times \mathbb{F}_2^m .$$

## 3-Spaces Criteria

### 3-space criteria

Let  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ , not be a permutation. If it is CCZ-equivalent to a permutation then  $\mathcal{Z}_F$  must contain at least 3 vector spaces of zeroes of dimension  $n$ .

## Projected Spaces Criteria

### Key observation

The projections

$$p : (x, y) \mapsto x \text{ and } p' : (x, y) \mapsto y$$

mapping  $\mathbb{F}_2^n \times \mathbb{F}_2^m$  to  $\mathbb{F}_2^n$  and  $\mathbb{F}_2^m$  respectively are **linear**.

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Thus, if  $G$  is CCZ-equivalent to a permutation then  $p(V)$  and  $p(V')$  are subspaces of  $\mathbb{F}_2^n$  whose span is  $\mathbb{F}_2^n$ .

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### Projected Spaces Criteria

If  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$  is CCZ-equivalent to a permutation, then there are at least two subspaces of dimension  $n/2$  in  $p(\mathcal{Z}_F)$  and in  $p'(\mathcal{Z}_F)$ .



## QAM

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None of them are CCZ-equivalent to a permutation

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Göloğlu's introduced APN functions

$$f_k : x \mapsto x^{2^k+1} + (x + x^{2^{n/2}})^{2^k+1}$$

for  $n = 4t$ . They have the *subspace property* of the Kim mapping.

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*Unfortunately,  $f_k$  are not equivalent to permutations on  $n = 4, 8$  and does not seem to be equivalent to one on  $n = 12$  (we say "it does not seem to be equivalent to a permutation" since checking the existence of CCZ-equivalent permutations **requires huge amount of computing and is infeasible on  $n = 12$ ; our program was still running at the time of writing**).*

## Göloğlu's Candidates (2/2)

$n$	cardinal proj.	time proj. (s)	time BasesExtraction (s)
12	1365	0.066	0.0012
16	21845	16.79	0.084
20	349525	10096.00	37.48

Time needed to show that  $f_k$  is **not** CCZ-equivalent to a permutation.

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## Conclusion

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- $CCZ = EA + \text{Twist}$ , both of which have a simple interpretation.
- Efficient criteria to know if a function is CCZ-equivalent to a permutation...
- ... implemented using a very efficient vector space extraction algorithm (not presented)

**The Fourier transform solves everything!**

## Open Problems

### EA-equivalence

How can we efficiently check the EA-equivalence of two functions?

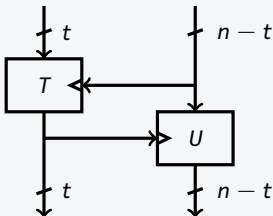
## Open Problems

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### Conjecture

If the CCZ-class of a permutation  $P$  is not reduced to the EA-classes of  $P$  and  $P^{-1}$ , then  $P$  has the following decomposition



where **both**  $T$  and  $U$  are keyed permutations.