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Optimality of linear optical Bell measurements

How much can ancillæ help?

Andrea Olivo^{1,2}, Frédéric Grosshans¹

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arXiv:1806.01243, Phys. Rev. A **98**, 042323



The task: unambiguous Bell measurement

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell Measurement

Projective measurement on the Bell basis

Unambiguous

Outcome never wrong, but can fail with probability $\mathcal{P}_{\text{fail}}$

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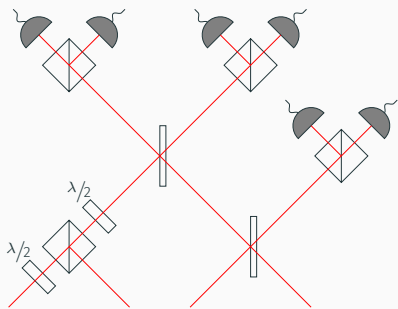
Outcome never wrong, but can fail with probability $\mathcal{P}_{\text{fail}}$

Quantum teleportation, dense coding, entanglement swapping...

The framework: static linear optics

Why? Because...

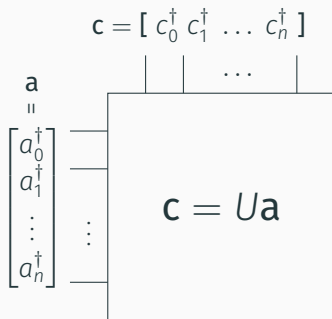
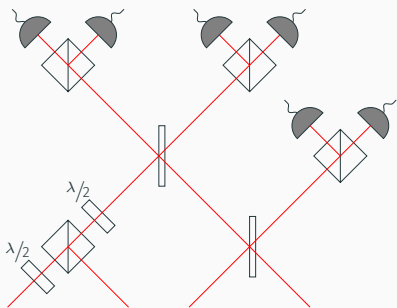
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The framework: static linear optics

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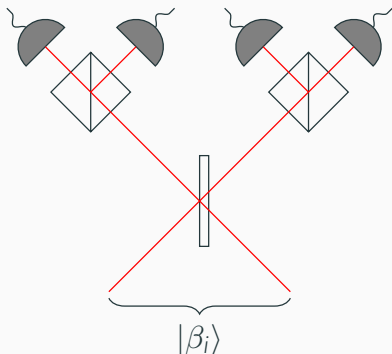
- ...experimentally easier: **no feedforward**
- ...simple mathematical framework to work with



State of the art: Bell measurement without ancilla

[Braunstein and Mann, 1995]

- Known $\mathcal{P}_{\text{succ}} = \frac{1}{2}$ scheme [Braunstein & Mann, 1995]
- **Optimal**, even with feedforward [Calsamiglia & Lütkenhaus, 2001]



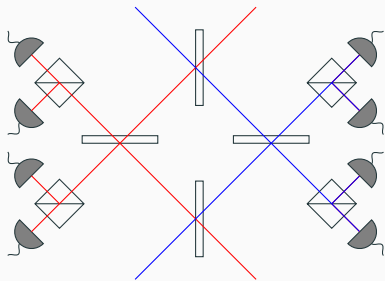
$$|\psi^+\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$|\psi^-\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle)$$

$$|\phi^\pm\rangle \longrightarrow \frac{i}{2}(|2000\rangle \pm |0200\rangle + |0020\rangle \pm |0002\rangle)$$

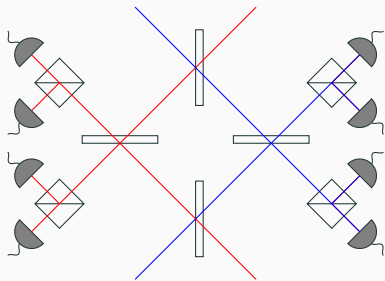
[Grice, 2011]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$ with an extra $|\phi^+\rangle$ as ancilla
- $\mathcal{P}_{\text{fail}} = 2^{-N}$ with GHZ-like states of $2^N - 2$ photons



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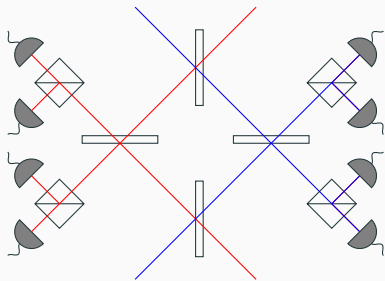


[Ewert and van Loock, 2014]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$ with 4 single photons
- $\mathcal{P}_{\text{succ}} = \frac{25}{32}$ with 12 single photons
- Cons: limited efficiency
- Pros: unentangled ancilla (for the two above)

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What can we say about their **optimality**?

Ideally, we want an **upper bound** on $\mathcal{P}_{\text{succ}}$ depending on the ancilla.

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We provide:

- An **analytical** bound for polarization-preserving interferometers

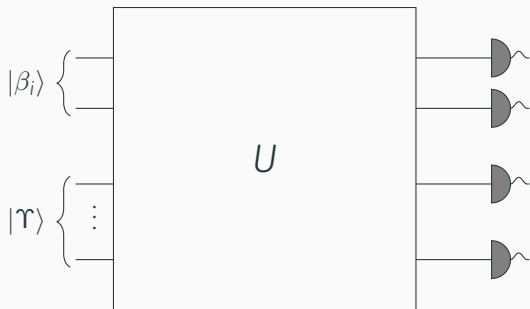
Ideally, we want an **upper bound** on $\mathcal{P}_{\text{succ}}$ depending on the ancilla.

We provide:

- An **analytical** bound for polarization-preserving interferometers
- A thorough **numerical search** for generic (small) interferometers

Analytical upper bound

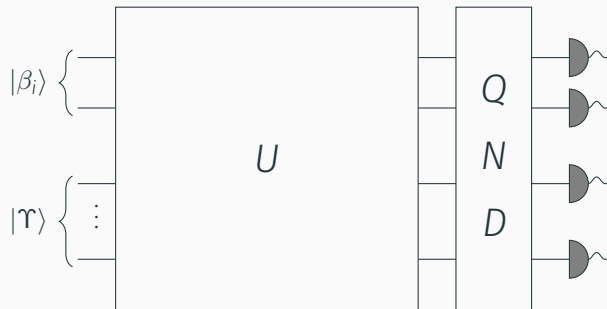
Overview of the proof



Input: $|\Psi_{\text{in}}\rangle = |\beta_i\rangle |\Upsilon\rangle$, with $|\Upsilon\rangle = \sum_{\lambda=0}^k v_{\lambda} |\Upsilon, \lambda\rangle$ a k -photon ancilla

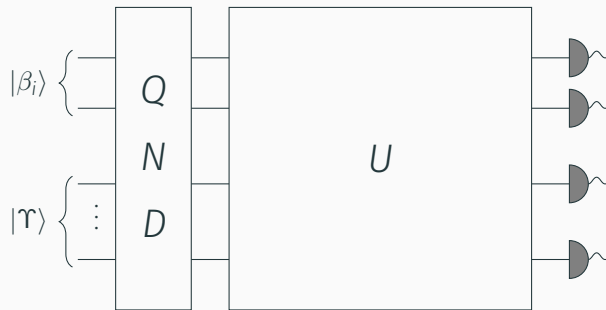
Each $|\Upsilon, \lambda\rangle$ is a state with λ horizontally polarized photons

Overview of the proof



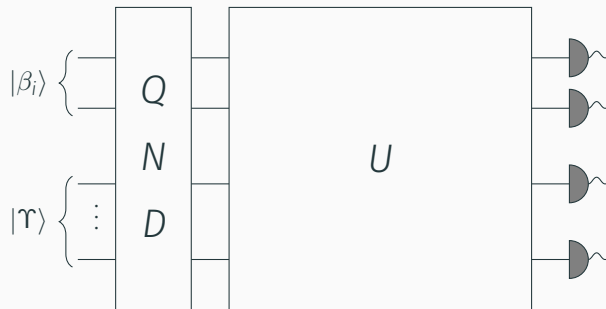
Output statistics unchanged under a projective measurement of the horizontally-polarized photon number of $|\psi_{\text{out}}\rangle$

Overview of the proof



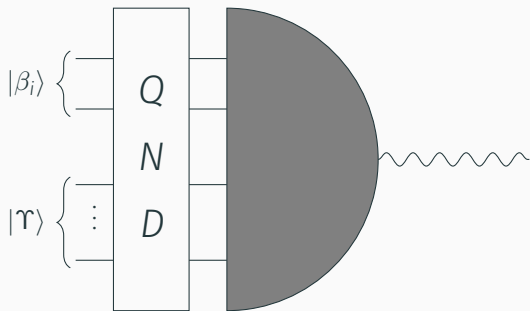
Measurement commutes \iff polarization-preserving network

Overview of the proof



$|\psi^+\rangle$, $|\psi^-\rangle$ and $|\phi^\pm\rangle$ are mapped in **three orthogonal subspaces** $\forall \lambda$

Overview of the proof



Non-orthogonal post-measurement states $|\Lambda^\pm\rangle$ corresponding to $|\phi^+\rangle$ and $|\phi^-\rangle$ can be discriminated with optimal $\mathcal{P}_{\text{succ}} \leq 1 - |\langle \Lambda^+ | \Lambda^- \rangle|$.

Ancilla polarization-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{2} \left(\max_{\lambda \text{ even}} |v_{\lambda}|^2 + \max_{\lambda \text{ odd}} |v_{\lambda}|^2 \right) \quad (1)$$

(looser) Photon number-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{\lceil k+1 \rceil_{\text{even}}} \quad (2)$$

N.B. The Grice's schemes **saturate** both bounds!

Linear network optimizer

Polynomial representation

Second quantization, n -modes interferometer.

Input/output state \longrightarrow polynomial in the input/output mode operators:

$$|\psi_{\text{in}}\rangle = P_{\text{in}}(a_1^\dagger, \dots, a_n^\dagger) |0\rangle \quad |\psi_{\text{out}}\rangle = P_{\text{out}}(c_1^\dagger, \dots, c_n^\dagger) |0\rangle$$

Action of the interferometer on $|\psi_{\text{in}}\rangle \longrightarrow$ **unitary transformation** U acting on the mode operators:

$$a_i^\dagger = \sum_{j=1}^n u_{ij} c_j^\dagger$$

Measurement of $|\psi_{\text{out}}\rangle$ by an array of PNRD.

Detection event \longrightarrow a configuration of clicks at the output.

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Above steps are automatic symbolic computation. Then:

3. Functions $p_i^e(U)$ are hardcoded in C
4. A f.o.m. $f(U)$ is numerically optimized over $U(n)$

An example: no ancilla on $|\phi^+\rangle$

$$n = 4, k = 0 \quad P_{\text{in}} = \frac{1}{\sqrt{2}}(a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger)$$

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$$P_{\text{out}} = \frac{1}{\sqrt{2}} \left(\sum_{j_1} u_{1j_1} c_{j_1}^\dagger \right) \left(\sum_{j_2} u_{3j_2} c_{j_2}^\dagger \right) + \frac{1}{\sqrt{2}} \left(\sum_{j_3} u_{2j_3} c_{j_3}^\dagger \right) \left(\sum_{j_4} u_{4j_4} c_{j_4}^\dagger \right)$$

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↓

| | |
|------|------------------------------------------------------------------------|
| 2000 | $u_{11}u_{31} + u_{21}u_{41}$ |
| 0200 | $u_{12}u_{32} + u_{22}u_{42}$ |
| 0020 | $u_{13}u_{33} + u_{23}u_{43}$ |
| 0002 | $u_{14}u_{34} + u_{24}u_{44}$ |
| 1100 | $(u_{11}u_{32} + u_{12}u_{31} + u_{21}u_{42} + u_{22}u_{41})/\sqrt{2}$ |
| 1010 | $(u_{11}u_{33} + u_{13}u_{31} + u_{21}u_{43} + u_{23}u_{41})/\sqrt{2}$ |
| 1001 | $(u_{11}u_{34} + u_{14}u_{31} + u_{21}u_{44} + u_{24}u_{41})/\sqrt{2}$ |
| 0110 | $(u_{12}u_{33} + u_{13}u_{32} + u_{22}u_{43} + u_{23}u_{42})/\sqrt{2}$ |
| 0101 | $(u_{12}u_{34} + u_{14}u_{32} + u_{22}u_{44} + u_{24}u_{42})/\sqrt{2}$ |
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Optimization: a case for symbolic computation

Why not using pen and paper for $p_i^e(U)$?

For (small!) $\frac{3}{4}$ Grice's scheme:

| | |
|----------------------------------|--------|
| Total number of functions | 1320 |
| Independent | just 5 |

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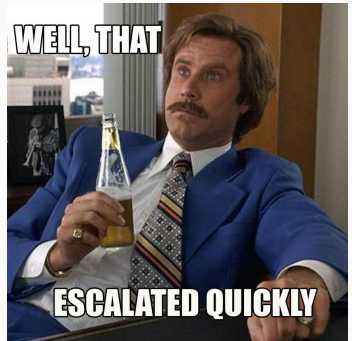
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|---------------------------|--------|
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| Independent | just 5 |

But 2nd iteration, $\frac{7}{8}$:

| | |
|---------------------------|------------------|
| Total number of functions | 490314 |
| Independent | 22 |
| Total number of terms | $1,8 \cdot 10^6$ |



Results

Simplest type of ancilla, no known scheme

- Polarization-preserving bound predicts $\mathcal{P}_{\text{succ}} \leq \frac{1}{2}$
- Numerical search confirms the result

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We find **no advantage** using just one extra photon.

More single, unentangled photons

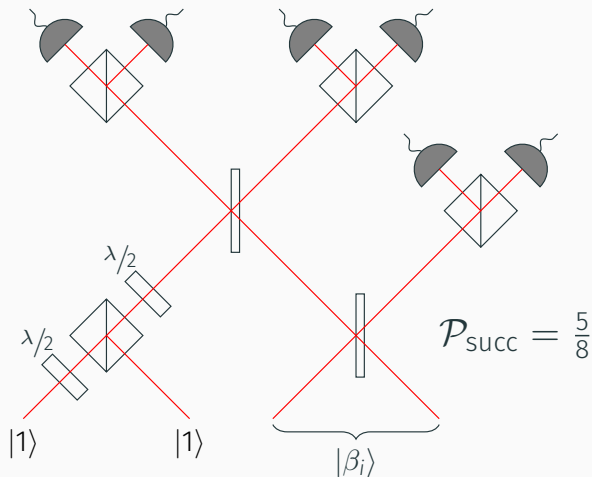
| k photons | Pol. Pres. bound | Num. search | Explicit scheme |
|-------------|------------------|-------------|-----------------|
| 2 | $5/8$ | $5/8$ | $5/8$ |
| 4 | $3/4$ | $3/4$ | $3/4$ |
| 6 | $13/16$ | $3/4$ | — |
| 8 | $25/32$ | — | $49/64$ |
| 12 | $13/16$ | — | $25/32$ |

More single, unentangled photons

| k photons | Pol. Pres. bound | Num. search | Explicit scheme |
|-------------|------------------|-------------|-----------------|
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| 4 | 3/4 | 3/4 | 3/4 |
| 6 | 13/16 | 3/4 | — |
| 8 | 25/32 | — | 49/64 |
| 12 | 13/16 | — | 25/32 |

just 2 single photons beat 50% limit!

The “half”-Ewert & van Loock scheme



Also independently found by Ewert & van Loock

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 - Pol. pres. bound predicts $\mathcal{P}_{\text{fail}} \gtrsim \frac{1}{\sqrt{\pi k}} > \frac{1}{k}$
- Grice's $\frac{7}{8}$ scheme needs $n = 16, k = 6$: barely out of reach

Conclusion

What can we say on the optimal $\mathcal{P}_{\text{succ}}$ of unambiguous Bell measurement?

- **Upper bound** for polarization-preserving interferometers, saturated by known schemes
- Hybrid **numerical/symbolical search**
 - Confirms optimality of (some) known schemes
 - New **2-photon scheme** with $\mathcal{P}_{\text{succ}} = \frac{5}{8}$
- Explored several ancillæ $\rightarrow \frac{3}{4}$ stays the best (for small networks)

Open questions

We have [automated second quantization simulator](#), up to 10 modes and 8 photons.

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We have **automated second quantization simulator**, up to 10 modes and 8 photons.

- Include noise. But how to generalize “unambiguous”?
- Adapt for **state generation** problem

Thank you