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# Optimality of linear optical Bell measurements. How much can ancillae help?

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# Optimality of linear optical Bell measurements

How much can ancillæ help?

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Andrea Olivo<sup>1,2</sup>, Frédéric Grosshans<sup>1</sup>

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<sup>2</sup>Laboratoire Aimé Cotton, Université Paris Saclay



# The task: unambiguous Bell measurement

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

## Bell Measurement

Projective measurement on the Bell basis

## Unambiguous

Outcome never wrong, but can fail with probability  $\mathcal{P}_{\text{fail}}$

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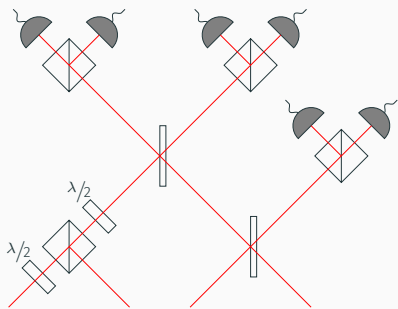
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Quantum teleportation, dense coding, entanglement swapping...

# The framework: static linear optics

Why? Because...

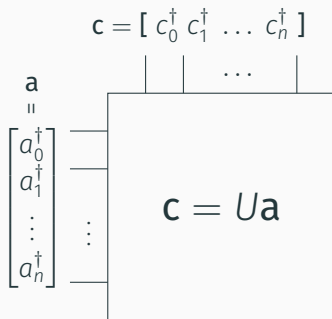
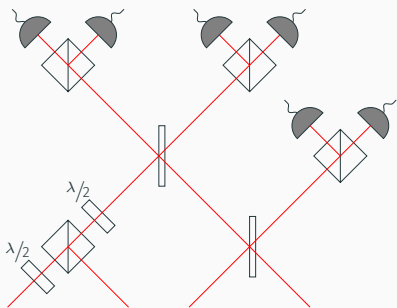
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# The framework: static linear optics

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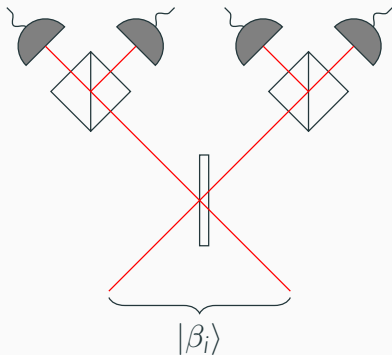
- ...experimentally easier: **no feedforward**
- ...simple mathematical framework to work with



# State of the art: Bell measurement without ancilla

[Braunstein and Mann, 1995]

- Known  $\mathcal{P}_{\text{succ}} = \frac{1}{2}$  scheme [Braunstein and Mann, 1995]
- **Optimal**, even with feedforward [Calsamiglia and Lütkenhaus, 2001]



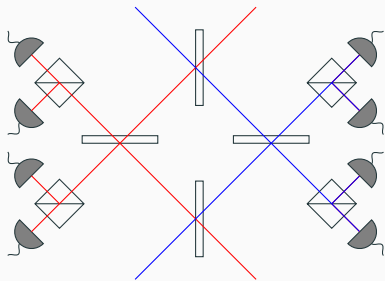
$$|\psi^+\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$|\psi^-\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle)$$

$$|\phi^\pm\rangle \longrightarrow \frac{i}{2}(|2000\rangle \pm |0200\rangle + |0020\rangle \pm |0002\rangle)$$

[Grice, 2011]

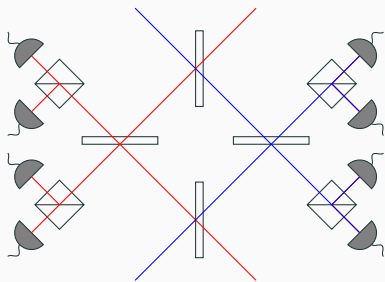
- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$  with an extra  $|\phi^+\rangle$  as ancilla
- $\mathcal{P}_{\text{fail}} = 2^{-N}$  with GHZ-like states of  $2^N - 2$  photons





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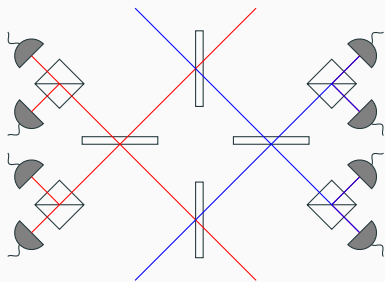


[Ewert and van Loock, 2014]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$  with 4 single photons
- $\mathcal{P}_{\text{succ}} = \frac{25}{32}$  with 12 single photons
- Cons: limited efficiency
- Pros: unentangled ancilla (for the two above)

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What can we say about their **optimality**?

Ideally, we want an **upper bound** on  $\mathcal{P}_{\text{succ}}$  depending on the ancilla.

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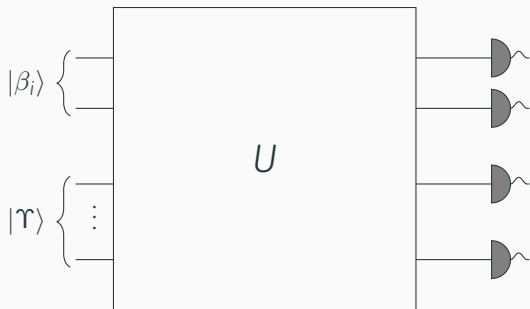
We provide:

- An **analytical** bound for polarization-preserving interferometers
- A thorough **numerical search** for generic (small) interferometers

## Analytical upper bound

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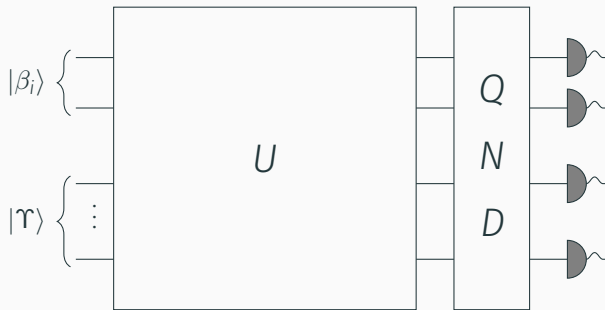
## Overview of the proof



Input:  $|\Psi_{\text{in}}\rangle = |\beta_i\rangle |\Upsilon\rangle$ , with  $|\Upsilon\rangle = \sum_{\lambda=0}^k v_{\lambda} |\Upsilon, \lambda\rangle$  a  $k$ -photon ancilla

Each  $|\Upsilon, \lambda\rangle$  is a state with  $\lambda$  horizontally polarized photons

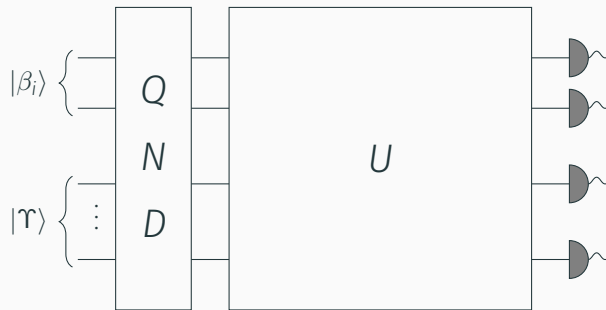
## Overview of the proof



Output statistics unchanged under a projective measurement of the horizontally-polarized photon number of  $|\Psi_{\text{out}}\rangle$

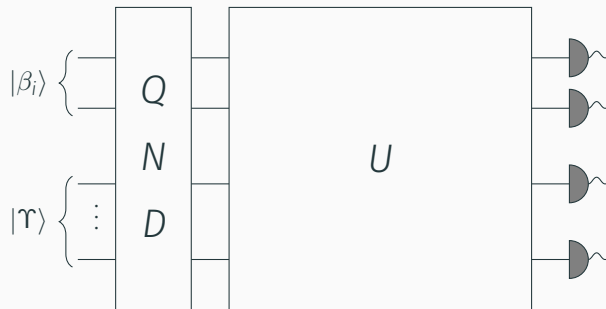


# Overview of the proof



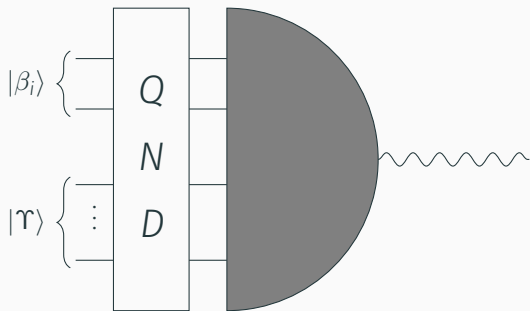
Measurement commutes  $\iff$  polarization-preserving network

# Overview of the proof



$|\psi^+\rangle$ ,  $|\psi^-\rangle$  and  $|\phi^\pm\rangle$  are mapped in **three orthogonal subspaces**  $\forall \lambda$

# Overview of the proof



Non-orthogonal post-measurement states  $|\Lambda^\pm\rangle$  corresponding to  $|\phi^+\rangle$  and  $|\phi^-\rangle$  can be discriminated with optimal  $\mathcal{P}_{\text{succ}} \leq 1 - |\langle \Lambda^+ | \Lambda^- \rangle|$ .

Ancilla polarization-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{2} \left( \max_{\lambda \text{ even}} |v_{\lambda}|^2 + \max_{\lambda \text{ odd}} |v_{\lambda}|^2 \right) \quad (1)$$

(looser) Photon number-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{\lceil k+1 \rceil_{\text{even}}} \quad (2)$$

**N.B.** The Grice's schemes **saturate** both bounds!

# Linear network optimizer

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# Polynomial representation

Second quantization,  $n$ -modes interferometer.

Input/output state  $\longrightarrow$  polynomial in the input/output mode operators:

$$|\psi_{\text{in}}\rangle = P_{\text{in}}(a_1^\dagger, \dots, a_n^\dagger) |0\rangle \quad |\psi_{\text{out}}\rangle = P_{\text{out}}(c_1^\dagger, \dots, c_n^\dagger) |0\rangle$$

Action of the interferometer on  $|\psi_{\text{in}}\rangle \longrightarrow$  **unitary transformation**  $U$  acting on the mode operators:

$$a_i^\dagger = \sum_{j=1}^n u_{ij} c_j^\dagger$$

Measurement of  $|\psi_{\text{out}}\rangle$  by an array of PNRD.

**Detection event**  $\longrightarrow$  a configuration of clicks at the output.

Task: maximize  $\mathcal{P}_{\text{succ}}$  over all possible  $n$ -modes interferometers, using  $k$ -photon ancilla  $|\Upsilon\rangle$

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1. For each  $|\psi_{\text{in}}\rangle = |\beta_i\rangle |\Upsilon\rangle \longrightarrow 2^{\text{nd}}$  quantization output polynomial



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Above steps are **automatic symbolic** computation. Then:

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4. A f.o.m.  $f(U)$  is **numerically** optimized over  $U(n)$

## An example: no ancilla on $|\phi^+\rangle$

$$n = 4, k = 0 \quad P_{\text{in}} = \frac{1}{\sqrt{2}}(a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger)$$

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2000	$u_{11}u_{31} + u_{21}u_{41}$
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# Optimization: a case for symbolic computation

Why not using pen and paper for  $p_i^e(U)$ ?

For (small!)  $\frac{3}{4}$  Grice's scheme:

<b>Total number of functions</b>	1320
<b>Independent</b>	just 5

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<b>Total number of functions</b>	1320
<b>Independent</b>	just 5

But 2<sup>nd</sup> iteration,  $\frac{7}{8}$ :

<b>Total number of functions</b>	490314
<b>Independent</b>	22
<b>Total number of terms</b>	$1,8 \cdot 10^6$

# Results

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Simplest type of ancilla, no known scheme

- Polarization-preserving bound predicts  $\mathcal{P}_{\text{succ}} \leq \frac{1}{2}$
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We find **no advantage** using just one extra photon.

## More single, unentangled photons

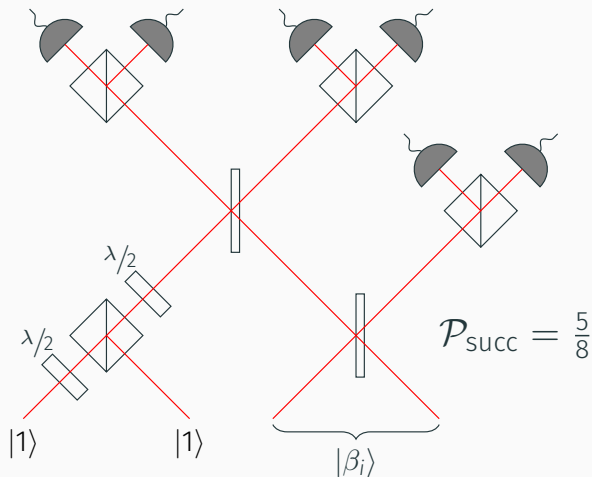
$k$ photons	Pol. Pres. bound	Num. search	Explicit scheme
2	$5/8$	$5/8$	$5/8$
4	$3/4$	$3/4$	$3/4$
6	$13/16$	$3/4$	—
8	$25/32$	—	$49/64$
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just 2 single photons beat 50% limit!

# The “half”-Ewert & van Loock scheme



Also independently found by Ewert & van Loock



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  - Pol. pres. bound predicts  $\mathcal{P}_{\text{fail}} \gtrsim \frac{1}{\sqrt{\pi k}} \geq \frac{1}{k}$
- Grice's  $\frac{7}{8}$  scheme needs  $n = 16, k = 6$ : barely out of reach

## Conclusion

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What can we say on the optimal  $\mathcal{P}_{\text{succ}}$  of unambiguous Bell measurement?

- **Upper bound** for polarization-preserving interferometers, saturated by known schemes
- Hybrid **numerical/symbolical search**
  - Confirms optimality of (some) known schemes
  - New **2-photon scheme** with  $\mathcal{P}_{\text{succ}} = \frac{5}{8}$
- Explored several ancillæ  $\rightarrow \frac{3}{4}$  stays the best (for small networks)

Thank you