



Optimality of linear optical Bell measurements. How much can ancillae help?

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Optimality of linear optical Bell measurements

How much can ancillæ help?

Andrea Olivo^{1,2}, Frédéric Grosshans¹

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The task: unambiguous Bell measurement

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell Measurement

Projective measurement on the
Bell basis

Unambiguous

Outcome never wrong, but can fail
with probability $\mathcal{P}_{\text{fail}}$

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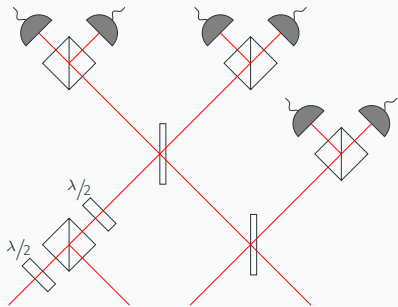
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Quantum teleportation, dense coding, entanglement swapping...

The framework: static linear optics

Why? Because...

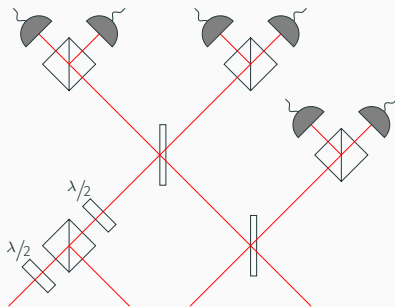
- ...experimentally easier: no feedforward



The framework: static linear optics

Why? Because...

- ...experimentally easier: **no feedforward**
- ...simple mathematical framework to work with

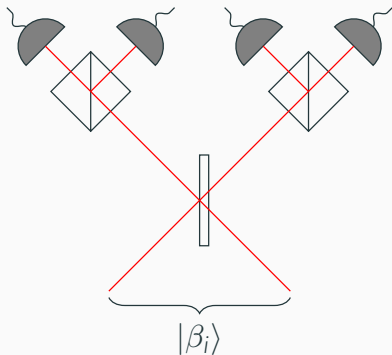


$$\mathbf{c} = [c_0^\dagger \ c_1^\dagger \ \dots \ c_n^\dagger]$$
$$\mathbf{a} = \begin{bmatrix} a_0^\dagger \\ a_1^\dagger \\ \vdots \\ a_n^\dagger \end{bmatrix}$$
$$\mathbf{c} = U\mathbf{a}$$

State of the art: Bell measurement without ancilla

[Braunstein and Mann, 1995]

- Known $\mathcal{P}_{\text{succ}} = \frac{1}{2}$ scheme [Braunstein and Mann, 1995]
- **Optimal**, even with feedforward [Calsamiglia and Lütkenhaus, 2001]



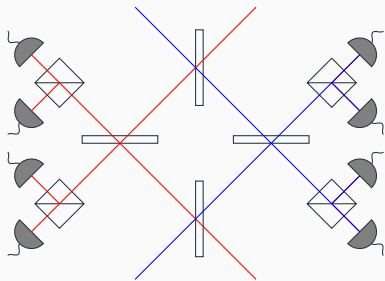
$$|\psi^+\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$|\psi^-\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle)$$

$$|\phi^\pm\rangle \longrightarrow \frac{i}{2}(|2000\rangle \pm |0200\rangle + |0020\rangle \pm |0002\rangle)$$

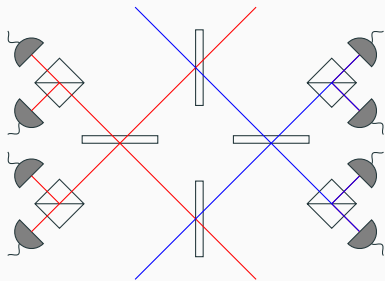
[Grice, 2011]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$ with an extra $|\phi^+\rangle$ as ancilla
- $\mathcal{P}_{\text{fail}} = 2^{-N}$ with GHZ-like states of $2^N - 2$ photons



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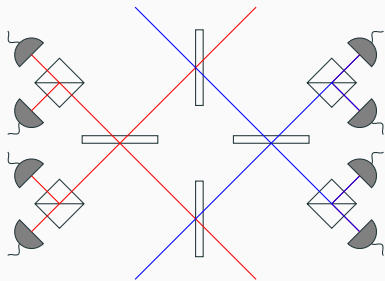


[Ewert and van Loock, 2014]

- $\mathcal{P}_{\text{succ}} = \frac{3}{4}$ with 4 single photons
- $\mathcal{P}_{\text{succ}} = \frac{25}{32}$ with 12 single photons
- Cons: limited efficiency
- Pros: **unentangled** ancilla (for the two above)

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What can we say about their **optimality**?

Ideally, we want an **upper bound** on $\mathcal{P}_{\text{succ}}$ depending on the ancilla.

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We provide:

- An **analytical** bound for polarization-preserving interferometers

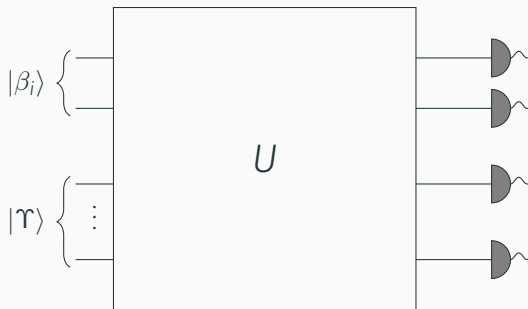
Ideally, we want an **upper bound** on $\mathcal{P}_{\text{succ}}$ depending on the ancilla.

We provide:

- An **analytical** bound for polarization-preserving interferometers
- A thorough **numerical search** for generic (small) interferometers

Analytical upper bound

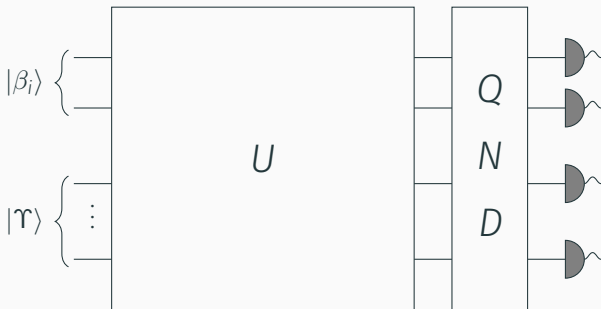
Overview of the proof



Input: $|\Psi_{\text{in}}\rangle = |\beta_i\rangle |\Upsilon\rangle$, with $|\Upsilon\rangle = \sum_{\lambda=0}^k v_{\lambda} |\Upsilon, \lambda\rangle$ a k -photon ancilla

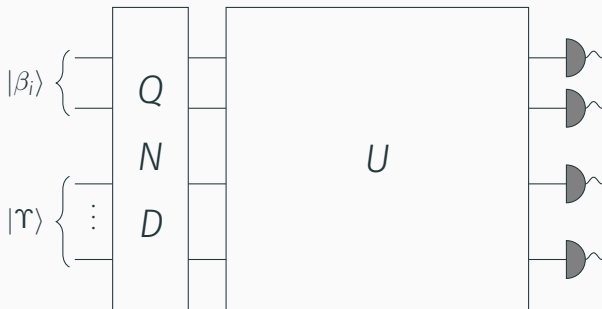
Each $|\Upsilon, \lambda\rangle$ is a state with λ horizontally polarized photons

Overview of the proof



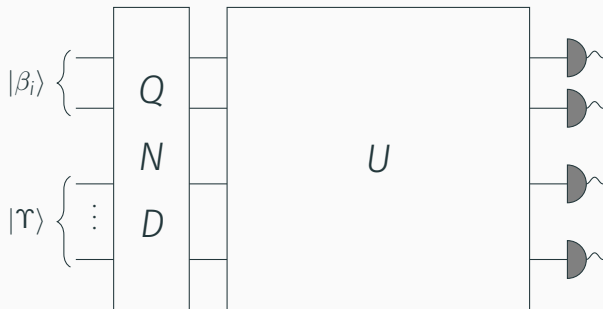
Output statistics unchanged under a projective measurement of the
horizontally-polarized photon number of $|\psi_{\text{out}}\rangle$

Overview of the proof



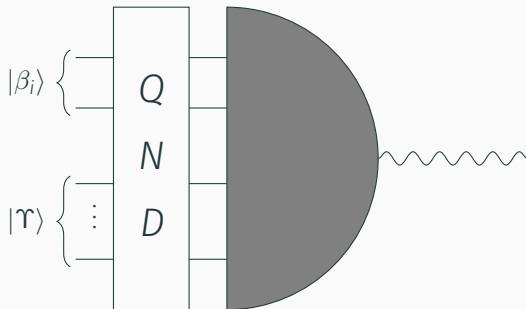
Measurement commutes $\iff$ polarization-preserving network

Overview of the proof



$|\psi^+\rangle$, $|\psi^-\rangle$ and $|\phi^\pm\rangle$ are mapped in three orthogonal subspaces $\forall \lambda$

Overview of the proof



Non-orthogonal post-measurement states $|\Lambda^\pm\rangle$ corresponding to $|\phi^+\rangle$ and $|\phi^-\rangle$ can be discriminated with optimal $\mathcal{P}_{\text{succ}} \leq 1 - |\langle \Lambda^+ | \Lambda^- \rangle|$.

Ancilla polarization-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{2} \left(\max_{\lambda \text{ even}} |v_{\lambda}|^2 + \max_{\lambda \text{ odd}} |v_{\lambda}|^2 \right) \quad (1)$$

(looser) Photon number-based upper bound:

$$\mathcal{P}_{\text{fail}} \geq \frac{1}{\lceil k+1 \rceil_{\text{even}}} \quad (2)$$

N.B. The Grice's schemes **saturate** both bounds!

Linear network optimizer

Polynomial representation

Second quantization, n -modes interferometer.

Input/output state $\longrightarrow$ polynomial in the input/output mode operators:

$$|\psi_{\text{in}}\rangle = P_{\text{in}}(a_1^\dagger, \dots, a_n^\dagger) |0\rangle \quad |\psi_{\text{out}}\rangle = P_{\text{out}}(c_1^\dagger, \dots, c_n^\dagger) |0\rangle$$

Action of the interferometer on $|\psi_{\text{in}}\rangle \longrightarrow$ unitary transformation U acting on the mode operators:

$$a_i^\dagger = \sum_{j=1}^n u_{ij} c_j^\dagger$$

Measurement of $|\psi_{\text{out}}\rangle$ by an array of PNRD.

Detection event $\longrightarrow$ a configuration of clicks at the output.

The method

Task: maximize $\mathcal{P}_{\text{succ}}$ over all possible n -modes interferometers, using k -photon ancilla $|\Upsilon\rangle$

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Above steps are automatic symbolic computation. Then:

3. Functions $p_i^e(U)$ are hardcoded in C
4. A f.o.m. $f(U)$ is numerically optimized over $U(n)$

An example: no ancilla on $|\phi^+\rangle$

$$n = 4, k = 0 \quad P_{\text{in}} = \frac{1}{\sqrt{2}}(a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger)$$

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$\downarrow$

$$P_{\text{out}} = \frac{1}{\sqrt{2}} \left(\sum_{j_1} u_{1j_1} c_{j_1}^\dagger \right) \left(\sum_{j_2} u_{3j_2} c_{j_2}^\dagger \right) + \frac{1}{\sqrt{2}} \left(\sum_{j_3} u_{2j_3} c_{j_3}^\dagger \right) \left(\sum_{j_4} u_{4j_4} c_{j_4}^\dagger \right)$$

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↓

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0002	$u_{14}u_{34} + u_{24}u_{44}$
1100	$(u_{11}u_{32} + u_{12}u_{31} + u_{21}u_{42} + u_{22}u_{41})/\sqrt{2}$
1010	$(u_{11}u_{33} + u_{13}u_{31} + u_{21}u_{43} + u_{23}u_{41})/\sqrt{2}$
1001	$(u_{11}u_{34} + u_{14}u_{31} + u_{21}u_{44} + u_{24}u_{41})/\sqrt{2}$
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Optimization: a case for symbolic computation

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For (small!) $\frac{3}{4}$ Grice's scheme:

Total number of functions	1320
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For (small!) $\frac{3}{4}$ Grice's scheme:

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Independent	just 5

But 2nd iteration, $\frac{7}{8}$:

Total number of functions	490314
Independent	22
Total number of terms	$1,8 \cdot 10^6$

Results

Ancilla: a single photon

Simplest type of ancilla, no known scheme

- Polarization-preserving bound predicts $\mathcal{P}_{\text{succ}} \leq \frac{1}{2}$
- Numerical search confirms the result

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We find no advantage using just one extra photon.

More single, unentangled photons

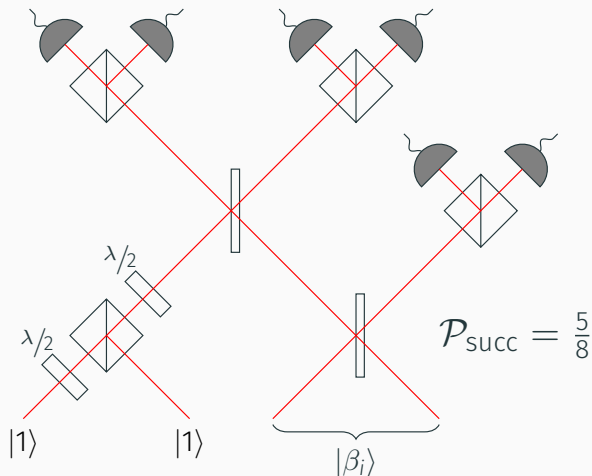
k photons	Pol. Pres. bound	Num. search	Explicit scheme
2	$5/8$	$5/8$	$5/8$
4	$3/4$	$3/4$	$3/4$
6	$13/16$	$3/4$	—
8	$25/32$	—	$49/64$
12	$13/16$	—	$25/32$

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k photons	Pol. Pres. bound	Num. search	Explicit scheme
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4	3/4	3/4	3/4
6	13/16	3/4	—
8	25/32	—	49/64
12	13/16	—	25/32

just 2 single photons beat 50% limit!

The “half”-Ewert & van Loock scheme



Also independently found by Ewert & van Loock

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- Grice's $\frac{7}{8}$ scheme needs $n = 16, k = 6$: barely out of reach

Conclusion

What can we say on the optimal $\mathcal{P}_{\text{succ}}$ of unambiguous Bell measurement?

- Upper bound for polarization-preserving interferometers, saturated by known schemes
- Hybrid numerical/symbolical search
 - Confirms optimality of (some) known schemes
 - New 2-photon scheme with $\mathcal{P}_{\text{succ}} = \frac{5}{8}$
- Explored several ancillæ $\rightarrow \frac{3}{4}$ stays the best (for small networks)

Thank you