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▶ To cite this version:

Jean-Daniel Boissonnat, Siargey Kachanovich, Mathijs Wintraecken. Triangulating submanifolds: An elementary and quantified version of Whitney's method. 2018. hal-01950149v1

HAL Id: hal-01950149 https://inria.hal.science/hal-01950149v1

Preprint submitted on 10 Dec 2018 (v1), last revised 12 May 2023 (v2)

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Triangulating submanifolds: An elementary and quantified version of Whitney's method

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– Abstract –

We quantize Whitney's construction to prove the existence of a triangulation for any C^2 manifold,

so that we get an algorithm with explicit bounds. We also give a new elementary proof, which is 3 completely geometric. 4

2012 ACM Subject Classification Theory of computation \rightarrow Computational geometry

Keywords and phrases Triangulations, Manifolds, Coxeter triangulations

Related Version A full version of this paper is available at https://hal.inria.fr/hal-????? ??

Funding This work has been partially funded by the European Research Council under the European Union's ERC Grant Agreement number 339025 GUDHI (Algorithmic Foundations of Geometric Understanding in Higher Dimensions).

Lines 483

1 Introduction

The question whether every C^1 manifold admits a triangulation was of great importance to topologists in the first half of the 20th century. This question was answered in the affirmative by Cairns [11], see also Whitehead [26]. However the first proofs were complicated and not 8 very geometric, let alone algorithmic. It was Whitney [27, Chapter IV], who eventually gave an insightful geometric constructive proof. Here, we will be reproving Theorem 12A of [27, 10 Section IV.12], in a more quantitative/algorithmic fashion for C^2 manifolds: 11

▶ Theorem 1. Every compact n-dimensional C^2 manifold \mathcal{M} embedded in \mathbb{R}^d with reach 12 $\operatorname{rch}(\mathcal{M})$ admits a triangulation. 13

By more quantitative, we mean that instead of being satisfied with the existence of constants 14 that are used in the construction, we want to provide explicit bounds in terms of the reach 15 of the manifold, which we shall assume to be positive. The reach was introduced by Federer 16

[19], as the minimal distance between a set \mathcal{M} and its medial axis. The medial axis consists 17



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LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

00:2 An elementary and quantified version of Whitney's triangulation method

¹⁸ of points in ambient space that do not have a unique closest point on \mathcal{M} . It is not too ¹⁹ difficult to generalize the precise quantities to the setting where the manifold is $C^{1,1}$ at a ²⁰ small cost, see Appendix B.

Note that Theorem 1 implies that any C^1 manifolds admits a triangulation. This is because any C^1 manifold can be smoothed (see for example [20]) and Whitney's own embedding theorem (see [27, Section IV.1]) gives an embedding in \mathbb{R}^d .

Triangulations in computational geometry/topology are most often based on Voron-24 oi/Delaunay triangulations of the input point set, see for example [3, 4, 9, 12, 15]. Whitney's 25 construction is of a quite different nature. He uses an ambient triangulation and construct 26 the triangulation of the manifold $\mathcal M$ based on the intersections of $\mathcal M$ with this triangu-27 lation. In this paper, we have chosen this ambient triangulation $\tilde{\mathcal{T}}$ to be (a perturbation 28 of) a Coxeter triangulation \mathcal{T} of type \tilde{A}_d . A Coxeter triangulation of type \tilde{A}_d is Delaunay 29 protected, a concept we'll recall in detail in Section 4. Delaunay protection gives that the 30 triangulation is stable under perturbations. This property simplifies the proof, which in fact 31 was one of the motivations for our choice. Moreover, Coxeter triangulations can be stored 32 very compactly, in contrast with previous work [9, 12] on Delaunay triangulations. 33

The approach of the proof of correctness of the method, that we present in this paper, 34 focuses on proving that after perturbing the ambient triangulation the intersection of each d-35 simplex in the triangulation $\tilde{\mathcal{T}}$ with \mathcal{M} is a slightly deformed *n*-dimensional convex polytope. 36 The triangulation K of \mathcal{M} consists of a barycentric subdivision of a straightened version of 37 these polytopes. This may remind the reader of the general result on CW-complexes, see 38 [21], which was exploited by Edelsbrunner and Shah [18] for their triangulation result. This 39 interpretation of Whitney's triangulation method is different from the Whitney's original 40 proof where the homeomorphism is given by the closest point projection and uses techniques 41 which we also exploited in [8]. Here we construct 'normals' and a tubular neighbourhood 42 for K that is compatible with the ambient triangulation $\tilde{\mathcal{T}}$ and prove that the projection 43 along these 'normals' is a homeomorphism. We believe that the tubular neighbourhood we 44 construct is of independent interest. Because we have a bound on the size of the tubular 45 neighbourhood of K and \mathcal{M} lies in this neighbourhood, we automatically bound the Haus-46 dorff distance between the two. A bound on the difference between the normals of K and 47 $\mathcal M$ is also provided. Thanks to our choice of ambient triangulation and our homeomorphism 48 proof, this entire paper is elementary in the sense that no topological results are needed, all 49 arguments are geometrical. 50

In addition to the more quantitative/algorithmic approach, the purely geometrical homeomorphism proof, the link with the closed ball property, the tubular neighbourhood for the triangulation K, and a bound on the Hausdorff distance, we also give different proofs for a fair number of Whitney's intermediate results.

The algorithm described in this paper is being implemented at the moment. The results will be reported in a companion paper.

All proofs of statements that are not recalled from other papers are given in the appendix.

2 The algorithm and overview

⁵⁹ 2.1 The algorithm

⁶⁰ The algorithm takes as input an *n*-dimensional C^2 manifold $\mathcal{M} \subset \mathbb{R}^d$ with reach rch(\mathcal{M}),

and outputs the triangulation K of \mathcal{M} . The algorithm based on Whitney's construction consists of two parts

62 consists of two parts.



Figure 1 The two parts of the algorithm: Part 1, where we perturb the vertices of the ambient triangulation, is depicted on top. Part 2, where the triangulation is constructed from the points of intersection of \mathcal{M} and the edges, is depicted below.

Part 1. This part of the algorithm handles the ambient triangulation and consists of the following steps:

⁶⁸ Choose a Coxeter triangulation \mathcal{T} of type \tilde{A}_d of \mathbb{R}^d that is sufficiently fine (as determined ⁶⁹ by the longest edge length L, see (8)).

- Perturb the vertices of \mathcal{T} slightly (see (14)) into a $\tilde{\mathcal{T}}$ (with the same combinatorial 70 structure), such that all simplices in $\tilde{\mathcal{T}}$ of dimension at most d-n-1 are sufficiently 71 far away from the manifold, see (11). This is done as follows: One maintains a list of 72 vertices and simplices of $\tilde{\mathcal{T}}_i$, starting with an empty list and adding perturbed vertices, 73 keeping the combinatorial structure of \mathcal{T} intact. This means that if $\tau = \{v_{j_1}, \ldots, v_{j_k}\}$ 74 is a simplex in \mathcal{T} and $\tilde{v}_{j_1}, \ldots, \tilde{v}_{j_k} \in \tilde{\mathcal{T}}_i$, where \tilde{v}_i denotes the perturbed vertex v_i , then 75 $\tilde{\tau} = \{\tilde{v}_{j_1}, \ldots, \tilde{v}_{j_k}\}$ is a simplex in $\tilde{\mathcal{T}}_i$. To this list, one first adds all vertices of \mathcal{T} that are 76 very far from \mathcal{M} , as well as the simplices with these vertices (see Case 1 of Section 5.2). 77 For a vertex v_i that is relatively close to \mathcal{M} (Case 2), one goes through the following 78 procedure. We first pick a point $p \in \mathcal{M}$ that is not too far from v_i . We then consider 79 all $\tau'_{j} \subset \tilde{\mathcal{T}}_{i-1}$ of dimension at most d-n-2, such that the join $v_{i} * \tau'_{j}$ lies in $\tilde{\mathcal{T}}_{i}$. For all 80 such τ'_j we consider span $(\tau'_j, T_p\mathcal{M})$ and we pick our perturbed v_i , that is \tilde{v}_i , such that it 81 lies sufficiently far from the union of these spans (see (27)). 82

00:4 An elementary and quantified version of Whitney's triangulation method

Note that we only require limited knowledge of the manifold. Given a vertex v_i we need 83 to be able to find a point on \mathcal{M} that is close to v_i or know if v_i is far from \mathcal{M} and we 84 need access to $T\mathcal{M}$ in a finite sufficiently dense set of points (so that for every point v_i that 85 is close to \mathcal{M} we have a linear approximation of \mathcal{M}). We assume we have two oracles for 86 the two operations. There are no fundamental difficulties in including small uncertainties 87 in our knowledge of the close points or the tangent spaces, but the analysis would be more 88 complicated. If we can sample \mathcal{M} densely finding close points is algorithmically not difficult. 89 Methods to estimate the tangent space have been described in [1]. 90

Complexity of part 1. The complexity of the perturbation (per vertex) of the al-91 gorithm is dominated by the number of simplices τ'_i that we have to consider. This number 92 is bounded by the number of simplices of at most dimension d - n - 2 in the star of a 93 vertex in a Coxeter triangulation plus 1, see (3) below. The number of simplices in turn is 94 bounded by $(d-n)^d d^{(d-n)}$, see claim 15. This compares favourably with the complexity 95 of the perturbation method in [5] for Delaunay triangulations, which is of order $\mathcal{O}(2^{d^2})$. 96 A full analysis of the complexity of the algorithm, including basic operations on Coxeter 97 triangulations, will be reported upon in a separate paper. 98

Part 2. The construction of the triangulation of \mathcal{M} is now straightforward barycentric subdivision, for each $\tau^k \in \tilde{\mathcal{T}}$, of dimension k that contains a part of \mathcal{M} , we pick a point $v(\tau^k)$ in τ^k , see (19). For any sequence $\tau^{d-n} \subset \tau^{d-n+1} \subset \cdots \subset \tau^d$, such that all simplices in the sequence contain a part of \mathcal{M} we add a simplex $\{v(\tau^{d-n}), \ldots, v(\tau^d)\}$ to a simplicial complex K. If we have done this for all simplices that contain \mathcal{M} , K is a triangulation of \mathcal{M} .

For this second part we need an oracle that is able to tell us if the intersection between \mathcal{M} and $\tau^{d-n} \in \tilde{\mathcal{T}}$ is non-empty and if so, gives us the point of intersection. As we'll see in Section 6.1, it would in fact suffice to be able to find intersections between tangent planes and simplices.

¹⁰⁹ 2.2 Outline and overview of the proof

This paper is dedicated to the correctness proof of the algorithm presented above. After some background sections dedicated to manifolds of positive reach and Coxeter triangulations and their stability under perturbations, we continue with the perturbation algorithm.

The ambient triangulation. In Section 5, we both give the details of the perturbation of the vertices.

The triangulation of \mathcal{M} . In Section 6, the triangulation K of \mathcal{M} is defined and an important quality bound for the simplices is given.

The triangulation proof. Section 7 is dedicated to proving that K is a triangulation of 117 \mathcal{M} . The proof is quite different from the approach Whitney described, which uses the closest 118 point projection onto \mathcal{M} . Here we construct a tubular neighbourhood and 'normals' around 119 the triangulation K, which is adapted to the ambient triangulation $\tilde{\mathcal{T}}$. We then prove that 120 the projection using these 'normals' gives a piecewise smooth homeomorphism from $\tau^d \cap \mathcal{M}$ 121 to $\tau^d \cap K$, where $\tau^d \in \tilde{\mathcal{T}}$ is d-dimensional. Because the construction is compatible on the 122 faces of d dimensional simplices the global result immediately follows. A more detailed 123 overview of the homeomorphism proof is given in Section 7. 124

¹²⁵ **3** Manifolds, tangent spaces, distances and angles

In this section, we discuss some general results that will be of use. The manifold $\mathcal{M} \subset \mathbb{R}^d$ is a compact C^2 manifold with reach rch(\mathcal{M}).

We adhere as much as possible to the same notation as used in [10]. The tangent bundle will be denoted by $T\mathcal{M}$, while the tangent space at a point p is written as $T_p\mathcal{M}$. Similarly, $N\mathcal{M}$ is the normal bundle and $N_p\mathcal{M}$ a normal space. Distances on the manifold will be indicated by $d_{\mathcal{M}}(\cdot, \cdot)$, while we write $d(\cdot, \cdot)$ for distances in the ambient Euclidean space, and $|\cdot|$ for the length of vectors. A ball centred at x with radius r is denoted by B(x, r).

The closest point projection of point x in the ambient space, such that $d(x, \mathcal{M}) < \operatorname{rch}(\mathcal{M})$, onto \mathcal{M} is denoted by $\pi_{\mathcal{M}}(x)$. The orthogonal projection onto the tangent $T_p\mathcal{M}$ is denoted by $\pi_{T_p\mathcal{M}}(x)$.

We will use a result from [10], which improves upon previous works such as Niyogi, Smale and Weinberger [23]:

Lemma 2 (Lemma 6 and Corollary 3 of [10]). Suppose that \mathcal{M} is C^2 and let $p, q \in \mathcal{M}$, then

$$_{^{140}} \qquad \angle(T_p\mathcal{M}, T_q\mathcal{M}) \le \frac{d_{\mathcal{M}}(p, q)}{\operatorname{rch}(\mathcal{M})} \qquad and \qquad \sin\left(\frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2}\right) \le \frac{|p-q|}{\operatorname{rch}(\mathcal{M})}$$

We now prove that the projection onto the tangent space is a diffeomorphism in a neighbourhood of size the reach of the manifold. This improves upon previous results by Niyogi, Smale, and Weinberger [23] in terms of the size of the neighbourhood, and is a more quantitative version of results by Whitney [27].

We first recall some notation. Similarly to [10], we let $C(T_p\mathcal{M}, r_1, r_2)$ denote the 'filled cylinder' given by all points that project orthogonally onto a ball of radius r_1 in $T_p\mathcal{M}$ and whose distance to this ball is at most r_2 . We write $\mathring{C}(T_p\mathcal{M}, r_1, r_2)$ for the open cylinder. We now have:

Lemma 3. Suppose that \mathcal{M} is C² and $p \in \mathcal{M}$, then for all $r < \operatorname{rch}(\mathcal{M})$, the projection $\pi_{T_p\mathcal{M}}$ onto the tangent space $T_p\mathcal{M}$, restricted to $\mathcal{M} \cap \mathring{C}(T_p\mathcal{M}, r, \operatorname{rch}(\mathcal{M}))$ is a diffeomorphism onto the open ball $B_{T_p\mathcal{M}}(r)$ of radius r in $T_p\mathcal{M}$.

It is clear by considering the sphere that this result is tight. See Appendix B for some remarks on these results in the $C^{1,1}$ setting.

▶ **Definition 4.** We shall write π_p as an abbreviation for the restriction (of the domain) of $\pi_{T_p\mathcal{M}}$ to $\mathcal{M} \cap \mathring{C}(T_p\mathcal{M}, \operatorname{rch}(\mathcal{M}), \operatorname{rch}(\mathcal{M}))$ and π_p^{-1} for its inverse.

¹⁵⁶ We now also immediately have a quantitative version of Lemma IV.8a of [27]:

▶ Lemma 5. Suppose that \mathcal{M} is C^2 , then for all $r < \operatorname{rch}(\mathcal{M})$

 $d(p, M \setminus C(T_p\mathcal{M}, r, \operatorname{rch}(\mathcal{M}))) = d(p, M \setminus \pi_p^{-1}(B_{T_p\mathcal{M}}(r))) \ge r.$ (1)

¹⁵⁹ We shall also need the following bound on the (local) distance between a tangent space ¹⁶⁰ and the manifold.

Lemma 6 (Distance to Manifold, Lemma 11 of [10]). Let \mathcal{M} be a manifold of positive reach. Suppose that $w \in T_p \mathcal{M}$ and $|w - p| < \operatorname{rch}(\mathcal{M})$. Let $\pi_p^{-1}(w)$ be as in Definition 4. Then

$$|\pi_p^{-1}(w) - w| \le \left(1 - \sqrt{1 - \left(\frac{|w - p|}{\operatorname{rch}(\mathcal{M})}\right)^2}\right) \operatorname{rch}(\mathcal{M}).$$

¹⁶⁴ This is attained for the sphere of radius $rch(\mathcal{M})$.

¹⁶⁵ **4** Coxeter triangulations, Delaunay protection and stability

Coxeter triangulations [14] of Euclidean space play a significant role in our work. They combine many of the advantages of cubes, with the advantages of triangulations. They are also attractive from the geometrical perspective, because they provide simplices with very good quality and some particular Coxeter triangulations are Delaunay protected and thus very stable Delaunay triangulations. We will now very briefly introduce both the concepts of Coxeter triangulations and Delaunay protection, but refer to [13] for more details on Coxeter triangulations and to [5, 6] for Delaunay protection.

Definition 7. A monohedral¹ triangulation is called a *Coxeter triangulation* if all its d-simplices can be obtained by consecutive orthogonal reflections through facets of the d-simplices in the triangulation.

This definition imposes very strong constraints on the geometry of the simplices, implying that there are only a small number of such triangulations in each dimension. Most of these triangulations are part of 4 families for which there is one member for (almost) every dimension d. We will focus on one such family, \tilde{A}_d , which is Delaunay protected.

¹⁸¹ **Protection.**

▶ **Definition 8.** The *protection* of a *d*-simplex σ in a Delaunay triangulation on a point set *P* is the minimal distance of points in $P \setminus \sigma$ to the circumscribed ball of σ :

$$\delta(\sigma) = \inf_{p \in P \setminus \sigma} d(p, B(\sigma))$$
, where $B(\sigma)$ is the circumscribed ball of σ .

The protection δ of a Delaunay triangulation \mathcal{T} is the infimum over the *d*-simplices of the triangulation: $\delta = \inf_{\sigma \in \mathcal{T}} \delta(\sigma)$. A triangulation with a positive protection is called *protected*.

The proof that \tilde{A}_d triangulations are protected can be found in [13, Section 6]. We shall denote the triangulation of this type by \mathcal{T} .

Stability. In the triangulation proof below we need that a perturbation $\tilde{\mathcal{T}}$ of our initial ambient triangulation (\mathcal{T} of type \tilde{A}_d) is still a triangulation of \mathbb{R}^d . Because Whitney did not use a protected Delaunay triangulation he needs a non-trivial topological argument to establish this, see [27, Appendix Section II.16]. The argument for stability of triangulations for \tilde{A} type Coxeter triangulations is much simpler, because it is a Delaunay triangulation and δ -protected, see [13]. Before we can recall this result we need to introduce some notation:

The minimal altitude or height, denoted by min alt, is the minimum over all vertices of the altitude, that is the distance from a vertex to the affine hull of the opposite face. $t(\tau)$ denotes the thickness of a simplex τ , that is the ratio of the minimal altitude to the

maximal edge length. We write $t(\mathcal{T})$ for thickness of any simplex in \mathcal{T} .

We can think of the vertices of \mathcal{T} as an (ϵ, μ) -net. Here μ is the separation (for Coxeter triangulations, the shortest edge length in \mathcal{T}), and ϵ the sampling density (which is the circumradius of the simplices in the Coxeter triangulation). We write μ_0 for the normalized separation, that is $\mu = \mu_0 \epsilon$.

For any complex K, L(K) denotes the longest edge length in K. We use the abbreviations $L = L(\mathcal{T})$ and $\tilde{L} = L(\tilde{\mathcal{T}})$.

²⁰⁵ Theorem 4.14 of [5] immediately gives:

¹⁷³ ¹ A triangulation of \mathbb{R}^d is called *monohedral* if all its *d*-simplices are congruent.

Corollary 9. The triangulation \mathcal{T} is (combinatorially) stable under a $\tilde{c}L$ -perturbation, meaning that $d(v_i - \tilde{v}_i) \leq \tilde{c}L$, as long as

$$\tilde{c}L \le \frac{t(\mathcal{T})\mu_0}{18d}\delta.$$
(2)

209 We claim the following concerning the behaviour of \tilde{c}

Claim 10.

210

$$\tilde{c} \le \frac{t(\sigma)\mu_0}{18d} \frac{\delta}{L} \le \sqrt{2} \frac{\sqrt{d^2 + 2d + 24} - \sqrt{d^2 + 2d}}{9d^{3/2}\sqrt{d + 2}(d + 1)} \sim \frac{\sqrt{32}}{3d^4}$$

Thickness and angles. The quality of simplices and the control over the alignment of the simplices with the manifold is an essential part of the triangulation proof, for which we need two basic results. Similar statements can be found in Section IV.14 and Section IV.15 of [27].

²¹⁵ We remind ourselves of the following:

▶ Lemma 11 (Thickness under distortion [17, Lemma 7]). Suppose that $\sigma = \{v_0, \ldots, v_k\}$ and $\tilde{\sigma} = \{\tilde{v}_0, \ldots, \tilde{v}_k\}$ are two k-simplices in \mathbb{R}^d such that $||v_i - v_j| - |\tilde{v}_i - \tilde{v}_j|| \le c_0 L(\sigma)$ for all $0 \le i < j \le k$. If $c_0 \le \frac{t(\sigma)^2}{4}$, then $t(\tilde{\sigma}) \ge \frac{4}{5\sqrt{k}}(1 - \frac{4c_0}{t(\sigma)^2})t(\sigma)$.

²¹⁹ We can now state a variation of Whitney's alignment result, see [27, Section IV.15].

Lemma 12 (Whitney's angle bound). Suppose σ is a *j*-simplex of \mathbb{R}^d , *j* < *d*, whose vertices all lie within a distance d_{max} from a *k*-dimensional affine space $A_0 \subset \mathbb{R}^d$ with *k* ≥ *j*. Then

222
$$\sin \angle (\operatorname{aff}(\sigma), A_0) \le \frac{(j+1) d_{max}}{\min \operatorname{alt}(\sigma)}.$$

Simplices in a star in a triangulation of type \tilde{A}_d . The precise number of simplices in the star of a vertex plays an important role in the volume estimates in Section 5. We will now give an explicit bound on this number.

In general the (d - k)-faces of a Voronoi cell correspond to the k-faces in the Delaunay dual. The triangulation \mathcal{T} is Delaunay and the dual of a vertex is a permutahedron, see [13]. We now remind ourselves of the following definition, see [2], and lemma:

▶ **Definition 13.** Let S(d, k) be the Stirling number of the second kind, which is the number of ways to partition a set of *d* elements into *k* non-empty subsets, that is

231
$$S(d,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{i} (k-j)^d.$$

L

▶ Corollary 14 (Corollary 3.15 of [22]). The number of (d+1-k)-faces of the permutahedron is k!S(d+1,k).

By duality, the lemma immediately gives us the number N_k of k-faces that contain a given vertex in \mathcal{T} , $N_k = k!S(d+1,k)$. We also write

236
$$N_{\leq k} = 2 + \sum_{j=1}^{n} j! S(d+1,j),$$
 (3)

which is an upper bound on the total number of faces of dimension less or equal to k that contain a given vertex. We have added 2 because we want to have a safety margin if we have to consider the empty set (as will be apparent in (16)), and have a strict inequality. We now claim the following:

▶ Claim 15. We have $N_{\leq k} \leq k^d d^k$.

00:8 An elementary and quantified version of Whitney's triangulation method

²⁴² **5** Perturbing the ambient triangulation

This section is dedicated to the perturbation of the Coxeter triangulation such that the manifold is sufficiently far from the simplices of dimension at most d - n - 1 in $\tilde{\mathcal{T}}$.

- In Section 5.1, we prove that it is possible to perturb the points as described in the second step of part 1 of the algorithm. This involves a significant amount of volume estimates, which are now completely quantized. Indicate how fine the ambient triangulation \mathcal{T} has
- to be compared to $rch(\mathcal{M})$
- In Section 5.2, we define the perturbation and prove that this in fact gives a triangulation
 for which the low dimensional simplices lie sufficiently far from the manifold.
- The proofs of the results in Section 5.2 rely on Appendix A. We shall indicate the corresponding sections in Whitney [27] between brackets in the title of the section, when appropriate.

²⁵³ 5.1 The complex $\tilde{\mathcal{T}}$ (Section IV.18 of [27])

Before we can dive into the algorithmic construction of the perturbed complex $\tilde{\mathcal{T}}$, we need to fix some constants and give some explicit bounds on the constants.

Balls and exclusion volumes. Let $B^d(r)$ be any ball in \mathbb{R}^d of radius r. We now define $\bar{\rho}_1 > 0$ as follows: For any two parallel (d-1)-hyperplanes whose distance apart is less than $2\bar{\rho}_1 r$, the intersection of the slab between the two hyperplanes with the ball $B^d(r)$ is denoted by S. Now, $\bar{\rho}_1$ is the largest number such that the volume (vol) of any S satisfies

vol(
$$\mathcal{S}$$
) $\leq \frac{\operatorname{vol}(B^d(r))}{2N_{\leq d-n-1}},$

with $N_{\leq d-n}$ as in (3). A precise bound on $\bar{\rho}_1$ can be given, see Remark 35 in Appendix C. We will use an easier bound ρ_1 , at the cost of weakening the result:

²⁶³ ► Lemma 16. We have

$$_{4} \qquad \bar{\rho}_{1} \ge \rho_{1} = \begin{cases} \frac{2^{2k-2}(k!)^{2}}{\pi(2k)!N_{\le d-n-1}} & \text{if } d = 2k\\ \frac{(2k)!}{2^{2k+2}k!(k-1)!N_{\le d-n-1}} & \text{if } d = 2k-1. \end{cases}$$

$$\tag{4}$$

265 Note that

26

266
$$\rho_1 \sim \frac{1}{\sqrt{d}N_{\le d-n-1}}.$$

The coarseness of \mathcal{T} . As mentioned, we perturb the vertices of a Coxeter triangulation. The maximal distance between an unperturbed vertex v_i and the associated perturbed vertex \tilde{v}_i is $\tilde{c}L$. We will assume that \tilde{c} satisfies the bounds below in (14).

We are now ready to introduce the demands on the triangulation of ambient space. We start by bounding the scale of the Coxeter triangulation \mathcal{T} by bounding the longest edge length. We do this by giving some constants. We define α_1 and α_k by a recursion relation as follows

$$\alpha_1 = \frac{4}{3}\rho_1 \tilde{c} \qquad \qquad \frac{2}{3}\alpha_{k-1}\tilde{c}\rho_1 = \alpha_k, \tag{5}$$

that is $\alpha_k = \frac{2^{k+1}}{3^k} \rho_1^k \tilde{c}^k$. These definitions play an essential role in the volume estimates for the perturbation of the vertices, that are necessary to guarantee quality. Note that α_k is extremely small. In particular, we shall have that

$$\alpha_k \le \frac{1}{18^k},\tag{6}$$

²⁷⁹ because of the way that \tilde{c} will be chosen, see (15), ρ_1 is also very small. Furthermore we ²⁸⁰ notice that $\alpha_k < \alpha_{k-1}$. To make sure the formulae do not become too big, we introduce the ²⁸¹ notation

$$\zeta = \frac{8}{15\sqrt{d} \begin{pmatrix} d \\ d-n \end{pmatrix} \cdot (1+2\tilde{c})} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right) t(\mathcal{T}).$$

$$\tag{7}$$

Note that ζ depends on both the ambient and intrinsic dimension and the perturbation parameter \tilde{c} . We set the coarseness of the ambient triangulation by demanding that Lsatisfies

$$(1 - \sqrt{1 - \left(\frac{6L(\mathcal{T})}{\operatorname{rch}(\mathcal{M})}\right)^2} \operatorname{rch}(\mathcal{M}) = \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2} \zeta^{2n} L,$$
(8)

287 or equivalently

$${}_{288} \qquad \frac{L}{\operatorname{rch}(\mathcal{M})} = \frac{2\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}}{\left(\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}\right)^2 + 6^2}.$$
(9)

289 Note that

$${}_{290} \qquad \frac{L}{\operatorname{rch}(\mathcal{M})} < \frac{(\alpha_{d-n})^{4+2n}}{54(n+1)^2} \zeta^n < \frac{(\alpha_{d-n})^2}{54}, \qquad \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2} \zeta^{2n} < \frac{(\alpha_{d-n})^2}{3} \le \frac{\alpha_{d-n}}{3}, \tag{10}$$

²⁹¹ which will be often used below to simplify expressions.

Remark. We have to choose the right hand side in (8) very small, because the bounds on
the quality of the simplices that will make up the triangulations are very weak. The details
of these estimates can be found in Lemma 26.

 295 (d - n - 1)-skeleton safe triangulations. We shall denote the simplices by τ and 296 σ . We will use lower indices to distinguish simplices, while upper indices will stress the 297 dimension, for example τ_i^k is a simplex of dimension k.

▶ Definition 17 ((d - n - 1)-skeleton safe triangulations). We say that a perturbated triangulation $\tilde{\mathcal{T}}$ of \mathcal{T} in \mathbb{R}^d is (d-n-1)-skeleton safe with respect to the *n*-dimensional manifold \mathcal{M} if

$$_{301} \qquad d(\tau^k, \mathcal{M}) > \alpha_k L, \tag{11}$$

302 for all faces τ^k in $\tilde{\mathcal{T}}$, with $k \leq d - n - 1$, and

$$\tilde{L} < \frac{13}{12}L \tag{12}$$

$$_{304} t(\tilde{\mathcal{T}}) \ge \frac{4}{5\sqrt{d}} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right) t(\mathcal{T}). (13)$$

SoCG 2019

00:10 An elementary and quantified version of Whitney's triangulation method

³⁰⁵ 5.2 Perturbing the vertices (Section IV.18)

We now discuss the details of the perturbation scheme that we described in the algorithm section. The perturbation scheme follows Whitney and is inductive.

Construction of $\tilde{\mathcal{T}}$. Let v_1, v_2, \ldots be the vertices of \mathcal{T} , we are going to inductively choose new vertices $\tilde{v}_1, \tilde{v}_2, \ldots$ for $\tilde{\mathcal{T}}$, with

$$|v_i - \tilde{v}_i| \le \tilde{c}L = \min\left\{\frac{t(\mathcal{T})\mu_0}{18d}\delta, \frac{1}{24}t(\mathcal{T})^2L\right\},$$
(14)

using the notation of Section 4. Notice that because $t(\mathcal{T}) \leq 1$, by definition of the thickness of a simplex, we have

$$\tilde{c} \le \frac{1}{24}.\tag{15}$$

³¹⁴ With this bound we immediately have that (12) is satisfied. We also claim the following:

▶ Claim 18. $\tilde{\mathcal{T}}$ has the same combinatorial structure as \mathcal{T} . Moreover, (13) is satisfied.

We now give the scheme where the vertices are perturbed inductively. Suppose that the vertices $\tilde{v}_1, \ldots, \tilde{v}_{i-1}$ have been determined, and thus the complex $\tilde{\mathcal{T}}_{i-1}$ with these vertices. A simplex $\{\tilde{v}_{j_1}\ldots\tilde{v}_{j_k}\}$ lies in $\tilde{\mathcal{T}}_{i-1}$ if and only if $\{v_{j_1}\ldots v_{j_k}\}$ lies in \mathcal{T} . We shall now find \tilde{v}_i and thus $\tilde{\mathcal{T}}_i$ so that for any $\tau^k \in \tilde{\mathcal{T}}_i$ of dimension $k \leq d-n-1$, (11) is satisfied. We distinguish two cases:

³²¹ **Case 1.** The distance $d(v_i, \mathcal{M})$ is greater than $\frac{3}{2}L$. In this case we choose $\tilde{v}_i = v_i$. The ³²² inequality (11) now follows, because $\tilde{L} < (1+2\tilde{c})L$ and thus by the triangle inequality any ³²³ simplex in $\tilde{\mathcal{T}}$ with vertex v_i is at least distance $(\frac{1}{2} - 2\tilde{c})L > \frac{5}{12}L$ from the manifold, which ³²⁴ is much larger than $\alpha_k L < \frac{1}{18^k}L$.

³²⁵ **Case 2.** The distance $d(v_i, \mathcal{M})$ is smaller than $\frac{3}{2}L$. Let p be a point in \mathcal{M} such that ³²⁶ $d(v_i, p) < \frac{3}{2}L$. Let

$$\tau_0'(=\emptyset), \tau_1', \dots, \tau_\nu'$$
 (16)

be the simplices of $\tilde{\mathcal{T}}_{i-1}$ such that the joins $\tau_j = \tau'_j * \tilde{v}_i$ are simplices of $\tilde{\mathcal{T}}$, and $\dim(\tau'_j * \tilde{v}_i) \leq d - n - 1$ (and thus $\dim(\tau'_j) \leq d - n - 2$), with $0 \leq j \leq \nu$. We note that $\nu \leq N_{\leq d-n-1}$, with $N_{\leq k}$ as defined in (3). We now consider the span, denoted by $\operatorname{span}(\tau'_j, T_p\mathcal{M})$, for all $0 \leq j \leq \nu$. Note that the dimension of $\operatorname{span}(\tau'_j, T_p\mathcal{M})$ is at most (d - n - 2) + n + 1 = d - 1. We now claim the following:

Solution 19. We can pick \tilde{v}_i such that it lies sufficiently far from each $span(\tau'_j, T_p\mathcal{M})$, that is

$$_{335} \qquad d(\tilde{v}_i, span(\tau'_i, T_p\mathcal{M})) \ge \rho_1 \tilde{c}L, \tag{17}$$

while it is not too far from v_i , that is $|\tilde{v}_i - v_i| \leq \tilde{c}L$.

The following claim completes Case 2 and establishes that the triangulation is (d-n-1)skeleton safe:

 \bullet Claim 20. (11) is satisfied.

We emphasize that in the perturbation of the points it suffices to look at the tangent spaces at specific points, making this constructive proof an algorithm.

$_{342}$ 6 Constructing the triangulation of \mathcal{M}

Section 6.1 gives geometric consequences of the perturbation we discussed in the previous section. Most importantly we shall see that the intersections of simplices in $\tilde{\mathcal{T}}$ with \mathcal{M} and those of simplices in $\tilde{\mathcal{T}}$ with the tangent space of \mathcal{M} at a nearby point are equivalent. Here we again rely on Appendix A. The triangulation K of \mathcal{M} is defined in Section 6.2.

$_{_{347}}$ 6.1 The geometry of the intersection of simplices in $\tilde{\mathcal{T}}$ and \mathcal{M}

In this section, we discuss the geometry of simplices in $\tilde{\mathcal{T}}$ in relation to \mathcal{M} . We follow [27, Section IV.19], with the usual exceptions of the use of Coxeter triangulations, the thickness, and the reach to quantify the results. The proofs below also differ in a fair number of places from the original.

We first establish a lower bound on the distance between $T_p\mathcal{M}$ and simplices in the (d-n-1)-skeleton of $\mathcal{T}^{\mathcal{P}}$ that are close to p.

Lemma 21. Let $p \in \mathcal{M}$ and suppose that $\tau^k \in \tilde{\mathcal{T}}$, with $k \leq d - n - 1$, such that $\tau^k \subset B(p, 6L)$, then

$$_{356} \qquad d(\tau^k, T_p\mathcal{M}) > \frac{2}{3}\alpha_k L.$$

We can now examine the relation between intersections with the manifold and nearby tangent spaces.

Lemma 22. Suppose that \mathcal{M} intersects $\tau^k \in \tilde{\mathcal{T}}$. Let $p \in \mathcal{M}$, such that $\tau^k \subset B(p, 6(L),$ then $T_p\mathcal{M}$ intersects τ^k .

³⁶¹ We can now bound the angle between simplices and tangent spaces.

Lemma 23. Suppose that \mathcal{M} intersects $\tau^k \in \tilde{\mathcal{T}}$ and τ^k has dimension d - n, that is k = d - n. Let $p \in \mathcal{M}$, such that $\tau^k \subset B(p, 6L)$, then

$$\sin \angle (\operatorname{aff}(\tau^k), T_p \mathcal{M}) \ge \frac{2d(T_p \mathcal{M}, \partial \tau^k)}{L + 2\tilde{c}L} \ge \frac{\frac{4}{3}\alpha_k L}{L + 2\tilde{c}L} \ge \frac{16}{10}\alpha_k.$$

Below we investigate the relation between intersections of tangent spaces and simplices
and intersections between the manifold and simplices. We combine two statements of Section
IV.19 in the following lemma. The proof is quite different from the original.

Lemma 24. If $p \in M$, $τ^k \in \tilde{T}$ and $τ^k \subset B(p, 6L)$, and moreover T_pM intersects $τ^k$, then $k \ge d-n$ and M intersects $τ^k$. If k = d-n this point is unique, which in particular means that every simplex of dimension d-n contains at most one point of M

Finally, we study the faces of a simplex that intersects \mathcal{M} . This is essential for the barycentric subdivision in part 2 of the algorithm. The proof is identical to the original.

▶ Lemma 25. If \mathcal{M} intersects $\tau = \{v_0, \ldots, v_r\} \in \tilde{\mathcal{T}}$, then for each $v_i \in \tau$, there exists some (d-n)-face τ' of τ such that $v_i \in \tau'$ and τ' intersects \mathcal{M} .

³⁷⁵ 6.2 The triangulation of \mathcal{M} : The complex K (Section IV.20 of [27])

The construction of the complex follows Section IV.20 of [27].

In each simplex τ of $\tilde{\mathcal{T}}$ that intersects \mathcal{M} we choose a point $v(\tau)$ and construct a complex

³⁷⁸ K with these points as vertices. The construction goes via barycentric subdivision. For each ³⁷⁹ sequence $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_k$ of distinct simplices in $\tilde{\mathcal{T}}$ such that τ_0 intersects \mathcal{M} ,

380
$$\sigma^k = \{v(\tau_0), \dots, v(\tau_k)\}$$
 (18)

will be a simplex of K. The definition of $v(\tau)$ depends on the dimension of τ :

If τ is a simplex of dimension d - n, then there is an unique point of intersection with \mathcal{M} , due to Lemma 24. We define $v(\tau)$ to be this unique point.

If τ has dimension greater than d - n, then we consider the faces $\tau_1^{d-n}, \ldots, \tau_j^{d-n}$ of τ of dimension d - n that intersect \mathcal{M} . These faces exist thanks to Lemma 25. We now define $v(\tau)$ as follows:

$$v(\tau) = \frac{1}{j} \left(v(\tau_1^{d-n}) + \dots + v(\tau_j^{d-n}) \right).$$
(19)

Remark. We stress that thanks to Lemma 24, choosing the point $v(\tau^{d-n})$ to be the point of intersection with $T_p\mathcal{M}$, assuming p is sufficiently close, locally gives the same combinatorial structure as intersections with \mathcal{M} . We also stress that for the combinatorial structure it does not really matter where \mathcal{M} intersects a simplex of $\tilde{\mathcal{T}}$, as long as it does.

We can now prove the following bound on the altitudes of the simplices we constructed in this manner.

Lemma 26. Let σ^n be a top dimensional simplex as defined in (18), then

395
$$\min \operatorname{alt}(\sigma^n) > \zeta^n (\alpha_{d-n-1})^n \tilde{L},$$

where min alt denotes the minimal altitude or height, and we used the notation ζ , as defined in (7).

7 The triangulation proof

Given the triangulation $\tilde{\mathcal{T}}$, we want to prove that the intersection of $\mathcal{M} \cap \tau^d$ is homeomorphic to the triangulated polytope described in Section 6.2. This immediately gives a global homeomorphism between the triangulation and the manifold.

The homeomorphism we discuss in this section differs greatly from Whitney's own approach. Firstly, he used the closest point projection as a map (which does not respect simplices, meaning that the point in K and its projection may lie in different simplices of $\tilde{\mathcal{T}}$. Secondly, to prove that this map is a homeomorphism, he uses what has become known as Whitney's lemma in much the same way as in [8].

The great advantage of our approach to the homeomorphism proof is that it is extremely explicit and it is elementary in the sense that it does not rely on topological results. We also need precise bounds on the angles, which do not require deep theory, but are quite intricate.

Because we work with an ambient triangulation of type \tilde{A} and we do not perturb too much, the simplices of $\tilde{\mathcal{T}}$ are Delaunay. The homeomorphism from $\mathcal{M} \cap \tau^d$ to the triangulated polytope $K \cap \tau^d$, with K as defined in Section 6.2 and $\tau^d \in \tilde{\mathcal{T}}$, gives that the intersection of any simplex in $\tilde{\mathcal{T}}$ with \mathcal{M} is a topological ball of the appropriate dimension. This may remind the reader of the closed ball property of Edelsbrunner and Shah [18].

⁴¹⁵ **Overview homeomorphism proof.** The homeomorphism proof consists of three steps:

For each maximal simplex $\tau \in \tilde{\mathcal{T}}$ we provide a 'tubular neighbourhood' for $K \cap \tau$ adapted to τ . By this we mean that, for each point \bar{p} in $K \cap \tau$, we designate a 'normal' space $\mathcal{N}_{\bar{p}}$ that has dimension equal to the codimension of \mathcal{M} and K and is transversal to $K \cap \tau$. Moreover, these directions shall be chosen in a sufficiently controlled and smooth way, so that every point x in τ that is sufficiently close to K has a unique point \bar{p} on $K \cap \tau$ such that $x - \bar{p} \in \mathcal{N}_{\bar{p}}$.

We give conditions that enforce that the 'normal' spaces $\mathcal{N}_{\bar{p}}$ intersect \mathcal{M} transversely. More precisely, we prove that the angle between $\tilde{N}_{\bar{p}}$ and $N_q \mathcal{M}$, for any $q \in \mathcal{M} \cap \tau$ is upper bounded by a quantity strictly less than 90 degrees.

We conclude that the projection along $\mathcal{N}_{\bar{p}}$ gives a homeomorphism from \mathcal{M} to K.

426 7.1 Constructing the tubular neighbourhood

We now give the construction of the 'tubular neighbourhood' of K. We use two results from the previous sections:

The normal space is almost constant, see Lemma 2, near a simplex $\tau \in \tilde{\mathcal{T}}$, because it is small. So $T\mathcal{M}$ and $N\mathcal{M}$ near p are well approximated by $T_p\mathcal{M}$ and $N_p\mathcal{M}$.

The angles between the normal space and those faces τ_k^{d-n} of dimension d-n that intersect \mathcal{M} are bounded from below by Lemma 23.

This means that the orthogonal projection map $\pi_{\operatorname{aff}(\tau_k^{d-n})\to N_p\mathcal{M}} = \pi_{\tau_k^{d-n}}$ from the affine hull aff (τ_k^{d-n}) to $N_p\mathcal{M}$ is a (linear) bijection. Now note that the orthonormal basis e_1, \ldots, e_{d-n} of $N_p\mathcal{M}$ induces a (generally not orthonormal) basis of $\operatorname{aff}(\tau_k^{d-n})$ under the inverse image of the map $\pi_{\tau_k^{d-n}}$, denoted by $\{\pi_{\tau_k^{d-n}}^{-1}(e_j) \mid j=1,\ldots,d-n\}$.

⁴³⁷ We can now define the normal spaces for the complex K. We first do this for the vertices ⁴³⁸ $v(\tau^{d-n})$ (these vertices lie on \mathcal{M}), secondly for general vertices of K (these vertices do not ⁴³⁹ necessarily lie on \mathcal{M}) and finally using barycentric coordinates for arbitrary points in K.

Having defined a basis of $\operatorname{aff}(\tau_k^{d-n})$ for the vertices $v(\tau_k^{d-n})$, as defined in Section 6.2, we can construct a basis of a (d-n)-dimensional affine space, through each $v(\tau)$. The construction is similar to that of Section 6.2. If τ has dimension greater than d-n, then we consider the faces $\tau_1^{d-n}, \ldots, \tau_j^{d-n}$ of τ of dimension d-n that intersect \mathcal{M} . For each $e_i \in N_p \mathcal{M}$, we write

$$_{445} \qquad N_{v(\tau)}(e_i) = \frac{1}{j} \left(\pi_{\tau_1^{d-n}}^{-1}(e_i) + \dots + \pi_{\tau_j^{d-n}}^{-1}(e_i) \right). \tag{20}$$

⁴⁴⁶ Using these vectors we can define the space $\mathcal{N}_{v(\tau)} = \operatorname{span}(N_{v(\tau)}(e_i)).$

Let $\sigma^n = \{v(\tau_0^{d-n}), \dots, v(\tau_n^d)\}$ be a simplex of K. For any point \bar{p} in σ^n , with barycentric coordinates $\lambda = (\lambda_0, \dots, \lambda_n)$, we define

449
$$N_{\bar{p}}(e_i) = \lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda_n N_{v(\tau_n^{d-n})}(e_i).$$
(21)

⁴⁵⁰ By defining $\mathcal{N}_{\bar{p}} = \operatorname{span}(N_{\bar{p}}(e_i))$, we get affine spaces for each point in each $\sigma^n \in K$.

⁴⁵¹ \triangleright Remark. By construction, these spaces are consistent on the faces of simplices in K as ⁴⁵² well as with the boundaries of the d dimensional simplices in $\tilde{\mathcal{T}}$.

453 7.2 The size of the tubular neighbourhoods and the homeomorphism

In this section, we establish the size of the neighbourhood of K as defined by $\mathcal{N}_{\bar{p}}$.

The following angle estimate is an essential part of the estimate of the size of the neighbourhood of the triangulation K.

Lemma 27. Suppose that $p \in \mathcal{M}$, $\tau^d \subset B(p, 6L)$ and $\sigma^n \in K$ such that $\sigma^n \subset \tau^d$, where we regard σ^n and τ^d as subsets of \mathbb{R}^d . Then the angle between $T_p\mathcal{M}$ and aff(σ^n) is bounded as follows:

460
$$\sin \angle (\operatorname{aff}(\sigma^n), T_p\mathcal{M}) \le \frac{(\alpha_{d-n})^{4+n}}{6(n+1)} \zeta^n.$$

461 With this we can give a bound on the size of the neighbourhood of K.

Lemma 28. Let $\bar{p}, \bar{q} \in \sigma^n$, with barycentric coordinates $\lambda = (\lambda_0, ..., \lambda_n), \lambda' = (\lambda'_0, ..., \lambda'_n)$ respectively. Suppose that $N_{\bar{p}}$ and $N_{\bar{q}}$ be as defined in Section 7.1. Suppose now that the intersection between $\bar{p} + N_{\bar{p}}$ and $\bar{q} + N_{\bar{q}}$ is non-empty. Here $\bar{p} + N_{\bar{p}}$ and $\bar{q} + N_{\bar{q}}$ denote the affine spaces that go through \bar{p}, \bar{q} and are parallel to $N_{\bar{p}}, N_{\bar{q}}$, respectively. If $x \in \bar{p} + N_{\bar{p}} \cap \bar{q} + N_{\bar{q}}$, then

467
$$d(x, \operatorname{aff}(\sigma^{n})) \ge \frac{\left(\frac{16}{10}\right)^{2} (\alpha_{d-n})^{4}}{n+1} \zeta^{n} (\alpha_{d-n-1})^{n} \tilde{L}.$$

Because, by construction, the N_p agree on the faces of the n-dimensional simplices in K, this provides a tubular neighbourhood for K of this size.

▶ Lemma 29. Suppose $\tau^d \in \tilde{\mathcal{T}}$ and that $\mathcal{M} \cap \tau^d \neq \emptyset$. Then, $\mathcal{M} \cap \tau^d$ lies in the tubular neighbourhood of $K \cap \tau^d$ as defined in Section 7.1.

Having established that \mathcal{M} lies in the tubular neighbourhood around K we can sensibly 472 speak about the projection from \mathcal{M} to K along the direction N. Because we also have that 473 the projection from \mathcal{M} to K in the direction \mathcal{N} (as defined in Section 7.1) is transversal 474 (Because $\pi/2$ minus the angle between $\mathcal{N}_{\bar{p}}$ and $N_p\mathcal{M}$, see (33), is much bigger than the 475 variation of the tangent/normal space as bounded by Lemma 2 and (9)) we see that $\mathcal{M} \cap \tau^d$ 476 is homeomorphic to $K \cap \tau^d$. By construction the projection map is compatible on the 477 boundaries of τ^d , so we also immediately have an explicit homeomorphism between \mathcal{M} and 478 K. This completes the proof of Theorem 1. We emphasize that along the way we have also 479 given a bounds on 480

- 481 the Hausdorff distance between \mathcal{M} and K (Lemmas 29 and 28)
- $_{482}$ = the quality (Lemma 26)
- = the variation of the tangent spaces ((33), Lemma 2 and (9) as mentioned above).

484 — References -

485	1	E. Aamari and C. Levrard. Non-Asymptotic Rates for Manifold, Tangent Space, and		
486		Curvature Estimation. ArXiv e-prints, May 2017. arXiv:1705.00989.		
487	2	Milton Abramowitz and Irene A. Stegun. Handbook of mathematical functions : with		
488		formulas, graphs, and mathematical tables. National Bureau of Standards, 1970.		
490	3	N Amenta and M W Bern Surface reconstruction by Voronoi filtering In SoCC pages		

Amenta and M. W. Bern. Surface reconstruction by Voronoi filtering. In SoCG, pages 39–48, 1998. URL: citeseer.ist.psu.edu/amenta98surface.html.

- 00:15
- 4 N. Amenta, M. W. Bern, and M. Kamvysselis. A new Voronoi-based surface reconstruction
 algorithm. In ACM SIGGRAPH, pages 415–421, 1998.
- ⁴⁹³ **5** J.-D. Boissonnat, R. Dyer, and A. Ghosh. The stability of Delaunay triangulations. *Inter-*⁴⁹⁴ *national Journal of Computational Geometry & Applications*, 23(04n05):303–333, 2013.
- J.-D. Boissonnat, R. Dyer, and A. Ghosh. Delaunay stability via perturbations. International Journal of Computational Geometry & Applications, 24(02):125–152, 2014.
- Jean-Daniel Boissonnat, Frédéric Chazal, and Mariette Yvinec. Geometric and Topological Inference. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2018. doi:10.1017/9781108297806.
- Jean-Daniel Boissonnat, Ramsay Dyer, Arijit Ghosh, and Mathijs Wintraecken. Local Criteria for Triangulation of Manifolds. In Bettina Speckmann and Csaba D. Tóth, editors, 34th International Symposium on Computational Geometry (SoCG 2018), volume 99 of Leibniz International Proceedings in Informatics (LIPIcs), pages 9:1–9:14, Dagstuhl, Germany, 2018. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik. URL: http://drops. dagstuhl.de/opus/volltexte/2018/8722, doi:10.4230/LIPIcs.SoCG.2018.9.
- Jean-Daniel Boissonnat and Arijit Ghosh. Manifold reconstruction using tangential
 Delaunay complexes. Discrete & Computational Geometry, pages 221–267, 2014.
- Jean-Daniel Boissonnat, André Lieutier, and Mathijs Wintraecken. The Reach, Metric Distortion, Geodesic Convexity and the Variation of Tangent Spaces. In Bettina Speckmann and Csaba D. Tóth, editors, 34th International Symposium on Computational Geometry (SoCG 2018), volume 99 of Leibniz International Proceedings in Informatics (LIPIcs), pages 10:1–10:14, Dagstuhl, Germany, 2018. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik. URL: http://drops.dagstuhl.de/opus/volltexte/2018/8723, doi: 10.4230/LIPIcs.SoCG.2018.10.
- S. S. Cairns. On the triangulation of regular loci. Annals of Mathematics. Second Series, 35(3):579–587, 1934. doi:10.2307/1968752.
- S-W. Cheng, T. K. Dey, and E. A. Ramos. Manifold Reconstruction from Point Samples.
 In Proc. ACM-SIAM Symp. Discrete Algorithms, pages 1018–1027, 2005.
- Aruni Choudhary, Siargey Kachanovich, and Mathijs Wintraecken. Coxeter triangula tions have good quality. HAL preprint, December 2017. URL: https://hal.inria.fr/
 hal-01667404.
- H.S.M. Coxeter. Discrete groups generated by reflections. Annals of Mathematics, pages 588–621, 1934.
- T. K. Dey. Curve and Surface Reconstruction; Algorithms with Mathematical Analysis.
 Cambridge University Press, 2007.
- J.J. Duistermaat and J.A.C. Kolk. *Multivariable Real Analysis II: Integration*. Cambridge
 University Press, 2004.
- R. Dyer, G. Vegter, and M. Wintraecken. Riemannian simplices and triangulations.
 Geometriae Dedicata, 179(1):91–138, 2015. (Preprint: arXiv:1406.3740). doi:10.1007/
 s10711-015-0069-5.
- H. Edelsbrunner and N. R. Shah. Triangulating topological spaces. In SoCG, pages 285–292, 1994.
- ⁵³³ **19** H. Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93(3):418–491, 1959.
- ⁵³⁴ 20 M. W. Hirsch. *Differential Topology*. Springer-Verlag, 1976.
- Albert T. Lundell and Stephen Weingram. *The Topology of CW Complexes*. Van Nostrand
 Reinhold Company, 1969.
- ⁵³⁷ 22 Maurice Maes and Bert Kappen. On the permutahedron and the quadratic placement ⁵³⁸ problem. *Philips Journal of Research*, 46(6):267–292, 1992.
- P. Niyogi, S. Smale, and S. Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete & Comp. Geom.*, 39(1-3):419-441, 2008.

- 24 B.C. Rennie and A.J. Dobson. On Stirling numbers of the second kind. *Journal of Combinatorial Theory*, 7(2):116 – 121, 1969. URL: http://www.sciencedirect.com/science/
 article/pii/S0021980069800451, doi:https://doi.org/10.1016/S0021-9800(69)
 80045-1.
- J. G. Wendel. Note on the gamma function. The American Mathematical Monthly, 545 55(9):563-564, 1948. URL: http://www.jstor.org/stable/2304460.
- J. H. C. Whitehead. On C¹-complexes. Annals of Mathematics, 41(4):809-824, 1940. URL:
 http://www.jstor.org/stable/1968861.
- 549 27 H. Whitney. Geometric Integration Theory. Princeton University Press, 1957.

550 A Some properties of affine spaces

In this appendix, we discuss two variants of lemmas from Appendix Section II.14 of [27], 551 that are essential in the building of the triangulation, see Section 6.1 in particular. Both 552 lemmas are due to Whitney. However, in both cases, the statement is different, because we 553 prefer to work directly with angles and use the thickness as our quality measure. In the first 554 case, the proof we provide differs significantly from the original. The first lemma will allow 555 us to prove that if $T_p\mathcal{M}$ intersects a simplex $\tau \in \mathcal{T}$ and p and τ are not too far from each 556 other then \mathcal{M} intersects τ and vice versa. The second result is essential in proving that 557 the perturbation of the vertices as described in Section 2.1 part 1, gives a triangulation for 558 which the low dimensional simplices are sufficiently far away from the manifold. 559

We start with a variation on Lemma 14a of Appendix Section II.14 of [27].

▶ Lemma 30. Let σ be an s-simplex and A_0 an affine n-dimensional subspace in \mathbb{R}^d . Assume that $s + n \ge d$ and

$$_{563} \qquad d(A_0,\sigma) < d(A_0,\partial\sigma).$$

560

Then s + n = d, A intersects σ in a single point and

$$\sin \angle \operatorname{aff}(\sigma) A_0 \ge 2d(A_0, \partial \sigma)/L(\sigma)$$

Proof of Lemma 30. Choose $p \in \sigma$ and $q \in A_0$ such that

567
$$|p-q| = d(A_0, \sigma).$$

Now suppose that there is a vector $v \neq 0$ that lies in the intersection of $\operatorname{aff}(\sigma)$ and A_0 . Then there exists some $c \in \mathbb{R}$ such that $p + cv \in \partial \sigma$. Because v lies in the intersection of $\operatorname{aff}(\sigma)$ and A_0 , we have that $q + cv \in A_0$. Clearly translation leaves distances invariant, so

571
$$d(A_0, \sigma) = |p - q| = |(p + cv) - (q + cv)| \ge d(A_0, \partial \sigma),$$

which clearly contradicts the assumption. This means we can conclude that there is no such v and therefore s + n = d.

Because there is no v in the intersection of $\operatorname{aff}(\sigma)$ and A_0 , there is a unique point \bar{p} in this intersection. We'll now show that $\bar{p} \in \sigma$. I'll assume that $\bar{p} \notin \sigma$. This means in particular that $q \neq \bar{p}$. Because $d(A_0, \sigma) < d(A_0, \partial \sigma)$, p - q is normal to $\operatorname{aff}(\sigma)$ and $p \in \sigma \setminus \partial \sigma$. Now consider the line from q to \bar{p} , which lies in A_0 . The distance from a point on this line to σ decreases (at least at first) as you go from q toward \bar{p} . This contradicts the definition of q. We conclude that $\bar{p} \in \sigma$.

Now suppose that l_0 is a line in A_0 that goes though \bar{p} . In order to derive a contradiction, we assume that

$$\sin \phi < 2d(A_0, \partial \sigma)/L(\sigma),$$

where $\sin \phi$ denotes the angle between l_0 and $\operatorname{aff}(\sigma)$. Denote by $\pi_{\operatorname{aff}(\sigma)}(l_0)$ the orthogonal/closest point projection on $\operatorname{aff}(\sigma)$ of l_0 . Because $\bar{p} \in \sigma \pi_{\operatorname{aff}(\sigma)}(l_0)$ intersects $\partial \sigma$ at a point \bar{q} and we may assume that $|\bar{p} - \bar{q}| \leq \frac{1}{2}L(\sigma)$ so that l_0 contains a point a distance

586
$$\frac{1}{2}L(\sigma)\sin\phi < \frac{1}{2}L(\sigma)2d(A_0,\partial\sigma)/L(\sigma) = d(A_0,\partial\sigma)$$

from $\partial \sigma$, a contradiction. Because l_0 was an arbitrary line in A_0 the result now follows.

The following is a variation on Lemma 14b of Appendix Section II.14 of [27]. The proof presented here is almost identical to the original.

Lemma 31. Let A_1 and A_2 be two affine subspaces in \mathbb{R}^d , with $A_1 \subset A_2$. Let τ be a simplex in A_2 , and let v be a point in $\mathbb{R}^d \setminus \tau$. Define J to be the join of τ and v. Then

592
$$d(J, A_1) \ge \frac{d(\tau, A_1)d(v, A_2)}{L(J)},$$
 (22)

where the distances between sets are defined as $d(B,C) = \inf_{x \in B, y \in C} |x-y|$ and L(J) denotes the longest edge length of an edge in J.



Figure 2 Notation for the proof of Lemma 31.

⁵⁹⁶ **Proof of Lemma 31.** Let us suppose that (22) is false. Let J^c be the truncated cone that ⁵⁹⁷ consists of all half-lines that start at a point of τ and pass through v. Then we may choose ⁵⁹⁸ $p_J \in J^c$ and $a_1 \in A_1$ so that

⁵⁹⁹ $|p_J - a_1| = d(J^c, A_1),$

00:18 An elementary and quantified version of Whitney's triangulation method

 $_{600}$ by the definition of J^c and the hypothesis we also have

601
$$d(J^c, A_1) \le d(J, A_1) < \frac{d(\tau, A_1)d(v, A_2)}{L(J)}.$$
 (23)

Now suppose that p_J lies on the half line that starts at $w \in \tau$ and goes through v. Because 602 $\tau \subset A_2$, we see that $d(v, A_2) \leq L_e(J)$. This means that (23) gives that $d(J^c, A_1) < d(\tau, A_1)$, 603 so that $p_J \neq w$. We now immediately see that the line segment $a_1 p_J$ is orthogonal to the 604 line that goes through w and v, which extends the half line we mentioned above. Let ℓ now 605 be the line that goes through a_1 and w, and $\pi_{\ell}(v) \in \ell$ the point that is closest to v. It 606 follows that $\pi_{\ell}(v)w$ is perpendicular to ℓ . Because a_1 is nearer to p_J than w, a_1 and $\pi_{\ell}(v)$ 607 are on the same side of w in ℓ . This means because two of the angles are the same (and 608 thus the third), that the triangles $p_I w a_1$ and $\pi_\ell(v) w v$ are similar. We now have that 609

610
$$d(J^c, A_1) = |p_J - a_1| = \frac{|a_1 - w||v - \pi_\ell(v)|}{|v - w|} \ge \frac{d(\tau, A_1)d(v, A_2)}{L(J)},$$

611 contradicting the hypothesis and thus proving the lemma.

<

\mathbf{B} **B Remark on the** $C^{1,1}$ **case**

⁶¹³ We now first discuss a simpler version of 3. The result in this case is weaker, but can be ⁶¹⁴ easily extended to the $C^{1,1}$ setting as we shall see below.

⁶¹⁵ The following consequence of Lemma 2 of is a stronger version of Lemma 5.4 of [23]:

• **Corollary 32.** Suppose \mathcal{M} is C^2 and $p \in \mathcal{M}$, then for all $0 < r < \frac{\operatorname{rch}(\mathcal{M})}{\sqrt{2}}$, the projection $\pi_{T_p\mathcal{M}}$ onto the tangent space $T_p\mathcal{M}$, restricted to $\mathcal{M} \cap B(p,r)$ is a diffeomorphism onto its image.

⁶¹⁹ **Proof of Corollary 32.** Let $q \in \mathcal{M}$ such that $|p-q| \leq r$, then the differential of the projection ⁶²⁰ map $\pi_{T_p\mathcal{M}}$ at q is non-degenerate, because, by Lemma 2, the angle $\angle(T_p\mathcal{M}, T_q\mathcal{M})$ is less ⁶²¹ than $\pi/2$. Because $\mathcal{M} \cap B(p, r)$ is a topological ball of the right dimension by Proposition ⁶²² 1 of [10], the result now follows.

 $_{623}$ Similarly to Lemma 2 we have for $C^{1,1}$ manifolds that:

▶ Lemma 33 (Theorem 3 of [10]). Now suppose that \mathcal{M} has positive reach, that is \mathcal{M} is at least $C^{1,1}$, and let $|p-q| \leq \operatorname{rch}(\mathcal{M})/3$, then

$$\sin \frac{\angle (T_p \mathcal{M}, T_q \mathcal{M})}{2} \le \frac{1 - \sqrt{1 - \alpha^2}}{\sqrt{\frac{\alpha^2}{4} - (\frac{\alpha^2}{2} + 1 - \sqrt{1 - \alpha^2})^2}},$$

62

628

627 where $\alpha = |p - q|/\operatorname{rch}(\mathcal{M})$.

This lemma gives us a corollary, which is the equivalent of Corollary 32:

Corollary 34. Suppose \mathcal{M} is $C^{1,1}$ and $p \in \mathcal{M}$, then for all $r < \frac{\operatorname{rch}(\mathcal{M})}{3}$, the projection $\pi_{T_p\mathcal{M}}$ onto the tangent space $T_p\mathcal{M}$, restricted to $\mathcal{M} \cap B(p,r)$ is a diffeomorphism onto its image.

These are in fact all the fundamental results that are needed to be able to extend to the $C^{1,1}$ setting.

Assuming the manifold is $C^{1,1}$ would lead to minor changes in the calculations in the proof of Lemma 24 and would in theory influence the final conclusion in Section 7.2. However, because we have a significant margin in the difference between $\pi/2$ minus the angle between $\mathcal{N}_{\bar{p}}$ and $N_p\mathcal{M}$ we would not need to change the constants in Section 7.2. The rest of proofs hold verbatim.

Remark 35. Because of symmetry the largest volume S can attain is when both delimiting hyperplanes are equidistant to the center of $B^d(r)$. The volume of S is given by the integral

$$^{642} \qquad r^d \int_{-\rho_1}^{\rho_1} \operatorname{vol}\left(B_{d-1}\left(\sqrt{1-h^2}\right)\right) \mathrm{d}h = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} r^d \int_{-\rho_1}^{\rho_1} \left(\sqrt{1-h^2}\right)^{d-1} \mathrm{d}h,$$

where $B_{d-1}(r)$ denotes the ball on dimension d-1 with radius r and Γ denotes the Euler gamma function. This integral can be expressed using special functions such as the hypergeometric function or beta functions. This gives an explicit value for $\bar{\rho}_1$.

646 **D** Notation

In the following table we give an overview of the notation used in this paper and compare
 it to Whitney's notation.

649	Notation	definition	Whitney's notation (if
			relevant)
650	\mathcal{M}	The manifold	M
651	$T\mathcal{M}, T_p\mathcal{M}$	The tangent bundle and the tangent space at	P_p
		p	
652	$N\mathcal{M}, N_p\mathcal{M}$	The normal bundle and normal space at \boldsymbol{p}	
653	$N_{\leq k}$	An upper bound on the total number of faces	Whitney does not dis-
		of dimension less or equal to k that contain a	tinguish dimensions and
		given vertex.	uses N as an upper
	1 (1 4)		bound. (no value given)
654	$\operatorname{rcn}(\mathcal{M})$	The reach to the manifold \mathcal{M}	
655	8	Slab between two hyperplanes intersected with a ball	Q
656	\mathcal{T}	The ambient Coxeter triangulation of type \tilde{A}	${\cal L}$ is the ambient trian-
			gulation, but is not a
	~		Coxeter triangulation
657	T	Perturbed ambient triangulation	L^*
658	v_i	Vertices of $\mathcal{T}_{\tilde{a}}$	p_i
659	v_i^*	Vertices of \mathcal{T}	p_i^*
660	$L(\cdot)$	Longest edge length	δ is the longest edge
			length of the ambient tri-
	$U(\mathbf{V}, \mathbf{v})$	A mainth annhard af maline mafa ant V	angulation L
661	$U(\Lambda, r)$	A heighbourhood of radius r of a set A	$U_r(\Lambda)$
662	$ ho_1$	volume fraction of the part of a ball inside a	$ ho_1$
662	01	Lower bound on \bar{a}_1 , see (4)	
003	δ^{p_1}	Protection	
665	0	Separation as in an $(\epsilon \mu)$ -net (the shortest	
005	μ	edge length in \mathcal{T} for Coxeter triangulations)	
666	e	The sampling density as in an (ϵ, μ) -net (the	
	-	circumradius of the simplices in the Coxeter	
		triangulation)	
667	μ_0	The normalized separation, that is $\mu = \mu_0 \epsilon$	

00:20 An elementary and quantified version of Whitney's triangulation method

668	A_i	Affine subspaces	P, P' and Q
669	L(K)	For any complex K , $L(K)$ denotes the longest	
		edge length in K	
670	L	$L = L(\mathcal{T})$	
671	\tilde{L}	$\tilde{L} = L(\tilde{\mathcal{T}})$	
672	$\tilde{c}L$	Perturbation radius of the vertices of \mathcal{T}	ρ
673	$ ilde{c}$	Normalized perturbation radius	$ ho^*$
674	$ au, \sigma$	Simplices. We have tried to reserve τ for \mathcal{T}	Same
		or $\tilde{\mathcal{T}}$ and σ for K. However for arbitrary sim-	
		plices (such as in Appendix A) we use arbit-	
		rary choices. Subscripts are used for indices	
		and superscripts for the dimension.	
675	$t(\sigma)$	Thickness of σ	
676	aff	The affine hull	
677	n	Dimension of \mathcal{M}	n
678	d	Ambient dimension (\mathbb{R}^{a})	m
679	$d(\cdot, \cdot)$	Euclidean distance between sets	
680	$d_{\mathcal{M}}(\cdot, \cdot)$	Distance on \mathcal{M}	
681	λ	barycentric coordinates	
682	K	Triangulation of \mathcal{M}	K
683	$B^d(c,r)$	A ball with in \mathbb{R}^d of dimension with centre c	$U_r(c)$
		and radius r , if we do not need to emphasize	
		the centre or radius or they are to be determ-	
		ined, these are suppressed from the notation	
684	$B_{T_p\mathcal{M}}(c,r)$	A ball in $T_p\mathcal{M}$, using the same conventions as	
		for $B^{-}(c,r)$	
685	$C(T_p\mathcal{M}, r_1, r_2)$	Open cylinder given by all points that project	
		orthogonally onto an open ball of radius r_1 in T . M and whose distance to this ball is at most	
		r_p , r_p and whose distance to this ball is at most r_p .	
686	$\pi_{\Lambda A}$	Closest point projection on \mathcal{M}	π^*
687	π_{T} M	Orthogonal projection on the tangent spaces	
	n 1 _p M	$T_p\mathcal{M}$	
688	π_p^{-1}	See Definition 4	
689	$\pi_{\mathrm{aff}(\tau_k^{d-n}) \to N_p \mathcal{M}}$	The orthogonal projection map from the affine	
	$=\pi_{\tau_k^{d-n}}$	hull $\operatorname{aff}(\tau_i^{a-n})$ to $N_p\mathcal{M}$.	
690	$N_{v(\tau)}(e_i)$	See (20)	
691	$\mathcal{N}_{ar{p}}$	The 'normal' space of K at \bar{p} , that is	
		$\operatorname{span}(N_{\bar{p}}(e_i))$	

Overview most important bounds

 $_{\tt 693}$ α_1 and α_k have been defined by a recursion relation as follows

694
$$\alpha_1 = \frac{4}{3}\rho_1 \tilde{c}$$
 $\frac{2}{3}\alpha_{k-1}\tilde{c}\rho_1 = \alpha_k,$ (5)

and thus $\alpha_k = \frac{2^{k+1}}{3^k} \rho_1^k \tilde{c}^k$. In particular we have the bound

$$_{696} \qquad \alpha_k \le \frac{1}{18^k}. \tag{6}$$

 $_{697}$ L satisfies

$$\int_{698} \left(1 - \sqrt{1 - \left(\frac{6L(\mathcal{T})}{\operatorname{rch}(\mathcal{M})}\right)^2}\right) \operatorname{rch}(\mathcal{M}) = \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2} \zeta^{2n} L$$
(8)

699 or equivalently

$$\frac{L}{\operatorname{rch}(\mathcal{M})} = \frac{2\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}}{\left(\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}\right)^2 + 6^2},\tag{9}$$

701 with

$$\zeta = \frac{8}{15\sqrt{d} \begin{pmatrix} d \\ d-n \end{pmatrix} \cdot (1+2\tilde{c})} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right) t(\mathcal{T}).$$
(7)

703 We often use

$$\frac{L}{\operatorname{rch}(\mathcal{M})} < \frac{(\alpha_{d-n})^{4+2n}}{54(n+1)^2} \zeta^n < \frac{(\alpha_{d-n})^2}{54}, \qquad \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2} \zeta^{2n} < \frac{(\alpha_{d-n})^2}{3} \le \frac{\alpha_{d-n}}{3}.$$
 (10)

The perturbation radius \tilde{c} satisfies

$$|v_i - \tilde{v}_i| \le \tilde{c}L = \min\left\{\frac{t(\mathcal{T})\mu_0}{18d}\delta, \frac{1}{24}t(\mathcal{T})^2L\right\},$$
(14)

707 from which it follows that

$$\tilde{c} \le \frac{1}{24}.$$
(15)

709 E Proofs

⁷¹⁰ **Proof of Lemma 3.** Apart from Lemma 2, we'll be using the following results from [10]: ⁷¹¹ For a minimizing geodesic γ on \mathcal{M} with length ℓ parametrized by arc length, with $\gamma(0) = p$ ⁷¹² and $\gamma(\ell) = q$

713
$$\angle \dot{\gamma}(0)\dot{\gamma}(t) \le \frac{t}{\operatorname{rch}(\mathcal{M})}.$$
 (24)

⁷¹⁴ If we also write $v_p = \dot{\gamma}(0)$ we have that

$$\gamma_{15} \qquad \langle \gamma(\ell), v_p \rangle = \int_0^\ell \frac{\mathrm{d}}{\mathrm{d}t} \left(\langle \gamma(t), v_p \rangle \right) \mathrm{d}t$$

716

717

$$= \int_{0}^{\ell} \langle \dot{\gamma}(t), t_{0} \rangle dt$$

$$\geq \int_{0}^{\ell} \cos\left(\frac{t}{\operatorname{rch}(\mathcal{M})}\right) dt \qquad (\text{using (24)})$$

$$= \operatorname{rch}(\mathcal{M}) \sin\left(\frac{\ell}{\operatorname{rch}(\mathcal{M})}\right)$$

718 719

as long as
$$\ell < \frac{1}{2} \operatorname{rch}(\mathcal{M})\pi$$
. Because $v_p \in T_p\mathcal{M}$ and $\gamma(\ell) = q$ we have that

 $\geq \operatorname{rch}(\mathcal{M}) \sin(\angle(T_p\mathcal{M}, T_q\mathcal{M}))),$

$$|p - \pi_{T_p \mathcal{M}}(q)| \ge \operatorname{rch}(\mathcal{M}) \sin(\angle(T_p \mathcal{M}, T_q \mathcal{M})).$$
(25)

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(using Lemma 2)

00:22 An elementary and quantified version of Whitney's triangulation method

This means in particular that for all q such that $|p - \pi_{T_p\mathcal{M}}(q)| < \operatorname{rch}(\mathcal{M})$ and $|q - \pi_{T_p\mathcal{M}}(q)| \le \operatorname{rch}(\mathcal{M})$ the angle between $T_p\mathcal{M}$ and $T_q\mathcal{M}$ is less than 90 degrees. Note that the condition on ℓ mentioned above is satisfied by a combination of Theorem 1 and Lemma 11 of [10].

Proof of Claim 10. Choudhary et al. [13, Appendix B] provide explicit values of all the quantities mentioned in Corollary 9 for a Coxeter triangulation of type \tilde{A} , with the exception of μ , which can be easily derived from a more general result. If we fix the scale (which in [13] we did by a convenient choice of coordinates for the vertices), they are

The value of μ easily follows from the general expression of edge lengths (see [13, Appendix B, \tilde{A}_d , item 5]) and is equal to $\mu = \sqrt{\frac{d}{d+1}}$. From (26), we get that $\mu_0 = \frac{\mu}{\epsilon} = \sqrt{\frac{12}{d+2}}$. The bound in (2) is therefore

$$\tilde{c} \leq \frac{t(\sigma)\mu_0}{18d} \frac{\delta}{L} = \begin{cases} \sqrt{2} \frac{\sqrt{d^2 + 2d + 2d} - \sqrt{d^2 + 2d}}{9d^{3/2}\sqrt{d + 2(d+1)}} & \text{if } d \text{ is odd,} \\ \sqrt{2(d+1)} \frac{\sqrt{d^2 + 2d + 24} - \sqrt{d^2 + 2d}}{9d^2(d+2)^{3/2}} & \text{if } d \text{ is even.} \end{cases}$$

$$\leq \sqrt{2} \frac{\sqrt{d^2 + 2d + 24} - \sqrt{d^2 + 2d}}{9d^{3/2}\sqrt{d + 2}(d+1)} \sim \frac{\sqrt{32}}{3d^4},$$

where we used that $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$ if x is close to zero.

Proof of lemma 12. We first notice that the barycentre c_b of a simplex σ^j is at least a distance min $\operatorname{alt}(\sigma^j)/(j+1)$ removed from the faces of the simplex. This means that the ball in aff (σ^j) centred on c with radius min $\operatorname{alt}(\sigma^j)/(j+1)$, denoted by $B_{\operatorname{aff}(\sigma^j)}(c, \min \operatorname{alt}(\sigma^j)/(j+1))$, is contained in σ^j . We now consider any line segment ℓ connecting a pair of antipodal points of $\partial B_{\operatorname{aff}(\sigma^j)}(c, \min \operatorname{alt}(\sigma^j)/(j+1))$. This line segment is contained in a d_{\max} neighbourhood of A_0 and thus

rag sin
$$\angle(\ell, A_0) \le \frac{(j+1) d_{\max}}{\min \operatorname{alt}(\sigma)}.$$

The result now follows, because ℓ is arbitrarily chosen.

Proof of Claim 15. Theorem 3 [24] gives us that for $d \ge 2$ and $1 \le j \le d-1$

⁷⁴⁶
$$\frac{1}{2}(j^2+j+2)j^{d-j-1}-1 \le S(d,j) \le \frac{1}{2}\binom{d}{j}j^{d-j}.$$

⁷⁴⁷ Furthermore, Stirlings theorem and the binomial theorem give that $j! \sim j^j$ and $\sum_{j=0}^k {d \choose j} \lesssim$ ⁷⁴⁸ d^k , respectively. We now see that

$$N_{\leq k} = 2 + \sum_{j=1}^{k} j! S(d+1,j) \lesssim \sum_{j=1}^{k} j! \binom{d+1}{j} j^{d+1-j} \lesssim k^{d} \sum_{j=1}^{k} \binom{d}{j} \lesssim k^{d} d^{k}$$

⁷⁵⁰ It is clear that if k is much smaller than d that then k^d dominates.

<

Proof of Lemma 16. We can bound the volume of the slab $\mathcal S$ by the cylinder with base 751 $B_{d-1}(r)$ and height $2\rho_1 r$, that is 752

753
$$2\rho_1 r^d \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}.$$

This means that 754

⁷⁵⁵
$$\frac{\operatorname{vol}(S)}{\operatorname{vol}(B^{d}(r))} < \frac{2\rho_{1}r\operatorname{vol}(B_{d-1}(r))}{\operatorname{vol}(B^{d}(r))}$$
⁷⁵⁶
$$= \frac{2\rho_{1}\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2}+1)}}{\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}}$$
⁷⁵⁷
$$= \frac{2\rho_{1}\Gamma(\frac{d}{2}+1)}{\sqrt{\pi}\Gamma(\frac{d-1}{2}+1)}$$

756

758
$$= \begin{cases} \frac{\pi(2k)!}{2^{2k-1}(k!)^2}\rho_1 & \text{if } d = 2k\\ \frac{2^{2k+1}k!(k-1)!}{(2k)!}\rho_1 & \text{if } d = 2k-1 \end{cases}$$

using the standard formulae for the volume of the ball, see for example [16, page 622]. Note 759 that the inequality is strict because $\rho_1 > 0$. We see that therefore ρ_1 may be chosen to be 760

761
$$\rho_1 = \begin{cases} \frac{2^{2k-2}(k!)^2}{\pi(2k)!N_{\leq d-n-1}} & \text{if } d = 2k\\ \frac{(2k)!}{2^{2k+2}k!(k-1)!N_{\leq d-n-1}} & \text{if } d = 2k-1. \end{cases}$$
(4)

From Wendel's bound on the ratio of Gamma functions [25], we immediately see that for a 762 fixed constant a, 763

764
$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a.$$

This means that 765

⁷⁶⁶
$$\frac{2\rho_1\Gamma\left(\frac{d}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}+1\right)} \sim \frac{2\rho\left(\frac{d}{2}+\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} \sim \sqrt{d}.$$

We now see that 767

$$\rho_1 \sim \frac{1}{\sqrt{d}N_{\leq d-n-1}}.$$

769

768

Proof of Claim 18. Because we assume that the perturbation is sufficiently small compared 770 to the protection, as given in the first condition of (14), (2) is satisfied and $\tilde{\mathcal{T}}$ will have exactly 771 the same combinatorial structure as \mathcal{T} . 772

By the third condition of (14) we have a lower bound on the quality of the simplices. To 773 be precise, we have that for any simplex τ in $\tilde{\mathcal{T}}$ 774

$$t(\tau) \ge \frac{4}{5\sqrt{d}} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2} \right) t(\mathcal{T}), \tag{13}$$

as a consequence of Lemma 11, the fact that if you perturb the vertices by $\tilde{c}L$ the edge 776 lengths are perturbed $2\tilde{c}$ (that is $2\tilde{c} = c_0$) and the fact that if $\sigma \subset \tau$, then $t(\sigma) \geq t(\tau)$. So 777 we have established (13). 778

00:24 An elementary and quantified version of Whitney's triangulation method

Proof of Claim 19. The argument is volumetric. Let us first introduce the notation U(X,r)779 for the set of all points $x \in \mathbb{R}^d$ such that $d(x, X) \leq r$, where X is any subset of \mathbb{R}^d . By 780 definition of ρ_1 , see 'Balls and exclusion volumes' in Section 5.1, and because the dimension 781 of span $(\tau'_i, T_p\mathcal{M})$ is at most d-1, we have that 782

vol
$$(B(v_i, \tilde{c}L) \cap U(\operatorname{span}(\tau'_j, T_p\mathcal{M}), \rho_1\tilde{c}L)) \le \frac{\operatorname{vol}(B^d(r))}{2N_{\le d-n-1}}.$$

It now follows that 784

vol
$$\left(B(v_i, \tilde{c}L) \setminus \left(\bigcup_{1 \le j \le \nu} U(\operatorname{span}(\tau'_j, T_p\mathcal{M}), \rho_1 \tilde{c}L) \right) \right)$$

$$\geq \operatorname{vol}(B(v_i, \tilde{c}L)) - \sum_{0 \le j \le \nu} \operatorname{vol}(B(v_i, \tilde{c}L) \cap U(\operatorname{span}(\tau'_j, T_p\mathcal{M}), \rho_1 \tilde{c}L))$$

787
$$> \operatorname{vol}(B(v_i, \tilde{c}L)) - \sum_{0 \le j \le \nu} \frac{\operatorname{vol}(B(v_i, \tilde{c}L))}{2N_{\le d-n-1}}$$
788
$$= B(v_i, \tilde{c}L) \left(1 - \frac{\nu + 1}{2N_{\le d-n-1}}\right)$$

788

$$\geq \frac{1}{2}B(v_i, \tilde{c}L),$$

where we used that $\nu \leq N_{\leq d-n-1}$ in the last line, by definition, as mentioned in the de-790 scription of Case 2. Because the volume is positive we know there exists a point \tilde{v}_i that 791 satisfies 792

$$_{793} \qquad d(\tilde{v}_i, \operatorname{span}(\tau'_j, T_p\mathcal{M})) > \rho_1 \tilde{c}L, \tag{27}$$

for all
$$1 \leq j \leq \nu$$
.

Proof of Claim 20. We first make use of the induction² hypothesis $d(\tau'_i, \mathcal{M}) > \alpha_{k-1}L$ to 796 find a bound on the distance from τ'_j to the tangent space $T_p\mathcal{M}$, then bound the distance 797 from $\tilde{v}_i * \tau'_j = \tau_j$ to $T_p \mathcal{M}$ based on this. For this argument to work, we have to assume that 798 τ'_{i} is not the empty set, that is $j \neq 0$. This case is handled separately at the end. 799

Because $d(\tau'_j, \mathcal{M}) > \alpha_{k-1}L$ and the ball in the tangent space $B_{T_p\mathcal{M}}(p, r)$ centred on p of 800 801 radius 6L = r, satisfies

$$B_{T_p\mathcal{M}}(p,r) \subset U\left(\mathcal{M}, \left(1 - \sqrt{1 - \left(\frac{r}{\operatorname{rch}(\mathcal{M})}\right)^2}\right)\operatorname{rch}(\mathcal{M})\right).$$

Thanks to Lemma 6, we have that 803

$$^{\text{804}} \qquad d(\tau'_j, B_{T_p\mathcal{M}}(p, r)) > \alpha_{k-1}L - \left(1 - \sqrt{1 - \left(\frac{r}{\operatorname{rch}(\mathcal{M})}\right)^2}\right)\operatorname{rch}(\mathcal{M}).$$

² In particular $\tau'_i \subset \tilde{\mathcal{T}}_i$. 795

This can be simplified: 805

 $d(\tau'_i, B_{T_p\mathcal{M}}(p, r))$

$$d(\tau'_{j}, B_{T_{p}\mathcal{M}}(p, r))$$

$$> \alpha_{k-1}L - \left(1 - \sqrt{1 - \left(\frac{r}{\operatorname{rch}(\mathcal{M})}\right)^{2}}\right)\operatorname{rch}(\mathcal{M})$$

$$> \alpha_{k-1}L - \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^{2}}\zeta^{2n}L \qquad (using (8))$$

808

$$> \alpha_{k-1}L - \frac{1}{3}\alpha_{d-n}L \qquad (using (10))$$

$$\geq \frac{2}{3}\alpha_{k-1}L.$$
 (because $\alpha_{k-1} > \alpha_k$)
(28)

811

810

Because $d(v_i, p) < \frac{3}{2}L$ and $\tilde{L} < L + 2\tilde{c}L$, and $\tilde{c} < \frac{1}{24}$, see (15), we have the very course 812 bound that 813

$$d(\tau'_i, p) \le 4L,$$
 (29)

by the triangle inequality. We thus find that 815

$$a_{16} \qquad d(\tau'_j, T_p\mathcal{M} \setminus B_{T_p\mathcal{M}}(p, r)) > 2L$$

This means that (28) holds for the entire tangent space, that is, 817

⁸¹⁸
$$d(\tau'_j, T_p \mathcal{M}) > \frac{2}{3} \alpha_{k-1} L.$$
 (30)

Lemma 31, with $A_1 = T_p \mathcal{M}$ and $A_2 = \operatorname{span}(\tau'_j, T_p \mathcal{M})$, now gives 819

$$d(\tau_j, T_p\mathcal{M}) \ge \frac{d(\tau'_j, T_p\mathcal{M})d(v_i, \operatorname{span}(\tau'_j, T_p\mathcal{M}))}{L + 2\tilde{c}L}$$

This can again be simplified 821

$$d(\tau_{j}, T_{p}\mathcal{M}) \geq \frac{d(\tau_{j}', T_{p}\mathcal{M})d(v_{i}, \operatorname{span}(\tau_{j}', T_{p}\mathcal{M}))}{L + 2\tilde{c}L}$$

$$\frac{(\frac{2}{3}\alpha_{k-1}L)\rho_{1}\tilde{c}L}{L + 2\tilde{c}L} \qquad (\text{thanks to (30) and (17)})$$

$$\frac{\frac{2}{3}L}{4L}\alpha_{k-1}\rho_{1}\tilde{c}L \qquad (\text{because } \tilde{c} \leq \frac{1}{24})$$

$$= \frac{4}{3}\alpha_{k}L. \qquad (\text{using the relation (5) for } \alpha_{k})$$

$$(31)$$

Similarly to (29), we have that 827

$$d(\tau_j, p) \le 4L < 6L.$$

We can go from the distance from τ_j to the tangent space, as given in (31), to the distance 829 to the manifold as follows. Because of Lemma 5 we can localize the results and Lemma 6830

00:26 An elementary and quantified version of Whitney's triangulation method

allows us to estimate the difference in distance to the manifold and the tangent space. This 831 gives 832

$$_{\text{833}} \qquad d(\tau_j, \mathcal{M}) > \frac{4}{3} \alpha_k L - \left(1 - \sqrt{1 - \left(\frac{6L}{\operatorname{rch}(\mathcal{M})}\right)^2}\right) \operatorname{rch}(\mathcal{M}).$$

This can be again simplified 834

$$d(\tau_j, \mathcal{M}) > \frac{4}{3} \alpha_k L - \left(1 - \sqrt{1 - \left(\frac{6L}{\operatorname{rch}(\mathcal{M})}\right)^2} \right) \operatorname{rch}(\mathcal{M})$$

$$> \frac{4}{3} \alpha_k L - \frac{1}{3} \alpha_{d-n} L \qquad (\text{using (8) and (10)})$$

$$\ge \alpha_k L. \qquad (\text{because } \alpha_k \ge \alpha_{d-n} \text{ if } k \le d-n-1 \text{ by (5)})$$

This completes the proof for the case where $j \neq 0$ or τ_j is non-empty. 838 For j = 0, (27) and Lemma 6 yield 839

$$_{\text{840}} \qquad d(\tau_j, \mathcal{M}) > \rho_1 \tilde{c} L - \left(1 - \sqrt{1 - \left(\frac{6L}{\operatorname{rch}(\mathcal{M})}\right)^2}\right) \operatorname{rch}(\mathcal{M})$$

We simplify 841

$$d(\tau_j, \mathcal{M}) > \rho_1 \tilde{c}L - \left(1 - \sqrt{1 - \left(\frac{6L}{\operatorname{rch}(\mathcal{M})}\right)^2}\right) \operatorname{rch}(\mathcal{M}).$$

$$> \rho_1 \tilde{c}L - \frac{1}{3}\alpha_{d-n}L \qquad (using (8) and (10))$$

$$s_{44} > \alpha_1 L.$$
 (by definition of (5))

845

The following proof differs from Whitney's proof. 846

Proof of Lemma 21. Because $\tau^k \subset B(p, 6L)$, the point in $T_p\mathcal{M}$ that is closest to τ lies in 847 $T_p\mathcal{M} \cap B(p,6L) = B_{T_p\mathcal{M}}(p,6L).$ We now see that 848

$$d(\tau^{k}, T_{p}\mathcal{M}) \geq d(\tau^{k}, B_{T_{p}\mathcal{M}}(p, 6L)) \qquad \text{(first sentence of the proof)}$$

$$\geq d(\tau^{k}, \mathcal{M}) - \left(1 - \sqrt{1 - \left(\frac{6L}{\operatorname{rch}(\mathcal{M})}\right)^{2}}\right) \operatorname{rch}(\mathcal{M}) \qquad \text{(Lemma 6)}$$

$$\sum_{k=1}^{851} > \alpha_k L - \frac{1}{3} \alpha_{d-n} L, \qquad (d(\tau^k, \mathcal{M}) > \alpha_k L \text{ and } (10))$$

$$\sum_{k=2}^{852} > \frac{2}{3} \alpha_k L, \qquad (\alpha_k \ge \alpha_{d-n} \text{ for } k \le d-n)$$

which completes the proof. 853

Proof of Lemma 22. Let $\bar{p} \in \mathcal{M} \cap \tau^k$. Lemma 3 (and (8), (10)) gives us that $\bar{p} \in \pi_p^{-1}$ 854 $(B_{T_p\mathcal{M}}(p, 6L))$, where we use the notation of Definition 4. Lemma 6 gives that 855

$$d(\bar{p}, T_p\mathcal{M}) \le \left(1 - \sqrt{1 - \left(\frac{6L}{\operatorname{rch}(\mathcal{M})}\right)^2}\right) \operatorname{rch}(\mathcal{M}) < \frac{1}{3}\alpha_{d-n}L.$$

Let $\check{\tau} \subset \tau^k$ be the face of smallest dimension such that $d(\check{\tau}, T_p\mathcal{M}) \leq \frac{2}{3}\alpha_{d-n}L$. This face exists thanks to the triangle inequality. By Lemma 21 we have the dim $(\check{\tau}) \geq d-n$. Lemma 30 implies that $\check{\tau}$ intersects $T_p\mathcal{M}$. The reason for this is the following; $\check{\tau}$ is the simplex of the smallest dimension such that $d(\check{\tau}, T_p\mathcal{M}) \leq \frac{2}{3}\alpha_k L$ meaning in particular that $d(\check{\tau}, T_p\mathcal{M}) <$ $d(\partial\check{\tau}, T_p\mathcal{M})$. Because $\check{\tau}$ is a face of τ^k , clearly $T_p\mathcal{M}$ intersects τ^k .

Proof of Lemma 23. This is an immediate consequence of Lemma 30, (14), and the previous lemmas.

Proof of Lemma 24. Let $\check{\tau}$ be a face of smallest dimension of τ^k such that $d(\check{\tau}, T_p\mathcal{M}) \leq \frac{2}{3}\alpha_n L$. Now Lemma 30 and Lemma 22 give that $\check{\tau}$ and $T_p\mathcal{M}$ have a unique point \bar{p} in common and the dimension of $\check{\tau}$ is d-n.

Thanks to Lemma 3, \mathcal{M} can be written as the graph of a function f, in a neighbourhood of at most size $\operatorname{rch}(\mathcal{M})$. We note that $f: T_p\mathcal{M} \simeq \mathbb{R}^n \to N_p\mathcal{M} \simeq \mathbb{R}^{d-n}$, where here we really think of the tangent and normal spaces as embedded in \mathbb{R}^d . Using the identification of $T_p\mathcal{M}$ with \mathbb{R}^n , we now define

$$F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^d : (\lambda, x) \mapsto (x, \lambda f(x)).$$

Note that $F(0, \cdot)$ gives a parametrization of $T_p \mathcal{M}$. Similarly, we can define $G : \mathbb{R}^{d-n} \to \mathbb{R}^d$ to be a linear (orthonormal) parametrization of $\operatorname{aff}(\check{\tau})$. We now consider the difference of the two functions $F - G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{d-n} = \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Thanks to Lemma 23 we have that

sin
$$\angle (\operatorname{aff}(\check{\tau}), T_p \mathcal{M}) \ge \frac{16}{10} \alpha_{d-n}$$

Lemma 2, and (10) give that for any $q \in B(p, 6L)$

$$\sin\left(\frac{\angle(T_p\mathcal{M},T_q\mathcal{M})}{2}\right) \le \frac{6L}{\operatorname{2rch}(\mathcal{M})} \le \frac{6}{2} \cdot \frac{(\alpha_{d-n})^2}{54} = \frac{1}{18}(\alpha_{d-n})^2.$$

It is clear that this also gives an upper bound on the angle between $T_p\mathcal{M}$ and the graph of $F(\lambda, \cdot)$ (denoted by graph $F(\lambda, \cdot)$)for all $\lambda \in [0, 1]$, due to linearity of the inner product. Because the upper bound on the angle between the tangent spaces is much smaller than the lower bound on $\angle(\operatorname{aff}(\tau^k), T_p\mathcal{M})$, aff $(\check{\tau})$ and T_q graph $F(\lambda, \cdot)$ span \mathbb{R}^d , for any $\lambda \in [0, 1]$ and $q \in B(p, 6L)$. The implicit function theorem and the fact that $\check{\tau}$ and $T_p\mathcal{M}$ have a unique point \bar{p} in common now give that the intersection \bar{p}_{λ} between graph $F(\lambda, \cdot) \cap B(p, 6L)$ and aff $(\check{\tau})$ exists and is unique, for all $\lambda \in [0, 1]$.

We can now use Lemmas 3, 6, and 23, to bound $|\bar{p} - \bar{p}_{\lambda}|$. The distance from the manifold to the tangent space (and thus the same holds for graph $F(\lambda, \cdot)$) is bounded from above by

$$\frac{(\alpha_{d-n})^{3+2n}}{3(n+1)}\zeta^{2n}L < \frac{1}{3}(\alpha_{d-n})^2L$$

due to (8), and (10), while $\sin \angle (\operatorname{aff}(\check{\tau}), T_p \mathcal{M}) \geq \frac{16}{10} \alpha_{d-n}$, so

⁸⁹⁰
$$|\bar{p} - \bar{p}_{\lambda}| \le \frac{\frac{1}{3}(\alpha_{d-n})^2 L}{\frac{16}{10}\alpha_{d-n}} \le \frac{1}{3}\alpha_{d-n}L.$$

This distance bound is smaller than the distance bound of \bar{p} to the boundary of $\check{\tau}$, due to Lemma 21. This means that $\bar{p}_{\lambda} \in \check{\tau}$, and in particular that \mathcal{M} intersects τ^k .

00:28 An elementary and quantified version of Whitney's triangulation method

Proof of Lemma 25. Take $p \in \mathcal{M} \cap \tau$. Let $\check{\tau}^k$ be a face of the smallest dimension of τ , with $v_i \in \check{\tau}^k$, that intersects $T_p\mathcal{M}$. Now assume that k > d - n. Let us write $\check{\tau}^{k-1}$ for the face of $\check{\tau}^k$ opposite v_i . Because the dimension of $\check{\tau}^k \cap T_p\mathcal{M}$ is at least 1, the intersection of $T_p\mathcal{M}$ and $\check{\tau}^{k-1}$ is non-empty.

Similarly to the first argument in the proof of Lemma 24, we see that $T_p\mathcal{M}$ intersects 897 some (d-n)-face of $\check{\tau}^{k-1}$. Thanks to Lemma 23 the angle between this (d-n)-face and 898 $T_n\mathcal{M}$ is bounded from below. Due to Lemma 21 the intersection lies in the interior of the 899 (d-n)-face. The angle bound and the fact that the intersection lies in the interior gives 900 that any simplex in \mathcal{T} that contains this (d-n)-face has points in the interior that lie in 901 $T_p\mathcal{M}$. In particular the interior of $\check{\tau}^k$ contains part of $T_p\mathcal{M}$. Because both the interior of $\check{\tau}^k$ 902 and $\check{\tau}^{k-1}$ contain points of $T_p\mathcal{M}$, linearity gives that $T_p\mathcal{M}$ must intersect $\partial\check{\tau}^k\setminus\check{\tau}^{k-1}$. From 903 this contradiction of the assumption, we conclude that k = d - n. 904

Lemma 24 finally says that \mathcal{M} intersects $\check{\tau}^k$, because $T_p \mathcal{M}$ does.

-

Proof of Lemma 26. This inequality relies on estimates on the barycentric coordinates, and Lemma 11. We first establish a bound on the barycentric coordinates of $v(\tau_i^{d-n})$ for some (d-n)-dimensional simplex $\tau_i^{d-n} \in \tilde{\mathcal{T}}$ that intersects \mathcal{M} . By Lemma 21, $v(\tau_i^{d-n})$ lies at least a distance $\frac{2}{3}\alpha_{d-n-1}L$ from the boundary $\partial \tau_i^{d-n}$ and the longest edge is at most $L + 2\tilde{c}L$. This means that all the barycentric coordinates λ_l with respect to (the vertices of) τ_i^{d-n} are at least

$${}_{912} \qquad \lambda_l(\tau_i^{d-n}) > \frac{2}{3}\alpha_{d-n-1}\frac{L}{L+2\,\tilde{c}\,L} = \frac{2}{3}\alpha_{d-n-1}\frac{1}{1+2\,\tilde{c}}.$$
(32)

Let τ^d now be a top dimensional simplex in $\tilde{\mathcal{T}}$ that intersects \mathcal{M} . Let $\tau_1^{d-n}, \ldots, \tau_j^{d-n}$ be the faces of τ^d that intersect \mathcal{M} . This means that d-n+1 barycentric coordinates with respect to τ^d of any $v(\tau_i^{d-n})$ satisfy the bound (32), while the other *n* coordinates are zero. This also means that for the barycentric coordinates with respect to τ^d of

917
$$v(\tau^k) = \frac{1}{j} \left(v(\tau_1^{d-n}) + \dots + v(\tau_j^{d-n}) \right),$$

918 for k > d - n, we have that

919 k+1 of the coordinates λ_l satisfy

$$\lambda_l > \frac{2}{3} \frac{1}{j} \frac{1}{1+2\tilde{c}} \alpha_{d-n-1}.$$

921 — The other d - k coordinates are zero.

⁹²² Note that $j \leq \binom{d}{d-n}$. This means that

$$g_{23} \qquad d(v(\tau^k), \partial \tau^k) \ge \frac{2}{3 \binom{d}{d-n} \cdot (1+2\,\tilde{c})} \alpha_{d-n-1} \min \operatorname{alt}(\tau^d).$$

924 We now have

926

$$d(v(\tau^k), \partial \tau^k) \ge \frac{2}{3\binom{d}{d-n} \cdot (1+2\tilde{c})} \alpha_{d-n-1} t(\tilde{\mathcal{T}}) \tilde{L}$$
 (by definition of the thickness)

$$\geq \frac{2}{3\binom{d}{d-n} \cdot (1+2\tilde{c})} \alpha_{d-n-1} \frac{4}{5\sqrt{d}} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right) t(\mathcal{T})\tilde{L}$$

(by the estimate (13))

$$\geq \frac{8}{15\sqrt{d} \binom{d}{d-n} \cdot (1+2\,\tilde{c})} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^2}\right) \alpha_{d-n-1} t(\mathcal{T})\tilde{L}$$

Using this estimate and the fact that σ^n defined through a sequence $\tau_0^{d-n} \subset \tau_1^{d-n+1} \subset \cdots \subset$ 928 τ_n^d we can give a lower bound on the minimal altitude of the simplex. 929

We are going to use the following easy observation on the minimal altitude simplices. 930 Suppose that: 931

- The simplex σ is the join of a point p and the simplex σ' . 932
- $= d(p, \operatorname{aff}(\sigma')) \ge d_{\min}.$ 933
- $= \min \operatorname{alt}(\sigma') \ge h'.$ 934
- The maximum edge length of σ is $L(\sigma)$. 935
- Then the min $\operatorname{alt}(\sigma) \geq \frac{h'd_{\min}}{L(\sigma)}$, as can be established by simple trigonometric arguments, as 937 illustrated in Figure 3



Figure 3 Both triangles are right angled. 936

This gives that min $\operatorname{alt}(\sigma^n)$ with σ^n as in (18) is lower bounded as follows: 939

940
$$\min \operatorname{alt}(\sigma^{n}) > \left(\frac{8}{15\sqrt{d} \begin{pmatrix} d \\ d-n \end{pmatrix} \cdot (1+2\tilde{c})} \left(1 - \frac{8\tilde{c}}{t(\mathcal{T})^{2}}\right) \alpha_{d-n-1} t(\mathcal{T})\right)^{n} \tilde{L}$$
941
$$= \zeta^{n} (\alpha_{d-n-1})^{n} \tilde{L},$$

which completes the proof. 942

Proof of Lemma 27. By Lemma 12 we have that 943

944
$$\sin \angle (\operatorname{aff}(\sigma^n), T_p\mathcal{M}) \le \frac{(n+1) d_{\max}}{\min \operatorname{alt}(\sigma^n)},$$

where d_{max} denotes the maximum distance of the vertices of σ^n to $T_p\mathcal{M}$. Lemma 26 gives 945 us the following bound 946

947
$$\min \operatorname{alt}(\sigma^n) > (\alpha_{d-n-1})^n \zeta^n L.$$

Finally, d_{max} is bounded thanks to (8). Combining these results yields 948

949
$$\sin \angle (\operatorname{aff}(\sigma^{n}), T_{p}\mathcal{M}) \leq \frac{(n+1)\frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^{2}}\zeta^{2n}L}{(\alpha_{d-n-1})^{n}\zeta^{n}\tilde{L}}$$

950 $\leq \frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^{n}.$

950

938

(because $\alpha_{d-n-1} < \alpha_{d-n}$, and $\tilde{L} \ge L$ because there are unperturbed simplices in $\tilde{\mathcal{T}}$)

951

Proof of Lemma 28. We first establish a bound on the angles between $\mathcal{N}_{\bar{p}}$ and $\mathcal{N}_{\bar{q}}$. Lemma 952 23 gives that for each τ^{d-n} 953

954
$$\sin \angle (\operatorname{aff}(\tau^{d-n}), T_p \mathcal{M}) \ge \frac{16}{10} \alpha_{d-n},$$

955 so that

956
$$\cos \angle (\pi_{\operatorname{aff}(\tau^{d-n}) \to N_p \mathcal{M}}^{-1}(e_j), e_j) \ge \frac{16}{10} \alpha_{d-n}$$

with e_i a basis vector of $N_p \mathcal{M}$ as in Section 7.1. This means that $|\pi_{\operatorname{aff}(\tau^{d-n})\to N_p \mathcal{M}}^{-1}(e_j)| \leq \frac{10}{16\alpha_{d-n}}$ and as a consequence of this and the triangle inequality we find that $|N_{v(\tau)}(e_j)|, |N_{\bar{p}'}(e_j)| \leq \frac{10}{16\alpha_{d-n}}$, for any \bar{p}' . This in turn gives us that

$$_{960} \qquad \angle (N_{\bar{p}'}(e_j), e_j) \le \arccos\left(\frac{16}{10}\alpha_{d-n}\right). \tag{33}$$

⁹⁶¹ The triangle inequality for angles (or points on the sphere) now implies that

⁹⁶²
$$\angle (N_{\bar{p}}(e_j), N_{\bar{q}}(e_j)), \angle (N_{v(\tau_i^{d-n})}(e_j), N_{v(\tau_l^{d-n})}(e_j)) \le 2 \arccos\left(\frac{16}{10}\alpha_{d-n}\right).$$

We can tighten this result for $\angle(N_{\bar{p}}(e_j), N_{\bar{q}}(e_j))$ by the use of the barycentric coordinates. We now consider the e_i component of both $\lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda_n N_{v(\tau_n^{d-n})}(e_i)$ and $\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i))$ is e_i , by construction. We are going to compare this with the length of $\lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda_n N_{v(\tau_n^{d-n})}(e_i)$ and $\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i)$ and $\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i)$. For estimates on these lengths we need to introduce the following notation: $(N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i))$ denotes the matrix whose columns are the vectors $N_{v(\tau_0^{d-n})}(e_i), \dots, N_{v(\tau_n^{d-n})}(e_i), \| \cdot \|_2$ denotes the operator 2-norm, and $\| \cdot \|_F$ the Frobenius norm. With this notation we can now derive the following bound:

971
$$|\lambda_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda_n N_{v(\tau_n^{d-n})}(e_i) - (\lambda'_0 N_{v(\tau_0^{d-n})}(e_i) + \dots + \lambda'_n N_{v(\tau_n^{d-n})}(e_i))|$$

972
$$= |(\lambda_0 - \lambda'_0)N_{v(\tau_0^{d-n})}(e_i) + \dots + (\lambda_n - \lambda'_n)N_{v(\tau_n^{d-n})}(e_i)|$$

973
$$= \left| \left(N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i) \right) \left(\begin{array}{c} \lambda_0 & \lambda_0 \\ \vdots \\ \lambda_n - \lambda'_n \end{array} \right) \right|$$

974
$$\leq \| (N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i)) \|_2 |\lambda - \lambda'|$$
975
$$\leq \| (N_{v(\tau_0^{d-n})}(e_i) \dots N_{v(\tau_n^{d-n})}(e_i)) \|_F |\lambda - \lambda'|$$

(because $\|\cdot\|_2 \leq \|\cdot\|_F$)

976

977

$$= \sqrt{|N_{v(\tau_0^{d-n})}(e_i)|^2 + \dots + |N_{v(\tau_n^{d-n})}(e_i)|^2 |\lambda - \lambda'|} \\ \le \frac{10\sqrt{n+1}}{16\,\alpha_{d-n}} |\lambda - \lambda'|.$$

We now turn our attention to the triangle, with edges $N_{\bar{p}}(e_j)$, $N_{\bar{q}}(e_j)$, and $N_{\bar{p}}(e_j) - N_{\bar{q}}(e_j)$ as depicted in Figure 4. We apply the sine rule to this triangle and find

$$\sin \angle (N_{\bar{p}}(e_j), N_{\bar{q}}(e_j)) \le \sin \phi_0 = \frac{\frac{10\sqrt{n+1}}{16\,\alpha_{d-n}} |\lambda - \lambda'| \frac{10}{16\,\alpha_{d-n}}}{1} = \left(\frac{10}{16\,\alpha_{d-n}}\right)^2 \sqrt{n+1} |\lambda - \lambda'|.$$
(34)

⁹⁸¹ Note that this can be tightened at the cost of introducing extra square roots.

We now consider the triangle $\bar{p}\bar{q}x$ and we want to estimate $|\bar{p}x|$, and $|\bar{q}x|$. This following estimate will use:

987 — the sine rule



Figure 4 The worst case for the angle between the vectors $N_{\bar{p}}(e_j)$, and $N_{\bar{q}}(e_j)$. We write ϕ_0 for an upper bound on $\angle(N_{\bar{p}}(e_j), N_{\bar{q}}(e_j))$. Moreover $\theta_0 \ge \arccos(\frac{16}{10}\alpha_{d-n})$. The length or bound on the length of two of the edges is also indicated in the figure.

- the fact that the distance between \bar{p} and \bar{q} , is at least $|\lambda \lambda'| \min \operatorname{alt}(\sigma) / \sqrt{n}$, thanks to Lemma 5.12 of [7]
- 990 Lemma 26 to bound min $\operatorname{alt}(\sigma)$
- ⁹⁹¹ Equation (34), which gives a bound on the angle $\angle \bar{p}x\bar{q}$, namely ϕ_0
- 992 Lemma 27, gives that

9

99

$$\angle(\operatorname{aff}(\sigma^{n})^{\perp}, N_{p}\mathcal{M}) \leq \operatorname{arcsin}\left(\frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^{n}\right) \leq \operatorname{arcsin}\left(\frac{(\alpha_{d-n})^{3}}{3}\right),\tag{35}$$

where $\operatorname{aff}(\sigma^n)^{\perp}$ denotes the space perpendicular to $\operatorname{aff}(\sigma^n)$. Because $e_j \in N_p \mathcal{M}$, combining this with Lemma 23 and the triangle inequality for angles yields

996
$$\angle (N_{\bar{p}'}(e_j), \operatorname{aff}(\sigma^n)^{\perp}) \leq \angle (N_{\bar{p}'}(e_j), e_j) + \angle (N_p \mathcal{M}, \operatorname{aff}(\sigma^n)^{\perp})$$
997
$$\leq \arccos\left(\frac{16}{10}\alpha_{d-n}\right) + \arcsin\left(\frac{(\alpha_{d-n})^3}{3}\right).$$
(36)

We need a lower bound on $\sin(\angle \bar{p}\bar{q}x)$ and $\sin(\angle \bar{q}\bar{p}x)$, that is

$$\sin \angle (N_{\bar{p}'}(e_j), \operatorname{aff}(\sigma^n)) = \cos \angle (N_{\bar{p}'}(e_j), \operatorname{aff}(\sigma^n)^{\perp})$$

1000 We also remind ourselves of the trigonometric identity

1001
$$\cos(\arccos(a) + \arcsin(b)) = a\sqrt{1 - b^2} - b\sqrt{1 - a^2}.$$

1002 Using (36) now gives

$$\geq \frac{16}{10}\alpha_{d-n} - \left(\frac{16}{10}\alpha_{d-n}\right)\frac{(\alpha_{d-n})^6}{3^2} - \frac{(\alpha_{d-n})^3}{3}$$

$$\geq \frac{10}{10} \alpha_{d-n} - \frac{5(\alpha_{d-n})^2}{9}$$

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(37)

00:32 An elementary and quantified version of Whitney's triangulation method

The considerations we summed up yield: 1008

1010

1011

$$\geq \frac{n+1}{n+1}$$
$$\geq \frac{\alpha_{d-n} \left(\frac{16}{10} \alpha_{d-n}\right)^2}{n+1} (\zeta \alpha_{d-n-1})^n \tilde{L}$$

Using (37) again yields that the distance from x to $\operatorname{aff}(\sigma^n)$ is bounded from below by 1012

¹⁰¹³
$$d(x, \operatorname{aff}(\sigma^n)) \ge \frac{(\alpha_{d-n})^2 \left(\frac{16}{10} \alpha_{d-n}\right)^2}{n+1} (\zeta \alpha_{d-n-1})^n \tilde{L}.$$

1014

Proof of Lemma 29. Consider $v(\tau^d) \subset K \cap \tau^d$, where we use the definition (19) and choose 1015 an arbitrary n dimensional simplex $\sigma^n \subset K \cap \tau^d$. Note that $v(\tau^d) \in K \cap \tau^d$. Thanks to 1016 Lemma 27, 1017

$$\sin \angle (\operatorname{aff}(\sigma^n), T_v \mathcal{M}) \le \frac{(\alpha_{d-n})^{4+n}}{6(n+1)} \zeta^n.$$

From this bound we conclude that 1019

$$d_H(T_v\mathcal{M} \cap B(v, 2L), \operatorname{aff}(\sigma^n) \cap B(v, 2L)) \le 2\frac{(\alpha_{d-n})^{4+n}}{6(n+1)}\zeta^n L,$$

where d_H denotes the Hausdorff distance. Because of Lemma 6 and (8), we have that 1021

¹⁰²²
$$d_H(T_v\mathcal{M}\cap B(v,2L),\pi_v^{-1}(B_{T_v\mathcal{M}}(v,2L))) \le \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2}\zeta^{2n}L,$$

where $B_{T_v\mathcal{M}}(v, 2L)$ denotes the ball in $T_v\mathcal{M}$ with radius 2L and centre v. This gives us 1023

$$\begin{array}{ll} {}_{1024} & d_H(\mathrm{aff}(\sigma^n) \cap B(v, 2L), \pi_v^{-1}(B_{T_v\mathcal{M}}(v, 2L))) \\ \\ {}_{1025} & \leq 2\frac{(\alpha_{d-n})^{4+n}}{6(n+1)} \zeta^n L + \frac{(\alpha_{d-n})^{4+2n}}{6(n+1)^2} \zeta^{2n} L & \text{(by the triangle inequality)} \\ \\ {}_{1026} & \leq \frac{(\alpha_{d-n})^{4+n} \zeta^n}{n+1} L. \end{array}$$

1026

Because $\mathcal{M} \cap \tau \subset \pi_v^{-1}(B_{T_v\mathcal{M}}(v, 2L))$ and the distance between $M \cap \tau$ and $\operatorname{aff}(\sigma^n)$ is small 1027 compared to the size of the neighbourhood of K given in Lemma 28, that is 1028

$$\frac{(\alpha_{d-n})^{4+n}\zeta^{n}}{n+1}L \le \frac{\left(\frac{16}{10}\right)^{2}(\alpha_{d-n})^{4}}{n+1}\zeta^{n}(\alpha_{d-n-1})^{n}\tilde{L},$$
(38)

 $\mathcal{M} \cap \tau$ is contained in this neighbourhood of K. 1030

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