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# Randomized incremental construction of Delaunay triangulations of nice point sets

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## 1 — Abstract —

2 *Randomized incremental construction* (RIC) is one of the most important paradigms for building  
3 geometric data structures. Clarkson and Shor developed a general theory that led to numerous  
4 algorithms that are both simple and efficient in theory and in practice.

5 Randomized incremental constructions are most of the time space and time optimal in the  
6 worst-case, as exemplified by the construction of convex hulls, Delaunay triangulations and ar-  
7 rangements of line segments. However, the worst-case scenario occurs rarely in practice and we  
8 would like to understand how RIC behaves when the input is nice in the sense that the associated  
9 output is significantly smaller than in the worst-case. For example, it is known that the Delaunay  
10 triangulations of nicely distributed points in  $\mathbb{E}^d$  or on polyhedral surfaces in  $\mathbb{E}^3$  has linear com-  
11 plexity, as opposed to a worst-case complexity of  $\Theta(n^{\lfloor d/2 \rfloor})$  in the first case and quadratic in the  
12 second. The standard analysis does not provide accurate bounds on the complexity of such cases  
13 and we aim at establishing such bounds in this paper. More precisely, we will show that, in the  
14 two cases above and variants of them, the complexity of the usual RIC is  $O(n \log n)$ , which is  
15 optimal. In other words, without any modification, RIC nicely adapts to good cases of practical  
16 value.

17 Along the way, we prove a probabilistic lemma for sampling without replacement, which may  
18 be of independent interest.

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19 **1 Introduction**

20 The *randomized incremental construction* (RIC) is an algorithmic paradigm introduced  
 21 by Clarkson and Shor [12], which has since found immense applicability in computational  
 22 geometry, e.g. [28, 27]. The general idea is to process the input points sequentially in a  
 23 random order, and to analyze the expected complexity of the resulting procedure. The  
 24 theory developed by Clarkson and Shor is quite general and led to numerous algorithms that  
 25 are simple and efficient, both in theory and in practice. On the theory side, randomized  
 26 incremental constructions are most of the time space and time optimal in the worst-case, as  
 27 exemplified by the construction of convex hulls, Delaunay triangulations and arrangements  
 28 of line segments. Randomized incremental constructions appear also to be very efficient in  
 29 practice, which, together with their simplicity, make them the most popular candidates for  
 30 implementations. Not surprisingly, the CGAL library includes several randomized incremental  
 31 algorithms, e.g. for computing Delaunay triangulations.

32 Experimental evidence has shown that randomized incremental constructions work well  
 33 beyond the worst-case, which is fortunate since worst-case situations are rare in applications.  
 34 This paper aims at extending the analysis of RIC to the case of *average-case* complexity.  
 35 More precisely, our goal is to understand how randomized incremental constructions behave  
 36 when the input is nice in the sense that the associated construction is significantly smaller  
 37 than in the worst-case.

38 We need a model of good point sets to describe the input data and analyze the algorithms.  
 39 This will be done through the notion of  $\varepsilon$ -nets, which have a long and rich history since their  
 40 introduction in the 1950's in the works of Kolmogorov and others on functional analysis and  
 41 topological vector spaces (see e.g. [32]).  $\varepsilon$ -nets have become ubiquitous in many theoretical as  
 42 well as applied areas, from geometry, functional analysis to probability theory and statistics,  
 43 where they are often used as countable or finite approximations of continuous spaces.

44 When we enforce such a hypothesis of “nice” distribution of the points in space, a volume  
 45 counting argument ensures that the local complexity of the Delaunay triangulation around a  
 46 vertex is bounded by a constant (dependent only on the dimension).  
 47

48 Unfortunately, to be able to control the complexity of the usual randomized incremental  
 49 algorithms [15, 10, 12, 3], it is not enough to control the final complexity of the Delaunay  
 50 triangulation. We need to control also the complexity of the triangulation of random subsets.  
 51 One might expect that a random subsample of size  $k$  of an  $\varepsilon$ -net is also an  $\varepsilon'$ -net for  $\varepsilon' = \varepsilon \sqrt{\frac{n}{k}}$ .  
 52 Actually this is not quite true, it may happen with reasonable probability that a ball of radius  
 53  $O(\varepsilon')$  contains  $\Omega(\log k / \log \log k)$  points or that a ball of radius  $\Omega(\varepsilon' \sqrt[4]{\log k})$  does not contain  
 54 any point. For the convenience of the reader, we briefly sketch the proofs in the Appendix  
 55 (Lemma 45). However, it can only be shown that such a subsample is an  $\left(\frac{\varepsilon'}{\log(1/\varepsilon)}\right)$ -covering  
 56 and an  $(\varepsilon' \log(1/\varepsilon))$ -packing, with high probability. Thus this approach can transfer the  
 57 complexity of an  $\varepsilon$ -net to the one of a random subsample of an  $\varepsilon$ -net but with an extra  
 58 multiplicative factor of  $\Omega(\log 1/\varepsilon) = \Omega(\log n)$ . It follows that, in the two cases we consider,  
 59 the standard analysis does not provide accurate bounds on the complexity of the (standard)  
 60 randomized incremental construction. Our results are based on proving that, *in expectation*,  
 61 the above bad scenarios occur rarely, and the algorithm achieves optimal run-time complexity.  
 62

63 **Related Work:** The Delaunay triangulations of nicely-distributed points have been stud-

64 ied since the 50's, e.g. in the work of Meijering [23], Gilbert [21], Miles [24] Møller [26],  
 65 Golin-Na [22], for Poisson-distributed points, Dwyer [17, 16] for uniformly distributed points,  
 66 Attali-Boissonnat [4], Attali-Boissonnat-Lieutier [5], Amenta-Attali-Devillers [2], and others  
 67 for  $(\varepsilon, \kappa)$ -samples, and Erickson [18, 19] for points with bounded spread (the ratio between  
 68 the maximum to minimum distance between any two points). Except for a few authors such  
 69 as Dwyer [17] and Erickson [19], most of the above results discuss only the combinatorial  
 70 aspects and not the algorithmic ones. For Poisson and uniformly distributed point samples,  
 71 we observe that the standard analysis of the RIC procedure immediately implies a bound on  
 72 the expected run-time, of the order of the expected number of simplices times a logarithmic  
 73 factor, which is optimal. However, for deterministic notions of nice distributions such as  $\varepsilon$   
 74 nets,  $(\varepsilon, \kappa)$  samples, and bounded spread point sets, the standard RIC analysis is not optimal,  
 75 since, as we observed, it gives at least an extra logarithmic factor for  $(\varepsilon, \kappa)$  samples and even  
 76 worse for bounded spread point-sets, as stated in an open problem by Erickson [19]. Miller,  
 77 Sheehy and Velingker [25] follow a very different approach, giving an algorithm to compute  
 78 the approximate Delaunay graph of a nicely-spaced superset of points for an arbitrary input  
 79 point-set, with optimal time complexity and a  $2^{O(d)}$ -dependence on the dimension. However  
 80 their algorithm is quite complicated and uses several subroutines that have varying difficulties  
 81 of implementation. The RIC, while having a worse  $2^{O(d^2)}$  dependence on the dimension  
 82 (which Miller et al. observe, may be impossible to avoid for computing the exact Delaunay  
 83 graph), computes the entire Delaunay triangulation of the given point set rather than a  
 84 superset, is easy to implement and works efficiently in practice.

85

86 **Our contribution:** We consider two main questions in this paper. First, we consider the  
 87 case of an  $\varepsilon$ -net in the periodic space of dimension  $d$ , which, as mentioned before, have linear  
 88 complexity instead of the worst-case  $\Theta(n^{\lfloor d/2 \rfloor})$ . The reason to consider a periodic space is to  
 89 avoid dealing with boundary effects that would distract us from the main point, and the fact  
 90 that periodic spaces are often used in practice, e.g. in simulations in astronomy, biomedical  
 91 computing, solid-state chemistry, condensed matter physics, etc. [11, 13, 29, 20, 33]. Following  
 92 this, we deal with  $\varepsilon$ -nets on a polyhedral surface of  $\mathbb{E}^3$ , which is also a commonly-occurring  
 93 practical scenario in e.g. surface reconstruction [1, 8], and has Delaunay triangulations with  
 94 linear, as opposed to the worst-case quadratic, complexity. In this case, the boundary effects  
 95 need to be explicitly controlled, which requires a more careful handling along with some new  
 96 ideas. In both cases, we establish tight bounds and show that the complexity of the usual  
 97 RIC is  $O(n \log n)$ , which is optimal. Hence, without any modification, the standard RIC  
 98 nicely adapts to the good cases above.

99 Our technical developments rely on a general bound for the probability of certain non-  
 100 monotone events in sampling without replacement, which may be of independent interest.

101

102 **Extensions** We also give some extensions of our results for periodic spaces. Our extensions  
 103 are in four directions: (i) a more general notion of well-distributed point sets, the  $(\varepsilon, \kappa)$   
 104 samples (ii) a different notion of subsampling - the Bernoulli or i.i.d. sample where each  
 105 point is selected to be in  $\mathcal{Y}$  independently of the others, with probability  $q = s/n$ , (iii) a  
 106 more general class of spaces - Euclidean  $d$ -orbifolds, and (iv) a more general class of metrics  
 107 - those having bounded-distortion with respect to the Euclidean metric. Precisely, for all  
 108 the above cases, we show that the Delaunay triangulation of a random subsample has a  
 109 linear size in expectation. We believe that our methods should work for an even larger class  
 110 of spaces, though this might require more delicate handling of boundary effects and other  
 111 features specific to the metric space under consideration.

112

113 **Outline** The rest of the paper is as follows. In Section 2, we define the basic concepts  
 114 of Delaunay triangulation,  $\varepsilon$ -net, flat torus and random samples. We state our results in  
 115 Section 3. In Section 4, we bound the size of the Delaunay triangulation of a uniform random  
 116 sample of a given size extracted from an  $\varepsilon$ -net on the flat torus  $\mathbb{T}^d$ . In Section 5, we analyse  
 117 the case when the uniform random subsample is drawn from an  $\varepsilon$ -net on a polyhedral surface  
 118 in  $\mathbb{E}^3$ . In Section 6, we use the size bounds established in Sections 4 and 5, to compute  
 119 the space and time complexity of the randomized incremental construction for constructing  
 120 Delaunay triangulations of  $\varepsilon$ -nets. Finally, in Section 7, we state and prove some extensions.  
 121 Proofs missing from the main sections are given in the Appendix.

## 122 **2 Background**

### 123 **2.1 Notations**

124 We denote by  $\Sigma(p, r)$ ,  $B(p, r)$  and  $B[p, r]$ , the sphere, the open ball, and the closed ball of  
 125 center  $p$  and radius  $r$  respectively. For  $x \in \mathbb{E}^2$ ,  $y \geq 0$ ,  $D(x, r)$  denotes the disk with center  
 126  $x$  and radius  $r$ , i.e. the set of points  $\{y \in \mathbb{E}^2 : \|y - x\| < r\}$ , and similarly  $D[x, r]$  denotes  
 127 the corresponding closed disk. The volume of the unit Euclidean ball of dimension  $d$  is  
 128 denoted  $V_d$  and the area of the boundary of such a ball is denoted  $S_{d-1}$ . It is known that  
 129  $V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$  and  $S_d = 2\pi V_{d-1}$ , where  $\Gamma(t) := \int_0^\infty e^{-x} x^{t-1} dx$ , ( $t > 0$ ) denotes the *Gamma*  
 130 *function*. For  $d \in \mathbb{Z}^+$ ,  $\Gamma(d+1) = d!$ . We note that  $2^d d^{-d/2} \leq V_d \leq 2^{4d} d^{-d/2}$  (see e.g. [34]).

131

132 For an event  $\mathcal{E}$  in some probability space  $\Omega$ , we use  $\mathbf{1}_{[\mathcal{E}]}$  to denote the indicator variable  
 133  $\mathbf{1}_{[\mathcal{E}]} = \mathbf{1}_{[\mathcal{E}]}(\omega)$  which is 1 whenever  $\omega \in \mathcal{E}$ , and zero otherwise. We use  $[n]$  to mean the  
 134 set  $\{1, 2, \dots, n\}$ . Given a discrete set  $A$ ,  $\#(A)$  denotes its cardinality and, for  $k \in \mathbb{Z}^+$ ,  $\binom{A}{k}$   
 135 denotes the set of  $k$ -sized subsets of  $A$ . Given an event  $A$  in some probability space,  $\mathbb{P}[A]$   
 136 denotes the probability of  $A$  occurring. For a random variable  $Z$  in a probability space,  $\mathbb{E}[Z]$   
 137 denotes the expected value of  $Z$ . Lastly,  $e = 2.7182\dots$  denotes the base of the natural  
 138 logarithm.

### 139 **2.2 $\varepsilon$ -nets**

140 A set  $\mathcal{X}$  of  $n$  points in a metric space  $\mathcal{M}$ , is an  $\varepsilon$ -packing if any pair of points in  $\mathcal{X}$  are at  
 141 least distance  $\varepsilon$  apart, and an  $\varepsilon$ -cover if each point in  $\mathcal{M}$  is at distance at most  $\varepsilon$  from some  
 142 point of  $\mathcal{X}$ .  $\mathcal{X}$  is an  $\varepsilon$ -net if it is an  $\varepsilon$ -cover and an  $\varepsilon$ -packing simultaneously.

143 The definition of an  $\varepsilon$ -net applies for any metric space. In the case of the Euclidean  
 144 metric, we can prove some additional properties. We shall use  $\|\cdot\|$  to denote the Euclidean  $\ell_2$   
 145 norm. The following lemmas are folklore.

146 **► Lemma 1** (Maximum packing size). *Any packing of the ball of radius  $r \geq \rho$  in dimension  $d$   
 147 by disjoint balls of radius  $\rho/2$  has a number of balls smaller than  $\left(\frac{3r}{\rho}\right)^d$ .*

148 **Proof.** Consider a maximal set of disjoint balls of radius  $\frac{\rho}{2}$  with center inside the ball  $B(r)$   
 149 of radius  $r$ . Then the balls with the same centers and radius  $\rho$  cover the ball  $B(r)$  (otherwise  
 150 it contradicts the maximality). By a volume argument we get that the number of balls is  
 151 bounded from above by  $\frac{V_d \times \left(r + \frac{\rho}{2}\right)^d}{V_d \times \left(\frac{\rho}{2}\right)^d} \leq \left(\frac{3r}{\rho}\right)^d$ . ◀

152 ► **Lemma 2** (Minimum cover size). *Any covering of a ball of radius  $r$  in dimension  $d$  by balls*  
 153 *of radius  $\rho$  has a number of balls greater than  $\left(\frac{r}{\rho}\right)^d$ .*

154 **Proof.** The volume argument gives a lower bound of  $\frac{V_d \times r^d}{V_d \times \rho^d} = \left(\frac{r}{\rho}\right)^d$ . ◀

155 For  $d \in \mathbb{Z}^+$ , the *flat  $d$ -torus*  $\mathbb{T}^d$  is the compact quotient group  $\mathbb{E}^d/\mathbb{Z}^d$ , with addition as  
 156 the group action. More generally, for  $k \in \mathbb{Z}^+$ , the flat torus of length  $k$  is  $\mathbb{T}_k^d := \mathbb{E}^d/(k\mathbb{Z})^d$ .

157 ► **Lemma 3** ( $\varepsilon$ -net size bounds). *Given  $\varepsilon \in (0, 1/2]$ , let  $\mathcal{X}$  be an  $\varepsilon$ -net over the flat torus  $\mathbb{T}^d$ .*  
 158 *Then,  $\#\mathcal{X} \in [d^{d/2}2^{-4d} \cdot \varepsilon^{-d}, d^{d/2}\varepsilon^{-d}]$ .*

159 **Proof.** Observe that, by the minimum distance property of the points in  $\mathcal{X}$ , the balls of radius  
 160  $\varepsilon/2$  centered around each point in  $\mathcal{X}$  are disjoint, and by a volume argument there can be at  
 161 most  $\frac{1}{V_d \times (\varepsilon/2)^d} \leq 2^{-d}d^{d/2}(\varepsilon/2)^{-d} = d^{d/2}\varepsilon^{-d}$  such balls in  $\mathbb{T}^d$ . The balls of radius  $\varepsilon$  centered  
 162 around each point in  $\mathcal{X}$  cover the space thus their number is at least  $\frac{1}{V_d \times \varepsilon^d} \geq d^{d/2}2^{-4d} \cdot \varepsilon^{-d}$ .  
 163 This completes the proof of the lemma. ◀

## 164 2.3 Delaunay triangulation

165 For simplicity of exposition and no real loss of generality, all finite point sets considered  
 166 in this paper will be assumed to be in *general position*, i.e. no set of  $d + 2$  points lie on a  
 167 sphere. Given a set  $\mathcal{X}$  in some ambient topological space, the *Delaunay complex* of  $\mathcal{X}$  is the  
 168 (abstract) simplicial complex with vertex set  $\mathcal{X}$  which is the nerve of the Voronoi diagram of  
 169  $\mathcal{X}$ , that is, a simplex  $\sigma$  (of arbitrary dimension) belongs to  $Del(\mathcal{X})$  iff the Voronoi cells of its  
 170 vertices have a non empty common intersection. Equivalently,  $\sigma$  can be circumscribed by an  
 171 empty ball, i.e. a ball whose bounding sphere contains the vertices of  $\sigma$  and whose interior  
 172 contains no points of  $\mathcal{X}$ .

173 The Delaunay complex is a triangulation if it triangulates the ambient space, or more  
 174 precisely, the Delaunay complex  $Del(\mathcal{X})$  of a point set  $\mathcal{X}$  over an ambient space  $\mathcal{M}$ , is said  
 175 to be a *Delaunay triangulation* of  $\mathcal{M}$  if there exists a homeomorphism between  $Del(\mathcal{X})$  and  
 176  $\mathcal{M}$ . Given a set  $\mathcal{X}$  in some ambient space  $\mathcal{M}$ , with its Delaunay complex  $Del(\mathcal{X})$ , the *star*  
 177 of a subset  $S \in \mathcal{X}$ , or *star*( $S$ ), is the set of all simplices in  $Del(\mathcal{X})$  which are incident to at  
 178 least one point in  $S$ . For a point  $p \in \mathcal{X}$ , we shall use the shorthand expression *star*( $p$ ) to  
 179 mean *star*( $\{p\}$ ). Given topological spaces  $\mathcal{S}$  and  $\mathcal{C}$ , and a continuous map  $\pi : \mathcal{C} \rightarrow \mathcal{S}$ ,  $\mathcal{C}$  is a  
 180 *covering space* of  $\mathcal{S}$  if  $\pi$  is such that for every point  $x \in \mathcal{S}$ , there exists an open neighbourhood  
 181  $U$  of  $x$ , such that the pre-image  $\pi^{-1}(U)$  is a disjoint union of open neighbourhoods in  $\mathcal{C}$ ,  
 182 each of which is homeomorphically mapped onto  $U$  by  $\pi$ . A covering  $\mathcal{C}$  of  $\mathcal{S}$  is  *$m$ -fold* or  
 183  *$m$ -sheeted* if the cardinality of the pre-image of each point  $x \in \mathcal{S}$  under the covering map is  
 184  $m$ .

185 For example,  $\mathbb{T}_k^d$  forms a  $k^d$ -sheeted *covering space* of  $\mathbb{T}^d$ , with the covering map  $x \mapsto x$   
 186 mod 1, the modulus operation being defined coordinate-wise. Caroli and Teillaud [11] showed

187 ► **Theorem 4** (Caroli-Teillaud [11]). *The Delaunay complex of any finite point set in  $\mathbb{T}^d$*   
 188 *having at least 1 point, embeds in the  $3^d$ -sheeted covering of  $\mathbb{T}^d$ . If the maximum circumradius*  
 189 *of a simplex is at most  $1/2$ , then the complex embeds in  $\mathbb{T}^d$  itself.*

190 Note that the above theorem implies that the Delaunay triangulation of any finite point  
 191 set in  $\mathbb{T}^d$  *always exists* in the  $3^d$ -sheeted covering of  $\mathbb{T}^d$ .

192

193 A key property of  $\varepsilon$ -nets is that their Delaunay triangulations have linear size.

194 ► **Lemma 5** (Talmor [30]). *Let  $\varepsilon \in (0, 1/2]$  be given, and let  $\mathcal{X}$  be an  $\varepsilon$ -net over  $\mathbb{T}^d$ . Then*  
 195 *the Delaunay triangulation of  $\mathcal{X}$ ,  $Del(\mathcal{X})$  has at most  $4^{d^2} \varepsilon^{-d}$  simplices.*

196 **Proof.** Observe that the circumradius of any simplex in  $Del(\mathcal{X})$  cannot be greater than  
 197  $\varepsilon$ , since this would imply the existence of a ball in  $\mathbb{T}^d$  of radius at least  $\varepsilon$ , containing  
 198 no points from  $\mathcal{X}$ . Therefore given a point  $p \in \mathcal{X}$ , any point which lies in a Delaunay  
 199 simplex incident to  $p$ , must be at most distance  $2\varepsilon$  from  $p$ . Again by a volume argument,  
 200 the number of such points is at most  $\frac{V_d \times (2\varepsilon + \varepsilon/2)^d}{V_d \times ((\varepsilon/2)^d)} = 5^d$ . Thus, the number of Delaunay  
 201 simplices of dimension at most  $d$  that contain  $p$ , is at most the complexity of the Delaunay  
 202 triangulation in  $\mathbb{T}^d$  on  $5^d$  vertices. This is at most  $(5^d)^{\lceil d/2 \rceil}$ . Thus we can conclude that  
 203 the number of simplices in  $Del(\mathcal{X})$  is at most the cardinality of  $\mathcal{X}$ , times the maximum  
 204 number of simplices incident to any given point  $p \in \mathcal{X}$ . Now using Lemma 3, we get that  
 205  $(5^d)^{\lceil d/2 \rceil} \cdot \#\mathcal{X} \leq (5^d)^{\frac{d+1}{2}} d^{d/2} \varepsilon^{-d} \leq 4^{d^2} \varepsilon^{-d}$ . ◀

206 **2.4 Randomized incremental construction and random subsamples**

207 For the algorithmic complexity aspects, we state a version of a standard theorem for the  
 208 RIC procedure, (see e.g. [14]). We first need a necessary condition for the theorem. When a  
 209 new point  $p$  is added to an existing triangulation, a *conflict* is defined to be a previously  
 210 existing simplex whose circumball contains  $p$ .

211 ► **Condition 6.** *At each step of the RIC, the set of simplices in conflict can be removed and*  
 212 *the set of newly introduced conflicts computed in time proportional to the number of conflicts.*

213 We now come to the general theorem on the algorithmic complexity of RIC using the  
 214 Clarkson-Shor technique (see e.g. Devillers [14] Theorem 5(1,2)).

215 ► **Theorem 7.** *Let  $F(s)$  denote the expected number of simplices that appear in the Delaunay*  
 216 *triangulation of a uniform random sample of size  $s$ , from a given point set  $P$ . Then, if*  
 217 *Condition 6 holds and  $F(s) = O(s)$ , we have*

- 218 (i) *The expected space complexity of computing the Delaunay triangulation is  $O(n)$ .*  
 219 (ii) *The expected time complexity of computing the Delaunay triangulation is  $\sum_{s=1}^n \frac{n-s}{s} =$*   
 220  *$O(n \log n)$ .*

221 A subset  $\mathcal{Y}$  of set  $\mathcal{X}$  is a *uniform random sample* of  $\mathcal{X}$  of size  $s$  if  $\mathcal{Y}$  is any possible subset  
 222 of  $\mathcal{X}$  of size  $s$  with equal probability. In case the multiplicity of a point in  $\mathcal{X}$  is greater than  
 223 1, the sample counts only one copy of the point; all other copies are present in  $\mathcal{Y}$  if and only  
 224 if the original point is present.

225 In order to work with uniform random samples, we shall prove a lemma on the uniformly  
 226 random sampling distribution or *sampling without replacement*, which is stated below, and  
 227 will be a key probabilistic component of our proofs. The lemma provides a bound on the  
 228 probability of a non-monotone compound event, that is, if the event holds true for a fixed  
 229 set of  $k$  points, there could exist supersets as well as subsets of the chosen set for which the  
 230 event does not hold. This may well be of general interest, as most natural contiguity results  
 231 with Bernoulli (i.e. independent) sampling, are for monotone events.

232 ► **Lemma 8.** *Given  $a, b, c \in \mathbb{Z}^+$ , with  $2b \leq a \leq c$ ,  $t \leq c$ . Let  $C$  be a set, and  $B$  and  $T$  two*  
 233 *disjoint subsets of  $C$ . If  $A$  is a random subset of  $C$ , chosen uniformly from all subsets*  
 234 *of  $C$  having size  $a$ , the probability that  $A$  contains  $B$  and is disjoint from  $T$ , is at most*  
 235  $\left(\frac{a}{c}\right)^b \left(1 - \frac{t}{c-b}\right)^{a-b} \leq \left(\frac{a}{c}\right)^b \cdot \exp\left(-\frac{at}{2c}\right)$ , *where  $a, b, c$  are the cardinalities of  $A, B$ , and  $C$*   
 236 *respectively, and the cardinality of  $T$  is at least  $t$ .*

237 **Proof.** The total number of ways of choosing the random sample  $A$  is  $\binom{c}{a}$ . The number  
 238 of ways of choosing  $A$  such that  $B \subset A$  and  $T \cap A = \emptyset$ , is  $\binom{c-b-t}{a-b}$ . Therefore the required  
 239 probability is

$$\begin{aligned}
 240 \quad \mathbb{P}[B \subset A, T \cap A = \emptyset] &= \frac{\binom{c-b-t}{a-b}}{\binom{c}{a}} \\
 241 &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=b}^{a-1} (a-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=b}^{a-1} (c-i)} \cdot \frac{\prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{a-b-1} (a-b-i)} \\
 242 &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=0}^{a-b-1} (c-b-i)} \\
 243 &= (a/c)^b \frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)} \left(1 - \frac{t}{c-b}\right)^{a-b} \left(\frac{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b-t})}{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b})}\right) \\
 244 &\leq (a/c)^b \left(1 - \frac{t}{c-b}\right)^{a-b},
 \end{aligned}$$

245 where in the last step, observe that for the product  $\frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)}$  for each  $i$ , the term  $(1-i/a)$   
 246 in the numerator is smaller than the corresponding term  $(1-i/c)$  in the denominator, since  
 247  $a \leq c$ . A similar observation holds for the product  $\left(\frac{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b-t})}{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b})}\right)$ .

248 Now, observe that  $\left(1 - \frac{t}{c-b}\right)^{a-b} \leq \exp\left(-\left(\frac{t(a-b)}{c-b}\right)\right) \leq \exp\left(-\left(\frac{at}{2c}\right)\right)$ , if  $b \leq a/2$  and  
 249  $b < c$ . ◀

### 250 **3 Results**

251 **Random samples of  $\varepsilon$ -nets in  $\mathbb{T}^d$ :** The following theorem gives a constant bound on the  
 252 expected size of  $star(p)$  for the Euclidean metric on the flat torus  $\mathbb{T}^d$ .

253 **► Theorem 9 (Euclidean metric).** *Given an  $\varepsilon$ -net  $\mathcal{X}$  in  $\mathbb{T}^d$  in general position, where  $\varepsilon \in (0, \frac{1}{4}]$ ,  
 254 the expected number of simplices incident to a point  $p \in \mathcal{X}$ ,  $\mathbb{E}[\sharp(star(p))]$  in the Delaunay  
 255 triangulation of a uniform random sample  $S \subset \mathcal{X}$  of size  $s \geq 4(2\sqrt{d})^d d^3 + 1$  containing  $p$ , is  
 256 less than  $2 \cdot 6^{d^2+d}$ .*

257 **Polyhedral Surfaces in  $\mathbb{E}^3$ :** A polyhedral surface  $\mathcal{S}$  in  $\mathbb{E}^3$  is a collection of a finite number  
 258 of polygons  $F \subset \mathcal{S}$ , called *facets*, which are pairwise disjoint or meet along an edge. In this  
 259 paper,  $\mathcal{S}$  will denote an arbitrary but fixed polyhedral surface, with  $\mathcal{C}$  facets, and having  
 260 total length of the boundaries of its faces  $L$  and total area of its faces  $A$ .

261 We show that the expected complexity of the Delaunay triangulation of a uniformly  
 262 random subsample of an  $\varepsilon$ -net on a polyhedral surface is linear in the size of the subsample:

263 **► Theorem 10.** *Let  $\varepsilon \in [0, 1]$ ,  $\mathcal{X}$  be an  $\varepsilon$ -net on a polyhedral surface  $\mathcal{S}$ , having  $n$  points and  
 264 let  $\mathcal{Y} \subset \mathcal{X}$  be a random sub-sample of  $\mathcal{X}$  having size  $s$ . Then the Delaunay triangulation  
 265  $Del(\mathcal{Y})$  of  $\mathcal{Y}$  on  $\mathcal{S}$  has  $O(s)$  simplices.*

266 **Algorithmic Bounds:** We next use the above combinatorial bounds to get the space and  
 267 time complexity of the randomized incremental construction of the Delaunay triangulation  
 268 of an  $\varepsilon$ -net on the flat  $d$ -torus or on a polyhedral surface in  $\mathbb{E}^3$ .

269 **► Theorem 11 (Randomized incremental construction).** *Let  $\varepsilon \in [0, 1/4]$ , and let  $\mathcal{X}$  be an  
 270  $\varepsilon$ -net in general position over (i) the flat  $d$ -dimensional torus  $\mathbb{T}^d$ , or (ii) a fixed polyhedral*



271 surface  $\mathcal{S} \subset \mathbb{E}^3$ , then the randomized incremental construction of the Delaunay triangulation  
 272 takes  $O(n \log n)$  expected time and  $O(n)$  expected space, where  $n = \sharp(\mathcal{X})$  and the constant in  
 273 the big  $O$  depends only on  $d$ , and not on  $n$  or  $\varepsilon$ . Further, at each step of the randomized  
 274 incremental construction, the Delaunay complex of the set  $\mathcal{Y}$  of already added points of  $\mathcal{X}$  is  
 275 a triangulation of the space.

276 **Extensions:** Finally, our extensions are stated and proved in Section 7.

## 277 4 Euclidean Metric on $\mathbb{T}^d$

278 In this section, we prove that a subsample  $\mathcal{Y}$  of a given size  $s$ , drawn randomly from an  $\varepsilon$ -net  
 279  $\mathcal{X} \subset \mathbb{T}^d$ , has a Delaunay triangulation in which the star of any given vertex has a constant  
 280 expected complexity. Hence, the expected complexity of the triangulation is linear in the size  
 281 of the subsample. The constant of proportionality is bounded by  $2^{cd^2}$ , where  $c$  is a constant  
 282 independent of  $\varepsilon$  and  $d$ .

### 283 Existence of Delaunay triangulation $Del(\mathcal{Y})$

284 In order to ensure we always have the Delaunay complex embedded in  $\mathbb{T}^d$ , we shall use  
 285 Theorem 4. Accordingly, we get two different regimes of the potential simplices in the  
 286 triangulation. When the circumradius of a potential simplex  $\sigma \in star(p)$  is at most  $1/4$ ,  
 287 then the simplex lies in a ball of radius at most  $1/2$  with center  $p$ . By Theorem 4, in this  
 288 regime the Delaunay complex  $star(p)$  embeds in the one-sheeted covering of  $\mathbb{T}^d$ . Therefore,  
 289 for a fixed set of vertices, there is a unique circumball. When the circumradius is greater  
 290 than  $1/4$ , the simplex is contained in a ball of radius  $> 1/2$  around  $p$ , and therefore  $star(p)$   
 291 embeds in the  $3^d$ -sheeted covering of  $\mathbb{T}^d$ , i.e.  $\mathbb{T}_3^d$ . In this case, each vertex has  $3^d$  copies, and  
 292 so for a given choice of  $d$  vertices together with  $p$ , one can have  $(3^d)^d$  circumballs.

### 293 Proof Framework

294 Now we set up the formal proof. Recall that  $n := \sharp(\mathcal{X})$ . Define  $q := \frac{s-1}{n-1}$ . Define  
 295  $\delta := \varepsilon \cdot \left(\frac{2d}{q}\right)^{1/d}$ . Let  $I_0 := [0, \delta)$ ,  $I_k := [2^{k-1}\delta, 2^k\delta)$  for  $k > 0$ . To bound the expected  
 296 complexity, we shall consider the probability of existence of potential  $d$ -simplices in  $\mathcal{X}$ ,  
 297 incident to  $p$  and having radius in the intervals  $I_k$ , as  $k$  ranges over  $\mathbb{Z}^+$ .

298 Throughout this section, we shall use  $\sigma$  to mean a  $d$ -simplex incident to  $p$ , with cir-  
 299 cumcentre  $c_\sigma$  and circumradius  $r_\sigma$ , and  $\tau$  to mean the set of vertices of  $\sigma \setminus \{p\}$ . To count  
 300 the number of simplices in  $star(p)$  with circumradius in  $I_k$ , let  $S_p(k)$  denote the set of  
 301 possible  $(d-1)$ -simplices with vertices in  $\mathcal{X}$ , such that for every  $\tau \in S_p(k)$ , the  $d$ -simplex  
 302  $\sigma := \tau \cup \{p\}$  has circumradius  $r_\sigma \in I_k$ . Set  $s_p(k) := \sharp(S_p(k))$ . Let  $n_k$  denote the min-  
 303 imum number of points of  $\mathcal{X}$  in the interior of the circumball of  $\sigma$ , over all  $\tau \in S_p(k)$ :  
 304  $n_k := \min_{\tau \in S_p(k)} \{\sharp(B(c_\sigma, r_\sigma) \cap \mathcal{X}) : \sigma = \tau \cup \{p\}\}$ . For  $\tau \in S_p(k)$ , let  $P_p(k)$  denote an  
 305 upper bound on the probability that  $\sigma = \tau \cup \{p\}$  appears in  $Del(\mathcal{Y})$ , that is,

$$306 \quad P_p(k) := \max_{\tau \in S_p(k)} \{\mathbb{P}[\sigma \in Del(\mathcal{Y})]\}.$$

307 Finally, let  $Z_p(k)$  denote the number of simplices  $\tau \in S_p(k)$  such that  $\sigma \in Del(\mathcal{Y})$ . The main  
 308 lemma in the proof is a bound on the expected complexity of the star of  $p$ , in terms of  $s_p(k)$   
 309 and  $P_p(k)$ .

310 **► Lemma 12.**  $\mathbb{E}[\sharp(star(p))] \leq \sum_{k \geq 0} \mathbb{E}[Z_p(k)] \leq \sum_{k \geq 0} s_p(k) \cdot P_p(k)$ .

311 **Proof.** For a simplex  $\sigma = \tau \cup \{p\}$ , with the vertex set of  $\tau$  in  $S_p(k)$ , let  $\mathbf{1}_{[\tau]}$  be the indicator  
 312 random variable which is 1 if  $\sigma \in \text{star}(p)$ , and zero otherwise. Then  $Z_p(k) := \sum_{\tau \in S_p(k)} \mathbf{1}_{[\tau]}$ ,  
 313 and  $\sharp(\text{star}(p)) = \sum_{k \geq 0} Z_p(k)$ .

314 Taking expectations over the random sample  $\mathcal{Y}$ , we get

$$\begin{aligned} 315 \quad \mathbb{E}[\sharp(\text{star}(p))] &= \sum_{k \geq 0} \mathbb{E}[Z_p(k)] \\ 316 \quad &= \sum_{k \geq 0} \sum_{\tau \in S_p(k)} \mathbb{E}[\mathbf{1}_{[\tau]}] \leq \sum_{k \geq 0} \sum_{\tau \in S_p(k)} P_d(k) = \sum_{k \geq 0} s_p(k) \cdot P_d(k). \end{aligned}$$

317

318 It only remains, therefore, to establish bounds on  $s_p(k)$  and  $P_p(k)$  as functions of  $k$ , and  
 319 finally to bound the sum  $\sum_{k \geq 0} s_p(k) \cdot P_p(k)$ .

320 Following the earlier discussion on the existence of the Delaunay triangulation  $\text{Del}(\mathcal{Y})$ ,  
 321 we shall split the sum  $\sum_{k \geq 0} \mathbb{E}[Z_p(k)]$  into the two regimes,  $0 \leq k \leq k_{\max}$ , and  $k > k_{\max}$ ,  
 322 where  $k_{\max}$  denotes  $\log_2 \frac{1}{4\delta}$ .

323 **Case I: Simplices with small circumradii**  $k \in [0, k_{\max}]$

324 In this regime, the circumradii  $r_\sigma$  of the potential simplices, are at most  $1/4$ , since recall that  
 325 by the definition of  $k_{\max}$ , we have  $r_\sigma \leq 2^{k_{\max}} \delta = 1/4$ . Therefore every set of  $d + 1$  vertices in  
 326  $\mathcal{Y}$ , has a unique circumball. We begin by establishing a bound on  $P_p(k)$ . First, we bound  $n_k$   
 327 from below using Lemma 2.

328 **► Lemma 13.** *Let  $\sigma$  be a simplex incident to  $p$ , having circumradius  $r_\sigma \in I_k$ ,  $k \geq 0$ .*

$$329 \quad n_k \geq \begin{cases} 0 & k = 0. \\ \left(\frac{2^{k-1}\delta}{\varepsilon}\right)^d & 0 < k \leq k_{\max}. \end{cases}$$

330 **Proof.** When  $k \leq k_{\max}$ , the radius of the circumball of a simplex  $\sigma = \tau \cup \{p\}$ ,  $\tau \in S_p(k)$ , is  
 331 at most  $2^{k_{\max}} \delta \leq 1/4 < 1/2$ . Applying Theorem 4, we work in the one-sheeted covering of  
 332  $\mathbb{T}^d$ . Using the fact that  $\mathcal{X}$  is an  $\varepsilon$ -covering, we apply Lemma 2 to get that  $n_k \geq (2^{k-1}\delta/\varepsilon)^d$ .  
 333

334 Now applying Lemma 8, we can bound  $P_p(k)$ .

335 **► Lemma 14.** *For  $k \geq 0$ ,  $P_p(k) \leq q^d \cdot \exp(-qn_k/2)$ .*

336 **Proof.** The simplex  $\sigma$  can be a Delaunay simplex only if (i) the set of its vertices is included  
 337 in the subsample  $\mathcal{Y}$ , and (ii) all points in  $B(c_\sigma, r_\sigma) \cap \mathcal{X}$  are excluded from  $\mathcal{Y}$ . The idea is  
 338 therefore, to use Lemma 13 to bound the number of points in  $B(c_\sigma, r_\sigma) \cap \mathcal{X}$  from below by  
 339  $n_k$ , and then upper-bound the probability that all these points are excluded from  $\mathcal{Y}$ .

340 This suggests applying Lemma 8, with the universe having  $c = n - 1$  elements, sample  
 341 size  $a = s - 1$ , included set having  $b = d$  elements, and excluded set having  $t = n_k$  elements.  
 342 We verify first that the conditions of the lemma are satisfied, i.e. (i)  $b \leq \min\{\frac{a}{2}, c - 1\}$ , since  
 343  $s \geq 4d + 1$ . Now applying the lemma, we get

$$\begin{aligned} 344 \quad \mathbb{P}[\sigma \in \text{Del}(\mathcal{Y})] &\leq \left(\frac{s-1}{n-1}\right)^d \exp\left(-n_k \cdot \left(\frac{s-1}{2(n-1)}\right)\right) \\ 345 \quad &= q^d \cdot \exp(-qn_k/2) \leq q^d \cdot \exp(-qn_k/2), \end{aligned}$$

346 where the equality was by the substitution  $q = \frac{s-1}{n-1}$ , and the last inequality followed from  
 347 the fact that  $d = b \leq c - 1 = n - 2$ .

## 23:10 RIC of Delaunay triangulations of nice point sets

348 Next, we shall upper bound  $s_p(k)$  from above. We first state a simple observation.

349 ► **Lemma 15.** *Let  $\sigma$  be a  $d$ -simplex incident to  $p$ , with circumcentre  $c_\sigma$  and circumradius*  
 350  *$r_\sigma$ . Then  $\sigma \subset B[p, 2r_\sigma]$  and  $B(c_\sigma, r_\sigma) \subset B(p, 2r_\sigma)$ .*

351 **Proof.** This follows simply from the triangle inequality. For the first statement, we have that  
 352 for any  $p' \in \sigma$ ,  $\|p, p'\| \leq \|p, c_\sigma\| + \|c_\sigma, p'\| = 2r_\sigma$ . The second statement follows by replacing  
 353 the above inequalities with strict inequalities for the points in the open ball  $B(c_\sigma, r_\sigma)$ . ◀

354 Now we can bound  $s_p(k)$  using the above observation together with Lemma 1.

355 ► **Lemma 16.** *For  $k \geq 0$ ,  $s_p(k) \leq \frac{(6 \cdot 2^k \delta / \varepsilon)^{d^2}}{d!}$ .*

356 **Proof.** Let  $\tau$  be an element of  $S_p(k)$ . Using Lemma 15 and the definition of  $S_p(k)$ , we  
 357 have that  $\sigma = \tau \cup \{p\} \subset B[p, 2^{k+1}\delta]$ . If  $k \leq k_{\max}$ , then  $2^{k+1}\delta \leq 1/2$ . Therefore applying  
 358 Theorem 4, we can work in the one-sheeted covering of  $\mathbb{T}^d$ . Now applying Lemma 1, the  
 359 number of points in  $B(p, 2^{k+1}\delta) \cap \mathcal{X}$  is at most  $(3 \cdot 2^{k+1}\delta/\varepsilon)^d = (6 \cdot 2^k \delta/\varepsilon)^d$ . Therefore, the  
 360 set of possible simplices incident to  $p$  and having vertices in  $B[p, 2^{k+1}\delta] \cap \mathcal{X}$  is at most the  
 361 set of all  $d$ -tuples of points in  $B[p, 2^{k+1}\delta] \cap \mathcal{X}$ , i.e. at most  $(6 \cdot 2^k \delta/\varepsilon)^{d^2}/d!$ . ◀

362 Next, using the above bounds on  $s_p(k)$  and  $n_k$ , we shall bound the sum  $\sum_{k=0}^{k_{\max}} \mathbb{E}[Z_p(k)]$   
 363 in the following three lemmas.

364 ► **Lemma 17.**  $\mathbb{E}[Z_p(0)] \leq 6^{d^2+d}$ .

365 **Proof.** Substituting the bounds on  $s_p(0)$  and  $P_p(0)$  proved in Lemmas 16, 14 and 13  
 366 respectively, we have

$$\begin{aligned} 367 \mathbb{E}[Z_p(0)] &= s_p(0) \cdot P_p(0) \\ 368 &= \frac{(6(\delta/\varepsilon))^{d^2}}{d!} \cdot q^d \leq \frac{6^{d^2} \cdot (2d)^d}{d!} \leq 6^{d^2} \cdot (2e)^d, \end{aligned}$$

369 where in the second step we used the definition of  $\delta$  to get  $q(\delta/\varepsilon)^d = 2d$ , and in the last step  
 370 we used Stirling's approximation  $d^d/d! \leq e^d$ , and that  $2e < 6$ . ◀

371 ► **Lemma 18.**  $\sum_{k=0}^{k_{\max}} \mathbb{E}[Z_p(k)] \leq (1 - (2/e)^6)^{-1} \cdot 6^{d^2+d}$ .

372 **Proof.** First, recall from Lemma 12 that  $\sum_{k \geq 1} \mathbb{E}[Z_p(k)] \leq \sum_{k \geq 0} s_p(k) \cdot P_p(k)$ . Now from  
 373 Lemmas 14, 13 and 16, we have that for all  $k \geq 0$ ,

374 (i)  $s_p(k) \leq \tilde{s}_p(k) := (6 \cdot 2^k \delta/\varepsilon)^{d^2}/d!$  and

375 (ii)  $P_p(k) \leq \tilde{P}_p(k) := \begin{cases} q^d, & k = 0 \\ q^d \cdot \exp(-q(2^{k-1}\delta/\varepsilon)^d/2), & k \in [1, k_{\max}] \end{cases}$ .

376 Observe that  $\sum_{k \geq 0} s_p(k) \cdot P_p(k) \leq \sum_{k \geq 0} \tilde{s}_p(k) \cdot \tilde{P}_p(k)$ . In order to bound  $\sum_{k \geq 0} s_p(k) \cdot P_p(k)$ ,  
 377 it therefore suffices to simply bound  $\sum_{k \geq 0} \tilde{s}_p(k) \cdot \tilde{P}_p(k)$ . For the rest of the proof therefore,  
 378 we shall focus on bounding this sum.

379 **When  $1 \leq k \leq k_{\max}$ :**

380 Consider the ratio of successive terms  $\frac{\tilde{s}_p(k+1) \cdot \tilde{P}_p(k+1)}{\tilde{s}_p(k) \cdot \tilde{P}_p(k)}$  of the sequence  $(\tilde{s}_p(k) \cdot \tilde{P}_p(k))_{k \geq 1}$ .  
 381 From Lemmas 16 and 13, we have

$$\begin{aligned}
 \frac{\tilde{s}_p(k+1) \cdot \tilde{P}_p(k+1)}{\tilde{s}_p(k) \cdot \tilde{P}_p(k)} &= \frac{(6 \cdot 2^{k+1} \delta / \varepsilon)^{d^2} / d!}{(6 \cdot 2^k \delta / \varepsilon)^{d^2} / d!} \times \frac{q^d \cdot \exp(-q(2^k \delta / \varepsilon)^d / 2)}{q^d \cdot \exp(-q(2^{k-1} \delta / \varepsilon)^d / 2)} \\
 &= 2^{d^2} \cdot \exp\left(-\frac{q}{2} \cdot ((2^{k-1} \delta / \varepsilon)^d) (2^d - 1)\right) \\
 &= 2^{d^2} \cdot \exp\left(-\frac{1}{2} \cdot (2^{(k-1)d} 2d) (2^d - 1)\right) \leq 2^4 e^{-6},
 \end{aligned}$$

385 where in the last step we used the definition of  $\delta = \varepsilon(2d/q)^{1/d}$ , i.e.  $q(\delta/\varepsilon)^d = 2d$ . The last  
 386 step follows by taking  $k = 1, d = 2$ , to get  $2^4 \cdot e^{-6} \leq (2/e)^6$ .

387 **When  $k = 0$ :**

388 In this case, the ratio  $\frac{\tilde{s}_p(1) \cdot \tilde{P}_p(1)}{\tilde{s}_p(0) \cdot \tilde{P}_p(0)} \leq 2^{d^2} \cdot \exp(-d \cdot 2^d)$ , which is at most  $(2/e)^6$  for  $d \geq 2$ .  
 389 Therefore for all  $0 \leq k \leq k_{\max} - 1$ , we have that  $\frac{\tilde{s}_p(k+1) \cdot \tilde{P}_p(k+1)}{\tilde{s}_p(k) \cdot \tilde{P}_p(k)} \leq (2/e)^6$ . Thus, the sum  
 390  $\sum_{k=0}^{k_{\max}} \mathbb{E}[Z_p(k)]$  is upper bounded by the sum of a geometric progression with leading term  
 391  $\tilde{s}_p(0) \cdot \tilde{P}_p(0) \leq 6^{d^2+d}$  and common ratio  $(2/e)^6$ , which is at most  $(1 - (2/e)^6)^{-1} \cdot 6^{d^2+d}$ . ◀

392 Lastly, we bound the expected number of simplices with exceptionally large circumradii,  
 393 i.e.  $\sum_{k > k_{\max}} \mathbb{E}[Z_p(k)]$ .

394 **Case II: Simplices with large circumradii  $k > k_{\max}$ .**

395 In this regime, the circumradii of the potential simplices are greater than  $1/4$ . Therefore by  
 396 Theorem 4, we shall work in the  $3^d$ -sheeted covering of  $\mathbb{T}^d$ .

397 ▶ **Lemma 19.**  $\sum_{k > k_{\max}} \mathbb{E}[Z_p(k)] \leq 5$ .

398 **Proof.** From Lemma 3, we have that  $n = \#(\mathcal{X}) \leq 2^{-d} d^{d/2} \varepsilon^{-d}$ . Therefore, by Lemma 2, any  
 399 ball  $B$  of radius at least  $2^{k_{\max}} \delta \geq 1/4$ , has at least  $(2^{k_{\max}} \delta / \varepsilon)^d = (1/4\varepsilon)^d \geq \left(\frac{n}{2^d \cdot d^{d/2}}\right)$  points  
 400 in its interior, i.e.  $\#(\text{int} B \cap \mathcal{X}) \geq \left(\frac{n}{(2\sqrt{d})^d}\right)$ . Here, since  $2^{k+1} \delta > 1/2$ , we shall use Theorem 4  
 401 and work in the  $3^d$ -sheeted covering of  $\mathbb{T}^d$ . The maximum number of  $d$ -tuples which can  
 402 possibly form a Delaunay  $d$ -simplex with  $p$ , is at most  $\binom{n-1}{d} \leq (n-1)^d / d!$ . Since we are  
 403 working in the  $3^d$ -sheeted covering space, each vertex of a simplex  $\tau \in S_p(k)$  can be chosen  
 404 from one of at most  $3^d$  copies in the covering space. Thus, each simplex in  $S_p(k)$  yields  
 405 less than  $3^{d^2}$  possible Delaunay spheres in  $\mathbb{T}^d$ . Therefore, the expected number of simplices  
 406 having radius at least  $2^{k_{\max}} \delta$ , is at most

$$\begin{aligned}
 \sum_{k > k_{\max}} \mathbb{E}[Z_p(k)] &= 3^{d^2} \frac{(n-1)^d}{d!} \cdot P_p(k) = 3^{d^2} \frac{(n-1)^d}{d!} \cdot q^d \cdot \exp\left(-\frac{q}{2} \cdot \frac{n-1}{(2\sqrt{d})^d}\right) \\
 &\leq 3^{d^2} \frac{(s-1)^d}{d!} \cdot \exp\left(-\frac{s-1}{2(2\sqrt{d})^d}\right). \tag{1}
 \end{aligned}$$

409 For  $s > s_0 = 4(2\sqrt{d})^d \cdot d^3 + 1$ , this function is decreasing in term of  $s$  and it is easy to check  
 410 that the value in  $s_0$  is smaller than 5. The lemma follows. ◀

411 Thus, by Lemmas 12, 18, and 19, the expected complexity of the star of  $p$  is at most  
 412  $(1 - (2/e)^6)^{-1} \cdot 6^{d^2+d} + 5 \leq 2 \cdot 6^{d^2+d}$  for  $d \geq 2$ , which completes the proof of Theorem 9.

413 **5 Polyhedral Surfaces in  $\mathbb{E}^3$** 

414 In this section, we introduce a partition of the sub-sample  $\mathcal{Y}$  into *boundary* and *interior*  
 415 points, to do a case analysis of the expected number of edges in the Delaunay triangulation  
 416  $Del(\mathcal{Y})$ , depending on whether the end-points of a potential Delaunay edge, are boundary or  
 417 interior points, and whether they lie on the same facet or on different facets.

418 **Main ideas:** Our overall strategy will be to mesh the proofs of Attali-Boissonnat [4] and  
 419 Theorem 9. Briefly, Attali and Boissonnat reduce the problem to counting the Delaunay edges  
 420 of the point sample, which they do by distinguishing between *boundary* and *interior* points  
 421 of a facet. For boundary points, they allow all possible edges. For interior points, the case  
 422 of edges with endpoints on the same facet is easy to handle, while geometric constructions  
 423 are required to handle the case of endpoints on different facets, or that of edges with one  
 424 endpoint in the interior and another on the boundary.  
 425

426  
 427 However, we shall need to introduce some new ideas to adapt our previous methods to  
 428 this setting. Firstly, an edge can have multiple balls passing through its endpoints and, as  
 429 soon as one of these balls is empty, the edge is in the triangulation. This is handled using  
 430 a geometric construction (see Lemma 28). Basically, the idea is to build a constant-sized  
 431 packing of a sphere centered on a given point, using large balls, such that any sphere of a  
 432 sufficiently large radius which passes through the point, must contain a ball from the packing.  
 433

434 Secondly, since we have randomly spaced points at the boundaries, boundary effects  
 435 could penetrate deep into the interior. To handle this, we introduce the notion of *levels* of a  
 436 surface, instead of the fixed strip around the boundary used in [4], and use a probabilistic,  
 437 rather than deterministic, classification of boundary and interior points. The new classifica-  
 438 tion is based on the level of a point and the radius of the largest empty disk passing through it.

439 Recall the definitions of  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{S}$  from Theorem 10. For a curve  $\Gamma$ ,  $l(\Gamma)$  denotes its  
 440 length. For a subset of a surface  $R \subset \mathcal{S}$ ,  $a(R)$  denotes the area of  $R$ . We next present some  
 441 general lemmas, which will be needed in the proofs of the main lemmas. For sets  $A, B \subset \mathbb{E}^3$ ,  
 442  $A \oplus B$  denotes the Minkowski sum of  $A$  and  $B$ , i.e. the set  $\{x + y : x \in A, y \in B\}$ . For  
 443 convenience, the special case  $A \oplus B(0, r)$  shall be denoted by  $A \oplus r$ . Throughout this section,  
 444 we shall use  $\kappa$  to denote the maximum number of points of an  $\varepsilon$ -net in a disk of radius  $2\varepsilon$ ,  
 445 which is at most  $6^d = 36$  (using Lemma 1 with  $r = 2\varepsilon$  and  $\rho = \varepsilon$ ), we and define  $q := \frac{\kappa}{n}$ , and  
 446  $\delta := \varepsilon/\sqrt{q}$ .

447 **Level sets, Boundary points and Interior points**

448 We now introduce some definitions which will play a central role in the analysis. First we define  
 449 the notion of levels. Given facet  $F \in \mathcal{S}$  and  $k \geq 0$ , define the *level set*  $L_{\leq k} := F \cap (\partial F \oplus 2^k \delta)$ .  
 450  $L_{=k} := L_{\leq k} \setminus L_{\leq k-1}$ . For  $x \in \mathcal{X}$ , the *level* of  $x$ , denoted  $Lev(x)$ , is  $k$  such that  $x \in L_{=k}$ .  
 451 Let  $L_{\leq k}(\mathcal{X}), L_{=k}(\mathcal{X})$  denote  $L_{\leq k} \cap \mathcal{X}, L_{=k} \cap \mathcal{X}$  respectively. Note that for  $x \in L_{=k}, k \geq 1$ ,  
 452 the distance  $d(x, \partial F) \in (2^{k-1}\delta, 2^k\delta]$ . Hence, if  $Lev(x) = k$ ,  $D(x, 2^{k-1}\delta) \subset F$ . For  $k = 0$ ,  
 453  $d(x, \partial F) \in [0, \delta]$ .  
 454

455 Next, we define a bi-partition of the point set into boundary and interior points. Given  
 456  $x \in F$  having  $Lev(x) = k$ , then  $x \in Bd_F(\mathcal{Y})$ , or  $x$  is a *boundary point*, if  $k = 0$  or if there  
 457 exists an empty disk (w.r.t.  $\mathcal{Y}$ ) of radius greater than  $2^{k-1}\delta$ , whose boundary passes through  
 458  $x$ .  $x \in Int_F(\mathcal{Y})$ , or an *interior point* if and only if  $x \in \mathcal{Y} \setminus Bd_F(\mathcal{Y})$ . In general,  $x \in Bd_S(\mathcal{Y})$

459 if  $x \in Bd_F(\mathcal{Y})$  for some  $F \in \mathcal{S}$ , and  $x \in Int_{\mathcal{S}}(\mathcal{Y})$  is defined similarly.

460

461 The above bi-partition induces a classification of potential edges, depending on whether  
 462 the end-points are boundary or interior points. Let  $E_1$  denote the set of edges in  $Del(\mathcal{Y})$   
 463 of the type  $\{x_1, x_2\} : x_1, x_2 \in Bd_{\mathcal{S}}(\mathcal{Y})$ . Let  $E_2$  denote the set of edges in  $Del(\mathcal{Y})$  of the  
 464 type  $\{x, y\} : x, y \in Int_F(\mathcal{Y})$ , for some  $F \in \mathcal{S}$ . Let  $E_3$  denote the set of edges in  $Del(\mathcal{Y})$  of  
 465 the type  $\{x, y\} : x, y \in Int_{\mathcal{S}}(\mathcal{Y})$ , such that  $x \in F, y \in F' \neq F$ . Let  $E_4$  denote the set of  
 466 edges  $\{x, y\}$  in  $Del(\mathcal{Y})$  of the type  $x \in Bd_{\mathcal{S}}(\mathcal{Y}), y \in Int(F)$ , where  $F$  is a facet in  $\mathcal{S}$  with  
 467 supporting plane  $P$ .

468 We have the following lemmas, to be proved in section 5.2.

469 ► **Lemma 20.**  $\mathbb{E} [\#(E_1)] \leq O(1) \cdot (\kappa^2 L^2 / A) \cdot s$ .

470 ► **Lemma 21.**  $\mathbb{E} [\#(E_2)] \leq c_4 \cdot \kappa s$ , where  $c_4 \leq 2 \cdot 10^5$ .

471 ► **Lemma 22.**  $\mathbb{E} [\#(E_3)] \leq c_4 \cdot (C - 1) \cdot \kappa s$ .

472 ► **Lemma 23.**  $\mathbb{E} [\#(E_4)] \leq O(1) \cdot \frac{\kappa^2 L^2}{A} \cdot s$ .

473 Given the above lemmas, the proof of Theorem 10 follows easily.

474 **Proof of Theorem 10.** As in [4] (Section 4), by Euler's formula, the number of tetrahedra  
 475  $t(Del(\mathcal{Y}))$  in the Delaunay triangulation of  $\mathcal{S}$ , is at most  $e(Del(\mathcal{Y})) - \#(\mathcal{Y}) = e(Del(\mathcal{Y})) - s$ ,  
 476 where  $e(Del(\mathcal{Y}))$  is the number of edges in the Delaunay triangulation. Therefore, it suffices  
 477 to count the edges of  $Del(\mathcal{Y})$ . Next, observe that any point  $x \in \mathcal{Y}$  is either a boundary or an  
 478 interior point, that is  $Bd_{\mathcal{S}}(\mathcal{Y}) \sqcup Int_{\mathcal{S}}(\mathcal{Y}) = \mathcal{Y}$ . An edge in  $Del(\mathcal{Y})$ , therefore, can be either  
 479 between two points in  $Bd_{\mathcal{S}}(\mathcal{Y})$ , or two points in  $Int_{\mathcal{S}}(\mathcal{Y})$ , or between a point in  $Bd_{\mathcal{S}}(\mathcal{Y})$  and  
 480 another in  $Int_{\mathcal{S}}(\mathcal{Y})$ . The case of a pair of points in  $Int_{\mathcal{S}}(\mathcal{Y})$  is further split based on whether  
 481 the points belong to the same facet of  $\mathcal{S}$  or different facets. Thus using the above exhaustive  
 482 case analysis, the proof follows simply by summing the bounds. ◀

483 Before proving Lemmas 20- 23, we first present a few technical lemmas.

## 484 5.1 Some Technical Lemmas

485 The following geometric and probabilistic lemmas prove certain properties of  $\varepsilon$ -nets on  
 486 polyhedral surfaces, random subsets, etc., as well as exploit the notion of boundary and  
 487 interior points to get an exponential decay for boundary effects penetrating into the interior.

488 ► **Proposition 24** ([4]). *Let  $F$  be a facet of  $\mathcal{S}$ . For any Borel set  $R \subset F$ , we have*

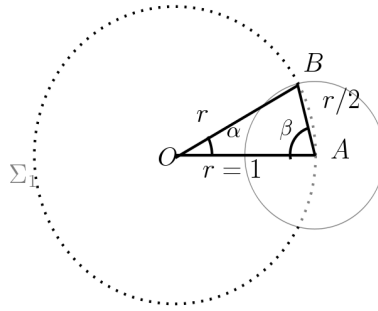
489 
$$\left( \frac{a(R)}{4\pi\varepsilon^2} \right) \leq \#(R \cap \mathcal{X}) \leq \left( \frac{\kappa \cdot a(R \oplus \varepsilon)}{\pi\varepsilon^2} \right), \text{ and therefore,} \quad (2)$$

490 
$$\left( \frac{A}{4\pi\varepsilon^2} \right) \leq \#(\mathcal{S} \cap \mathcal{X}) = n. \quad (3)$$

491 ► **Proposition 25** ([4]). *Let  $F$  be a facet of  $\mathcal{S}$ , let  $\Gamma \subset F$  be a curve contained in  $F$ , and  
 492  $k \in \mathbb{N}$ . Then*

493 
$$\#((\Gamma \oplus k\varepsilon) \cap \mathcal{X}) \leq \left( \frac{(2k+1)^2}{k} \right) \kappa \frac{l(\Gamma)}{\varepsilon} \quad (4)$$

494 
$$\leq \left( 9k\kappa \frac{l(\Gamma)}{\varepsilon} \right), \text{ when } k \geq 1. \quad (5)$$



497 ■ **Figure 1** Angle covered by disk of radius 1/2 is  $= 2\alpha$ .

495 ► **Lemma 26.** *Given a circle  $\Sigma_1 \subset \mathbb{E}^2$  of unit radius centered at the origin, seven disks*  
 496 *having centers in  $\Sigma_1$  and radius 1/2, are necessary and sufficient to cover  $\Sigma_1$ .*

498 **Proof.** Let  $D_2$  denote a disk of radius 1/2, having its center lying on the circle  $\Sigma_1$ . Let  $2\alpha$   
 499 denote the angle subtended by  $\Sigma_1 \cap D_2$  on the center of  $\Sigma_1$ . By symmetry, the angle  $\angle OAB$   
 500 in figure 1 is  $\alpha$ . Applying the sine law to the triangle  $OAB$ , we get

$$\begin{aligned}
 501 \quad \frac{1}{\sin \beta} &= \frac{1/2}{\sin \alpha}, \\
 502 \quad \Rightarrow \quad \frac{1}{\sin(\pi/2 - \alpha/2)} &= \frac{1/2}{\sin \alpha}, \\
 503 \quad \Rightarrow \quad \frac{1}{\cos(\alpha/2)} &= \frac{1/2}{\sin \alpha}, \\
 504 \quad \Rightarrow \quad 4 \sin(\alpha/2) &= 1.
 \end{aligned}$$

505 Therefore,  $\alpha = 2 \arcsin(1/4)$ , or  $2\alpha = 4 \arcsin(1/4)$ .  
 506 Thus, one disk covers an angle of  $4 \arcsin(1/4)$ , and so the number of required disks having  
 507 radius 1/2, is at least  $\frac{2\pi}{4 \arcsin(1/4)} \approx 6.21 < 7$ .

508  
 509 It is now easy to see that we can place the disks on the boundary in a greedy manner,  
 510 such that they cover the maximum possible angle, except of course the last disk which may  
 511 have some overlap with the first disk. Thus seven disks would suffice as well. ◀

512 ► **Lemma 27 (Level Size).**  $\#(L_{=k} \cap \mathcal{X}) \leq \#(L_{\leq k} \cap \mathcal{X}) \leq 9\kappa L \left(\frac{2^k \delta}{\varepsilon^2}\right)$ .

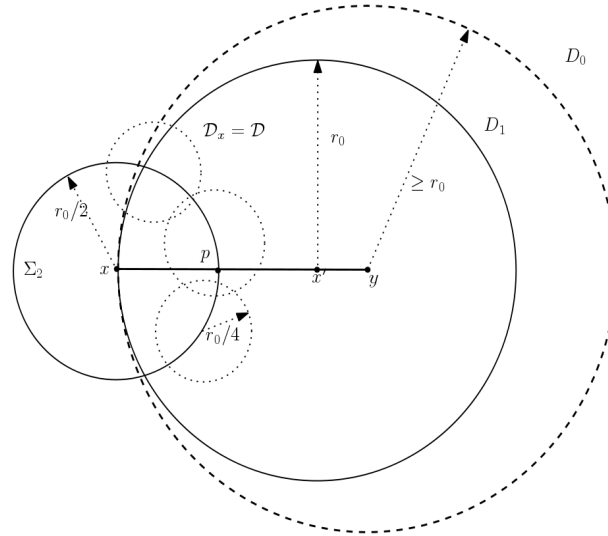
513 **Proof.** The first inequality is obvious, as  $L_{=k} \subseteq L_{\leq k}$ . The proof of the second inequality  
 514 follows by applying Proposition 25 over the boundaries of the facets. For a fixed facet  $F \in \mathcal{S}$ ,  
 515 we get

$$516 \quad \# \left( \left( \partial F \oplus \left( \frac{2^k \delta}{\varepsilon} \right) \varepsilon \right) \cap \mathcal{X} \right) \leq \left( \frac{9\kappa 2^k \delta}{\varepsilon} \right) \cdot \left( \frac{l(\partial F)}{\varepsilon} \right).$$

517 Summing over all  $F \in \mathcal{S}$ , we get

$$518 \quad \#(L_{\leq k} \cap \mathcal{X}) = \sum_{F \in \mathcal{S}} \# \left( (\partial F \oplus 2^k \delta) \cap \mathcal{X} \right) \leq \left( \frac{9\kappa 2^k \delta}{\varepsilon^2} \right) \sum_{F \in \mathcal{S}} l(\partial F) = \left( \frac{9\kappa 2^k \delta}{\varepsilon^2} \right) \cdot L.$$

519 ◀



520 ■ **Figure 2** For Lemma 28

521 ► **Lemma 28.** *Let  $F$  be a facet of  $\mathcal{S}$  with supporting plane  $P$ , and  $x \in F$  with  $\text{Lev}(x) > 0$ .  
522 Then there exists a collection  $\mathcal{D}_x$  of at most  $c_B = 7$  disks in  $F$ , such that*

- 523 (i) *Each  $D \in \mathcal{D}_x$  is contained in  $F$ ,*  
524 (ii) *Each  $D \in \mathcal{D}_x$  has radius  $r_0/4$ , where  $r_0 = 2^k \delta$  and  $k \in \mathbb{N}$  such that  $0 \leq k < \text{Lev}(x)$ , and*  
525 (iii) *Any disk  $D \subset P$  of radius at least  $r_0$ , such that  $x \in \partial D$ , contains at least one disk in  $\mathcal{D}_x$ .*

526 **Proof.** Let  $D_0 = D(y, r)$  for some  $y \in P$ ,  $r \geq r_0$ , be a disk such that  $x \in \partial D_0$ . Let  
527  $D_1 = D(x', r_0)$  be the unique disk with centre  $x'$  on the line  $xy$ , radius  $r_0$ , and having  
528  $x \in \partial D_1$ . Note that

- 529 (a)  $D_1 \subseteq D_0$  by construction, and,  
530 (b)  $x' \in F$ , since  $r_0 = 2^k \delta \leq 2^{\text{Lev}(x)-1} \delta$ , so that  $x' \in D(x, r_0) \subset F$ .

531 Consider  $\Sigma_2 = \Sigma(x, r_0/2)$ , and let  $p = xx' \cap \Sigma_2$ , that is, the point  $p$  lies on the line  
532  $xx'$ , at distance  $r_0/2$  from  $x$  (and therefore from  $x'$  as well). We shall build a minimal  
533 covering  $\mathcal{D}$  of the circle  $\Sigma_2$ , by disks centered in  $\Sigma_2$ , having radius  $r_0/4$ . From Lemma 26,  
534 we get  $\sharp(\mathcal{D}) = 7$ . Let  $D' \in \mathcal{D}$  be a disk in the covering. Then by the triangle inequality,  
535  $D' \subset D(x, r_0/2 + r_0/4) \subset D(x, r_0)$ . As before, by the definitions of  $\text{Lev}(x)$  and  $r_0$ , this  
536 implies  $D' \subset F$ . Thus  $\mathcal{D}$  satisfies conditions (i) and (ii) of the lemma. Further, since  $\mathcal{D}$  is a  
537 covering of  $\Sigma_2$ , there exists  $D_p \in \mathcal{D}$  such that  $p \in D_p$ . Therefore, the disk  $D_p \subset D_1 \subset D_0$ ,  
538 and  $D_p \subset F$ . Thus  $D_p \in \mathcal{D}$  satisfies condition (iii). Now taking  $\mathcal{D}_x = \mathcal{D}$  completes the proof  
539 of the lemma. ◀

540 ► **Lemma 29** (Decay lemma). *Given  $x_1, \dots, x_t \in \mathcal{X}$ , such that  $\text{Lev}(x_i) > 0$ ,  $1 \leq i \leq t$ , then  
541 for all  $0 \leq k_i < \text{Lev}(x_i)$ , with  $r_i^* := 2^{k_i} \delta$ , the probability of the event*

542 
$$E := \{\forall i \in [t] : \exists D_i = D(y_i, r_i) : r_i \geq r_i^*, x_i \in \mathcal{Y}, x_i \in \partial D_i \text{ and } \text{int}(D_i) \cap \mathcal{Y} = \emptyset\},$$

543 is given by

544 
$$\mathbb{P}[E] \leq \begin{cases} q^t, & \text{if } k_{\max} = 0, \\ c_1 \cdot q^t \cdot \exp(-c_2 \cdot 2^{2k_{\max}}), & \text{if } k_{\max} > 0, \end{cases}$$



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545 where  $c_1 = c_B^t$ ,  $c_2 \geq 2^{-7}$ , and  $k_{\max} := \max_i \{k_i\}$ . Thus

$$546 \quad \mathbb{P}[E] \leq c_1 \cdot q^t \cdot \exp(-c_2 \cdot 2^{2k_{\max}}), \quad k_{\max} \geq 0.$$

547 **Proof.** Firstly, consider the case where  $k_{\max} = 0$ , i.e. all the  $k_i$ 's are zero. In this case we  
548 simply upper bound the probability of the event  $E$ , by the probability of including all the  
549 points  $x_1, \dots, x_t$  in  $\mathcal{Y}$ . By Lemma 8, this is at most  $q^t$ .

550 We next come to the case when  $k_{\max} > 0$ . Since for all  $i \in [t]$ ,  $k_i < Lev(x_i)$ , we can apply  
551 Lemma 28 for each  $i$ , with  $k = k_i$ , to conclude that for each  $i$ , there exists a collection  $\mathcal{D}_i$   
552 of at most  $c_B$  disks of radius  $r_i^*/4$ , such that any disk having radius greater than  $r_i^*$  and  
553 passing through  $x_i$ , must contain some disk  $D_i^* \in \mathcal{D}_i$ . Let  $T$  denote the set  $\prod_{i=1}^t \mathcal{D}_i$ . By the  
554 union bound over the set  $T$ , we get

$$555 \quad \begin{aligned} \mathbb{P}[E] &\leq \mathbb{P}[\exists B \in T : \forall i \in [t], B_i \cap \mathcal{Y} = \emptyset] \\ 556 &\leq \#(T) \cdot \mathbb{P}[\forall i \in [t], B_i \cap \mathcal{Y} = \emptyset], \end{aligned}$$

557 where  $B \in T$  is some fixed element of  $T$ .

558 Let  $j := \arg \max_{i \in [t]} k_i$ , so that  $k_{\max} = k_j$ . Now, the event  $E$  requires the set  $x_1, \dots, x_t$   
559 to be in the sample  $\mathcal{Y}$ , and the interiors of the disks  $D_i^*$  to be free from points in  $\mathcal{Y}$ . In  
560 particular, the disk  $D_j^*$  should not contain any points in  $\mathcal{Y}$ . Therefore applying Lemma 8 on  
561 the universe  $C = \mathcal{X}$ , the random sample  $A = \mathcal{Y}$ , the included subset  $B = \{x_1, \dots, x_t\}$ , and  
562 the excluded subset  $Z = D_j^* \cap \mathcal{X}$  of size at least  $z = \frac{\pi(r_j^*/4)^2}{4\pi\varepsilon^2}$ , we get

$$563 \quad \begin{aligned} \mathbb{P}[E] &\leq \#(T) \cdot \mathbb{P}[\forall B_i \in u, B_i \cap \mathcal{Y} = \emptyset] \\ 564 &\leq (c_B)^t \cdot q^t \cdot \exp\left(-\frac{1}{2}q \cdot \left(\frac{\pi(r_j^*/4)^2}{4\pi\varepsilon^2}\right)\right) \\ 565 &\leq c_1 q^t \cdot \exp(-2^{2k_{\max}-7}). \end{aligned}$$

566 where in the last step we used that  $q = s/n$ ,  $\delta = \sqrt{n/s} \cdot \varepsilon$ , and  $r_j^* = 2^{k_{\max}}\delta$ , and set  
567  $c_1 = c_B^t$ . ◀

568 **► Lemma 30 (Growth Lemma).** Given any point  $x \in \mathcal{S}$  in a facet  $F$ , and  $0 \leq k < Lev(x)$ ,  
569 we have

$$570 \quad (i) \quad 2^{2k-2}/q \leq \#(D(x, 2^k\delta) \cap \mathcal{X}) \leq 4 \cdot (2^{2k}/q).$$

$$571 \quad (ii) \quad 2^{2k-2} \leq \#(D(x, 2^k\delta) \cap \mathcal{Y}) \leq 4 \cdot (2^{2k}).$$

572 **Proof.** By the definition of  $Lev(x)$ , we have that  $D(x, 2^{Lev(x)-1}\delta) \subset F$ . Now the statement  
573 follows by the application of Proposition 24, as below

$$574 \quad \begin{aligned} \frac{\pi(2^k\delta)^2}{4\pi\varepsilon^2} &\leq \#(D(x, 2^k\delta) \cap \mathcal{X}) \leq \frac{\pi(2^k\delta + \varepsilon)^2}{\pi\varepsilon^2}, \text{ or,} \\ 575 \quad \frac{2^{2k-2}n}{s} &\leq \#(D(x, 2^k\delta) \cap \mathcal{X}) \leq \frac{2 \cdot 2^{2k}n}{s}, \end{aligned}$$

576 where we used that  $\delta := (\sqrt{\frac{n}{s}})\varepsilon$ . This gives the first statement of the lemma, using  $q = s/n$ .  
577 The second statement follows simply by taking expectation. ◀

578 **5.2 Proofs of Lemmas 20-23**

579 The proofs of Lemmas 20 and 21 now follow from the Decay and Growth lemmas, together  
580 with similar ideas as for the flat torus case.

581 **Proof of Lemma 20.** Let  $x_1, x_2 \in Bd_S(\mathcal{Y})$ . To bound the expected number of edges in  $E_1$ ,  
582 we simply bound the number of pairs  $(x_1, x_2) \in Bd_S(\mathcal{Y}) \times Bd_S(\mathcal{Y})$ . Let  $l_1 := Lev(x)$  and  
583  $l_2 := Lev(y)$ , and let  $l := \max_i(l_i)_{i=1}^2$ . By definition, if  $l = 0$ , then  $x_1, x_2 \in Bd_S(\mathcal{Y})$ . For  
584  $l \geq 1$ , we get that  $x_1 \in Bd_S(\mathcal{Y})$  and  $x_2 \in Bd_S(\mathcal{Y})$  only if there exists a disk of radius at  
585 least  $2^{l-1}\delta$  passing through  $x_1$  or  $x_2$ , and containing no points of  $\mathcal{Y}$ . Therefore to bound the  
586 probability that  $(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2$ , we can apply the Decay Lemma 29, with  $t = 2$ , for  
587  $i \in \{1, 2\}$ . We get

$$588 \quad \mathbb{P}[(x_1, x_2) \in E_1] \leq \mathbb{P}[(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2]$$

$$589 \quad \leq c_1 q^2 \cdot \exp(-c_2 \cdot 2^{2l-2}) \leq c_1 q^2 \cdot \exp(-c'_2 \cdot 2^{2l}), \quad (6)$$

590 where  $c'_2 = c_2/4 = 2^{-9}$ . Summing over all choices of levels of  $x_1$  and  $x_2$ , we have

$$591 \quad \mathbb{E}[\#(E_1)] \leq \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2 \geq 0} \#(L_{=l_2} \cap \mathcal{X}) \mathbb{P}[(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2].$$

592 By symmetry, it is enough to assume without loss of generality that  $l_1 \geq l_2$ , i.e.  $l = l_1$ . Thus,

$$593 \quad \mathbb{E}[\#(E_1)] \leq 2 \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2=0}^{l_1} \#(L_{=l_2} \cap \mathcal{X}) \mathbb{P}[(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2].$$

594 Applying equation (6) and the Level Size Lemma 27, we get

$$595 \quad \mathbb{E}[\#(E_1)] \leq 2 \sum_{l_1 \geq 0} \#(L_{\leq l_1} \cap \mathcal{X}) \sum_{l_2=0}^{l_1} \#(L_{\leq l_2} \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 \cdot 2^{2l_1})$$

$$596 \quad \leq 2c_1 q^2 \sum_{l_1 \geq 0} (9\kappa L \cdot (2^{l_1} \delta / \varepsilon^2)) \sum_{l_2=0}^{l_1} (9\kappa L \cdot (2^{l_2} \delta / \varepsilon^2)) \cdot \exp(-c'_2 \cdot 2^{2l_1})$$

$$597 \quad \leq 2c_1 q^2 (9\kappa L \left(\frac{\delta}{\varepsilon^2}\right))^2 \sum_{l_1 \geq 0} 2^{l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}) \sum_{l_2=0}^{l_1} 2^{l_2}.$$

598 Using the definitions of  $q$  and  $\delta$ , together with Proposition 24, and writing the terms  
599 outside the summation as  $N_1$ , we get  $N_1 := 2 \cdot c_1 q^2 (9\kappa L \left(\frac{\delta}{\varepsilon^2}\right))^2 = 2c_1 \cdot (9\kappa L)^2 \left(\frac{s}{n\varepsilon^2}\right) \leq$   
600  $4c_1 \cdot \left(\frac{4\pi(9\kappa L)^2}{A}\right) \cdot s$ , we continue

$$601 \quad \mathbb{E}[\#(E_1)] \leq N_1 \sum_{l_1 \geq 0} 2^{l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}) \cdot 2 \cdot 2^{l_1} \leq 2N_1 \sum_{l_1 \geq 0} 2^{2l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}).$$

602 The summation can be bounded using Lemma 46, to get

$$603 \quad \mathbb{E}[\#(E_1)] \leq 2N_1 \cdot \left(2 \cdot \frac{\log 1/c'_2}{2ec'_2}\right) = 2N_1 \cdot \frac{\log 1/c'_2}{e \cdot c'_2}.$$

604 Now substituting  $c'_2 = 2^{-9}$  gives  $\mathbb{E}[\#(E_1)] \leq 2 \cdot 10^4 \cdot c_1 \cdot \left(\frac{4\pi(9\kappa L)^2}{A}\right) \cdot s$ . ◀

605 **Proof of Lemma 21.** Let  $l$  denote  $\min\{Lev(x), Lev(y)\}$ . Observe that if  $l = 0$ , then either  $x$   
 606 or  $y$  is a boundary point, and hence we can assume  $l \geq 1$ . Let  $x' = \arg \min_{z \in \partial F} d(z, x)$ , and  
 607  $y' = \arg \min_{z \in \partial F} d(z, y)$ , i.e.  $x'$  is the closest point to  $x$  in  $\partial F$ , and similarly for  $y'$ . By the  
 608 definition of  $Bd_S(\mathcal{Y})$ , observe that  $d(x, y) \leq d(x, x') + d(x', y) \leq d(x, x') + d(y, y') \leq 2 \cdot 2^{l-1} \delta$ .  
 609 Hence we have that  $d(x, y) \leq 2^l \delta$ .

610 By the Growth Lemma 30, the expected number of Delaunay neighbours  $y$  of a point  $x$   
 611 such that  $d(x, y) \leq \delta$  is at most  $\mathbb{E}[D(x, \delta) \cap \mathcal{Y}] \leq q \cdot \frac{4}{q} = 4$ . Thus the expected number of  
 612 edges in  $E_2$  from pairs  $(x, y)$  with  $x, y \in Int_F(\mathcal{Y})$  for some  $F \in \mathcal{S}$ , and  $d(x, y) \leq \delta$ , is at most  
 613  $4 \cdot \#\mathcal{Y}$ . For longer-distance edges, let  $k \geq 1$  be such that  $2^{k-1} \delta \leq d(x, y) \leq 2^k \delta$ . Taking  
 614  $t = 2$ ,  $x_1 = x$ ,  $x_2 = y$ ,  $k_1 = k - 1$ , and  $k_2 \leq k_1$ , and applying the Decay Lemma 29, we get  
 615 that

$$616 \quad \mathbb{P}[\{x, y\} \in E_2] \leq c_1 q^2 \cdot \exp(-c_2 2^{2k-2}) \leq c_1 q^2 \cdot \exp(-c'_2 2^{2k}),$$

617 where  $c'_2 = c_2/4$ . Summing over all possible choices of  $l \geq 1$ , and  $k \leq l$ , we get

$$618 \quad \mathbb{E}[\#(E_2)] \leq \sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \sum_{k=1}^l \#(D(x, 2^k \delta) \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 2^{2k})$$

$$619 \quad \leq c_1 q^2 \left( \sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \sum_{k=1}^l \kappa 2^{2k} (\delta/\varepsilon)^2 \cdot \exp(-c'_2 2^{2k}) \right)$$

$$620 \quad \leq c_1 \kappa q \left( \frac{q \delta^2}{\varepsilon^2} \right) \cdot \left( \sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \right) \frac{2 \cdot (1/2) \cdot \log 1/c'_2}{ec'_2}$$

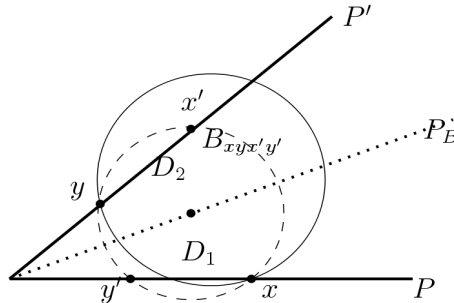
$$621 \quad \leq c_1 \kappa q \cdot c_3 \left( \sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \right)$$

$$622 \quad \leq c_1 \kappa q \cdot c_3 \cdot n = c_4 \cdot \kappa s.$$

623 where in the second step we applied the Growth Lemma 30, and in the third step we bounded  
 624 the sum  $\sum_{k \geq 0} 2^{2k} \exp(-c_2 2^{2k})$  using Lemma 46, and used that  $q = s/n = \varepsilon^2/\delta^2$ . Note that  
 625  $c_3 \leq 4 \cdot 10^3$ , and  $c_4 := c_1 \cdot c_3 \leq 2 \cdot 10^5$ .  $\blacktriangleleft$

626 For the proofs of Lemmas 22 and 23, we need some more geometric ideas of [4].

627 **Proof of Lemma 22.** Let  $x, y \in Int_S(\mathcal{Y})$ , where  $x \in F$  and  $y \in F'$ , for some  $F, F' \in \mathcal{S}$ . Let  
 628  $F'$  be fixed. To analyse this case, we shall first give a geometric construction of [4], and state  
 629 an observation from their proof.



630  $\blacksquare$  **Figure 3**  $x, y \in Int_S(\mathcal{Y})$ , on different facets  $F \subset P$ ,  $F' \subset P'$

631 ▶ **Construction 31** (Attali-Boissonnat [4]). *Let  $P$  and  $P'$  denote the supporting planes of*  
 632 *the facets  $F$  and  $F'$  respectively. Let  $P_B$  be the bisector plane of  $P$  and  $P'$ . We denote by*  
 633  *$x' \in P'$ , the reflection of  $x \in P$  with respect to  $P_B$ , and similarly by  $y' \in P$ , the reflection*  
 634 *of  $y \in P'$ . Let  $B = B_{xyx'y'}$  be the smallest ball in  $\mathbb{E}^3$  passing through  $x, y, x', y'$ , having*  
 635 *intersections  $D_1 = B \cap P$  and  $D_2 = B \cap P'$  with  $P$  and  $P'$  respectively.*

636 Attali and Boissonnat observed that

637 ▶ **Proposition 32** (Attali-Boissonnat [4]). *Any ball in  $\mathbb{E}^3$  having  $x$  and  $y$  on its boundary,*  
 638 *must contain either  $D_1$  or  $D_2$ .*

639 Therefore, if there exists a ball  $B \in \mathbb{E}^3$  such that  $x, y \in \partial B$ , and  $\text{int}(B) \cap \mathcal{Y} = \emptyset$ , then  
 640 either  $D_1 \cap \mathcal{Y} = \emptyset$ , or  $D_2 \cap \mathcal{Y} = \emptyset$ . We get

641 
$$\mathbb{P}[\{x, y\} \in E_3] \leq \mathbb{P}[\cup_{i=1}^2 \{D_i \cap \mathcal{Y} = \emptyset\}] \leq 2 \cdot \mathbb{P}[D_1 \cap \mathcal{Y} = \emptyset].$$

642 Observe that, as in Case II, we have  $x' \in D(y, 2^{Lev(y)}\delta)$ , since otherwise  $y \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$ . (Note  
 643 that our definition of boundary points allows us to ignore the fact that  $x'$  is not necessarily  
 644 a point in  $\mathcal{X}$ .) Further, the set  $\{x' \in D(y, 2^k\delta)\}$ ,  $0 < k < Lev(y)$ , is bounded in size by  
 645  $\#(D(y', 2^k\delta) \cap F \cap \mathcal{Y})$ . The rest of the analysis for the fixed facet  $F'$ , therefore follows as in  
 646 Case II. Summing over all  $F' \in \mathcal{S} \setminus \{F\}$ , we get  $\mathbb{E}[\#(E_3)] \leq c_4 \cdot (C - 1)\kappa s$ . ◀

647 Before proving Lemma 23, we briefly describe a construction, which will be central to our  
 648 analysis.

649 ▶ **Construction 33** (Attali-Boissonnat [4]). *Let  $P$  be a plane and  $Z$  be a finite set of points.*  
 650 *To each point  $x \in Z$ , assign the region  $V(x) = V_x(Z) \subset P$  of points  $y \in P$  such that the sphere*  
 651 *tangent to  $P$  at  $y$  and passing through  $x$  encloses no point of  $Z$ . Let  $\mathcal{V} := \{V(x) : x \in Z\}$ .*

652 We summarize some conclusions of Attali-Boissonnat regarding the construction. The  
 653 proofs of these propositions can be found in [4].

654 ▶ **Proposition 34.** (i)  $\mathcal{V}$  is a partition of  $P$ .

655 (ii) For each  $x \in Z$ ,  $V(x)$  is an intersection of regions that are either disks or complements  
 656 of disks.

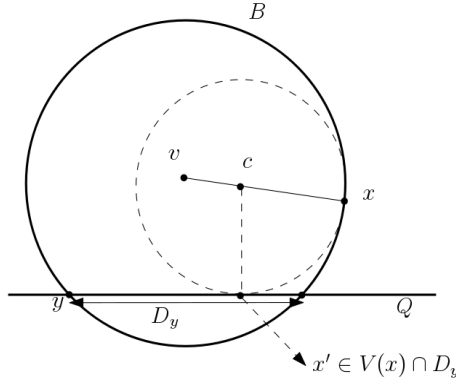
657 (iii) The total length of the boundary curves in  $\mathcal{V}$  is equal to the total length of the convex  
 658 boundaries.

659 **Proof.** The proofs are (i) and (ii) are easy.

660 (iii) Consider a point  $x \in Z$ , and let  $V(x)$  be the region corresponding to  $x$  in  $\mathcal{V}$ . By  
 661 Proposition 34 (ii),  $V(x) = (\cap_{D \in D_x} D) \cap (\cap_{\bar{D} \in C_x} \bar{D})$ , where  $D_x$  is a set of disks and  $C_x$  is  
 662 a set of complements of disks in the plane  $P$ . Let  $y \in \partial V(x)$ . Then if  $y \in \cap_{D \in D_x} D$ , then  
 663 there exists  $D_1 \in D_x$ , such that  $y \in \partial D_1$ , and so  $y$  is part of a convex segment in  $\partial V(x)$ .  
 664 Otherwise, there exists  $\bar{D}_2 \in C_x$ , such that  $y \in \partial \bar{D}_2$ . In this case, let  $V(z)$ ,  $z \in Z$ , denote  
 665 the region such that  $y \in \partial V(z)$ . Then  $D_2 \supset V(z)$ , and therefore  $y$  belongs to a convex  
 666 segment in  $\partial V(z)$ .

667 Thus, every point  $y \in \partial V(x)$  is convex either for  $V(x)$  or for a neighbouring region of  
 668  $V(x)$ , and so the total length of the convex boundary curves in  $\mathcal{V}$  gives the total length  
 669 of all the boundary curves. ◀

671 For the rest of this subsection, we shall apply Construction 33 on the plane  $P$ , and the  
 672 points in  $\text{Bd}_{\mathcal{S}}(\mathcal{Y})$  as  $Z$ . Let  $\mathcal{T} := \text{Int}_F(\mathcal{Y})$  for some facet  $F \in \mathcal{S}$ . Given  $x \in Z$ ,  $y \in P \setminus V(x)$ ,  
 673 let  $k_y = k_y(x)$  denote the least  $k \geq 0$  such that  $y \in \partial V(x) \oplus 2^k\delta$ .



674 ■ **Figure 4**  $x \in Z = Bd_S(\mathcal{Y})$ ,  $y \in \mathcal{T} = Int_F(\mathcal{Y})$ ,  $z \in V(x) \cap D_y$ .

675 ► **Proposition 35** (Attali-Boissonnat [4]). *Suppose there exists a ball  $B \subset \mathbb{E}^3$  and  $y \in P$ , such*  
 676 *that  $y, x \in \partial B$ , and  $B \cap \mathcal{T} = \emptyset$ . Then the disk  $D_y = P \cap B$  satisfies  $D_y \cap \mathcal{T} = \emptyset$ ,  $y \in \partial D_y$*   
 677 *and  $D_y \cap V_x \neq \emptyset$ .*

678 **Proof.** The first part of the proposition,  $D_y \cap \mathcal{T} = \emptyset$ , follows from the condition on  $B$ . For  
 679 the next part, note that  $y \in \partial D_y$ . Let  $v$  denote the center of the ball  $B$ , and let  $z$  be  
 680 a variable point on the line segment  $vx$ . Let  $B(z)$  denote the ball with center  $z$ , having  
 681  $x \in \partial B(z)$ . For  $z = v$ ,  $B(z) = B$  intersects  $P$ . For  $z = x$ ,  $B(z) = \{x\}$  does not intersect  $P$ .  
 682 Therefore there exists some value of  $z = c$  such that  $B(c)$  is tangential to  $P$  (see figure 4).  
 683 Let  $x'$  denote the point where  $B(c)$  touches  $P$ . Then  $x' \in D_y$ , since by Construction 33  
 684  $B(z) \subset B$  for all  $z$  in the segment  $vx$ , and hence  $B(c) \cap P \subset B \cap P$ . Also,  $x' \in V(x)$ , by the  
 685 definition of  $V(x)$ . Therefore we get  $x' \in D_y \cap V(x)$ . ◀

686 ► **Lemma 36.** *If  $\{x, y\} \in E_4$  with  $x \in Bd_S(\mathcal{Y})$ ,  $y \in Int(F)$ , then  $k_y \leq Lev(y)$ .*

687 **Proof.** Suppose  $\{x, y\} \in E_4$ . Then there exists a ball  $B \in \mathbb{E}^3$  with  $x, y \in \partial B$ , and  
 688  $int(B) \cap \mathcal{Y} = \emptyset$ . Therefore  $D_y := B \cap P$  also satisfies  $int(D_y) \cap \mathcal{Y} = \emptyset$ . By Proposition 35  
 689 we have that  $D_y \cap V(x) \neq \emptyset$ . Therefore,  $y \in V(x) \oplus 2r_y$ , where  $r_y$  is the radius of  $D_y$ . But  
 690 since  $y \in Int(F)$ , we have that any disk having  $y$  on its boundary and containing no point of  
 691  $\mathcal{Y}$  in its interior can have radius at most  $2^{Lev(y)-1}\delta$ . Therefore  $r_y \leq 2^{Lev(y)-1}\delta$ . Now taking  
 692  $k_y$  such that  $2^{k_y}\delta = 2r_y$ , we get that  $k_y \leq Lev(y)$ . ◀

693 Now we partition the pairs of vertices  $\{x, y\} \in E_4$  with  $x \in Bd_S(\mathcal{Y})$ , depending on  
 694 whether  $y \in V_F(x)$  or  $y \in \partial V_F(x) \oplus 2^{k_y}\delta$ . That is, given a facet  $F \in \mathcal{S}$ , let  $E_4(Int(F))$   
 695 denote the set of edges  $\{x, y\} \in E_4$  with  $y \in int(V_F(x))$ , and  $E_4(Bd(F))$  denote the set of  
 696 edges in  $E_4$  with  $y \in \partial(V_F(x)) \oplus 2^k\delta$ , for  $k \in [0, k_y]$ . Define  $E_4(Int) := \bigcup_{F \in \mathcal{S}} E_4(Int(F))$   
 697 and  $E_4(Bd) := \bigcup_{F \in \mathcal{S}} E_4(Bd(F))$  respectively.

698 **Lemma 23.** The proof follows from Lemmas 37 and 38, which bound the expected number  
 699 of edges in  $E_4(Int)$  and  $E_4(Bd)$  respectively. ◀

700 ► **Lemma 37.** *Given a facet  $F \in \mathcal{S}$ ,  $\mathbb{E}[E_4(Int(F))] \leq q \cdot \#(\mathcal{X} \cap F)$ . As a consequence,*  
 701  $\mathbb{E}[E_4(Int)] \leq s$ .

702 **Proof.** Let  $x \in \mathcal{X}$  and  $y \in \mathcal{X} \cap F$ . Let  $\mathcal{E}_{x,y}$  denote the event  $\{x, y\} \in E_4(Int(F))$ . Then  
 703  $\mathcal{E}_{x,y}$  can occur only if (i)  $x \in Bd_S(\mathcal{Y})$  and, (ii)  $y \in Int_S(Y) \cap V_F(x)$ . Fix a choice of  $\mathcal{Y}$ , say

704  $Y \in \binom{\mathcal{X}}{s}$ . Conditioning on this choice of  $\mathcal{Y}$ ,  $Bd_{\mathcal{S}}(\mathcal{Y})$  is a fixed set of points. The number of  
 705 pairs contributing to  $E_4(Int(F))$  is at most  $\sharp(\{(x, y) \in Y \times Y \mid x \in Bd_{\mathcal{S}}(Y), y \in V_F(x)\})$ .  
 706 The main observation is now that since  $\mathcal{V}$  restricted to  $F$  is a sub-division of  $F$ , for each  
 707  $y \in \mathcal{X} \cap F$ , there is a unique  $x = x_y \in Bd_{\mathcal{S}}(Y)$  such that  $y \in V_F(x)$ . Therefore we get

$$708 \quad E_4(Int(F)) \leq \sum_{V_F(x) \in \mathcal{V}: x \in Bd_{\mathcal{S}}(Y)} \sharp(V_F(x) \cap Y) \leq \sharp(Y \cap F).$$

709 Since the last bound holds for any choice of  $Y$ , taking expectation over all choices we get

$$710 \quad \mathbb{E}[E_4(Int(F))] \leq \mathbb{E}[\sharp(\mathcal{Y} \cap F)] = q \cdot \sharp(\mathcal{X} \cap F).$$

711 Now summing over all faces gives  $[E_4(Int)] \leq \mathbb{E}[\sharp(\mathcal{Y})] = s$ .

712 ◀

713 **► Lemma 38.** *Given a facet  $F \in \mathcal{S}$ ,  $\mathbb{E}[E_4(Bd(F))] \leq \left(O(1) \cdot \frac{\kappa^2 L \cdot l(\partial F) s}{A}\right)$ . As a consequence,*  
 714  $\mathbb{E}[E_4(Bd(\mathcal{S}))] \leq \left(O(1) \cdot \frac{\kappa^2 L^2 s}{A}\right)$ .

715 **Proof.** To compute the expected value of  $E_4(Bd(\mathcal{S}))$ , fix a face  $F \in \mathcal{S}$ . Consider a pair of  
 716 points  $x, y \in \mathcal{X}$ , such that  $y \in F$ . Let  $\mathcal{E}_{x,y}$  denote the event  $\{x, y\} \in E_4(Bd(F))$ .

717 The value of  $E_4(Bd)$  is the number of  $x, y \in \mathcal{X}$ , such that  $\mathcal{E}_{x,y}$  occurs. Taking expectations,

$$718 \quad \mathbb{E}[E_4(Bd(F))] \leq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{X} \cap F} \mathbb{P}[\mathcal{E}_{x,y}]. \quad (7)$$

719 Observe that  $\mathcal{E}_{x,y}$  occurs only if (i)  $x \in Bd_{\mathcal{S}}(\mathcal{Y})$  and (ii)  $k_y(x) \leq Lev(y)$ , by applying  
 720 Construction 33, on the plane  $P$ ,  $Z = Bd_{\mathcal{S}}(\mathcal{Y})$ , and  $\mathcal{T} = \mathcal{Y} \cap P$ , and using Proposition 35.  
 721 By Lemma 36,  $k_y(x) \in [0, Lev(y)]$ .

722 Let  $P_{k_1, k_2}$  denote the probability that  $\{x, y\} \in E_4(Bd(F))$ , with  $Lev(x) = k_1$ , and  
 723  $k_y(x) = k_2$ . Equation (7) can be rewritten in terms of  $k_1$  and  $k_2$  as

$$724 \quad \mathbb{E}[E_4(Bd(F))] \leq \sum_{k_1 \geq 0} \sharp(L_{=k_1} \cap \mathcal{X}) \sum_{k_2 \geq 0} \sum_{V_F \in \mathcal{V}} \sharp((\partial V_F \oplus 2^{k_2} \delta) \cap \mathcal{X}) \cdot P_{k_1, k_2}.$$

725 Applying the Decay Lemma 29 with  $t = 2$ ,  $x_1 = x$ ,  $x_2 = y$ ,  $k_1 = \max\{0, k_1 - 1\}$  (since  
 726  $x \in Bd_{\mathcal{S}}(\mathcal{Y})$ ), and  $k_2 = \max\{0, k_2 - 1\}$ , we get

$$727 \quad P_{k_1, k_2} \leq c_1 q^2 \cdot \exp(-f(k^*)),$$

728 where  $k^* := \max\{0, k_1 - 1, k_2 - 1\}$ , and  $f(k^*) = 0$  if  $k^* = 0$ , and  $c'_2 \cdot 2^{2k^*}$  otherwise, with  
 729  $c'_2 = c_2/4$ .

730 As in the proof of Lemma 20, we shall use symmetry to handle the case where  $k_1 \geq k_2$   
 731 and  $k_2 > k_1$  together. We get

$$732 \quad \mathbb{E}[E_4(Bd(F))] \leq 2 \sum_{k_1 \geq 0} \sharp(L_{=k_1} \cap \mathcal{X}) \sum_{k_2 \leq k_1} \sum_{V(x) \in \mathcal{V}} \sharp((\partial V(x) \oplus 2^{k_2} \delta) \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 2^{2k_1}).$$

733 By the Level Size Lemma 27, we get that  $\sharp(L_{=k_1} \cap \mathcal{X}) \leq \frac{9\kappa L 2^{k_1} \delta}{\varepsilon^2}$ . Using Proposition 25, we  
 734 get that  $\sharp(\{\partial V(x) \oplus 2^{k_2} \delta\} \cap \mathcal{Y}) \leq \frac{9\kappa \cdot l(\partial V(x)) 2^{k_2} \delta}{\varepsilon^2}$ . By Proposition 34 (iii), each boundary in  
 735 the partition  $\mathcal{V}$  is convex for some  $x \in Bd_{\mathcal{S}}(\mathcal{Y})$ . Therefore we need to sum  $l(\partial V(x))$  only  
 736 over the convex curves in  $\partial V(x)$ ,  $x \in Bd_{\mathcal{S}}(\mathcal{Y})$ . The length of these curves is at most  $l(\partial F)$ .  
 737 Thus we get

$$738 \quad \mathbb{E}[E_4(Bd(F))] \leq 2L \cdot l(\partial F) \cdot \left(\frac{9\kappa \delta q}{\varepsilon^2}\right)^2 \sum_{k_1 \geq 0} 2^{k_1} \sum_{k_2 \leq k_1} 2^{k_2} c_1 \cdot \exp(-c'_2 \cdot 2^{2k_1}).$$

739 Using Lemma 46, the above summation is bounded by a constant. This comes to  $c_1 \cdot$   
 740  $O(1) \left( \frac{(9\kappa)^2 L \cdot l(\partial F) \delta^2 q^2}{\varepsilon^4} \right) = O\left( \frac{\kappa^2 L \cdot l(\partial F) s}{A} \right)$ , where the last step followed from the lower bound  
 741 on  $n$  in Proposition 24 (3), and the identities  $q = s/n = \delta^2/\varepsilon^2$ . Summing  $y$  over all facets  $F$   
 742 in  $\mathcal{S}$ , we get  $\mathbb{E}[E_4(Bd(\mathcal{S}))] = \left( O(1) \cdot \frac{\kappa^2 L^2 s}{A} \right)$ . ◀

## 743 **6 Randomized Incremental Construction (Proof of Theorem 11)**

744 In this section, we show how theorems 9 and 10 imply bounds on the computational complexity  
 745 of constructing Delaunay triangulations of  $\varepsilon$ -nets. Our main tool shall be Theorem 7. However,  
 746 we need to show first that Condition 6 holds. The standard proof of this (see e.g. [12], [10],  
 747 also the discussion in [9](Section 2.2 D)) is sketched below.

748 Now we come to the proof of Theorem 11.

749 **Proof.** To verify that Condition 6 indeed holds in the Euclidean metric case, observe first  
 750 that the union  $\mathcal{C}_p$  of the simplices in conflict with a new point  $p$  is a connected set. Therefore,  
 751 walking on the adjacency graph of the simplices by rotating around the  $(d-2)$ -simplex shared  
 752 between two adjacent faces on the boundary of  $\mathcal{C}_p$ , is enough to yield the set of new conflicts.  
 753 This idea works directly when the Delaunay complex is embedded in the one-sheeted covering  
 754 of  $\mathbb{T}^d$ . In the  $3^d$ -sheeted covering, there can be at most  $3^{d^2}$  simplices formed using a given  
 755 set of  $d$  points and  $p$ , and we need to check each of these possible simplices. Thus the time  
 756 goes up by a multiplicative factor of  $3^{d^2}$ . However, as the increase is by a constant factor  
 757 depending only on the dimension, Condition 6 is still satisfied, albeit with a larger constant.  
 758 Now Theorem 7 can be applied to get the claimed result. ◀

## 759 **7 Euclidean Orbifolds and Bounded-Distortion Metrics**

760 In this section, we shall give some extensions of Theorem 9 and 11. The proofs of our  
 761 theorems follow by finding covering spaces of bounded multiplicity where the Delaunay  
 762 complex can be embedded, and generalizing Lemmas 16–19 to such spaces.

763 Given a space  $\mathcal{S}$ ,  $\varepsilon \in [0, 1]$  and  $\kappa \in \mathbb{Z}^+$ , an  $(\varepsilon, \kappa)$ -sample is a set of points for which any  
 764 ball of radius  $\varepsilon$  in  $\mathcal{S}$ , contains at least one point and at most  $\kappa$  points.

765 ▶ **Theorem 39.** *Theorems 9, 10 and 11 hold when the point set is an  $(\varepsilon, \kappa)$  sample.*

766 ▶ **Theorem 40.** *Theorems 9, 10 and 11 also work for the case when the random sample is  
 767 an i.i.d. sample with probability parameter  $q = s/n$ .*

768 **Proof Sketch.** The proof is on the lines of Theorems 9 and 10. It is only when computing  
 769  $P_p(k)$  that we make use of the fact that points are selected independently. Here, we get  
 770 directly that  $P_p(k) \leq q^d(1-q)^{n_k} \leq q^d \cdot \exp(-qn_k)$ . The rest of the proof follows as before. ◀

771 Coming to our results for Delaunay triangulations of Euclidean  $d$ -manifolds and embedded  
 772 metrics in  $\mathbb{T}^d$ , we need a few definitions first.

### 773 **Euclidean $d$ -orbifolds**

774 A  $d$ -dimensional Bieberbach group  $\mathcal{G}$  is a discrete group of isometries acting on  $\mathbb{E}^d$ . A  
 775  $d$ -orbifold  $\mathbb{E}^d/\mathcal{G}$  is the compact quotient space (i.e. collection of orbits) of  $\mathbb{E}^d$  acted on by a  
 776  $d$ -dimensional Bieberbach group  $\mathcal{G}$ . When the group action is free (i.e. has no fixed points),  
 777 the  $d$ -orbifold is a closed Euclidean  $d$ -manifold. Every Euclidean  $d$ -manifold is the quotient  
 778 space of some  $d$ -Bieberbach group acting on  $\mathbb{E}^d$  [7], [31]. For Euclidean  $d$ -orbifolds, we have:

779 ► **Theorem 41.** *Given a closed Euclidean  $d$ -orbifold  $\mathbb{M} = \mathbb{E}^d/\mathcal{G}$ , equipped with the Euclidean*  
 780 *metric, where  $\mathcal{G}$  is a  $d$ -Bieberbach group, there exists a covering space  $\mathcal{C}_{\mathbb{M}}$  with multiplicity*  
 781  *$m = m^*(\mathcal{G}, d)$ , such that the Delaunay complex on  $\mathbb{M}$  is a triangulation of  $\mathcal{C}_{\mathbb{M}}$ , and the*  
 782 *statements of Theorems 9 and 11 apply for  $\varepsilon$ -nets, for any  $\varepsilon \in [0, 1/4]$ .*

783 **Proof.** In this case, the existence of the covering space follows from the algorithmic version  
 784 of Bieberbach's theorem [6] by Caroli-Teillaud [11](Section 4).

785 ► **Theorem 42** (Bieberbach [6], Caroli-Teillaud [11]). *Every Euclidean  $d$ -orbifold  $\mathbb{M}$  has a*  
 786 *covering space using a number  $m_{\mathbb{M}}$  of sheets of a  $d$ -hyperparallelepiped  $\tilde{\mathbb{T}}_{\mathbb{M}}^d$ , where  $m_{\mathbb{M}}$  depends*  
 787 *only on  $d$ , such that the Delaunay triangulation of any point set on  $\mathbb{M}$  is the projection of*  
 788 *the Delaunay complex of the cover of the point set in the covering space.*

789 The proof of Theorem 41 is on similar lines as that of Theorem 9, except (i) we work with  
 790 the hyperparallelepiped  $\tilde{\mathbb{T}}_{\mathbb{M}}^d$ , and (ii) we need to take the effect of the multiplicity (i.e. the  
 791 number  $m_{\mathbb{M}}$  of sheets of  $\tilde{\mathbb{T}}_{\mathbb{M}}^d$  required for Theorem 42 to hold) into account for all simplices.  
 792 To handle (i), we observe that the volume of balls will change as the hyperparallelepiped is  
 793 no longer a hypercube. Thus, a factor of the volume of the unit hyperparallelepiped  $\tilde{\mathbb{T}}_{\mathbb{M}}^d$ , will  
 794 come into the estimates in Lemma 5. To handle the effect of multiplicity, we introduce an  
 795 extra multiplicative factor of  $m_{\mathbb{M}}^d$  in the bound of the number of possible  $d$ -simplices with  
 796 any fixed set of points (compared to Lemma 19). Additionally, we take into account that  
 797 the number of *distinct* points inside a potential simplex is at least a  $1/m_{\mathbb{M}}$ -fraction of the  
 798 number guaranteed by Lemma 13. This gives a worse bound for the expected complexity of  
 799 the star than in Theorem 9, but still a constant. ◀

### 800 Embedded metrics with bounded distortion

801 For a metric  $\mathfrak{d}$  on some domain  $\mathcal{S}$  embedded in  $\mathbb{E}^d$ , define its *distortion*  $\kappa_{\mathfrak{d}}$  (with respect to  
 802  $\mathbb{E}^d$ ) to be the minimum  $\lambda \geq 1$  such that  $\forall x, y \in \mathcal{S} : \frac{1}{\lambda}\|x - y\| \leq \mathfrak{d}(x, y) \leq \lambda\|x - y\|$ . A  $d \times d$   
 803 matrix  $M \in \mathbb{E}^d$  is *positive definite* if, for all  $x \neq 0 \in \mathbb{E}^d$ ,  $x^T M x > 0$ . For a positive definite  
 804 matrix  $M$ , define its *condition number*  $c_M$  to be the ratio of its maximum to its minimum  
 805 eigenvalue.

806 For embedded metrics with bounded distortion, we have:

807 ► **Theorem 43.** *Given a metric  $\mathfrak{d}$  over  $\mathbb{T}^d$  with distortion  $\kappa_{\mathfrak{d}} < \infty$ , there exists an integer*  
 808  *$m = m_{\mathfrak{d}} < (2\kappa_{\mathfrak{d}}\sqrt{d})^d$ , such that the Delaunay triangulation over  $(\mathbb{T}^d, \mathfrak{d})$  embeds in  $\mathbb{T}_m^d$*   
 809 *with the Euclidean metric. In particular, if  $\mathfrak{d}$  is of the form  $\mathfrak{d}(x, y) = \sqrt{(x - y)^T M (x - y)}$ ,*  
 810  *$x, y \in \mathbb{T}^d$ , where  $M \in \mathbb{E}^{d \times d}$  is a positive definite matrix having condition number at most  $c_M$ ,*  
 811 *then  $m \leq (2c_M\sqrt{d})^d$ . Hence, given any  $\varepsilon \in [0, 1/4]$ , the statements of Theorems 9 and 11*  
 812 *apply for  $\varepsilon$ -nets over the metric space  $(\mathbb{T}^d, \mathfrak{d})$ .*

813 For Theorem 43, we use a geometric condition of Caroli-Teillaud [11] (Criterion 3.11) to  
 814 explicitly bound the multiplicity of the covering space.

815 **Proof of Theorem 43.** The action of  $\mathbb{Z}^d$  on  $\mathbb{E}^d$  is defined by translation, i.e. for  $x \in \mathbb{E}^d$ ,  
 816  $g \in \mathbb{Z}^d$ ,  $gx := g \cdot x = g + x$ . For a finite point set  $P \in \mathbb{E}^d$ , let  $\Delta(\mathbb{Z}^d P)$  denote the largest  
 817 ball in  $\mathbb{E}^d$  containing no points from  $\mathbb{Z}^d P$ . Let  $\delta((k\mathbb{Z})^d)$  denote the minimum distance by  
 818 which a point in  $\mathbb{E}^d$  is translated by  $(k\mathbb{Z})^d$ . Finally, let  $\pi(\cdot)$  denote the projection map of the  
 819 covering space  $\mathbb{T}_k^d$  on to  $\mathbb{T}^d$ . We shall use the following geometric condition of Caroli and  
 820 Teillaud [11].

821 ► **Lemma 44.** *If  $\Delta(\mathbb{Z}^d P) < \delta((k\mathbb{Z})^d)/2$ , then for any finite  $Y \supset P$ , the projection*  
 822  *$\pi(\text{Del}(\mathbb{Z}^d Y))$  is a triangulation of  $\mathbb{T}^d$ .*



823 Now, observe that in the Euclidean metric, the diameter  $\Delta_{\|\cdot\|}(\mathbb{Z}^d P)$  of the largest ball not  
 824 containing any point from the set  $\mathbb{Z}^d P$  is at most  $\sqrt{d}$ , with equality holding when  $\sharp(P) = 1$ ,  
 825 and that  $\Delta(S') \leq \Delta(S)$  for any  $S' \supseteq S$ , since adding points can only decrease the diameter  
 826 of the largest empty ball. Therefore, in the metric  $\mathfrak{d}$ , we have that

$$827 \quad \Delta_{\mathfrak{d}}(\mathbb{Z}^d P) \leq \Delta_{\|\cdot\|}(\mathbb{Z}^d P) \cdot \left( \max_{x,y \in \mathcal{S}} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \right).$$

828 By the previously observed bound on  $\Delta_{\|\cdot\|}(\mathbb{Z}^d P)$ , we have

$$829 \quad \Delta_{\mathfrak{d}}(\mathbb{Z}^d P) \leq \left( \max_{x,y \in \mathcal{S}} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \right) \cdot \sqrt{d}.$$

830 Also, letting  $((k\mathbb{Z})^d)^*$  denote the non-identity elements of  $(k\mathbb{Z})^d$ ,

$$831 \quad \delta_{\mathfrak{d}}((k\mathbb{Z})^d) = \min_{x \in \mathcal{G}, g \in ((k\mathbb{Z})^d)^*} \mathfrak{d}(x, gx) = \min_{x \in \mathcal{G}, g \in ((k\mathbb{Z})^d)^*} \frac{\mathfrak{d}(x, gx)}{\|x - gx\|} \cdot \|x - gx\|.$$

832 For the flat torus  $\mathbb{T}_k^d$ ,  $\min_{x,g} \|x - gx\| = k$ . Therefore, in the metric  $\mathfrak{d}$ , the condition of  
 833 Lemma 44 is satisfied if

$$834 \quad \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \cdot \sqrt{d} < \frac{1}{2} \left( \min_{x \in P, g \in ((k\mathbb{Z})^d)^*} \frac{\mathfrak{d}(x, gx)}{\|x - gx\|} \cdot \|x - gx\| \right), \text{ or,}$$

$$835 \quad 2\sqrt{d} \cdot \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \cdot \max_{x' \in P, g \in ((k\mathbb{Z})^d)^*} \frac{\|x' - gx'\|}{\mathfrak{d}(x', gx')} < k, \text{ which is true if}$$

$$836 \quad 2\sqrt{d} \cdot \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \cdot \min_{x',y' \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x',y')}{\|x' - y'\|} < k, \text{ or,}$$

$$837 \quad 2\sqrt{d} \cdot \kappa_{\mathfrak{d}} < k,$$

838 since by definition, the distortion  $\kappa_{\mathfrak{d}}$  satisfies for all  $x, y \in \mathbb{Z}^d P$ ,  $\mathfrak{d}(x, y) \leq \kappa_{\mathfrak{d}} \|x - y\|$ , i.e.  
 839  $\kappa_{\mathfrak{d}} \geq \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|}$ , as well as  $\|x - y\| \leq \kappa_{\mathfrak{d}} \mathfrak{d}(x, y)$ , i.e.  $\kappa_{\mathfrak{d}} \geq \min_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|}$ .

840 Since the fundamental domain of  $\mathbb{T}_k^d$  contains  $k^d$  copies of the fundamental domain of  $\mathbb{T}^d$ ,  
 841 we have  $m \leq k^d$ , and so the first part of the theorem follows with  $m \leq (2\sqrt{d}\kappa_{\mathfrak{d}})^d$ . The second  
 842 part easily follows from the well-known linear algebraic facts that (a)  $\max_{\|x\|=1} \{x^\top Mx\} =$   
 843  $\max_{\|x\|=1} \{\|Ax\|\} = \sigma_{\max}(A)$ , and that (b)  $\min_{\|x\|=1} \{x^\top Mx\} = \sigma_{\max}(A^{-1}) = \sigma_{\min}(A)$  where  
 844  $A$  is such that  $M = A^\top A$  and  $\sigma_{\max}(A)$ ,  $\sigma_{\min}(A)$  are respectively the largest and the smallest  
 845 singular values of  $A$ . ◀

## 846 8 Conclusion and Remarks

847 In this paper, we analyzed the behaviour of the usual RIC algorithm for the Delaunay  
 848 triangulation of nice point sets, focussing on the cases where the ambient space is the flat  
 849  $d$ -torus or a polyhedral surface in  $\mathbb{E}^3$ . Similar questions can be asked for other spaces where  
 850 the Delaunay triangulation is known to have low complexity for "nice" point sets.

851 We leave for further research, a more general analysis of RIC of Delaunay triangulations  
 852 of cases such as polyhedral surfaces in higher dimensions, as well as extending the techniques  
 853 developed in this paper to the RIC of other geometric problems.

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940 [n-ball#Low\\_dimensions](https://en.wikipedia.org/wiki/Volume_of_an_n-ball#Low_dimensions).

## 942 **A** Appendix

### 943 Proofs from Section 1

- 944 ► **Lemma 45.** Given  $\varepsilon \in (0, 1]$ , an  $\varepsilon$ -net  $\mathcal{X}$  over  $\mathbb{T}^d$ , of  $n$  points, a uniformly random sample  
945  $S \subset \mathcal{X}$  of  $k$  points and  $\varepsilon' = \varepsilon \sqrt[d]{n/k}$ , then
- 946 (i) With high probability, there exists a ball of radius  $\varepsilon'$  with  $\Omega(\log k / \log \log k)$  points in  $S$ .
- 947 (ii) With probability at least a constant, there exists a ball of radius  $\Omega(\varepsilon' \log k)$  with no points  
948 in  $S$ .

949 **Sketch of Proof.** The following balls-and-bins and coupon-collector arguments which can  
950 be shown to prove this:

- 951 (i) Assuming the  $\varepsilon$ -net is over the unit cube or unit ball in  $\mathbb{E}^d$ , a volume argument gives  
 952 that there are  $\Omega(\varepsilon'^{-d}) = \Omega(k)$  disjoint balls of radius  $\varepsilon'$ , which will be our "bins". Now  
 953 choosing  $k$  points from the  $n$  points in the  $\varepsilon$ -net is akin to throwing  $k$  packets into  $k$  bins  
 954 - with high probability, the maximum load of a bin is  $\Omega(\frac{\log k}{\log \log k})$ , i.e. there exists a ball  
 955 of radius  $\varepsilon'$  with  $\Omega(\frac{\log k}{\log \log k})$  points.
- 956 (ii) Similarly, there are  $\Omega(\varepsilon' / \log k) = \Omega(k / \log k)$  balls of radius  $\varepsilon' \sqrt[d]{\log k}$  that cover the  $\varepsilon$ -net,  
 957 and choosing  $k$  points from this covering, can be thought of as an instance of the coupon  
 958 collector problem: we are allowed  $k$  draws (i.e. we choose  $k$  points), each draw is one of  
 959 the "coupons" (i.e. the balls in the covering) with equal probability, and we want the  
 960 probability that at least one coupon has not been collected, i.e. at least one ball from the  
 961 covering has no point chosen from it. Now the well-known lower tail of the number of  
 962 draws taken to collect all coupons in the coupon collector problem, gives that with  
 963 high probability, for  $k' = \Omega(k / \log k)$  coupons, and at most  $O(k' \log k') = k$  draws, all  
 964 coupons will not be collected, that is, some ball of radius  $\Omega(\varepsilon' \sqrt[d]{\log k})$  will be empty.  
 965 ◀

## 966 Proofs from Section 5

967 ▶ **Lemma 46.** *Given  $a > 0$ ,  $b \in (0, 1)$ , the sum  $\sum_{n \in \mathbb{Z}_+} 2^{an} \cdot \exp(-b \cdot 2^{an})$  is at most*  
 968  $\frac{2 \log_2(1/b)}{eab}$ .

969 **Proof of Lemma 46.** The proof follows from elementary calculus. The maximum term  
 970 is when  $2^{an} = 1/b$ , i.e.  $n = \frac{\log_2(1/b)}{a}$ , and evaluates to  $2^{an} \exp(-b \cdot 2^{an}) = \frac{1}{eb}$ , and terms  
 971 decrease exponentially afterwards. The sum is therefore upper bounded by twice the maximum  
 972 term, times the number of terms before the maximum, which is the claimed bound. ◀