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Randomized incremental construction of Delaunay triangulations of nice point sets

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1 — Abstract —

2 *Randomized incremental construction* (RIC) is one of the most important paradigms for building
3 geometric data structures. Clarkson and Shor developed a general theory that led to numerous
4 algorithms that are both simple and efficient in theory and in practice.

5 Randomized incremental constructions are most of the time space and time optimal in the
6 worst-case, as exemplified by the construction of convex hulls, Delaunay triangulations and ar-
7 rangements of line segments. However, the worst-case scenario occurs rarely in practice and we
8 would like to understand how RIC behaves when the input is nice in the sense that the associated
9 output is significantly smaller than in the worst-case. For example, it is known that the Delaunay
10 triangulations of nicely distributed points in \mathbb{E}^d or on polyhedral surfaces in \mathbb{E}^3 has linear com-
11 plexity, as opposed to a worst-case complexity of $\Theta(n^{\lfloor d/2 \rfloor})$ in the first case and quadratic in the
12 second. The standard analysis does not provide accurate bounds on the complexity of such cases
13 and we aim at establishing such bounds in this paper. More precisely, we will show that, in the
14 two cases above and variants of them, the complexity of the usual RIC is $O(n \log n)$, which is
15 optimal. In other words, without any modification, RIC nicely adapts to good cases of practical
16 value.

17 Along the way, we prove a probabilistic lemma for sampling without replacement, which may
18 be of independent interest.

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19 **1** Introduction

20 The *randomized incremental construction* (RIC) is an algorithmic paradigm introduced
 21 by Clarkson and Shor [11], which has since found immense applicability in computational
 22 geometry, e.g. [27, 26]. The general idea is to process the input points sequentially in a
 23 random order, and to analyze the expected complexity of the resulting procedure. The
 24 theory developed by Clarkson and Shor is quite general and led to numerous algorithms that
 25 are simple and efficient, both in theory and in practice. On the theory side, randomized
 26 incremental constructions are most of the time space and time optimal in the worst-case, as
 27 exemplified by the construction of convex hulls, Delaunay triangulations and arrangements
 28 of line segments. Randomized incremental constructions appear also to be very efficient in
 29 practice, which, together with their simplicity, make them the most popular candidates for
 30 implementations. Not surprisingly, the CGAL library includes several randomized incremental
 31 algorithms, e.g. for computing Delaunay triangulations.

32 Experimental evidence has shown that randomized incremental constructions work very
 33 well beyond the worst-case, which is fortunate since worst-case situations are rare in many
 34 applications. This paper aims at extending the analysis of RIC to the case of *average-*
 35 *case* complexity. More precisely, our goal is to understand how randomized incremental
 36 constructions behave when the input is nice in the sense that the associated construction is
 37 significantly smaller than in the worst-case.

38 We need a model of good point sets to describe the input data and analyze the algorithms.
 39 This will be done through the notion of ε -nets, which have a long and rich history since their
 40 introduction in the 1950's in the works of Kolmogorov and others on functional analysis and
 41 topological vector spaces (see e.g. [31]). ε -nets have become ubiquitous in many theoretical
 42 as well as applied areas, from geometry, functional analysis to probability theory, statistics
 43 etc., where they are often used as countable or finite approximations of continuous spaces.

44 When we enforce such a hypothesis of “nice” distribution of the points in space, a volume
 45 counting argument ensures that the local complexity of the Delaunay triangulation around a
 46 vertex is bounded by a constant (dependent only on the dimension).

47 Unfortunately, to be able to control the complexity of the usual randomized incremental
 48 algorithms [14, 9, 11, 3], it is not enough to control the final complexity of the Delaunay
 49 triangulation. We need to control also the complexity of the triangulation of random subsets.
 50 One might expect that a random subsample of size k of an ε -net is also an ε' -net for $\varepsilon' = \varepsilon \sqrt[d]{\frac{\pi}{k}}$.
 51 Actually this is not quite true, it may happen with reasonable probability that a ball of
 52 radius $O(\varepsilon')$ contains $\Omega(\log k / \log \log k)$ points or that a ball of radius $\Omega(\varepsilon' \sqrt[d]{\log k})$ does
 53 not contain any point. For the convenience of the reader, we briefly sketch the proofs in
 54 the Appendix (Lemma 43). However, it can only be shown that such a subsample is an
 55 $\left(\frac{\varepsilon'}{\log(1/\varepsilon)}\right)$ -covering and an $(\varepsilon' \log(1/\varepsilon))$ -packing, with high probability. Thus this approach
 56 can transfer the complexity of an ε -net to the one of a random subsample of an ε -net but
 57 with an extra multiplicative factor of $\Omega(\log 1/\varepsilon) = \Omega(\log n)$. It follows that, in the two cases
 58 we consider, the standard analysis does not provide accurate bounds on the complexity of
 59 the (standard) randomized incremental construction.

60
 61 **Related Work:** The Delaunay triangulations of nicely-distributed points have been stud-
 62 ied since the 50's, e.g. in the work of Meijering [22], Gilbert [20], Miles [23] Møller [25],
 63 Golin-Na [21], for Poisson-distributed points, Dwyer [16, 15] for uniformly distributed points,
 64 Attali-Boissonnat [4], Attali-Boissonnat-Lieutier [5], Amenta-Attali-Devillers [2], and others
 65 for (ε, κ) -samples, and Erickson [17, 18] for points with bounded spread (the ratio between

66 the maximum to minimum distance between any two points). Except for a few authors such
 67 as Dwyer [16] and Erickson [18], most of the above results discuss only the combinatorial
 68 aspects and not the algorithmic ones. For Poisson and uniformly distributed point samples,
 69 we observe that the standard analysis of the RIC procedure immediately implies a bound on
 70 the expected run-time, of the order of the expected number of simplices times a logarithmic
 71 factor, which is optimal. However, for deterministic notions of nice distributions such as ε
 72 nets, (ε, κ) samples, and bounded spread point sets, the standard RIC analysis is not optimal,
 73 since, as we observed, it gives at least an extra logarithmic factor for (ε, κ) samples and
 74 even worse for bounded spread point-sets, as stated in an open problem by Erickson [18].
 75 Miller, Sheehy and Velingker [24] follow a very different approach, giving an algorithm to
 76 compute the approximate Delaunay graph of nicely-spaced superset of points for an arbitrary
 77 input point-set, with optimal time complexity and a $2^{O(d)}$ -dependence on the dimension.
 78 However their algorithm is quite complicated and uses several subroutines that have varying
 79 difficulties of implementation. The RIC, while having a worse $2^{O(d^2)}$ dependence on the
 80 dimension (which Miller et al. observe, may be impossible to avoid for computing the exact
 81 Delaunay graph), computes the entire Delaunay triangulation of the given point set, i.e. not
 82 a superset, is easy to implement and works efficiently in practice.

83

84 **Our contribution:** We consider two main questions in this paper. First, we consider the
 85 case of an ε -net in the periodic space of dimension d , which, as mentioned before, have linear
 86 complexity instead of the worst-case $\Theta(n^{\lfloor d/2 \rfloor})$. The reason to consider a periodic space is to
 87 avoid dealing with boundary effects that would distract us from the main point, and the fact
 88 that periodic spaces are often used in practice, e.g. in simulations in astronomy, biomedical
 89 computing, solid-state chemistry, condensed matter physics, etc. [10, 12, 28, 19, 32]. Following
 90 this, we deal with ε -nets on a polyhedral surface of \mathbb{E}^3 , which is also a commonly-occurring
 91 practical scenario in e.g. surface reconstruction [1, 8], and has Delaunay triangulations with
 92 linear, as opposed to the worst-case quadratic, complexity. In this case, the boundary effects
 93 need to be explicitly controlled, which requires a more careful handling along with some new
 94 ideas. In both cases, we establish tight bounds and show that the complexity of the usual
 95 RIC is $O(n \log n)$, which is optimal. Hence, without any modification, the standard RIC
 96 nicely adapts to the good cases above.

97 Our main new technical development is a general bound for the probability of certain
 98 non-monotone events in sampling without replacement, which may be of independent interest.

99

100 **Extensions** We also give some extensions of our results for periodic spaces. Our extensions
 101 are in four directions: (i) a more general notion of well-distributed point sets, the (ε, κ)
 102 samples (ii) a different notion of subsampling - the Bernoulli or i.i.d. sample where each
 103 point is selected to be in \mathcal{Y} independently of the others, with probability $q = s/n$, (iii) a
 104 more general class of spaces - Euclidean d -orbifolds, and (iv) a more general class of metrics
 105 - those having bounded-distortion with respect to the Euclidean metric. Precisely, for all the
 106 above cases, we show that the Delaunay triangulation of a random subsample has a linear
 107 size in expectation. We believe that our methods should work for an even larger classes
 108 of spaces, though this might require more delicate handling of boundary effects and other
 109 features specific to the metric space under consideration.

110 **Outline** The rest of the paper is as follows. In Section 2, we define the basic concepts
 111 of Delaunay triangulation, ε -net, flat torus and random samples. We state our results in
 112 Section 3. In Section 4, we bound the size of the Delaunay triangulation of a uniform random
 113 sample of a given size extracted from an ε -net on the flat torus \mathbb{T}^d . In Section 5, we analyse

114 the case when the uniform random subsample is drawn from an ε -net on a polyhedral surface
 115 in \mathbb{E}^3 . In Section 6, we use the size bounds established in Sections 4 and 5, to compute
 116 the space and time complexity of the randomized incremental construction for constructing
 117 Delaunay triangulations of ε -nets. Finally, in Section 7, we state and prove some extensions.
 118 Proofs missing from the main sections are given in the Appendix.

119 2 Background

120 2.1 Notations

121 We denote by $\Sigma(p, r)$, $B(p, r)$ and $B[p, r]$, the sphere, the open ball, and the closed ball of
 122 center p and radius r respectively. For $x \in \mathbb{E}^2$, $y \geq 0$, $D(x, r)$ denotes the disk with center x
 123 and radius r , i.e. the set of points $\{y \in \mathbb{E}^2 : \|y - x\| < r\}$, and similarly $D[x, r]$ denotes the
 124 corresponding closed disk. The volume of the unit Euclidean ball of dimension d is denoted
 125 V_d and the area of the boundary of such a ball is denoted S_{d-1} . It is known that $V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$
 126 and $S_d = 2\pi V_{d-1}$, where $\Gamma(t) := \int_0^\infty e^{-x} x^{t-1} dx$, ($t > 0$) denotes the *Gamma function*. For
 127 $d \in \mathbb{Z}^+$, $\Gamma(d+1) = d!$. We note that $2^d d^{-d/2} \leq V_d \leq 2^{4d} d^{-d/2}$ (see e.g. [33]). For an event \mathcal{E}
 128 in some probability space Ω , we use $\mathbf{1}_{[\mathcal{E}]}$ to denote the indicator variable $\mathbf{1}_{[\mathcal{E}]} = \mathbf{1}_{[\mathcal{E}]}(\omega)$ which
 129 is 1 whenever $\omega \in \mathcal{E}$, and zero otherwise. We use $[n]$ to mean the set $\{1, 2, \dots, n\}$. Given a
 130 discrete set A , $\#(A)$ denotes its cardinality and, for $k \in \mathbb{Z}^+$, $\binom{A}{k}$ denotes the set of k -sized
 131 subsets of A . Given an event A in some probability space, $\mathbb{P}[A]$ denotes the probability of A
 132 occurring. For a random variable Z in a probability space, $\mathbb{E}[Z]$ denotes the expected value
 133 of Z . Lastly, $e = 2.7182\dots$ denotes the base of the natural logarithm.

134 2.2 ε -nets

135 A set \mathcal{X} of n points in a metric space \mathcal{M} , is an ε -packing if any pair of points in \mathcal{X} are at
 136 least distance ε apart, and an ε -cover if each point in \mathcal{M} is at distance at most ε from some
 137 point of \mathcal{X} . \mathcal{X} is an ε -net if it is an ε -cover and an ε -packing simultaneously.

138 The definition of ε -net applies for any metric space. In the case of the Euclidean metric,
 139 we can prove some additional properties. We shall use $\|\cdot\|$ to denote the Euclidean ℓ_2 norm.

140 ► **Lemma 1** (Maximum packing size). *Let $\rho \leq 1$, then any packing of the ball of radius $r \geq \rho$
 141 in dimension d by disjoint balls of radius $\rho/2$ has a number of balls smaller than $\left(\frac{3r}{\rho}\right)^d$.*

142 ► **Lemma 2** (Minimum cover size). *Any covering of a ball of radius r in dimension d by balls
 143 of radius ρ has a number of balls greater than $\left(\frac{r}{\rho}\right)^d$.*

144 For $d \in \mathbb{Z}^+$, the *flat d -torus* \mathbb{T}^d is the compact quotient group $\mathbb{E}^d/\mathbb{Z}^d$, with addition as
 145 the group action. More generally, for $k \in \mathbb{Z}^+$, the flat torus of length k is $\mathbb{T}_k^d := \mathbb{E}^d/(k\mathbb{Z})^d$.

146 ► **Lemma 3** (ε -net size bounds). *Given $\varepsilon \in (0, 1]$, let \mathcal{X} be an ε -net over the flat torus \mathbb{T}^d .
 147 Then, $\#(\mathcal{X}) \in [d^{d/2} 2^{-3d} \cdot \varepsilon^{-d}, d^{d/2} \varepsilon^{-d}]$.*

148 2.3 Delaunay triangulation

149 For simplicity of exposition and no real loss of generality, all finite point sets considered in
 150 this paper will be assumed to be in *general position*, i.e. no set of $d+2$ points lie on a sphere.
 151 Given a set \mathcal{X} in Euclidean space \mathbb{E}^d in general position, the *Delaunay triangulation* of \mathcal{X} , or
 152 $Del(\mathcal{X})$, is the geometric simplicial complex with vertex set \mathcal{X} such that any d -simplex σ can

153 be circumscribed by an empty ball, i.e. a ball whose bounding sphere contains the vertices
 154 of σ and whose interior contains no points of \mathcal{X} . It is known that such a triangulation exists,
 155 is unique, and is embedded in \mathbb{E}^d under the general position assumption. Given a set \mathcal{X} in
 156 Euclidean space \mathbb{E}^d , having its Delaunay triangulation $Del(\mathcal{X})$, the *star* of a subset $S \in \mathcal{X}$,
 157 or $star(S)$, is the set of all simplices in $Del(\mathcal{X})$ which are incident to at least one point in
 158 S . For a point $p \in \mathcal{X}$, we shall use the shorthand expression $star(p)$ to mean $star(\{p\})$,
 159 and $star^2(p)$ to mean $star(star(\{p\}))$ respectively. Given topological spaces \mathcal{S} and \mathcal{C} , and
 160 a continuous map $\pi : \mathcal{C} \rightarrow \mathcal{S}$, \mathcal{C} is a *covering space* of \mathcal{S} if π is such that for every point
 161 $x \in \mathcal{S}$, there exists an open neighbourhood U of x , such that the pre-image $\pi^{-1}(U)$ is a
 162 disjoint union of open neighbourhoods in \mathcal{C} , each of which is homeomorphically mapped onto
 163 U by π . The cardinality of the pre-image of any point $x \in \mathcal{S}$ under the covering map π is
 164 the *multiplicity* of x . A covering space of \mathcal{S} , with a covering map having multiplicity m is
 165 referred to as an *m-sheeted covering* of \mathcal{S} .

166 For example, \mathbb{T}_k^d forms a k^d -sheeted *covering space* of \mathbb{T}^d , with the covering map $x \mapsto x$
 167 mod 1, the modulus operation being defined coordinate-wise. Caroli and Teillaud [10] showed

168 ► **Theorem 4** (Caroli-Teillaud [10]). *The Delaunay triangulation of any finite point set in \mathbb{T}^d*
 169 *embeds in the 3^d -sheeted covering of \mathbb{T}^d . If the maximum circumradius of a simplex is at*
 170 *most $1/2$, then the triangulation embeds in the one-sheeted covering of \mathbb{T}^d .*

171 A key property of ε -nets is that their Delaunay triangulations have linear size.

172 ► **Lemma 5** ([29]). *Let $\varepsilon \in (0, 1]$ be given, and let \mathcal{X} be an ε -net over \mathbb{T}^d . Then the Delaunay*
 173 *triangulation of \mathcal{X} , $Del(\mathcal{X})$ has at most $4^{d^2} \varepsilon^{-d}$ simplices.*

174 2.4 Randomized incremental construction and random subsamples

175 For the algorithmic complexity aspects, we state a version of a standard theorem for the
 176 RIC procedure, stated in [13]). We first need a necessary condition for the theorem. When
 177 a new point p is added to an existing triangulation, a *conflict* is defined to be a previously
 178 existing simplex whose circumball contains p .

179 ► **Condition 6.** *At each step of the RIC, the set of newly introduced conflicts can be computed*
 180 *in time proportional to the number of conflicts.*

181 In the Euclidean case, Condition 6 is verified to hold by observing that the set of simplices
 182 in conflict with the new point is a connected set, and therefore walking on the adjacency
 183 graph of the simplices is enough to yield the set of introduced conflicts (see e.g. [11]). We
 184 now come to the general theorem on the algorithmic complexity of RIC.

185 ► **Theorem 7** (Devillers [13] Theorem 5(1,2)). *Let $F(s)$ denote the expected number of*
 186 *simplices that appear in the d -skeleton of the Delaunay complex of a uniform random sample*
 187 *of size s , from a given point set P . Then, if Condition 6 holds and $F(s) = O(s)$, we have*
 188 *(i) The expected space complexity is $O(n)$.*
 189 *(ii) The expected time complexity is $\sum_{s=1}^n \frac{n-s}{s} = O(n \log n)$.*

190 Thus bounding the complexity of incremental algorithms reduces to bounding the expected
 191 complexity of the Delaunay triangulation of random subsets of ε -nets of a given size.

192 A subset \mathcal{Y} of set \mathcal{X} is a *uniform random sample* of \mathcal{X} of size s if \mathcal{Y} is any possible subset
 193 of \mathcal{X} of size s with equal probability. In case the multiplicity of a point in \mathcal{X} is greater than
 194 1, the sample counts only one copy of the point; all other copies are present in \mathcal{Y} if and only
 195 if the original point is present.

196 In order to work with uniform random samples, we shall prove a lemma on the uniformly
 197 random sampling distribution or *sampling without replacement*, which is stated below, and
 198 shall be a key probabilistic component of our proofs. The lemma provides a bound on the
 199 probability of a non-monotone compound event, that is, if the event holds true for a fixed
 200 set of k points, there could exist supersets as well as subsets of the chosen set for which the
 201 event does not hold. This may well be of general interest, as most natural contiguity results
 202 with Bernoulli (i.e. independent) sampling, are for monotone events.

203 ► **Lemma 8.** *Given $a, b, c \in \mathbb{Z}^+$, with $b \leq \min(\frac{a}{2}, c - 1)$, the probability that the random
 204 sample A having cardinality a of a set C of cardinality c , contains the included subset B
 205 having cardinality b , and is disjoint from the excluded subset T , which is disjoint from B and
 206 has cardinality at least t , is at most $(\frac{a}{c})^b \left(1 - \frac{t}{c-b}\right)^{a-b} \leq (\frac{a}{c})^b \cdot \exp\left(-\frac{at}{2c}\right)$,*

207 As a consequence of Lemma 8, it can be shown that for finite sampling spaces, the j -th
 208 moment is essentially the j -th power of the first moment, for sufficiently small j .

209 ► **Lemma 9** (*j -th moment*). *Given $a, b, c \in \mathbb{Z}^+$, and a set C of size c , a fixed subset $B \subseteq C$
 210 of size $b \leq c$ and a uniformly random sample A of size a , then for $j \leq c/2$, the j -th moment
 211 of the random variable $X := \#(A \cap B)$ i.e. $\mathbb{E}[X^j]$, satisfies $\mathbb{E}[X^j] \leq (\mathbb{E}[X])^j$.*

212 **3 Results**

213 **ε -nets in \mathbb{T}^d :** The following theorem gives a constant bound on the expected size of $star(p)$
 214 and $star^2(p)$, for the Euclidean metric on the flat torus \mathbb{T}^d .

215 ► **Theorem 10** (Euclidean metric). *Given an ε -net \mathcal{X} in \mathbb{T}^d in general position, where
 216 $\varepsilon \in (0, \frac{1}{4}]$, the following holds.*

217 (i) *The expected number of simplices incident to a point $p \in \mathcal{X}$, $\mathbb{E}[\#(star(p))]$ in the Delaunay
 218 triangulation of a uniform random sample $S \subset \mathcal{X}$ of size $s \geq 4(2\sqrt{d})^d d^3 + 1$ containing p ,
 219 is less than $2 \cdot 6^{d^2+d}$.*

220 (ii) *The expected number of simplices incident to the simplices in $star(p)$, $\mathbb{E}[\#(star^2(p))]$, is
 221 also a constant.*

222 **Polyhedral Surfaces in \mathbb{E}^3 :** A *polyhedral surface* \mathcal{S} in \mathbb{E}^3 is a collection of a finite number
 223 of polygons $F \subset \mathcal{S}$, called *facets*, which are pairwise disjoint or meet along an edge. In this
 224 paper, \mathcal{S} will denote an arbitrary but fixed polyhedral surface, with \mathcal{C} facets, and having
 225 total length of the boundaries of its faces L and total area of its faces A .

226 We next show that the expected complexity of the Delaunay triangulation of a uniformly
 227 random subsample of an ε -net on a polyhedral surface, is linear in the size of the subsample.

228 ► **Theorem 11.** *Given $\varepsilon \in [0, 1]$, and \mathcal{X} an ε -net on \mathcal{S} , having n points, let $\mathcal{Y} \subset \mathcal{X}$ be a
 229 random sub-sample of \mathcal{X} having size s . Then the Delaunay triangulation $Del(\mathcal{Y})$ of \mathcal{Y} on \mathcal{S}
 230 has $O(s)$ simplices. Further, for a random point $p \in \mathcal{Y}$, $\mathbb{E}[\#(star^2(p))] = O(1)$.*

231 **Algorithmic Bounds:** We next use the above combinatorial bounds to get the space and
 232 time complexity of the randomized incremental construction of the Delaunay triangulation
 233 of an ε -net, on the flat d -torus or on a polyhedral surface in \mathbb{E}^3 .

234 ► **Theorem 12** (Randomized incremental construction). *Given $\varepsilon \in [0, 1/4]$, and an ε -net
 235 \mathcal{X} in general position over (i) a flat torus \mathbb{T}^d , or (ii) a fixed polyhedral surface $\mathcal{S} \subset \mathbb{E}^3$,
 236 then the randomized incremental construction of the Delaunay triangulation takes $O(n \log n)$
 237 expected time and $O(n)$ expected space, where $n = \#(\mathcal{X})$ and the constant in the big O is*

238 independent of n . Further, at each step of the randomized incremental construction, the
 239 Delaunay triangulation of the set \mathcal{Y} of already added points of \mathcal{X} embeds in \mathbb{T}^d .

240 **Extensions:** Finally, our extensions are stated and proved in Section 7.

241 **4 Euclidean Metric on \mathbb{T}^d**

242 In this section, we prove that a subsample \mathcal{Y} of a given size s , drawn randomly from an ε -net
 243 $\mathcal{X} \subset \mathbb{T}^d$, has a Delaunay triangulation in which the star of any given vertex has a constant
 244 expected complexity. Hence, the expected complexity of the triangulation is linear in the size
 245 of the subsample. The constant of proportionality is bounded by 2^{cd^2} , where c is a constant
 246 independent of ε and d .

247 We begin by setting up the proof framework. We recall that $n := \sharp(\mathcal{X})$. Define $q := \frac{s-1}{n-1}$.
 248 Define $\delta := \varepsilon \cdot \left(\frac{2d}{q}\right)^{1/d}$. Let $I_0 := [0, \delta)$, $I_k := [2^{k-1}\delta, 2^k\delta)$ for $k > 0$. To bound the expected
 249 complexity, we shall consider the probability of existence of potential d -simplices in \mathcal{X} ,
 250 incident to p and having radius in the intervals I_k , as k ranges over \mathbb{Z}^+ .

251 Throughout this section, we shall use σ to mean a d -simplex incident to p , with cir-
 252 cumcentre c_σ and circumradius r_σ , and τ to mean the set of vertices of $\sigma \setminus \{p\}$. To count
 253 the number of simplices in $star(p)$ with circumradius in I_k , let $S_p(k)$ denote the set of
 254 possible $(d-1)$ -simplices with vertices in \mathcal{X} , such that for every $\tau \in S_p(k)$, the d -simplex
 255 $\sigma := \tau \cup \{p\}$ has circumradius $r_\sigma \in I_k$. Set $s_p(k) := \sharp(S_p(k))$. Let n_k denote the min-
 256 imum number of points of \mathcal{X} in the interior of the circumball of σ , over all $\tau \in S_p(k)$:
 257 $n_k := \min_{\tau \in S_p(k)} \{\sharp(B(c_\sigma, r_\sigma) \cap \mathcal{X}) : \sigma = \tau \cup \{p\}\}$. For $\tau \in S_p(k)$, let $P_p(k)$ denote an
 258 upper bound on the probability that $\sigma = \tau \cup \{p\}$ appears in $Del(\mathcal{Y})$, that is,

$$259 \quad P_p(k) := \max_{\tau \in S_p(k)} \{\mathbb{P}[\sigma \in Del(\mathcal{Y})]\}.$$

260 Finally, let $Z_p(k)$ denote the subset of simplices $\tau \in S_p(k)$ such that $\sigma \in Del(\mathcal{Y})$.

261 The main lemma in the proof is a bound on the expected complexity of the star of p , in
 262 terms of $s_p(k)$ and $P_p(k)$.

263 **► Lemma 13.** $\mathbb{E}[\sharp(star(p))] \leq \sum_{k \geq 0} \mathbb{E}[Z_p(k)] \leq \sum_{k \geq 0} s_p(k) \cdot P_p(k)$.

264 It only remains, therefore, to establish bounds on $s_p(k)$ and $P_p(k)$ as functions of k ,
 265 and finally to bound the sum $\sum_{k \geq 0} s_p(k) \cdot P_p(k)$. In order to ensure we always have a
 266 triangulation embedded in \mathbb{T}^d , we shall use Theorem 4. Accordingly, we get two different
 267 regimes of the simplices in the triangulation. When the circumradius of a potential simplex
 268 $\sigma \in star(p)$ is at most $1/4$, then the simplex lies in a ball of radius at most $1/2$ with center p .
 269 By Theorem 47, in this regime the triangulation embeds in the one-sheeted covering of \mathbb{T}^d .
 270 Therefore, for a fixed set of vertices, there is a unique circumball. When the circumradius is
 271 greater than $1/4$, the simplex is contained in a ball of radius $> 1/2$ around p , and therefore
 272 embeds in the 3^d -sheeted covering of \mathbb{T}^d , i.e. \mathbb{T}_3^d . In this case, each vertex has 3^d copies, and
 273 so for a given choice of d vertices of a simplex, one can have $(3^d)^d$ circumballs. We shall
 274 therefore split the sum $\sum_{k \geq 0} \mathbb{E}[Z_p(k)]$ into the two regimes, $0 \leq k \leq k_{\max}$, and $k > k_{\max}$,
 275 where k_{\max} denotes $\log_2 \frac{1}{4\delta}$. We first discuss the case when $0 \leq k \leq k_{\max}$.

276
 277 **Simplices with small circumradii:** $k \in [0, k_{\max}]$. We begin by establishing a bound on
 278 $P_p(k)$. Applying Lemma 8,

279 **► Lemma 14.** $P_p(k) \leq q^d \cdot \exp(-qn_k/2)$.

280 Next, we shall upper bound $s_p(k)$ from above. We first state a simple observation.

281 ► **Lemma 15.** *Let σ be a d -simplex incident to p , with circumcentre c_σ and circumradius*
 282 *r_σ . Then $\sigma \subset B[p, 2r_\sigma]$ and $B(c_\sigma, r_\sigma) \subset B(p, 2r_\sigma)$.*

283 Now we can bound $s_p(k)$ using the above observation together with Lemma 1.

284 ► **Lemma 16.** *For $k \geq 0$, $s_p(k) \leq \frac{(6 \cdot 2^k \delta / \varepsilon)^{d^2}}{d!}$.*

285 Finally, we bound n_k from below using Lemma 2.

286 ► **Lemma 17.** *Let σ be a simplex incident to p , having circumradius $r_\sigma \in I_k$, $k \geq 0$.*

$$287 \quad n_k \geq \begin{cases} 0 & k = 0. \\ \left(\frac{2^{k-1}\delta}{\varepsilon}\right)^d & 0 < k \leq k_{\max}. \end{cases}$$

288 **Proof.** When $k \leq k_{\max}$, the radius of the circumball of a simplex $\sigma = \tau \cup \{p\}$, $\tau \in S_p(k)$, is
 289 at most $2^{k_{\max}}\delta \leq 1/4 < 1/2$. Applying Theorem 4, we work in the one-sheeted covering of
 290 \mathbb{T}^d . Using the fact that \mathcal{X} is an ε -covering, we apply Lemma 2 to get that $n_k \geq (2^{k-1}\delta/\varepsilon)^d$.
 291 ◀

292 Next, using the above bounds on $s_p(k)$ and n_k , we shall bound the sum $\sum_{k=0}^{k_{\max}} \mathbb{E}[Z_p(k)]$
 293 in the following couple of lemmas.

294 ► **Lemma 18.** $\mathbb{E}[Z_p(0)] \leq 6^{d^2+d}$.

295 **Proof.** Substituting the bounds on $s_p(0)$ and $P_p(0)$ proved in Lemmas 16, 14 and 17
 296 respectively, we have

$$297 \quad \begin{aligned} \mathbb{E}[Z_p(0)] &= s_p(0) \cdot P_p(0) \\ 298 \quad &= \frac{(6(\delta/\varepsilon))^{d^2}}{d!} \cdot q^d \leq \frac{6^{d^2} \cdot (2d)^d}{d!} \leq 6^{d^2} \cdot (2e)^d, \end{aligned}$$

299 where in the second step we used the definition of δ to get $q(\delta/\varepsilon)^d = 2d$, and in the last step
 300 we used Stirling's approximation $d^d/d! \leq e^d$. Since $2e < 6$, this proves the lemma. ◀

301 ► **Lemma 19.** $\sum_{k=0}^{k_{\max}} \mathbb{E}[Z_p(k)] \leq (1 - (2/e)^6)^{-1} \cdot 6^{d^2+d}$.

302 Lastly, we need to bound the expected number of simplices with exceptionally large
 303 circumradii, i.e. $\sum_{k>k_{\max}} \mathbb{E}[Z_p(k)]$.

304 **Simplices with large circumradii: $k > k_{\max}$.**

305 ► **Lemma 20.** $\sum_{k>k_{\max}} \mathbb{E}[Z_p(k)] \leq 5$.

306 Thus, by Lemmas 13, 19, and 20, the expected complexity of the star of p is at most
 307 $(1 - (2/e)^6)^{-1} \cdot 6^{d^2+d} + 5 \leq 2 \cdot 6^{d^2+d}$ for $d \geq 2$, which completes the proof of Theorem 10 (i).

308 **Theorem 10(ii) (Proof Sketch).** The proof will build on the previous part. The idea is to
 309 consider potential pairs of simplices (σ, τ) , with $\sigma \in \text{star}(p)$ and $\tau \in \text{star}^2(p)$, such that
 310 $\sigma \setminus \{p\} \cap \tau \neq \emptyset$, and bound from above the probability that a pair (σ, τ) can exist in the
 311 Delaunay complex. As in the proof of part (i) of the Theorem, we shall sum over all possible
 312 ranges of pairs of circumradii of the simplex-pairs. We need to (a) upper bound the number
 313 of potential simplex-pairs, in a given range of pairs of circumradii, which we do using a
 314 product bound over the individual ranges, and (b) lower bound the number of points excluded
 315 from being selected in the subsample, if a given simplex-pair is to be realised in the Delaunay
 316 complex. The key idea is to use the larger circumradius of the given simplex-pair for the
 317 lower bound in (b). ◀

5 Polyhedral Surfaces in \mathbb{E}^3

In this section, we introduce a partition of the sub-sample \mathcal{Y} into *boundary* and *interior* points, to do a case analysis of the expected number of edges in the Delaunay triangulation $Del(\mathcal{Y})$, depending on whether the end-points of a potential Delaunay edge, are boundary or interior points, and whether they lie on the same facet or on different facets.

Main ideas: Our overall strategy will be to mesh the proofs of Attali-Boissonnat [4] and Theorem 10. Briefly, Attali and Boissonnat reduce the problem to counting the Delaunay edges of the point sample, which they do by distinguishing between *boundary* and *interior* points of a facet. For boundary points, they allow all possible edges. For interior points, the case of edges with endpoints on the same facet is easy to handle, while geometric constructions are required to handle the case of endpoints on different facets, or that of edges with one endpoint in the interior and another on the boundary. However, we shall need to introduce some new ideas to adapt our previous methods to this setting.

Firstly, an edge can have multiple balls passing through its endpoints and, as soon as one of these balls is empty, the edge is in the triangulation. This is handled using a geometric construction (see Lemma 29). Basically, the idea is to build a constant-sized packing of a sphere centered on a given point, using large balls, such that any sphere of a sufficiently large radius which passes through the point, must contain a ball from the packing.

Secondly, since we have randomly spaced points at the boundaries, boundary effects could penetrate deep into the interior. To handle this, we introduce the notion of *levels* of a surface, instead of the fixed strip around the boundary used in [4], and use a probabilistic, rather than deterministic, classification of boundary and interior points. The new classification is based on the level of a point and the radius of the largest empty disk passing through it.

For a curve Γ , $l(\Gamma)$ denotes its length. For a subset of a surface $R \subset \mathcal{S}$, $a(R)$ denotes the area of R . We next present some general lemmas, which will be needed in the proofs of the main lemmas. For sets $A, B \subset \mathbb{E}^3$, $A \oplus B$ denotes the Minkowski sum of A and B , i.e. the set $\{x + y : x \in A, y \in B\}$. Throughout this section, we shall use κ to denote the maximum number of points of an ε -net in a disk of radius 2ε , which is at most $3^d = 9$, and define $q := \frac{\kappa}{n}$, and $\delta := \varepsilon/\sqrt{q}$.

Given facet $F \in \mathcal{S}$ and $k \geq 0$, define the *level set* $L_{\leq k} := F \cap (\partial F \oplus 2^k \delta)$. $L_{=k} := L_{\leq k} \setminus L_{\leq k-1}$. For $x \in \mathcal{X}$, the *level* of x , denoted $Lev(x)$, is k such that $x \in L_{=k}$. Let $L_{\leq k}(\mathcal{X}), L_{=k}(\mathcal{X})$ denote $L_{\leq k} \cap \mathcal{X}, L_{=k} \cap \mathcal{X}$ respectively. Note that for $x \in L_{=k}, k \geq 1$, the distance $d(x, \partial F) \in (2^{k-1}\delta, 2^k\delta]$. Hence, if $Lev(x) = k, D(x, 2^{k-1}\delta) \subset F$. For $k = 0, d(x, \partial F) \in [0, \delta]$.

We now introduce a bi-partition of the point set, which will be crucial to the analysis of the main proof. Given $x \in F$ having $Lev(x) = k$, then $x \in Bd_F(\mathcal{Y})$, or x is a *boundary point*, if $k = 0$ or if there exists an empty disk (w.r.t. \mathcal{Y}) of radius greater than $2^{k-1}\delta$ passing through x . $x \in Int_F(\mathcal{Y})$, or an *interior point* if and only if $x \in \mathcal{Y} \setminus Bd_F(\mathcal{Y})$. In general, $x \in Bd_{\mathcal{S}}(\mathcal{Y})$ if $x \in Bd_F(\mathcal{Y})$ for some $F \in \mathcal{S}$, and $x \in Int_{\mathcal{S}}(\mathcal{Y})$ is defined similarly.

Let E_1 denote the set of edges in $Del(\mathcal{Y})$ of the type $\{x_1, x_2\} : x_1, x_2 \in Bd_{\mathcal{S}}(\mathcal{Y})$. Let E_2 denote the set of edges in $Del(\mathcal{Y})$ of the type $\{x, y\} : x, y \in Int_F(\mathcal{Y})$, for some $F \in \mathcal{S}$. Let E_3 denote the set of edges in $Del(\mathcal{Y})$ of the type $\{x, y\} : x, y \in Int_{\mathcal{S}}(\mathcal{Y})$, such that $x \in F, y \in F' \neq F$. Let E_4 denote the set of edges $\{x, y\}$ in $Del(\mathcal{Y})$ of the type $x \in Bd_{\mathcal{S}}(\mathcal{Y}), y \in Int(F)$, where F is a facet in \mathcal{S} with supporting plane P .

We have the following lemmas.

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365 ▶ **Lemma 21.** $\mathbb{E} [\sharp(E_1)] \leq O(1) \cdot (\kappa^2 L^2 / A) \cdot s.$

366 ▶ **Lemma 22.** $\mathbb{E} [\sharp(E_2)] \leq c_4 \cdot \kappa s$, where $c_4 \leq 2 \cdot 10^5$.

367 ▶ **Lemma 23.** $\mathbb{E} [\sharp(E_3)] \leq c_4 \cdot (C - 1) \cdot \kappa s.$

368 ▶ **Lemma 24.** $\mathbb{E} [\sharp(E_4)] \leq O(1) \cdot \frac{\kappa^2 L^2}{A} s.$

369 Given the above lemmas, the proof of Theorem 11 follows easily.

370 **Proof of Theorem 11.** As in [4] (Section 4), by Euler's formula, the number of tetrahedra
 371 $t(\text{Del}(\mathcal{Y}))$ in the Delaunay triangulation of \mathcal{S} , is at most $e(\text{Del}(\mathcal{Y})) - \sharp(\mathcal{Y}) = e(\text{Del}(\mathcal{Y})) - s$,
 372 where $e(\text{Del}(\mathcal{Y}))$ is the number of edges in the Delaunay triangulation. Therefore, it suffices
 373 to only count the edges of $\text{Del}(\mathcal{Y})$. Next, observe that any point $x \in \mathcal{Y}$ is either a boundary
 374 or an interior point, that is $Bd_{\mathcal{S}}(\mathcal{Y}) \sqcup \text{Int}_{\mathcal{S}}(\mathcal{Y}) = \mathcal{Y}$. An edge in $\text{Del}(\mathcal{Y})$, therefore, can
 375 be either between two points in $Bd_{\mathcal{S}}(\mathcal{Y})$, or two points in $\text{Int}_{\mathcal{S}}(\mathcal{Y})$, or between a point in
 376 $Bd_{\mathcal{S}}(\mathcal{Y})$ and another in $\text{Int}_{\mathcal{S}}(\mathcal{Y})$. The case of a pair of points in $\text{Int}_{\mathcal{S}}(\mathcal{Y})$ is further split
 377 based on whether the points belong to the same facet of \mathcal{S} or different facets. Thus using
 378 the above exhaustive case analysis, the proof follows simply by summing the bounds. The
 379 proof of part (ii) is by similar extensions, as sketched in the proof of Theorem 10 (ii). ◀

380 It only remains to prove Lemmas 21- 24. We first present a few technical lemmas, which
 381 will be required for the analysis of all the cases.

382 5.1 Some Technical Lemmas

383 ▶ **Proposition 25** ([4]). *Let F be a facet of \mathcal{S} . For any Borel set $R \subset F$, we have*

$$384 \quad \left(\frac{a(R)}{4\pi\varepsilon^2} \right) \leq \sharp(R \cap \mathcal{X}) \leq \left(\frac{\kappa \cdot a(R \oplus \varepsilon)}{\pi\varepsilon^2} \right), \text{ and therefore,} \quad (1)$$

$$385 \quad \left(\frac{A}{4\pi\varepsilon^2} \right) \leq \sharp(\mathcal{S} \cap \mathcal{X}) = n \leq \left(\frac{A}{\pi\varepsilon^2} \right), \quad (2)$$

386 where the last inequality in (2) holds for all $A \geq \pi\varepsilon^2$.

387 ▶ **Proposition 26** ([4]). *Let F be a facet of \mathcal{S} , let $\Gamma \subset F$ be a curve contained in F , and
 388 $k \in \mathbb{N}$. Then*

$$389 \quad \sharp((\Gamma \oplus k\varepsilon) \cap \mathcal{X}) \leq \left(\frac{(2k+1)^2}{k} \right) \kappa \frac{l(\Gamma)}{\varepsilon} \quad (3)$$

$$390 \quad \leq \left(9k\kappa \frac{l(\Gamma)}{\varepsilon} \right), \text{ when } k \geq 1. \quad (4)$$

391 ▶ **Lemma 27.** *Given a circle $\Sigma_1 \subset \mathbb{E}^2$ of unit radius centered at the origin, seven disks
 392 having centers in Σ_1 and radius $1/2$, are necessary and sufficient to cover Σ_1 .*

393 ▶ **Lemma 28** (Level Size). $\sharp(L_{=k} \cap \mathcal{X}) \leq \sharp(L_{\leq k} \cap \mathcal{X}) \leq 9\kappa L \left(\frac{2^k \delta}{\varepsilon^2} \right).$

395 ▶ **Lemma 29.** *Given a point $x \in \mathcal{S}$ in the facet F with supporting plane P with $\text{Lev}(x) > 0$,
 396 and $k \in \mathbb{N}$ such that $0 \leq k < \text{Lev}(x)$, with $r_0 = 2^k \delta$, there exists a collection \mathcal{D}_x of at most
 397 $c_B = 7$ disks in F , such that*

398 (i) *Each $D \in \mathcal{D}_x$ is contained in F .*

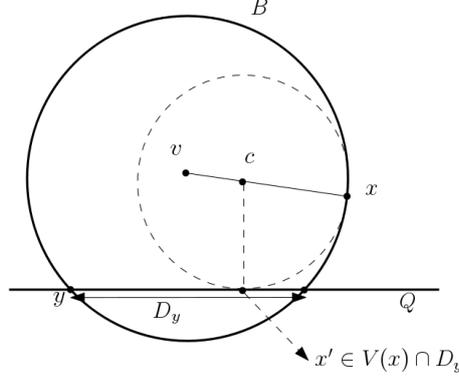
399 (ii) *Each $D \in \mathcal{D}_x$ has radius $r_0/4$,*

400 (iii) *Any disk $D \subset P$ of radius at least r_0 , such that $x \in \partial D$, contains at least one disk in \mathcal{D}_x .*

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422(iii) The total length of the boundary curves in \mathcal{V} is equal to the total length of the convex
 423 boundaries.

424 For the rest of this subsection, we shall apply Construction 32 on the plane P , and the
 425 points in $Bd_{\mathcal{S}}(\mathcal{Y})$ as Z . Let $\mathcal{T} := \text{Int}_F(\mathcal{Y})$ for some facet $F \in \mathcal{S}$. Given $x \in Z, y \in P \setminus V(x)$,
 426 let $k_y = k_y(x)$ denote the least $k \geq 0$ such that $y \in \partial V(x) \oplus 2^k \delta$.



427 ■ **Figure 2** $x \in Z = Bd_{\mathcal{S}}(\mathcal{Y}), y \in \mathcal{T} = \text{Int}_F(\mathcal{Y}), z \in V(x) \cap D_y$.

428 ► **Proposition 34** (Attali-Boissonnat [4]). Suppose there exists a ball $B \subset \mathbb{E}^3$ and $y \in P$, such
 429 that $y, x \in \partial B$, and $B \cap \mathcal{T} = \emptyset$. Then the disk $D_y = P \cap B$ satisfies $D_y \cap \mathcal{T} = \emptyset, y \in \partial D_y$
 430 and $D_y \cap V_x \neq \emptyset$.

431 ► **Lemma 35.** If $\{x, y\} \in E_4$ with $x \in Bd_{\mathcal{S}}(\mathcal{Y}), y \in \text{Int}(F)$, then $k_y \leq \text{Lev}(y)$.

432 Now we partition the pairs of vertices $\{x, y\} \in E_4$ with $x \in Bd_{\mathcal{S}}(\mathcal{Y})$, depending on
 433 whether $y \in V_F(x)$ or $y \in \partial V_F(x) \oplus 2^k \delta$. That is, given a facet $F \in \mathcal{S}$, let $E_4(\text{Int}(F))$
 434 denote the set of edges $\{x, y\} \in E_4$ with $y \in \text{int}(V_F(x))$, and $E_4(Bd(F))$ denote the set of
 435 edges in E_4 with $y \in \partial(V_F(x)) \oplus 2^k \delta$, for $k \in [0, k_y]$. Define $E_4(\text{Int}) := \bigcup_{F \in \mathcal{S}} E_4(\text{Int}(F))$
 436 and $E_4(Bd) := \bigcup_{F \in \mathcal{S}} E_4(Bd(F))$ respectively.

437 **Lemma 24.** The proof follows from Lemmas 36 and 37, which bound the expected number
 438 of edges in $E_4(\text{Int})$ and $E_4(Bd)$ respectively. ◀

439 ► **Lemma 36.** Given a facet $F \in \mathcal{S}, \mathbb{E}[E_4(\text{Int}(F))] \leq q \cdot \#(\mathcal{X} \cap F)$. As a consequence,
 440 $\mathbb{E}[E_4(\text{Int})] \leq s$.

441 ► **Lemma 37.** Given a facet $F \in \mathcal{S}, \mathbb{E}[E_4(Bd(F))] \leq \left(O(1) \cdot \frac{\kappa^2 L \cdot l(\partial F) s}{A}\right)$. As a consequence,
 442 $\mathbb{E}[E_4(Bd(\mathcal{S}))] \leq \left(O(1) \cdot \frac{\kappa^2 L^2 s}{A}\right)$.

443 6 Randomized Incremental Construction (Proof of Theorem 12)

444 In this section, we show how the results in Sections 4 and 5 imply bounds on the computational
 445 complexity of constructing Delaunay triangulations of ε -nets. Our main tool shall be
 446 Theorem 7. However, we need to show first that Condition 6 holds, which we do in the
 447 following lemma. In the setting of Theorem 10 and Theorem 11, given a subset $\mathcal{Y} \subset \mathcal{X}$, and
 448 a point $q \in \mathcal{Y}$, let $V_{\mathcal{Y}}$ denote the Voronoi diagram of \mathcal{Y} , and $V_{\mathcal{Y}}(q)$ denote the cell of q in $V_{\mathcal{Y}}$.

449 ▶ **Lemma 38.** *If $\mathcal{Y} \subset \mathcal{X}$ is a uniformly random subset of size s , and $p \in \mathcal{X} \setminus \mathcal{Y}$ is a uniformly*
 450 *random point from $\mathcal{X} \setminus \mathcal{Y}$, then the set of conflicts introduced by adding p to the Voronoi*
 451 *diagram of \mathcal{Y} , can be computed in constant expected time.*

452 **Proof.** To find the conflicts introduced by adding the point p to \mathcal{Y} , it suffices to examine
 453 only the faces of the Voronoi cells neighboring the point q such that $p \in V_{\mathcal{Y}}(q)$ (i.e. the
 454 *parent* of p), since each conflict gives rise to at most one face of $V_{\mathcal{Y} \cup \{p\}}(p)$. The number of
 455 faces of $V_{\mathcal{Y}}(q)$ is at most the faces of the neighbouring cells of $V_{\mathcal{Y} \cup \{p\}}(p)$, which is equal to
 456 $star^2(p)$. If $s \leq 4(2\sqrt{d})^d \cdot d^3 + 1$, $star^2(p)$ is at most $s^d = O(1)$, otherwise by Theorem 10(ii),
 457 $star^2(p)$ is at most $O(1)$. A similar argument also works for the polyhedral setting, using
 458 Theorem 11(ii). ◀

459 Theorem 12 now follows directly by applying Theorems 10(i) and 11(i) to the result of
 460 Theorem 7.

461 7 Euclidean Orbifolds and Bounded-Distortion Metrics

462 In this section, we shall give some extensions of Theorem 10 and 12. Given a space \mathcal{S} ,
 463 $\varepsilon \in [0, 1]$ and $\kappa \in \mathbb{Z}^+$, an (ε, κ) -sample is a set of points for which any ball of radius ε in \mathcal{S} ,
 464 contains at least one point and at most κ points.

465 ▶ **Theorem 39.** *Theorems 10 and 11 hold when the point set is an (ε, κ) sample.*

466 ▶ **Theorem 40.** *Theorems 10 and 11 also work for the case when the random sample is an*
 467 *i.i.d. sample with probability parameter $q = s/n$.*

468 **Proof Sketch.** The proof is on the lines of Theorems 10 and 11. It is only when computing
 469 $P_p(k)$ that we make use of the fact that points are selected independently. Here, we get
 470 directly that $P_p(k) \leq q^d(1-q)^{n_k} \leq q^d \cdot \exp(-qn_k)$. The rest of the proof follows as before. ◀

471 Coming to our results for Delaunay triangulations of Euclidean d -manifolds and embedded
 472 metrics in \mathbb{T}^d , we need a few definitions first.

473 Euclidean d -orbifolds

474 A d -dimensional Bieberbach group \mathcal{G} is a discrete group of isometries acting on \mathbb{E}^d . A
 475 d -orbifold \mathbb{E}^d/\mathcal{G} is the compact quotient space (i.e. collection of orbits) of \mathbb{E}^d acted on by a
 476 d -dimensional Bieberbach group \mathcal{G} . When the group action is free (i.e. has no fixed points),
 477 the d -orbifold is a closed Euclidean d -manifold. Every Euclidean d -manifold is the quotient
 478 space of some d -Bieberbach group acting on \mathbb{E}^d [7], [30]. For Euclidean d -orbifolds, we have:

479 ▶ **Theorem 41.** *Given a closed Euclidean d -orbifold $\mathbb{M} = \mathbb{E}^d/\mathcal{G}$, equipped with the Euclidean*
 480 *metric, where \mathcal{G} is a d -Bieberbach group, there exists a covering space $\mathcal{C}_{\mathbb{M}}$ with multiplicity*
 481 *$m = m^*(\mathcal{G}, d) < \infty$, such that the Delaunay triangulation of \mathcal{M} embeds in $\mathcal{C}_{\mathbb{M}}$, and the*
 482 *statements of Theorems 10 and 12 apply for ε -nets, for any $\varepsilon \in [0, 1/4]$.*

483 Embedded metrics with bounded distortion

484 For a metric \mathfrak{d} on some domain \mathcal{S} embedded in \mathbb{E}^d , define its *distortion* $\kappa_{\mathfrak{d}}$ (with respect to
 485 \mathbb{E}^d) to be the minimum $\lambda \geq 1$ such that $\forall x, y \in \mathcal{S} : \frac{1}{\lambda}\|x - y\| \leq \mathfrak{d}(x, y) \leq \lambda\|x - y\|$. A $d \times d$
 486 matrix $M \in \mathbb{E}^d$ is *positive definite* if, for all $x \neq 0 \in \mathbb{E}^d$, $x^\top M x > 0$. For a positive definite
 487 matrix M , define its *condition number* c_M to be the ratio of its maximum to its minimum
 488 eigenvalue. For embedded metrics with bounded distortion, we have:

489 ► **Theorem 42.** *Given a metric \mathfrak{d} over \mathbb{T}^d with distortion $\kappa_{\mathfrak{d}} < \infty$, there exists an integer*
 490 *$m = m_{\mathfrak{d}} < (2\kappa_{\mathfrak{d}}\sqrt{d})^d$, such that the Delaunay triangulation over $(\mathbb{T}^d, \mathfrak{d})$ embeds in \mathbb{T}_m^d*
 491 *with the Euclidean metric. In particular, if \mathfrak{d} is of the form $\mathfrak{d}(x, y) = \sqrt{(x - y)^\top M(x - y)}$,*
 492 *$x, y \in \mathbb{T}^d$, where $M \in \mathbb{E}^{d \times d}$ is a positive definite matrix having condition number at most c_M ,*
 493 *then $m \leq (2c_M\sqrt{d})^d$. Hence, given any $\varepsilon \in [0, 1/4]$, the statements of Theorems 10 and 12*
 494 *apply for ε -nets over the metric space $(\mathbb{T}^d, \mathfrak{d})$.*

495 The proofs of Theorems 41 and 42 follow by finding covering spaces of bounded multiplicity
 496 where the Delaunay triangulation can be embedded, and generalizing Lemmas 16–20 to
 497 such spaces. In the case of Theorem 41, the existence of the covering space follows from
 498 the algorithmic version of Bieberbach’s theorem [6] by Caroli-Teillaud [10](Section 4). For
 499 Theorem 42, we use a geometric condition of Caroli-Teillaud [10] (Criterion 3.11) to explicitly
 500 bound the multiplicity of the covering space. The proofs are sketched in the Appendix.

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584 **A** Appendix585 **Proofs from Section 1**

586 ► **Lemma 43.** *Given $\varepsilon \in (0, 1]$, an ε -net \mathcal{X} over \mathbb{T}^d , of n points, a uniformly random sample*
 587 *$S \subset \mathcal{X}$ of k points and $\varepsilon' = \varepsilon \sqrt[d]{n/k}$, then*

- 588 (i) *With high probability, there exists a ball of radius ε' with $\Omega(\log k / \log \log k)$ points in S .*
 589 (ii) *With probability at least a constant, there exists a ball of radius $\Omega(\varepsilon' \log k)$ with no points*
 590 *in S .*

591 **Sketch of Proof.** The following balls-and-bins and coupon-collector arguments which can
 592 be shown to prove this:

- 593 (i) Assuming the ε -net is over the unit cube or unit ball in \mathbb{E}^d , a volume argument gives
 594 that there are $\Omega(\varepsilon'^{-d}) = \Omega(k)$ disjoint balls of radius ε' , which will be our “bins”. Now
 595 choosing k points from the n points in the ε -net is akin to throwing k packets into k bins
 596 - with high probability, the maximum load of a bin is $\Omega(\frac{\log k}{\log \log k})$, i.e. there exists a ball
 597 of radius ε' with $\Omega(\frac{\log k}{\log \log k})$ points.
 598 (ii) Similarly, there are $\Omega(\varepsilon' / \log k) = \Omega(k / \log k)$ balls of radius $\varepsilon' \sqrt[d]{\log k}$ that cover the ε -net,
 599 and choosing k points from this covering, can be thought of as an instance of the coupon
 600 collector problem: we are allowed k draws (i.e. we choose k points), each draw is one of
 601 the “coupons” (i.e. the balls in the covering) with equal probability, and we want the
 602 probability that at least one coupon has not been collected, i.e. at least one ball from the
 603 covering has no point chosen from it. Now the well-known lower tail of the number of
 604 draws taken to collect all coupons in the the coupon collector problem, gives that with
 605 high probability, for $k' = \Omega(k / \log k)$ coupons, and at most $O(k' \log k') = k$ draws, all
 606 coupons will not be collected, that is, some ball of radius $\Omega(\varepsilon' \sqrt[d]{\log k})$ will be empty.
 607 ◀

608 **Proofs from Section 2**

609 **Proof of Lemma 1.** Consider a maximal set of disjoint balls of radius $\frac{\rho}{2}$ with center inside
 610 the ball $B(r)$ of radius r . Then the balls with the same centers and radius ρ cover the ball
 611 $B(r)$ (otherwise it contradicts the maximality). By a volume argument we get that the
 612 number of balls is bounded from above by $\frac{V_d \times (r + \frac{\rho}{2})^d}{V_d \times (\frac{\rho}{2})^d} \leq (\frac{3r}{\rho})^d$ for $\rho < 1$. ◀

613 **Proof of Lemma 2.** The volume argument gives a lower bound of $\frac{V_d \times r^d}{V_d \times \rho^d} = (\frac{r}{\rho})^d$. ◀

614 **Proof of Lemma 3.** Observe that, by the minimum distance property of the points in \mathcal{X} , the
 615 balls of radius $\varepsilon/2$ centered around each point in \mathcal{X} , are disjoint, and by a volume argument
 616 there can be at most $\frac{1}{V_d \times (\varepsilon/2)^d} \leq 2^{-d} d^{d/2} (\varepsilon/2)^{-d} = d^{d/2} \varepsilon^{-d}$ such balls in \mathbb{T}^d . The balls
 617 of radius ε centered around each point in \mathcal{X} cover the space thus their number is at least
 618 $\frac{1}{V_d \times \varepsilon^d} \geq d^{d/2} 2^{-3d} \cdot \varepsilon^{-d}$. This completes the proof of the lemma. ◀

619 **Lemma 5.** Observe that the circumradius of any simplex in $Del(\mathcal{X})$ cannot be greater than
 620 ε , since this would imply the existence of a ball in \mathbb{T}^d of radius at least ε , containing no points
 621 from \mathcal{X} . Therefore given a point $p \in \mathcal{X}$, any point which lies in a Delaunay simplex incident
 622 to p , must be at most distance 2ε from p . Again by a volume argument, the number of such
 623 points is at most $\frac{V_d \times (2\varepsilon + \varepsilon/2)^d}{V_d \times ((\varepsilon/2)^d)} = 5^d$. Thus, the number of Delaunay simplices of dimension at
 624 most d that contain p , is at most the complexity of the d -skeleton of the Delaunay complex

625 in \mathbb{T}^d on 5^d vertices. This is at most $(5^d)^{\lceil d/2 \rceil}$. Thus we can conclude that the number of
 626 simplices in $\text{Del}(\mathcal{X})$ is at most the cardinality of \mathcal{X} , times the maximum number of simplices
 627 incident to any given point $p \in \mathcal{X}$, or $(5^d)^{\lceil d/2 \rceil} \cdot \#\mathcal{X} \leq (5^d)^{\frac{d+1}{2}} d^{d/2} \varepsilon^{-d} \leq 4^{d^2} \varepsilon^{-d}$. ◀

628 **Proof of Lemma 9.** The j -th moment $\mathbb{E}[X^j]$ is the sum over all ordered j -tuples $l =$
 629 $(l_1, l_2, \dots, l_j) \in B^j$ of the probability that a j -tuple gets chosen in the random sample
 630 A^j . Fix $l \in B^j$. If any element of B gets repeated in the tuple, then $\mathbb{P}[l \text{ is selected}] = 0$.
 631 Assuming therefore that for all $i \in [j]$, $l_i \neq l_j$, and applying Lemma 8 with the included
 632 subset l , and the excluded subset as the empty set, this probability can be upper bound as
 633 $(\frac{a}{c})^j$. Summing over all b^j tuples in B^j , gives us that $\mathbb{E}[X^j] \leq (\frac{ab}{c})^j = (\mathbb{E}[X])^j$. ◀

634 **Proof of Lemma 8.** The total number of ways of choosing the random sample A is $\binom{c}{a}$. The
 635 number of ways of choosing A such that $B \subset A$ and $T \cap A = \emptyset$, is $\binom{c-b-t}{a-b}$. Therefore the
 636 required probability is

$$\begin{aligned}
 637 \quad \mathbb{P}[B \subset A, T \cap A = \emptyset] &= \frac{\binom{c-b-t}{a-b}}{\binom{c}{a}} \\
 638 &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=b}^{a-1} (a-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=b}^{a-1} (c-i)} \cdot \frac{\prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{a-b-1} (a-b-i)} \\
 639 &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=0}^{a-b-1} (c-b-i)} \\
 640 &= (a/c)^b \frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)} \left(1 - \frac{t}{c-b}\right)^{a-b} \left(\frac{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b-t})}{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b})}\right) \\
 641 &\leq (a/c)^b \left(1 - \frac{t}{c-b}\right)^{a-b},
 \end{aligned}$$

642 where in the last step, observe that for the product $\frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)}$ for each i , the term $(1-i/a)$
 643 in the numerator is smaller than the corresponding term $(1-i/c)$ in the denominator, since
 644 $a \leq c$. A similar observation holds for the product $\left(\frac{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b-t})}{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b})}\right)$.

645 Now, observe that $\left(1 - \frac{t}{c-b}\right)^{a-b} \leq \exp\left(-\left(\frac{t(a-b)}{c-b}\right)\right) \leq \exp\left(-\left(\frac{at}{2c}\right)\right)$, if $b \leq a/2$ and
 646 $b < c$. ◀

647 Proofs from Section 4

648 **Proof of Lemma 13.** For a simplex $\sigma = \tau \cup \{p\}$, with the vertex set of τ in $S_p(k)$, let
 649 $\mathbf{1}_{[\tau]}$ be the indicator random variable which is 1 if $\sigma \in \text{star}(p)$, and zero otherwise. Then
 650 $Z_p(k) := \sum_{\tau \in S_p(k)} \mathbf{1}_{[\tau]}$, and $\#\text{star}(p) = \sum_{k \geq 0}^{k_{\max}} Z_p(k)$.

651 Taking expectations over the random sample \mathcal{Y} , we get

$$\begin{aligned}
 652 \quad \mathbb{E}[\#\text{star}(p)] &= \sum_{k \geq 0}^{k_{\max}} \mathbb{E}[Z_p(k)] \\
 653 &= \sum_{k \geq 0}^{k_{\max}} \sum_{\tau \in S_p(k)} \mathbb{E}[\mathbf{1}_{[\tau]}] \leq \sum_{k \geq 0}^{k_{\max}} \sum_{\tau \in S_p(k)} P_d(k) = \sum_{k \geq 0}^{k_{\max}} s_p(k) \cdot P_d(k).
 \end{aligned}$$

654 ◀

655 **Proof of Lemma 14.** The simplex σ can be a Delaunay simplex only if (i) the set of its
 656 vertices is included in the subsample \mathcal{Y} , and (ii) all points in $B(c_\sigma, r_\sigma) \cap \mathcal{X}$ are excluded from
 657 \mathcal{Y} . The idea is therefore, to use Lemma 17 to bound the number of points in $B(c_\sigma, r_\sigma) \cap \mathcal{X}$
 658 from below by n_k , and then upper-bound the probability that all these points are excluded
 659 from \mathcal{Y} .

660 This suggests applying Lemma 8, with the universe having $c = n - 1$ elements, sample
 661 size $a = s - 1$, included set having $b = d$ elements, and excluded set having $t = n_k$ elements.
 662 We verify first that the conditions of the lemma are satisfied, i.e. (i) $b \leq \min\{\frac{a}{2}, c - 1\}$, since
 663 $s \geq 4d + 1$. Now applying the lemma, we get

$$664 \quad \mathbb{P}[\sigma \in \text{Del}(\mathcal{Y})] \leq \left(\frac{s-1}{n-1}\right)^d \exp\left(-n_k \cdot \left(\frac{s-1}{2(n-1)}\right)\right)$$

$$665 \quad = q^d \cdot \exp(-qn_k/2) \leq q^d \cdot \exp(-qn_k/2),$$

666 where the equality was by the substitution $q = \frac{s-1}{n-1}$, and the last inequality followed from
 667 the fact that $d = b \leq c - 1 = n - 2$. \blacktriangleleft

668 **Proof of Lemma 15.** This follows simply from the triangle inequality. For the first statement,
 669 we have that for any $p' \in \sigma$, $\|p, p'\| \leq \|p, c_\sigma\| + \|c_\sigma, p'\| = 2r_\sigma$. The second statement follows
 670 by replacing the above inequalities with strict inequalities for the points in the open ball
 671 $B(c_\sigma, r_\sigma)$. \blacktriangleleft

672 **Proof of Lemma 16.** Let τ be an element of $S_p(k)$. Using Lemma 15 and the definition of
 673 $S_p(k)$, we have that $\sigma = \tau \cup \{p\} \subset B[p, 2^{k+1}\delta]$. If $k \leq k_{\max}$, then $2^{k+1}\delta \leq 1/2$. Therefore
 674 applying Theorem 4, we can work in the one-sheeted covering of \mathbb{T}^d . Now applying Lemma 1,
 675 the number of points in $B(p, 2^{k+1}\delta) \cap \mathcal{X}$ is at most $(3 \cdot 2^{k+1}\delta/\varepsilon)^d = (6 \cdot 2^k\delta/\varepsilon)^d$. Therefore,
 676 the set of possible simplices incident to p and having vertices in $B[p, 2^{k+1}\delta] \cap \mathcal{X}$ is at most
 677 the set of all d -tuples of points in $B[p, 2^{k+1}\delta] \cap \mathcal{X}$, i.e. at most $(6 \cdot 2^k\delta/\varepsilon)^{d^2}/d!$. \blacktriangleleft

678 **Proof of Lemma 19.** For $k \geq 1$, consider the ratio of successive terms $\frac{\mathbb{E}[Z_p(k+1)]}{\mathbb{E}[Z_p(k)]} = \frac{s_p(k+1) \cdot P_p(k+1)}{s_p(k) \cdot P_p(k)}$
 679 in Lemma 13. From Lemmas 16 and 17, we have

$$680 \quad \frac{s_p(k+1) \cdot P_p(k+1)}{s_p(k) \cdot P_p(k)} = \frac{(6 \cdot 2^{k+1}\delta/\varepsilon)^{d^2}/d!}{(6 \cdot 2^k\delta/\varepsilon)^{d^2}/d!} \times \frac{q^d \cdot \exp(-q(2^k\delta/\varepsilon)^d/2)}{q^d \cdot \exp(-q(2^{k-1}\delta/\varepsilon)^d/2)}$$

$$681 \quad = 2^{d^2} \cdot \exp\left(-\frac{q}{2} \cdot ((2^{k-1}\delta/\varepsilon)^d) (2^d - 1)\right)$$

$$682 \quad = 2^{d^2} \cdot \exp\left(-\frac{1}{2} \cdot (2^{(k-1)d} 2d) (2^d - 1)\right),$$

683 where in the last step we used the definition of $\delta = \varepsilon(2d/q)^{1/d}$, so that $q(\delta/\varepsilon)^d = 2d$. The last
 684 expression is minimized when $k = 1$, $d = 2$, to get $2^{d^2} \cdot \exp(d(2^d - 1)) = 2^4 \cdot e^{-6} \leq (2/e)^6$.
 685 For $k = 0$, the ratio $\frac{\mathbb{E}[Z_p(1)]}{\mathbb{E}[Z_p(0)]} \leq 2^{d^2} \cdot \exp(-d \cdot 2^d)$, which is at most $(2/e)^6$ for $d \geq 2$.

686 Therefore for all $0 \leq k \leq k_{\max} - 1$, we have that $\frac{\mathbb{E}[Z_p(k+1)]}{\mathbb{E}[Z_p(k)]} \leq (2/e)^6$. Thus, the sum
 687 $\sum_{k=0}^{k_{\max}} \mathbb{E}[Z_p(k)]$ is upper bounded by the sum of a geometric progression with leading term
 688 $\mathbb{E}[Z_p(0)] \leq 6^{d^2+d}$ and common ratio $(2/e)^6$, which is at most $(1 - (2/e)^6)^{-1} \cdot 6^{d^2+d}$. \blacktriangleleft

689 **Proof of Lemma 20.** From Lemma 3, we have that $n = \sharp(\mathcal{X}) \leq 2^{-d} d^{d/2} \varepsilon^{-d}$. Therefore, by
 690 Lemma 2, any ball B of radius at least $2^{k_{\max}}\delta \geq 1/4$, has at least $(2^{k_{\max}}\delta/\varepsilon)^d = (1/4\varepsilon)^d \geq$
 691 $\left(\frac{n}{2^d \cdot d^{d/2}}\right)$ points in its interior, i.e.

$$692 \quad \sharp(\text{int}B \cap \mathcal{X}) \geq \left(\frac{n}{(2\sqrt{d})^d}\right).$$

693 Here, since $2^{k+1}\delta > 1/2$, we shall use Theorem 4 and work in the 3^d -sheeted covering of \mathbb{T}^d .
 694 The maximum number of d -tuples which can possibly form a Delaunay d -simplex with p , is
 695 at most $\binom{n-1}{d} \leq (n-1)^d/d!$. Since the multiplicity of the covering map is 3^d , each vertex
 696 of a simplex $\tau \in S_p(k)$ can be chosen from one of at most 3^d copies in the covering space.
 697 Thus, each simplex in $S_p(k)$ yields less than 3^{d^2} possible Delaunay spheres in \mathbb{T}^d . Therefore,
 698 the expected number of simplices having radius at least $2^{k_{\max}}\delta$, is at most

$$\begin{aligned}
 \sum_{k > k_{\max}} \mathbb{E}[Z_p(k)] &= 3^{d^2} \frac{(n-1)^d}{d!} \cdot P_p(k) \\
 &= 3^{d^2} \frac{(n-1)^d}{d!} \cdot q^d \cdot \exp\left(-\frac{q}{2} \cdot \frac{n-1}{(2\sqrt{d})^d}\right) \\
 &\leq 3^{d^2} \frac{(s-1)^d}{d!} \cdot \exp\left(-\frac{s-1}{2(2\sqrt{d})^d}\right). \tag{5}
 \end{aligned}$$

702 For $s > s_0 = 4(2\sqrt{d})^d \cdot d^3 + 1$, this function is decreasing in term of s and it is easy to check
 703 that the value in s_0 is smaller than 5. Thus we have $\sum_{k > k_{\max}} \mathbb{E}[Z_p(k)] \leq 5$. ◀

704 **Proofs from Section 5**

705 ▶ **Lemma 44.** Given $a > 0$, $b \in (0, 1)$, the sum $\sum_{n \in \mathbb{Z}_+} 2^{an} \cdot \exp(-b \cdot 2^{an})$ is at most
 706 $\frac{2 \log_2(1/b)}{eab}$.

707 **Proof of Lemma 44.** The proof follows from elementary calculus. The maximum term
 708 is when $2^{an} = 1/b$, i.e. $n = \frac{\log_2(1/b)}{a}$, and evaluates to $2^{an} \exp(-b \cdot 2^{an}) = \frac{1}{eb}$, and terms
 709 decrease exponentially afterwards. The sum is therefore upper bounded by twice the maximum
 710 term, times the number of terms before the maximum, which is the claimed bound. ◀

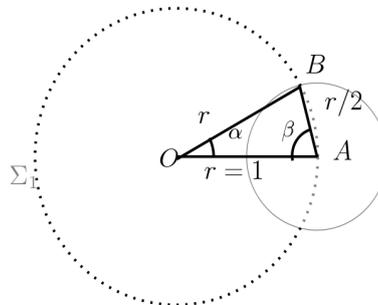
711 **Proof of Lemma 28.** The first inequality is obvious, as $L_{=k} \subseteq L_{\leq k}$. The proof of the second
 712 inequality follows by applying Proposition 26 over the boundaries of the facets. For a fixed
 713 facet $F \in \mathcal{S}$, we get

$$\# \left(\left(\partial F \oplus \left(\frac{2^k \delta}{\varepsilon} \right) \varepsilon \right) \cap \mathcal{X} \right) \leq \left(\frac{9\kappa 2^k \delta}{\varepsilon} \right) \cdot \left(\frac{l(\partial F)}{\varepsilon} \right).$$

715 Summing over all $F \in \mathcal{S}$, we get

$$\#(L_{\leq k} \cap \mathcal{X}) = \sum_{F \in \mathcal{S}} \#((\partial F \oplus 2^k \delta) \cap \mathcal{X}) \leq \left(\frac{9\kappa 2^k \delta}{\varepsilon^2} \right) \sum_{F \in \mathcal{S}} l(\partial F) = \left(\frac{9\kappa 2^k \delta}{\varepsilon^2} \right) \cdot L.$$

717 ◀



Proof of Lemma 27.

718 ■ **Figure 3** Angle covered by disk of radius $1/2$ is $= 2\alpha$

719 Let D_2 denote a disk of radius $1/2$, having its center lying on the circle Σ_1 . Let 2α denote
 720 the angle subtended by $\Sigma_1 \cap D_2$ on the center of Σ_1 . By symmetry, the angle $\angle OAB$ in
 721 figure 3 is α . Applying the sine law to the triangle OAB , we get

$$722 \quad \frac{1}{\sin \beta} = \frac{1/2}{\sin \alpha}, \text{ or,}$$

$$723 \quad \frac{1}{\sin(\pi/2 - \alpha/2)} = \frac{1/2}{\sin \alpha}, \text{ or,}$$

$$724 \quad \frac{1}{\cos(\alpha/2)} = \frac{1/2}{\sin \alpha}, \text{ or,}$$

$$725 \quad 4 \sin(\alpha/2) = 1.$$

726 Therefore, $\alpha = 2 \arcsin(1/4)$, or $2\alpha = 4 \arcsin(1/4)$.
 727 Thus, one disk covers an angle of $4 \arcsin(1/4)$, and so the number of required disks having
 728 radius $1/2$, is at least $\frac{2\pi}{4 \arcsin(1/4)} \approx 6.21 < 7$.

729 It is now easy to see that we can place the disks on the boundary in a greedy manner,
 730 such that they cover the maximum possible angle, except of course the last disk which may
 731 have some overlap with the first disk. Thus seven disks would suffice as well. ◀

732 **Proof of Lemma 31.** By the definition of $Lev(x)$, we have that $D(x, 2^{Lev(x)-1}\delta) \subset F$. Now
 733 the statement follows by the application of Proposition 25, as below

$$734 \quad \frac{\pi(2^k\delta)^2}{4\pi\varepsilon^2} \leq \#(D(x, 2^k\delta) \cap \mathcal{X}) \leq \frac{\pi(2^k\delta + \varepsilon)^2}{\pi\varepsilon^2}, \text{ or,}$$

$$735 \quad \frac{2^{2k-2}n}{s} \leq \#(D(x, 2^k\delta) \cap \mathcal{X}) \leq \frac{2 \cdot 2^{2k}n}{s},$$

736 where we used that $\delta := (\sqrt{\frac{n}{s}})\varepsilon$. This gives the first statement of the lemma, using $q = s/n$.
 737 The second statement follows simply by taking expectation. ◀

738 **Proof of Lemma 29.** Let $D_0 = D(y, r)$ for some $y \in P$, $r \geq r_0$, be a disk such that $x \in \partial D_0$.
 739 Let $D_1 = D(x', r_0)$ be the unique disk with centre x' on the line xy , radius r_0 , and having
 740 $x \in \partial D_1$. Note that

- 741 (a) $D_1 \subseteq D_0$ by construction, and,
- 742 (b) $x' \in F$, since $r_0 = 2^k\delta \leq 2^{Lev(x)-1}\delta$, so that $x' \in D(x, r_0) \subset F$.

743 Consider $\Sigma_2 = \Sigma(x, r_0/2)$, and let $p = xx' \cap \Sigma_2$, that is, the point p lies on the line
 744 xx' , at distance $r_0/2$ from x (and therefore from x' as well). We shall build a minimal
 745 covering \mathcal{D} of the circle Σ_2 , by disks centered in Σ_2 , having radius $r_0/4$. From Lemma 27,
 746 we get $\#(\mathcal{D}) = 7$. Let $D' \in \mathcal{D}$ be a disk in the covering. Then by the triangle inequality,
 747 $D' \subset D(x, r_0/2 + r_0/4) \subset D(x, r_0)$. As before, by the definitions of $Lev(x)$ and r_0 , this
 748 implies $D' \subset F$. Thus \mathcal{D} satisfies conditions (i) and (ii) of the lemma. Further, since \mathcal{D} is a
 749 covering of Σ_2 , there exists $D_p \in \mathcal{D}$ such that $p \in D_p$. Therefore, the disk $D_p \subset D_1 \subset D_0$,
 750 and $D_p \subset F$. Thus $D_p \in \mathcal{D}$ satisfies condition (iii). Now taking $\mathcal{D}_x = \mathcal{D}$ completes the proof
 751 of the lemma. ◀

752 **Proof of Lemma 30.** Firstly, consider the case where $k_{\max} = 0$, i.e. all the k_i 's are zero.
 753 In this case we simply upper bound the probability of the event E , by the probability of
 754 including all the points x_1, \dots, x_t in \mathcal{Y} . By Lemma 8, this is at most q^t .

755 We next come to the case when $k_{\max} > 0$. Since for all $i \in [t]$, $k_i < Lev(x_i)$, we can apply
 756 Lemma 29 for each i , with $k = k_i$, to conclude that for each i , there exists a collection \mathcal{D}_i
 757 of at most c_B disks of radius $r_i^*/4$, such that any disk having radius greater than r_i^* and
 758

759 passing through x_i , must contain some disk $D_i^* \in \mathcal{D}_i$. Let T denote the set $\prod_{i=1}^t \mathcal{D}_i$. By the
 760 union bound over the set T , we get

$$\begin{aligned} 761 \quad \mathbb{P}[E] &\leq \mathbb{P}[\exists B \in T : \forall i \in [t], B_i \cap \mathcal{Y} = \emptyset] \\ 762 &\leq \#(T) \cdot \mathbb{P}[\forall i \in [t], B_i \cap \mathcal{Y} = \emptyset], \end{aligned}$$

763 where $B \in T$ is some fixed element of T .

764 Let $j := \arg \max_{i \in [t]} k_i$, so that $k_{\max} = k_j$. Now, the event E requires the set x_1, \dots, x_t
 765 to be in the sample \mathcal{Y} , and the interiors of the disks D_i^* to be free from points in \mathcal{Y} . In
 766 particular, the disk D_j^* should not contain any points in \mathcal{Y} . Therefore applying Lemma 8 on
 767 the universe $C = \mathcal{X}$, the random sample $A = \mathcal{Y}$, the included subset $B = \{x_1, \dots, x_t\}$, and
 768 the excluded subset $Z = D_j^* \cap \mathcal{X}$ of size at least $z = \frac{\pi(r_j^*/4)^2}{4\pi\varepsilon^2}$, we get

$$\begin{aligned} 769 \quad \mathbb{P}[E] &\leq \#(T) \cdot \mathbb{P}[\forall B_i \in u, B_i \cap \mathcal{Y} = \emptyset] \\ 770 &\leq (c_B)^t \cdot q^t \cdot \exp\left(-\frac{1}{2}q \cdot \left(\frac{\pi(r_j^*/4)^2}{4\pi\varepsilon^2}\right)\right) \\ 771 &\leq c_1 q^t \cdot \exp(-2^{2k_{\max}-7}). \end{aligned}$$

772 where in the last step we used that $q = s/n$, $\delta = \sqrt{n/s} \cdot \varepsilon$, and $r_j^* = 2^{k_{\max}}\delta$, and set
 773 $c_1 = c_B^t$. ◀

774 **Proof of Lemma 21.** Let $x_1, x_2 \in Bd_S(\mathcal{Y})$. To bound the expected number of edges in E_1 ,
 775 we simply bound the number of pairs $(x_1, x_2) \in Bd_S(\mathcal{Y}) \times Bd_S(\mathcal{Y})$. Let $l_1 := Lev(x)$ and
 776 $l_2 := Lev(y)$, and let $l := \max_i (l_i)_{i=1}^2$. By definition, if $l = 0$, then $x_1, x_2 \in Bd_S(\mathcal{Y})$. For
 777 $l \geq 1$, we get that $x_1 \in Bd_S(\mathcal{Y})$ and $x_2 \in Bd_S(\mathcal{Y})$ only if there exists a disk of radius at
 778 least $2^{l-1}\delta$ passing through x_1 or x_2 , and containing no points of \mathcal{Y} . Therefore to bound the
 779 probability that $(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2$, we can apply the Decay Lemma 30, with $t = 2$, for
 780 $i \in \{1, 2\}$. We get

$$\begin{aligned} 781 \quad \mathbb{P}[(x_1, x_2) \in E_1] &\leq \mathbb{P}[(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2] \\ 782 &\leq c_1 q^2 \cdot \exp(-c_2 \cdot 2^{2l-2}) \leq c_1 q^2 \cdot \exp(-c'_2 \cdot 2^{2l}), \end{aligned} \quad (6)$$

783 where $c'_2 = c_2/4 = 2^{-9}$. Summing over all choices of levels of x_1 and x_2 , we have

$$784 \quad \mathbb{E}[\#(E_1)] \leq \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2 \geq 0} \#(L_{=l_2} \cap \mathcal{X}) \mathbb{P}[(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2].$$

785 By symmetry, it is enough to assume without loss of generality that $l_1 \geq l_2$, i.e. $l = l_1$. Thus,

$$786 \quad \mathbb{E}[\#(E_1)] \leq 2 \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2=0}^{l_1} \#(L_{=l_2} \cap \mathcal{X}) \mathbb{P}[(x_1, x_2) \in (Bd_S(\mathcal{Y}))^2].$$

787 Applying equation (6) and the Level Size Lemma 28, we get

$$\begin{aligned} 788 \quad \mathbb{E}[\#(E_1)] &\leq 2 \sum_{l_1 \geq 0} \#(L_{\leq l_1} \cap \mathcal{X}) \sum_{l_2=0}^{l_1} \#(L_{\leq l_2} \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 \cdot 2^{2l_1}) \\ 789 &\leq 2c_1 q^2 \sum_{l_1 \geq 0} (9\kappa L \cdot (2^{l_1} \delta / \varepsilon^2)) \sum_{l_2=0}^{l_1} (9\kappa L \cdot (2^{l_2} \delta / \varepsilon^2)) \cdot \exp(-c'_2 \cdot 2^{2l_1}) \\ 790 &\leq 2c_1 q^2 (9\kappa L \left(\frac{\delta}{\varepsilon^2}\right))^2 \sum_{l_1 \geq 0} 2^{l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}) \sum_{l_2=0}^{l_1} 2^{l_2}. \end{aligned}$$

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791 Using the definitions of q and δ , together with Proposition 25, and writing the terms
 792 outside the summation as N_1 , we get $N_1 := 2 \cdot c_1 q^2 (9\kappa L (\frac{\delta}{\varepsilon^2}))^2 = 2c_1 \cdot (9\kappa L)^2 (\frac{s}{n\varepsilon^2}) \leq$
 793 $4c_1 \cdot (\frac{4\pi(9\kappa L)^2}{A}) \cdot s$, we continue

$$794 \quad \mathbb{E} [\#(E_1)] \leq N_1 \sum_{l_1 \geq 0} 2^{l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}) \cdot 2 \cdot 2^{l_1} \leq 2N_1 \sum_{l_1 \geq 0} 2^{2l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}).$$

795 The summation can be bounded using Lemma 44, to get

$$796 \quad \mathbb{E} [\#(E_1)] \leq 2N_1 \cdot \left(2 \cdot \frac{\log 1/c'_2}{2ec'_2}\right) = 2N_1 \cdot \frac{\log 1/c'_2}{e \cdot c'_2}.$$

797 Now substituting $c'_2 = 2^{-9}$ gives $\mathbb{E} [\#(E_1)] \leq 2 \cdot 10^4 \cdot c_1 \cdot (\frac{4\pi(9\kappa L)^2}{A}) \cdot s$. ◀

798 **Proof of Lemma 22.** Let l denote $\min\{Lev(x), Lev(y)\}$. Observe that if $l = 0$, then either x
 799 or y is a boundary point, and hence we can assume $l \geq 1$. Let $x' = \arg \min_{z \in \partial F} d(z, x)$, and
 800 $y' = \arg \min_{z \in \partial F} d(z, y)$, i.e. x' is the closest point to x in ∂F , and similarly for y' . By the
 801 definition of $Bd_{\mathcal{S}}(\mathcal{Y})$, observe that $d(x, y) \leq d(x, x') + d(x', y) \leq d(x, x') + d(y, y') \leq 2 \cdot 2^{l-1} \delta$.
 802 Hence we have that $d(x, y) \leq 2^l \delta$.

803 By the Growth Lemma 31, the expected number of Delaunay neighbours y of a point x
 804 such that $d(x, y) \leq \delta$ is at most $\mathbb{E}[D(x, \delta) \cap \mathcal{Y}] \leq q \cdot \frac{4}{q} = 4$. Thus the expected number of
 805 edges in E_2 from pairs (x, y) with $x, y \in Int_F(\mathcal{Y})$ for some $F \in \mathcal{S}$, and $d(x, y) \leq \delta$, is at most
 806 $4 \cdot \#(\mathcal{Y})$. For longer-distance edges, let $k \geq 1$ be such that $2^{k-1} \delta \leq d(x, y) \leq 2^k \delta$. Taking
 807 $t = 2$, $x_1 = x$, $x_2 = y$, $k_1 = k - 1$, and $k_2 \leq k_1$, and applying the Decay Lemma 30, we get
 808 that

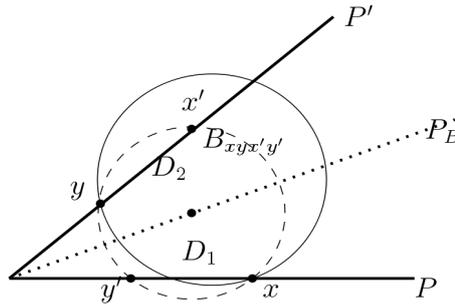
$$809 \quad \mathbb{P}[\{x, y\} \in E_2] \leq c_1 q^2 \cdot \exp(-c_2 2^{2k-2}) \leq c_1 q^2 \cdot \exp(-c'_2 2^{2k}),$$

810 where $c'_2 = c_2/4$. Summing over all possible choices of $l \geq 1$, and $k \leq l$, we get

$$\begin{aligned} 811 \quad \mathbb{E} [\#(E_2)] &\leq \sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \sum_{k=1}^l \#(D(x, 2^k \delta) \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 2^{2k}) \\ 812 &\leq c_1 q^2 \left(\sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \sum_{k=1}^l \kappa 2^{2k} (\delta/\varepsilon)^2 \cdot \exp(-c'_2 2^{2k}) \right) \\ 813 &\leq c_1 \kappa q \left(\frac{q\delta^2}{\varepsilon^2} \right) \cdot \left(\sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \right) \frac{2 \cdot (1/2) \cdot \log 1/c'_2}{ec'_2} \\ 814 &\leq c_1 \kappa q \cdot c_3 \left(\sum_{l \geq 1} \#(L_{=l} \cap \mathcal{X}) \right) \\ 815 &\leq c_1 \kappa q \cdot c_3 \cdot n = c_4 \cdot \kappa s. \end{aligned}$$

816 where in the second step we applied the Growth Lemma 31, and in the third step we bounded
 817 the sum $\sum_{k \geq 0} 2^{2k} \exp(-c_2 2^{2k})$ using Lemma 44, and used that $q = s/n = \varepsilon^2/\delta^2$. Note that
 818 $c_3 \leq 4 \cdot 10^3$, and $c_4 := c_1 \cdot c_3 \leq 2 \cdot 10^5$. ◀

819 **Proof of Lemma 23.** Let $x, y \in Int_{\mathcal{S}}(\mathcal{Y})$, where $x \in F$ and $y \in F'$, for some $F, F' \in \mathcal{S}$. Let
 820 F' be fixed. To analyse this case, we shall first give a geometric construction of [4], and state
 821 an observation from their proof.



822 ■ **Figure 4** $x, y \in \text{Int}_S(\mathcal{Y})$, on different facets $F \subset P, F' \subset P'$

823 ► **Construction 45** (Attali-Boissonnat [4]). Let P and P' denote the supporting planes of
 824 the facets F and F' respectively. Let P_B be the bisector plane of P and P' . We denote by
 825 $x' \in P'$, the reflection of $x \in P$ with respect to P_B , and similarly by $y' \in P$, the reflection
 826 of $y \in P'$. Let $B = B_{xyx'y'}$ be the smallest ball in \mathbb{E}^3 passing through x, y, x', y' , having
 827 intersections $D_1 = B \cap P$ and $D_2 = B \cap P'$ with P and P' respectively.

828 Attali and Boissonnat observed that

829 ► **Proposition 46** (Attali-Boissonnat [4]). Any ball in \mathbb{E}^3 having x and y on its boundary,
 830 must contain either D_1 or D_2 .

831 Therefore, if there exists a ball $B \in \mathbb{E}^3$ such that $x, y \in \partial B$, and $\text{int}(B) \cap \mathcal{Y} = \emptyset$, then
 832 either $D_1 \cap \mathcal{Y} = \emptyset$, or $D_2 \cap \mathcal{Y} = \emptyset$. We get

833
$$\mathbb{P}[\{x, y\} \in E_3] \leq \mathbb{P}[\cup_{i=1}^2 \{D_i \cap \mathcal{Y} = \emptyset\}] \leq 2 \cdot \mathbb{P}[D_1 \cap \mathcal{Y} = \emptyset].$$

834 Observe that, as in Case II, we have $x' \in D(y, 2^{Lev(y)}\delta)$, since otherwise $y \in \text{Bd}_S(\mathcal{Y})$. (Note
 835 that our definition of boundary points allows us to ignore the fact that x' is not necessarily
 836 a point in \mathcal{X} .) Further, the set $\{x' \in D(y, 2^k\delta)\}$, $0 < k < Lev(y)$, is bounded in size by
 837 $\sharp(D(y', 2^k\delta) \cap F \cap \mathcal{Y})$. The rest of the analysis for the fixed facet F' , therefore follows as in
 838 Case II. Summing over all $F' \in \mathcal{S} \setminus \{F\}$, we get $\mathbb{E}[\sharp(E_3)] \leq c_4 \cdot (C - 1)\kappa_s$. ◀

839 **Proof of Proposition 34.** The first part of the proposition, $D_y \cap \mathcal{T} = \emptyset$, follows from the
 840 condition on B . For the next part, note that $y \in \partial D_y$. Let v denote the center of the ball
 841 B , and let z be a variable point on the line segment vx . Let $B(z)$ denote the ball with
 842 center z , having $x \in \partial B(z)$. For $z = v$, $B(z) = B$ intersects P . For $z = x$, $B(z) = \{x\}$ does
 843 not intersect P . Therefore there exists some value of $z = c$ such that $B(c)$ is tangential to
 844 P (see figure 2). Let x' denote the point where $B(c)$ touches P . Then $x' \in D_y$, since by
 845 Construction 32 $B(z) \subset B$ for all z in the segment vx , and hence $B(c) \cap P \subset B \cap P$. Also,
 846 $x' \in V(x)$, by the definition of $V(x)$. Therefore we get $x' \in D_y \cap V(x)$. ◀

847 **Proof of Lemma 35.** Suppose $\{x, y\} \in E_4$. Then there exists a ball $B \in \mathbb{E}^3$ with $x, y \in \partial B$,
 848 and $\text{int}(B) \cap \mathcal{Y} = \emptyset$. Therefore $D_y := B \cap P$ also satisfies $\text{int}(D_y) \cap \mathcal{Y} = \emptyset$. By Proposition 34
 849 we have that $D_y \cap V(x) \neq \emptyset$. Therefore, $y \in V(x) \oplus 2r_y$, where r_y is the radius of D_y . But
 850 since $y \in \text{Int}(F)$, we have that any disk having y on its boundary and containing no point of
 851 \mathcal{Y} in its interior can have radius at most $2^{Lev(y)-1}\delta$. Therefore $r_y \leq 2^{Lev(y)-1}\delta$. Now taking
 852 k_y such that $2^{k_y}\delta = 2r_y$, we get that $k_y \leq Lev(y)$. ◀

853 **Proof of Proposition 33(iii)** Consider a point $x \in Z$, and let $V(x)$ be the region corresponding to x in \mathcal{V} . By Proposition 33 (ii), $V(x) = (\cap_{D \in D_x} D) \cap (\cap_{\bar{D} \in C_x} \bar{D})$, where D_x is a set of disks and C_x is a set of complements of disks in the plane P . Let $y \in \partial V(x)$. Then if $y \in \cap_{D \in D_x} D$, then there exists $D_1 \in D_x$, such that $y \in \partial D_1$, and so y is part of a convex segment in $\partial V(x)$. Otherwise, there exists $\bar{D}_2 \in C_x$, such that $y \in \partial \bar{D}_2$. In this case, let $V(z)$, $z \in Z$, denote the region such that $y \in \partial V(z)$. Then $D_2 \supset V(z)$, and therefore y belongs to a convex segment in $\partial V(z)$. Thus, every point $y \in \partial V(x)$ is convex either for $V(x)$ or for a neighbouring region of $V(x)$, and so the total length of the convex boundary curves in \mathcal{V} gives the total length of all the boundary curves.

863

864 **Proof of Lemma 36.** Let $x \in \mathcal{X}$ and $y \in \mathcal{X} \cap F$. Let $\mathcal{E}_{x,y}$ denote the event $\{x, y\} \in E_4(\text{Int}(F))$. Then $\mathcal{E}_{x,y}$ can occur only if (i) $x \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$ and, (ii) $y \in \text{Int}_{\mathcal{S}}(Y) \cap V_F(x)$. Fix a choice of \mathcal{Y} , say $Y \in \binom{\mathcal{X}}{s}$. Conditioning on this choice of \mathcal{Y} , $\text{Bd}_{\mathcal{S}}(\mathcal{Y})$ is a fixed set of points. The number of pairs contributing to $E_4(\text{Int}(F))$ is at most $\#\{(x, y) \in Y \times Y \mid x \in \text{Bd}_{\mathcal{S}}(Y), y \in V_F(x)\}$. The main observation is now that since \mathcal{V} restricted to F is a sub-division of F , for each $y \in \mathcal{X} \cap F$, there is a unique $x = x_y \in \text{Bd}_{\mathcal{S}}(Y)$ such that $y \in V_F(x)$. Therefore we get

$$871 \quad E_4(\text{Int}(F)) \leq \sum_{V_F(x) \in \mathcal{V}: x \in \text{Bd}_{\mathcal{S}}(Y)} \#(V_F(x) \cap Y) \leq \#(Y \cap F).$$

872 Since the last bound holds for any choice of Y , taking expectation over all choices we get

$$873 \quad \mathbb{E}[E_4(\text{Int}(F))] \leq \mathbb{E}[\#(\mathcal{Y} \cap F)] = q \cdot \#(\mathcal{X} \cap F).$$

874 Now summing over all faces gives $[E_4(\text{Int})] \leq \mathbb{E}[\#(\mathcal{Y})] = s$.

875

876 **Proof of Lemma 37.** To compute the expected value of $E_4(\text{Bd}(\mathcal{S}))$, fix a face $F \in \mathcal{S}$. Consider a pair of points $x, y \in \mathcal{X}$, such that $y \in F$. Let $\mathcal{E}_{x,y}$ denote the event $\{x, y\} \in E_4(\text{Bd}(F))$.

879 The value of $E_4(\text{Bd})$ is the number of $x, y \in \mathcal{X}$, such that $\mathcal{E}_{x,y}$ occurs. Taking expectations,

$$880 \quad \mathbb{E}[E_4(\text{Bd}(F))] \leq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{X} \cap F} \mathbb{P}[\mathcal{E}_{x,y}]. \quad (7)$$

881 Observe that $\mathcal{E}_{x,y}$ occurs only if (i) $x \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$ and (ii) $k_y(x) \leq \text{Lev}(y)$, by applying Construction 32, on the plane P , $Z = \text{Bd}_{\mathcal{S}}(\mathcal{Y})$, and $\mathcal{T} = \mathcal{Y} \cap P$, and using Proposition 34. By Lemma 35, $k_y(x) \in [0, \text{Lev}(y)]$.

884 Let P_{k_1, k_2} denote the probability that $\{x, y\} \in E_4(\text{Bd}(F))$, with $\text{Lev}(x) = k_1$, and $k_y(x) = k_2$. Equation (7) can be rewritten in terms of k_1 and k_2 as

$$886 \quad \mathbb{E}[E_4(\text{Bd}(F))] \leq \sum_{k_1 \geq 0} \#(L_{=k_1} \cap \mathcal{X}) \sum_{k_2 \geq 0} \sum_{V_F \in \mathcal{V}} \#((\partial V_F \oplus 2^{k_2} \delta) \cap \mathcal{X}) \cdot P_{k_1, k_2}.$$

887 Applying the Decay Lemma 30 with $t = 2$, $x_1 = x$, $x_2 = y$, $k_1 = \max\{0, k_1 - 1\}$ (since $x \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$), and $k_2 = \max\{0, k_2 - 1\}$, we get

$$889 \quad P_{k_1, k_2} \leq c_1 q^2 \cdot \exp(-f(k^*)),$$

890 where $k^* := \max\{0, k_1 - 1, k_2 - 1\}$, and $f(k^*) = 0$ if $k^* = 0$, and $c'_2 \cdot 2^{2k^*}$ otherwise, with $c'_2 = c_2/4$.

891

892 As in the proof of Lemma 21, we shall use symmetry to handle the case where $k_1 \geq k_2$
 893 and $k_2 > k_1$ together. We get

$$894 \mathbb{E}[E_4(Bd(F))] \leq 2 \sum_{k_1 \geq 0} \#(L_{=k_1} \cap \mathcal{X}) \sum_{k_2 \leq k_1} \sum_{V(x) \in \mathcal{V}} \#((\partial V(x) \oplus 2^{k_2} \delta) \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 2^{2k_1}).$$

895 By the Level Size Lemma 28, we get that $\#(L_{=k_1} \cap \mathcal{X}) \leq \frac{9\kappa L 2^{k_1} \delta}{\varepsilon^2}$. Using Proposition 26, we
 896 get that $\#(\{\partial V(x) \oplus 2^{k_2} \delta\} \cap \mathcal{Y}) \leq \frac{9\kappa \cdot l(\partial V(x)) 2^{k_2} \delta}{\varepsilon^2}$. By Proposition 33 (iii), each boundary in
 897 the partition \mathcal{V} is convex for some $x \in Bd_S(\mathcal{Y})$. Therefore we need to sum $l(\partial V(x))$ only
 898 over the convex curves in $\partial V(x)$, $x \in Bd_S(\mathcal{Y})$. The length of these curves is at most $l(\partial F)$.
 899 Thus we get

$$900 \mathbb{E}[E_4(Bd(F))] \leq 2L \cdot l(\partial F) \cdot \left(\frac{9\kappa \delta q}{\varepsilon^2}\right)^2 \sum_{k_1 \geq 0} 2^{k_1} \sum_{k_2 \leq k_1} 2^{k_2} c_1 \cdot \exp(-c'_2 \cdot 2^{2k_1}).$$

901 Using Lemma 44, the above summation is bounded by a constant. This comes to $c_1 \cdot$
 902 $O(1) \left(\frac{(9\kappa)^2 L \cdot l(\partial F) \delta^2 q^2}{\varepsilon^4}\right) = O\left(\frac{\kappa^2 L \cdot l(\partial F) s}{A}\right)$, where the last step followed from the lower bound
 903 on n in Proposition 25 (2), and the identities $q = s/n = \delta^2/\varepsilon^2$. Summing y over all facets F
 904 in \mathcal{S} , we get $\mathbb{E}[E_4(Bd(\mathcal{S}))] = \left(O(1) \cdot \frac{\kappa^2 L^2 s}{A}\right)$. ◀

905 Proofs from Section 7

906 **Proof of Theorem 41.** From Bieberbach's theorem [6], and Caroli-Teillaud's construction [10]
 907 (Section 4), it follows that

908 ▶ **Theorem 47** (Bieberbach [6], Caroli-Teillaud [10](Section 4)). *Every Euclidean d -orbifold*
 909 \mathbb{M} *has a covering space using a finite number $m_{\mathbb{M}}$ of sheets of a d -hyperparallelepiped $\tilde{\mathbb{T}}_{\mathbb{M}}^d$,*
 910 *such that the Delaunay triangulation of any point set on \mathbb{M} is the projection of the Delaunay*
 911 *triangulation of the cover of the point set in the covering space.*

912 The proof of Theorem 41 is on similar lines as that of Theorem 10, except (i) we work with
 913 the hyperparallelepiped $\tilde{\mathbb{T}}_{\mathbb{M}}^d$, and (ii) we need to take the effect of the multiplicity (i.e. the
 914 number $m_{\mathbb{M}}$ of sheets of $\tilde{\mathbb{T}}_{\mathbb{M}}^d$ required for Theorem 47 to hold) into account for all simplices.
 915 To handle (i), we observe that the volume of balls will change as the hyperparallelepiped is
 916 no longer a hypercube. Thus, a factor of the volume of the unit hyperparallelepiped $\tilde{\mathbb{T}}_{\mathbb{M}}^d$, will
 917 come into the estimates in Lemma 5. To handle the effect of multiplicity, we introduce an
 918 extra multiplicative factor of $m_{\mathbb{M}}^d$ in the bound of the number of possible d -simplices with
 919 any fixed set of points (compared to Lemma 20). Additionally, we take into account that
 920 the number of *distinct* points inside a potential simplex is at least a $1/m_{\mathbb{M}}$ -fraction of the
 921 number guaranteed by Lemma 17. This gives a worse bound for the expected complexity of
 922 the star than in Theorem 10, but still a constant. ◀

923 **Proof of Theorem 42.** The action of \mathbb{Z}^d on \mathbb{E}^d is defined by translation, i.e. for $x \in \mathbb{E}^d$,
 924 $g \in \mathbb{Z}^d$, $gx := g \cdot x = g + x$. For a finite point set $P \in \mathbb{E}^d$, let $\Delta(\mathbb{Z}^d P)$ denote the largest
 925 ball in \mathbb{E}^d containing no points from $\mathbb{Z}^d P$. Let $\delta((k\mathbb{Z})^d)$ denote the minimum distance by
 926 which a point in \mathbb{E}^d is translated by $(k\mathbb{Z})^d$. Finally, let $\pi(\cdot)$ denote the projection map of the
 927 covering space \mathbb{T}_k^d on to \mathbb{T}^d . We shall use the following geometric condition of Caroli and
 928 Teillaud [10].

929 ▶ **Lemma 48.** *If $\Delta(\mathbb{Z}^d P) < \delta((k\mathbb{Z})^d)/2$, then for any finite $Y \supset P$, the projection*
 930 $\pi(\text{Del}(\mathbb{Z}^d Y))$ *is a triangulation of \mathbb{T}^d .*

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931 Now, observe that in the Euclidean metric, the diameter $\Delta_{\|\cdot\|}(\mathbb{Z}^d P)$ of the largest ball not
 932 containing any point from the set $\mathbb{Z}^d P$ is at most \sqrt{d} , with equality holding when $\sharp(P) = 1$,
 933 and that $\Delta(S') \leq \Delta(S)$ for any $S' \supseteq S$, since adding points can only decrease the diameter
 934 of the largest empty ball. Therefore, in the metric \mathfrak{d} , we have that

$$935 \quad \Delta_{\mathfrak{d}}(\mathbb{Z}^d P) \leq \Delta_{\|\cdot\|}(\mathbb{Z}^d P) \cdot \left(\max_{x,y \in \mathcal{S}} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \right).$$

936 By the previously observed bound on $\Delta_{\|\cdot\|}(\mathbb{Z}^d P)$, we have

$$937 \quad \Delta_{\mathfrak{d}}(\mathbb{Z}^d P) \leq \left(\max_{x,y \in \mathcal{S}} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \right) \cdot \sqrt{d}.$$

938 Also, letting $((k\mathbb{Z})^d)^*$ denote the non-identity elements of $(k\mathbb{Z})^d$,

$$939 \quad \delta_{\mathfrak{d}}((k\mathbb{Z})^d) = \min_{x \in \mathcal{G}, g \in ((k\mathbb{Z})^d)^*} \mathfrak{d}(x, gx) = \min_{x \in \mathcal{G}, g \in ((k\mathbb{Z})^d)^*} \frac{\mathfrak{d}(x, gx)}{\|x - gx\|} \cdot \|x - gx\|.$$

940 For the flat torus \mathbb{T}_k^d , $\min_{x,g} \|x - gx\| = k$. Therefore, in the metric \mathfrak{d} , the condition of
 941 Lemma 48 is satisfied if

$$942 \quad \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \cdot \sqrt{d} < \frac{1}{2} \left(\min_{x \in P, g \in ((k\mathbb{Z})^d)^*} \frac{\mathfrak{d}(x, gx)}{\|x - gx\|} \cdot \|x - gx\| \right), \text{ or,}$$

$$943 \quad 2\sqrt{d} \cdot \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \cdot \max_{x' \in P, g \in ((k\mathbb{Z})^d)^*} \frac{\|x' - gx'\|}{\mathfrak{d}(x', gx')} < k, \text{ which is true if}$$

$$944 \quad 2\sqrt{d} \cdot \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|} \cdot \min_{x',y' \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x',y')}{\|x' - y'\|} < k, \text{ or,}$$

$$945 \quad 2\sqrt{d} \cdot \kappa_{\mathfrak{d}} < k,$$

946 since by definition, the distortion $\kappa_{\mathfrak{d}}$ satisfies for all $x, y \in \mathbb{Z}^d P$, $\mathfrak{d}(x, y) \leq \kappa_{\mathfrak{d}} \|x - y\|$, i.e.
 947 $\kappa_{\mathfrak{d}} \geq \max_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|}$, as well as $\|x - y\| \leq \kappa_{\mathfrak{d}} \mathfrak{d}(x, y)$, i.e. $\kappa_{\mathfrak{d}} \geq \min_{x,y \in \mathbb{Z}^d P} \frac{\mathfrak{d}(x,y)}{\|x-y\|}$.

948 Since the fundamental domain of \mathbb{T}_k^d contains k^d copies of the fundamental domain of \mathbb{T}^d ,
 949 we have $m \leq k^d$, and so the first part of the theorem follows with $m \leq (2\sqrt{d}\kappa_{\mathfrak{d}})^d$. The second
 950 part easily follows from the well-known linear algebraic facts that (a) $\max_{\|x\|=1} \{x^\top Mx\} =$
 951 $\max_{\|x\|=1} \{\|Ax\|\} = \sigma_{\max}(A)$, and that (b) $\min_{\|x\|=1} \{x^\top Mx\} = \sigma_{\max}(A^{-1}) = \sigma_{\min}(A)$ where
 952 A is such that $M = A^\top A$ and $\sigma_{\max}(A)$, $\sigma_{\min}(A)$ are respectively the largest and the smallest
 953 singular values of A . ◀