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# A direct proof of the equivalence of side conditions for strictly positive real matrix transfer functions

Augusto Ferrante Alexander Lanzon Bernard Brogliato

**Abstract**—This brief note proves in a direct way that two different side conditions, which have been used in the literature to characterize strictly positive real matrix transfer functions in the frequency domain, are equivalent.

## I. INTRODUCTION

The frequency domain conditions characterizing the fact that a matrix transfer function  $F$  is strictly positive real involve a positivity constraint at infinite frequency. This constraint — usually referred to as *side condition* — has been a source of confusion and controversy in the literature for more than a decade. As pointed out in [3], the side conditions used in [6], [7], [8] were incorrect as they had some inconsistencies. To fix the problem, [3] proposed a new condition at infinite frequency, i.e.

$$\lim_{\omega \rightarrow \infty} \omega^{2\rho} \det (F(j\omega) + F(-j\omega)^\top) > 0 \quad (1)$$

where  $\rho$  is the dimension of  $\ker (F(\infty) + F(\infty)^\top)$ .

On the other hand, a different, but equally valid, condition at infinite frequency was proposed the second edition of the book by Khalil published in 1996 (see [5, Lemma 10.1]); such a condition, that reads as follows, was recently used in [4] to establish a counterpart result for negative imaginary systems (see [4, Remark 1]):  $\exists \delta > 0, \sigma_0 > 0$  such that

$$\underline{\sigma} [\omega^2 (F(j\omega) + F(-j\omega)^\top)] \geq \sigma_0 \quad \forall |\omega| \geq \delta. \quad (2)$$

This note is devoted to the analysis of the two side-conditions (1) and (2). We will prove that while they are in general not equivalent at infinite frequency, they are indeed equivalent under the other conditions guaranteeing that  $F$  is strictly positive real. Hence both conditions at infinite frequency are equally valid. While this could be deduced from [3] and [5], our results provide a direct proof of such equivalency.

**Notation:** Let the set of real (resp. complex) numbers be denoted by  $\mathbb{R}$  (resp.  $\mathbb{C}$ ) and the corresponding sets of matrices

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of dimension  $m \times n$  be denoted by  $\mathbb{R}^{m \times n}$  (resp.  $\mathbb{C}^{m \times n}$ ). Given  $M \in \mathbb{C}^{m \times m}$ ,  $M^*$  denotes the complex conjugate transpose of matrix  $M$  (i.e., if  $M = A + jB$  for real matrices  $A$  and  $B$ , then  $M^* = A^\top - jB^\top$ ). A matrix  $M$  is said to be Hermitian if  $M = M^*$  and  $M > 0$  denotes that the matrix  $M$  is Hermitian and positive definite. The smallest singular value of  $M$  is denoted by  $\underline{\sigma}(M)$ . We recall that the singular values of a positive semi-definite Hermitian matrix are its nonzero eigenvalues [2, p.649].

## II. MAIN RESULT

The following definition, adapted from [4, Definitions 1 and 2], is the standard definition for strictly positive real systems. It essentially states that a transfer function matrix  $F(s)$  is strictly positive real if for some  $\epsilon > 0$ , the transfer function matrix  $F(s - \epsilon)$  is positive real and  $F(s) + F(-s)^\top$  has full normal rank. See also [4, Lemma 2] for an equivalent re-characterization.

**Definition 2.1:** Let  $F : \mathbb{C} \rightarrow \mathbb{C}^{m \times m}$  be a real transfer function. Then  $F(s)$  is said to be **Strictly Positive Real (SPR)** if there exists a real scalar  $\epsilon > 0$  such that  $F(s)$  is analytic in  $\{s \in \mathbb{C} : \Re\{s\} > -\epsilon\}$ ,  $F(s) + F(s)^* \geq 0$  for all  $s \in \{s \in \mathbb{C} : \Re\{s\} > -\epsilon\}$  and  $F(s) + F(-s)^\top$  has full normal rank. SPR matrix transfer functions can be characterized in the frequency domain by three conditions: the first two are conditions 1 and 2 in the next proposition, the third is the *side condition* and it has been stated in two different manners: side condition (3a) in Proposition 2.1 can be found in [5], [4], while side condition (3b) can be found in [3]. These side conditions can be interpreted as apparently different conditions on how  $(F(j\omega) + F(-j\omega)^\top)$  approaches zero for sufficiently large  $|\omega|$  in directions where it loses rank.

**Proposition 2.1:** Let  $F : \mathbb{C} \rightarrow \mathbb{C}^{m \times m}$  be a real, rational, proper transfer function such that the following two conditions hold:

- 1)  $F(s)$  has no poles in  $\{s \in \mathbb{C} : \Re\{s\} \geq 0\}$ ;
- 2)  $F(j\omega) + F(-j\omega)^\top > 0$  for all  $\omega \in \mathbb{R}$ .

Then, the following two side conditions are equivalent:

- 3a)  $\exists \delta > 0, \sigma_0 > 0 :$

$$\underline{\sigma} [\omega^2 (F(j\omega) + F(-j\omega)^\top)] \geq \sigma_0 \quad \forall |\omega| \geq \delta \quad (3)$$

- 3b)  $\lim_{\omega \rightarrow \infty} \omega^{2\rho} \det[F(j\omega) + F(-j\omega)^\top] \neq 0 \quad (4)$

where  $\rho = \dim \ker (F(\infty) + F(\infty)^\top)$ .

**Proof.** Since  $F(s)$  is proper, let  $F(s) = C(sI - A)^{-1}B + D$  for some state-space realization  $(A, B, C, D)$ . Near  $s = \infty$ , the expansion

$$F(s) = F_0 + F_1/s + F_2/s^2 + \dots \quad (5)$$

holds (with  $F_0 = F(\infty) = D$ ), so that

$$F(j\omega) + F(-j\omega)^\top = Q + \frac{H}{\omega} + \frac{K(\omega)}{\omega^2} \quad (6)$$

with  $Q = D + D^\top$ ,  $H = (F_1 - F_1^\top)/j$  and  $K(\omega) = \sum_{i=1}^{\infty} \left( \frac{F_{i+1}}{j^{i+1}\omega^{i-1}} + \frac{F_{i+1}^\top}{(-j)^{i+1}\omega^{i-1}} \right)$ . Then it is easy to see that

$$\lim_{\omega \rightarrow \infty} K(\omega) = K_0 \in \mathbb{R}^{m \times m},$$

with  $K_0 := -F_2 - F_2^\top$ . If  $Q$  is positive definite, in view of the Hermitian symmetry of both  $H$  and  $K(\omega)$ , both conditions 3a) and 3b) are clearly satisfied and the result is obvious.

Assume that  $Q$  is singular with rank  $m - \rho$ . Since  $Q$  is symmetric, there exists an orthogonal matrix  $U$  such that  $UQU^\top = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$  with  $Q_1 \in \mathbb{R}^{(m-\rho) \times (m-\rho)}$  nonsingular. Hence, without changing the essence of the problem, we assume that  $Q$  has the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (7)$$

with  $Q_1 \in \mathbb{R}^{(m-\rho) \times (m-\rho)}$  nonsingular, and hence  $Q_1 > 0$ . We partition  $H$  and  $K(\omega)$  accordingly, as follows:

$$H = \begin{bmatrix} H_1 & H_{12} \\ H_{21} & H_2 \end{bmatrix}, \quad K(\omega) = \begin{bmatrix} K_1(\omega) & K_{12}(\omega) \\ K_{21}(\omega) & K_2(\omega) \end{bmatrix}. \quad (8)$$

Thus

$$[F(j\omega) + F(-j\omega)^\top]_{22} = \frac{1}{\omega} (H_2 + \frac{K_2(\omega)}{\omega}). \quad (9)$$

Since  $H = (F_1 - F_1^\top)/j$ , we have that: (i)  $H$  is Hermitian, so that  $H_2$  is Hermitian as well, and (ii) the elements on the diagonal of  $H$  are zero, so that also the elements on the diagonal of  $H_2$  are zero and hence  $H_2$  is traceless. Therefore,  $H_2$  has only real eigenvalues and the sum of the eigenvalues of  $H_2$  is zero. Hence, either  $H_2 = 0$  or  $H_2$  has at least a negative eigenvalue. But for  $\omega$  sufficiently large, and by continuity of the eigenvalues as functions of  $\omega$ , the eigenvalues of  $H_2 + \frac{K_2(\omega)}{\omega}$  are arbitrarily close to those of  $H_2$ , so that if  $H_2 \neq 0$ , then  $[F(j\omega) + F(-j\omega)^\top]_{22}$  has a negative eigenvalue for a sufficiently large  $\omega$ , and this is against our assumptions because  $[F(j\omega) + F(-j\omega)^\top] > 0$  for all  $\omega \in \mathbb{R}$  so, in turn,  $[F(j\omega) + F(-j\omega)^\top]_{22} > 0$  for all  $\omega \in \mathbb{R}$ . In conclusion,  $H_2 = 0$ .

Now let

$$\Phi(\omega) = \begin{bmatrix} \Phi_1(\omega) & \Phi_{12}(\omega) \\ \Phi_{21}(\omega) & \Phi_2(\omega) \end{bmatrix} = F(j\omega) + F(-j\omega)^\top.$$

By continuity, as  $\omega \rightarrow \infty$ ,  $m - \rho$  of the eigenvalues of  $\Phi$ , i.e., the eigenvalues of  $\Phi_1(j\omega)$ , tend to the eigenvalues of  $Q_1$  (that are strictly positive) and the remaining  $\rho$  eigenvalues tend to zero. Let  $\lambda(\omega)$  be one of the eigenvalues of  $\Phi$  that tends to zero as  $\omega \rightarrow \infty$ . We now show that  $\lambda(\omega)$  tends to zero at least as fast as  $1/\omega^2$ . In fact, provided that  $\omega$  is large enough so that  $\lambda(\omega)$  is not an eigenvalue of  $\Phi_1(\omega)$ , then  $\lambda(\omega)$  must be an eigenvalue of

$$R(\omega) = \Phi_2(\omega) - \Phi_{21}(\omega)[\Phi_1(\omega) - \lambda(\omega)I]^{-1}\Phi_{12}(\omega) \quad (10)$$

because through Schur complements (e.g. [1])

$$\det[\Phi(\omega) - \lambda(\omega)I] = \det[\Phi_1(\omega) - \lambda(\omega)I] \det[R(\omega) - \lambda(\omega)I].$$

Using the fact that  $\Phi_1(\omega) - \lambda(\omega)I = Q_1 + \frac{H_1}{\omega} + \frac{K_1(\omega)}{\omega^2} - \lambda(\omega)I$ , and using [1, Fact 9.9.43],  $R(\omega)$  in (10) can be written equivalently as

$$R(\omega) = \Phi_2(\omega) - \Phi_{21}(\omega)[Q_1^{-1} + \Delta(\omega)]\Phi_{12}(\omega),$$

where  $\Delta(\omega) \rightarrow 0$  as  $\omega \rightarrow \infty$ , since  $\lambda(\omega) \rightarrow 0$  as  $\omega \rightarrow \infty$ . Finally, using (6) (7) and (8), we have  $R(\omega) = \frac{1}{\omega^2} [K_2(\omega) - P(\omega)]$  where

$$P(\omega) := (H_{21} + \frac{K_{21}(\omega)}{\omega})(Q_1^{-1} + \Delta(\omega))(H_{12} + \frac{K_{12}(\omega)}{\omega}).$$

Since for  $\omega$  sufficiently large  $[K_2(\omega) - P(\omega)]$  is bounded, then the eigenvalues of  $R(\omega)$  tend to zero at least as fast as  $1/\omega^2$ .

Then  $\Phi(\omega)$  has  $m$  strictly positive eigenvalues for all  $\omega \in \mathbb{R}$ ,  $\rho$  of which are going down to zero at least as fast as  $1/\omega^2$  as  $\omega \rightarrow \infty$ . The remaining  $(m - \rho)$  eigenvalues tend to the strictly positive eigenvalues of  $Q_1$  as  $\omega \rightarrow \infty$ . Now, order the eigenvalues of  $\Phi(\omega)$  in non-decreasing size.

Since  $\det(\cdot)$  is the product of eigenvalues, then  $\omega^{2\rho} \det[\Phi(\omega)]$  is the same as the product of  $(\omega^2 \lambda_1), \dots, (\omega^2 \lambda_\rho), \lambda_{\rho+1}, \dots, \lambda_m$  because for each  $i \in \{1, \dots, \rho\}$ ,  $\lambda_i$  tends to zero at least as fast as  $1/\omega^2$ . Then, we have that the side condition (3b):

$$\lim_{\omega \rightarrow \infty} \omega^{2\rho} \det[\Phi(\omega)] \neq 0$$

is equivalent to (since  $\lambda_{\rho+1}, \dots, \lambda_m$  do not tend to zero)

$$\lim_{\omega \rightarrow \infty} (\omega^2 \lambda_i) \neq 0 \text{ for all } i \in \{1, \dots, \rho\},$$

which is equivalent to

$$\lim_{\omega \rightarrow \infty} \omega^2 \lambda_1 > 0,$$

which, finally, is equivalent to the first side condition (3a). The proposition is proved.  $\square$

*Examples.* Consider the case of  $F(s) = \frac{1}{s+1} \begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix}$ . In this case  $\rho = 1$  and, by direct computation, we see that

$$\lim_{\omega \rightarrow \infty} \omega^{2\rho} \det[F(j\omega) + F(-j\omega)^\top] = \lim_{\omega \rightarrow \infty} \frac{8\omega^2 + 4\omega^4}{(1 + \omega^2)^2} = 4 \neq 0.$$

On the other hand, we have

$$\underline{\sigma} [\omega^2 (F(j\omega) + F(-j\omega)^\top)] = \frac{2\omega^2}{1 + \omega^2}$$

which is clearly greater than 1 for all  $|\omega| > 1$ .

The pathological case, corresponding to the situation in which some of the eigenvalues of the spectrum go to zero faster than  $\frac{1}{\omega^2}$ , as  $\omega$  tends to infinity, is more interesting: let

$$F(s) = \begin{bmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+2}{(s+1)} \end{bmatrix}. \text{ In this case } \rho = 1 \text{ and by direct computation, we easily see that}$$

$$\lim_{\omega \rightarrow \infty} \omega^{2\rho} \det[F(j\omega) + F(-j\omega)^\top] = \lim_{\omega \rightarrow \infty} \frac{\omega^2(16 + 8\omega^2)}{(1 + \omega^2)^3} = 0.$$

The same conclusion is obtained with the other side condition which, in this case, reads:

$$\underline{\sigma} [\omega^2 (F(j\omega) + F(-j\omega)^\top)] = \frac{4\omega^2}{(1 + \omega^2)^2}$$

which is clearly not bounded away from zero as  $|\omega|$  diverges.

Finally, the trivial example  $F(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  immediately shows that, if we do not assume the first two conditions of Proposition 2.1, the two side conditions are not necessarily equivalent.

### III. CONCLUSION

This note shows that two different side conditions used in the control literature to characterize strictly positive real matrix transfer functions are equivalent.

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