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► To cite this version:

Frédéric Blanqui, Guillaume Genestier, Olivier Hermant. Dependency Pairs Termination in Dependent Type Theory Modulo Rewriting. FSCD, Jun 2019, Dortmund, Germany. 10.4230/LIPIcs.FSCD.2019.9. hal-01943941v2

HAL Id: hal-01943941 https://inria.hal.science/hal-01943941v2

Submitted on 20 May 2019 (v2), last revised 15 Oct 2019 (v4)

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Dependency Pairs Termination in Dependent Type Theory Modulo Rewriting

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Abstract

Dependency pairs are a key concept at the core of modern automated termination provers for 10 first-order term rewriting systems. In this paper, we introduce an extension of this technique for 11 a large class of dependently-typed higher-order rewriting systems. This extends previous results 12 by Wahlstedt on the one hand and the first author on the other hand to strong normalization and 13 14 non-orthogonal rewriting systems. This new criterion is implemented in the type-checker DEDUKTI.

2012 ACM Subject Classification Theory of computation \rightarrow Equational logic and rewriting; Theory 15 of computation \rightarrow Type theory 16

Keywords and phrases Termination, Higher-Order Rewriting, Dependent Types, Dependency Pairs 17

Digital Object Identifier 10.4230/LIPIcs... 18

1 Introduction 19

Termination, that is, the absence of infinite computations, is an important problem in 20 software verification, as well as in logic. In logic, it is often used to prove cut elimination and 21 consistency. In automated theorem provers and proof assistants, it is often used (together 22 with confluence) to check decidability of equational theories and type-checking algorithms. 23 This paper introduces a new termination criterion for a large class of programs whose 24

operational semantics can be described by higher-order rewriting rules [33] typable in the 25 $\lambda \Pi$ -calculus modulo rewriting ($\lambda \Pi / \mathcal{R}$ for short). $\lambda \Pi / \mathcal{R}$ is a system of dependent types where 26 types are identified modulo the β -reduction of λ -calculus and a set \mathcal{R} of rewriting rules given 27 by the user to define not only functions but also types. It extends Barendregt's Pure Type 28 System (PTS) λP [3], the logical framework LF [16] and Martin-Löf's type theory. It can 29 encode any functional PTS like System F or the Calculus of Constructions [10]. 30

Dependent types, introduced by de Bruijn in AUTOMATH, subsume generalized algebraic 31 data types (GADT) used in some functional programming languages. They are at the core of 32 many proof assistants and programming languages: COQ, TWELF, AGDA, LEAN, IDRIS, ... 33 Our criterion has been implemented in DEDUKTI, a type-checker for $\lambda \Pi / \mathcal{R}$ that we will 34 use in our examples. The code is available in [12] and could be easily adapted to a subset of 35 other languages like AGDA. As far as we know, this tool is the first one to automatically 36 check termination in $\lambda \Pi / \mathcal{R}$, which includes both higher-order rewriting and dependent types. 37 This criterion is based on dependency pairs, an important concept in the termination 38 of first-order term rewriting systems. It generalizes the notion of recursive call in first-39 order functional programs to rewriting. Namely, the dependency pairs of a rewriting rule 40 $f(l_1,\ldots,l_p) \to r$ are the pairs $(f(l_1,\ldots,l_p),g(m_1,\ldots,m_q))$ such that $g(m_1,\ldots,m_q)$ is a 41 subterm of r and q is a function symbol defined by some rewriting rules. Dependency pairs 42 have been introduced by Arts and Giesl [2] and have evolved into a general framework for 43 termination [13]. It is now at the heart of many state-of-the-art automated termination 44



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LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

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⁴⁵ provers for first-order rewriting systems and HASKELL, JAVA or C programs.

⁴⁶ Dependency pairs have been extended to different simply-typed settings for higher-order

rewriting: Combinatory Reduction Systems [23] and Higher-order Rewriting Systems [29],
with two different approaches: dynamic dependency pairs include variable applications [24],
while static dependency pairs exclude them by slightly restricting the class of systems that
can be considered [25]. Here, we use the static approach.

In [37], Wahlstedt considered a system slightly less general than $\lambda \Pi / \mathcal{R}$ for which he 51 provided conditions that imply the weak normalization, that is, the existence of a finite 52 reduction to normal form. In his system, \mathcal{R} uses matching on constructors only, like in 53 the languages OCAML or HASKELL. In this case, \mathcal{R} is orthogonal: rules are left-linear (no 54 variable occurs twice in a left-hand side) and have no critical pairs (no two rule left-hand side 55 instances overlap). Wahlstedt's proof proceeds in two modular steps. First, he proves that 56 typable terms have a normal form if there is no infinite sequence of function calls. Second, 57 he proves that there is no infinite sequence of function calls if \mathcal{R} satisfies Lee, Jones and 58 Ben-Amram's size-change termination criterion (SCT) [26]. 59

In this paper, we extend Wahlstedt's results in two directions. First, we prove a stronger normalization property: the absence of infinite reductions. Second, we assume that \mathcal{R} is locally confluent, a much weaker condition than orthogonality: rules can be non-left-linear and have joinable critical pairs.

In [5], the first author developed a termination criterion for a calculus slightly more general than $\lambda \Pi / \mathcal{R}$, based on the notion of computability closure, assuming that type-level rules are orthogonal. The computability closure of a term $f(l_1, \ldots, l_p)$ is a set of terms that terminate whenever l_1, \ldots, l_p terminate. It is defined inductively thanks to deduction rules preserving this property, using a precedence and a fixed well-founded ordering for dealing with function calls. Termination can then be enforced by requiring each rule right-hand side to belong to the computability closure of its corresponding left-hand side.

⁷¹ We extend this work as well by replacing that fixed ordering by the dependency pair ⁷² relation. In [5], there must be a decrease in every function call. Using dependency pairs ⁷³ allows one to have non-strict decreases. Then, following Wahlstedt, SCT can be used to ⁷⁴ enforce the absence of infinite sequence of dependency pairs. But other criteria have been ⁷⁵ developed for this purpose that could be adapted to $\lambda \Pi/\mathcal{R}$.

76 Outline

⁷⁷ The main result is Theorem 11 stating that, for a large class of rewriting systems \mathcal{R} , the ⁷⁸ combination of β and \mathcal{R} is strongly normalizing on terms typable in $\lambda \Pi / \mathcal{R}$ if, roughly ⁷⁹ speaking, there is no infinite sequence of dependency pairs.

⁸⁰ The proof involves two steps. First, after recalling the terms and types of $\lambda \Pi / \mathcal{R}$ in ⁸¹ Section 2, we introduce in Section 3 a model of this calculus based on Girard's reducibility ⁸² candidates [15], and prove that every typable term is strongly normalizing if every symbol of ⁸³ the signature is in the interpretation of its type (Adequacy lemma). Second, in Section 4, we ⁸⁴ introduce our notion of dependency pair and prove that every symbol of the signature is in ⁸⁵ the interpretation of its type if there is no infinite sequence of dependency pairs.

In order to show the usefulness of this result, we give simple criteria for checking the conditions of the theorem. In Section 5, we show that *plain function passing* systems belong to the class of systems that we consider. And in Section 6, we show how to use size-change termination to obtain the termination of the dependency pair relation.

Finally, in Section 7 we compare our criterion with other criteria and tools and, in Section
8, we summarize our results and give some hints on possible extensions.

For lack of space, some proofs are given in an appendix at the end of the paper.

2 Terms and types

⁹⁴ The set \mathbb{T} of terms of $\lambda \Pi / \mathcal{R}$ is the same as those of Barendregt's λP [3]:

95

93

$$t \in \mathbb{T} = s \in \mathbb{S} \mid x \in \mathbb{V} \mid f \in \mathbb{F} \mid \forall x : t, t \mid tt \mid \lambda x : t, t$$

⁹⁶ where $S = \{TYPE, KIND\}$ is the set of sorts¹, V is an infinite set of variables and \mathbb{F} is a set of ⁹⁷ function symbols, so that S, V and \mathbb{F} are pairwise disjoint.

Furthermore, we assume given a set \mathcal{R} of rules $l \to r$ such that $FV(r) \subseteq FV(l)$ and l is of the form $f\vec{l}$. A symbol f is said to be defined if there is a rule of the form $f\vec{l} \to r$. In this paper, we are interested in the termination of

where \rightarrow_{β} is the β -reduction of λ -calculus and $\rightarrow_{\mathcal{R}}$ is the smallest relation containing \mathcal{R} and closed by substitution and context: we consider rewriting with syntactic matching only. Following [6], it should however be possible to extend the present results to rewriting with matching modulo $\beta\eta$ or some equational theory. Let SN be the set of terminating terms and, given a term t, let $\rightarrow(t) = \{u \in \mathbb{T} \mid t \rightarrow u\}$ be the set of immediate reducts of t.

A typing environment Γ is a (possibly empty) sequence $x_1 : T_1, \ldots, x_n : T_n$ of pairs of variables and terms, where the variables are distinct, written $\vec{x} : \vec{T}$ for short. Given an environment $\Gamma = \vec{x} : \vec{T}$ and a term U, let $\forall \Gamma, U$ be $\forall \vec{x} : \vec{T}, U$. The product arity ar(T) of a term T is the integer $n \in \mathbb{N}$ such that $T = \forall x_1 : T_1, \ldots, \forall x_n : T_n, U$ and U is not a product. Let \vec{t} denote a possibly empty sequence of terms t_1, \ldots, t_n of length $|\vec{t}| = n$, and FV(t) be the set of free variables of t.

For each $f \in \mathbb{F}$, we assume given a term Θ_f and a sort s_f , and let Γ_f be the environment such that $\Theta_f = \forall \Gamma_f, U$ and $|\Gamma_f| = \operatorname{ar}(\Theta_f)$.

The application of a substitution σ to a term t is written $t\sigma$. Given a substitution σ , let dom $(\sigma) = \{x | x\sigma \neq x\}$, FV $(\sigma) = \bigcup_{x \in \text{dom}(\sigma)} \text{FV}(x\sigma)$ and $[x \mapsto a, \sigma]$ $([x \mapsto a] \text{ if } \sigma \text{ is the}$ identity) be the substitution $\{(x, a)\} \cup \{(y, b) \in \sigma \mid y \neq x\}$. Given another substitution σ' , let $\sigma \to \sigma'$ if there is x such that $x\sigma \to x\sigma'$ and, for all $y \neq x, y\sigma = y\sigma'$.

The typing rules of $\lambda \Pi/\mathcal{R}$, in Figure 1, add to those of λP the rule (fun) similar to (var). Moreover, (conv) uses \downarrow instead of \downarrow_{β} , where $\downarrow = \rightarrow^* \ast \leftarrow$ is the joinability relation and \rightarrow^* the reflexive and transitive closure of \rightarrow . We say that t has type T in Γ if $\Gamma \vdash t : T$ is derivable. A substitution σ is well-typed from Δ to Γ , written $\Gamma \vdash \sigma : \Delta$, if, for all $(x:T) \in \Delta, \Gamma \vdash x\sigma : T\sigma$ holds.

The word "type" is used to denote a term occurring at the right-hand side of a colon in a typing judgment (and we usually use capital letters for types). Hence, KIND is the type of TYPE, Θ_f is the type of f, and s_f is the type of Θ_f . Common data types like natural numbers \mathbb{N} are usually declared in $\lambda \Pi$ as function symbols of type TYPE: $\Theta_{\mathbb{N}} =$ TYPE and $s_{\mathbb{N}} =$ KIND.

¹²⁹ The dependent product $\forall x : A, B$ generalizes the arrow type $A \Rightarrow B$ of simply-typed ¹³⁰ λ -calculus: it is the type of functions taking an argument x of type A and returning a term ¹³¹ whose type B may depend on x. If B does not depend on x, we sometimes simply write ¹³² $A \Rightarrow B$.

¹ Sorts refer here to the notion of sort in Pure Type Systems, not the one used in some first-order settings.

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Figure 1 Typing rules of $\lambda \Pi / \mathcal{R}$

$$(\text{weak}) \qquad \hline \Gamma, x : A \vdash b : B \qquad (\text{conv}) \qquad \hline \Gamma \vdash a : B \\ (\text{prod}) \qquad \hline \frac{\Gamma \vdash A : \text{TYPE} \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash (x : A)B : s} \qquad (\text{fun}) \qquad \hline \frac{\vdash \Theta_f : s_f}{\vdash f : \Theta_f}$$

Typing induces a hierarchy on terms [4, Lemma 47]. At the top, there is the sort KIND that is not typable. Then, comes the class K of kinds, whose type is KIND: $K = \text{TYPE} | \forall x : t, K$ where $t \in \mathbb{T}$. Then, comes the class of predicates, whose types are kinds. Finally, at the bottom lie (proof) objects whose types are predicates.

Example 1 (Filter function on dependent lists). To illustrate the kind of systems we consider, 137 we give an extensive example in the new DEDUKTI syntax combining type-level rewriting rules 138 (E1 converts datatype codes into DEDUKTI types), dependent types (\mathbb{L} is the polymorphic 139 type of lists parameterized with their length), higher-order variables (fil is a function 140 filtering elements out of a list along a boolean function f), and matching on defined function 141 symbols (fil can match a list defined by concatenation). Note that this example cannot be 142 represented in COQ or AGDA because of the rules using matching on app. And its termination 143 can be handled neither by [37] nor by [5] because the system is not orthogonal and has no 144 strict decrease in every recursive call. It can however be handled by our new termination 145 criterion and its implementation [12]. For readability, we removed the & which are used to 146 identify pattern variables in the rewriting rules. 147

```
148
     symbol Set: TYPE
                                    symbol arrow: Set \Rightarrow Set \Rightarrow Set
149
150
     symbol El: Set \Rightarrow TYPE
                                                   rule El (arrow a b) \rightarrow El a \Rightarrow El b
151
152
     symbol Bool: TYPE
                                        symbol true: Bool
                                                                         symbol false: Bool
153
     symbol Nat: TYPE
                                        symbol zero: Nat
                                                                         symbol s: Nat \Rightarrow Nat
154
155
                                                       set infix 1 "+" ≔ plus
     symbol plus: Nat \Rightarrow Nat \RightarrowNat
156
        rule zero + q \rightarrow q
                                                       rule (s p) + q \rightarrow s (p + q)
157
158
     symbol List: Set \Rightarrow Nat \Rightarrow TYPE
159
        symbol nil: ∀a, List a zero
160
         symbol cons:\foralla, El a \Rightarrow \forallp, List a p \Rightarrow List a (s p)
161
162
     symbol app: \forall a p, List a p \Rightarrow \forall q, List a q \Rightarrow List a (p+q)
163
                                                  q m \rightarrow m
        rule app a _ (nil _)
164
        rule app a _ (cons _ x p l) q m \rightarrow cons a x (p+q) (app a p l q m)
165
166
     symbol len_fil: \forall a, (El a \Rightarrow Bool) \Rightarrow \forall p, List a p \Rightarrow Nat
167
     \texttt{symbol len_fil_aux: Bool} \Rightarrow \forall \texttt{a, (El a} \Rightarrow \texttt{Bool}) \Rightarrow \forall \texttt{p, List a} p \Rightarrow \texttt{Nat}
168
        rule len_fil a f _ (nil _)
                                                             \rightarrow zero
169
        rule len_fil a f _ (cons _ x p l) \rightarrow len_fil_aux (f x) a f p l
170
```

```
rule len_fil a f _ (app _ p l q m)
171
              \rightarrow (len_fil a f p l) + (len_fil a f q m)
172
                                   afpl \rightarrow s (len_fil afpl)
       rule len_fil_aux true
173
       rule len_fil_aux false a f p l \rightarrow len_fil a f p l
174
175
     symbol fil: \da f p l, List a (len_fil a f p l)
176
     symbol fil_aux: \forallb a f, El a \Rightarrow \forallp l, List a (len_fil_aux b a f p l)
177
       rule fil a f _ (nil _)
                                              \rightarrow nil a
178
       rule fil a f _ (cons _ x p l)
                                             \rightarrow fil_aux (f x) a f x p l
179
       rule fil a f _ (app _ p l q m)
180
              \rightarrow app a (len_fil a f p l) (fil a f p l)
181
                         (len_fil a f q m) (fil a f q m)
182
       rule fil_aux false a f x p l \rightarrow fil a f p l
183
       rule fil_aux true
                               a f x p l
184
              \rightarrow cons a x (len_fil a f p l) (fil a f p l)
\frac{185}{186}
```

Assumptions: Throughout the paper, we assume that \rightarrow is locally confluent ($\leftarrow \rightarrow \subseteq \downarrow$) and preserves typing (for all Γ , A, t and u, if $\Gamma \vdash t : A$ and $t \rightarrow u$, then $\Gamma \vdash u : A$).

Note that local confluence implies that every $t \in SN$ has a unique normal form $t\downarrow$.

These assumptions are used in the interpretation of types (Definition 2) and the adequacy lemma (Lemma 5). Both properties are undecidable in general. For confluence, DEDUKTI can call confluence checkers that understand the HRS format of the confluence competition. For preservation of typing by reduction, it implements an heuristic [31].

¹⁹⁴ **3** Interpretation of types as reducibility candidates

We aim to prove the termination of the union of two relations, \rightarrow_{β} and $\rightarrow_{\mathcal{R}}$, on the set of 195 well-typed terms (which depends on \mathcal{R} since \downarrow includes $\rightarrow_{\mathcal{R}}$). As is well known, termination 196 is not modular in general. As a β step can generate an \mathcal{R} step, and vice versa, we cannot 197 expect to prove the termination of $\rightarrow_{\beta} \cup \rightarrow_{\mathcal{R}}$ from the termination of \rightarrow_{β} and $\rightarrow_{\mathcal{R}}$. The 198 termination of $\lambda \Pi / \mathcal{R}$ cannot be reduced to the termination of the simply-typed λ -calculus 199 either (as done for $\lambda \Pi$ alone in [16]) because of type-level rewriting rules like the ones defining 200 E1 in Example 1. Indeed, type-level rules enable the encoding of functional PTS like Girard's 201 System F, whose termination cannot be reduced to the termination of the simply-typed 202 λ -calculus [10]. 203

So, following Girard [15], to prove the termination of $\rightarrow_{\beta} \cup \rightarrow_{\mathcal{R}}$, we build a model of our calculus by interpreting types into sets of terminating terms. To this end, we need to find an interpretation [] having the following properties:

Because types are identified modulo conversion, we need $\llbracket \ \rrbracket$ to be invariant by reduction: if T is typable and $T \to T'$, then we must have $\llbracket T \rrbracket = \llbracket T' \rrbracket$.

As usual, to handle β -reduction, we need a product type $\forall x : A, B$ to be interpreted by the set of terms t such that, for all a in the interpretation of A, ta is in the interpretation of $B[x \mapsto a]$, that is, we must have $[\![\forall x : A, B]\!] = \Pi a \in [\![A]\!]$. $[\![B[x \mapsto a]]\!]$ where $\Pi a \in P.Q(a) = \{t \mid \forall a \in P, ta \in Q(a)\}$.

First, we define the interpretation of predicates (and TYPE) as the least fixpoint of a monotone function in a directed-complete (= chain-complete) partial order [28]. Second, we define the interpretation of kinds by induction on their size.

▶ Definition 2 (Interpretation of types). Let $\mathbb{I} = \mathcal{F}_p(\mathbb{T}, \mathcal{P}(\mathbb{T}))$ be the set of partial functions from \mathbb{T} to the powerset of \mathbb{T} . It is directed-complete wrt inclusion, allowing us to define \mathcal{I} as the least fixpoint of the monotone function $F : \mathbb{I} \to \mathbb{I}$ such that, if $I \in \mathbb{I}$, then: The domain of F(I) is the set D(I) of all the terminating terms T such that, if T reduces

to some product term $\forall x : A, B$ (not necessarily in normal form), then $A \in \text{dom}(I)$ and, for all $a \in I(A), B[x \mapsto a] \in \text{dom}(I)$.

- If $T \in D(I)$ and the normal form² of T is not a product, then F(I)(T) = SN.
- $If T \in D(I) and T \downarrow = \forall x : A, B, then F(I)(T) = \Pi a \in I(A). I(B[x \mapsto a]).$
- We now introduce $\mathcal{D} = D(\mathcal{I})$ and define the interpretation of a term T wrt to a substitution
- 225 σ , $[T]_{\sigma}$ (and simply [T] if σ is the identity), as follows:
- 226 $\llbracket s \rrbracket_{\sigma} = \mathcal{D} \text{ if } s \in \mathbb{S},$
- $\mathbb{I}_{227} \quad = \quad [\![\forall x : A, K]\!]_{\sigma} = \Pi a \in [\![A]\!]_{\sigma}. [\![K]\!]_{[x \mapsto a, \sigma]} \text{ if } K \in \mathbb{K} \text{ and } x \notin \operatorname{dom}(\sigma),$
- 228 $\llbracket T \rrbracket_{\sigma} = \mathcal{I}(T\sigma) \text{ if } T \notin \mathbb{K} \cup \{\text{KIND}\} \text{ and } T\sigma \in \mathcal{D},$
- 229 $\llbracket T \rrbracket_{\sigma} = SN$ otherwise.

²³⁰ A substitution σ is adequate wrt an environment Γ , $\sigma \models \Gamma$, if, for all $x : A \in \Gamma$, $x\sigma \in \llbracket A \rrbracket_{\sigma}$.

²³¹ A typing map Θ is adequate if, for all $f, f \in \llbracket \Theta_f \rrbracket$ whenever $\vdash \Theta_f : s_f \text{ and } \Theta_f \in \llbracket s_f \rrbracket$.

Let \mathbb{C} be the set of terms of the form $f\vec{t}$ such that $|\vec{t}| = \operatorname{ar}(\Theta_f), \vdash \Theta_f : s_f, \Theta_f \in [\![s_f]\!]$ and, if $\Gamma_f = \vec{x} : \vec{A}$ and $\sigma = [\vec{x} \mapsto \vec{t}]$, then $\sigma \models \Gamma_f$. (Informally, \mathbb{C} is the set of terms obtained by fully applying some function symbol to computable arguments.)

We can then prove that, for all terms T, [T] satisfies Girard's conditions of reducibility candidates, called computability predicates here, adapted to rewriting by including in neutral terms every term of the form $f\vec{t}$ when f is applied to enough arguments wrt \mathcal{R} [5]:

▶ Definition 3 (Computability predicates). A term is neutral if it is of the form $(\lambda x : A, t)u\vec{v}$, x \vec{v} or $f\vec{v}$ with, for every rule $f\vec{l} \rightarrow r \in \mathcal{R}$, $|\vec{l}| \leq |\vec{v}|$.

Let \mathbb{P} be the set of all the sets of terms S (computability predicates) such that (a) $S \subseteq SN$, (b) $\rightarrow (S) \subseteq S$, and (c) $t \in S$ if t is neutral and $\rightarrow (t) \subseteq S$.

Note that neutral terms satisfy the following key property: if t is neutral then, for all u, tu is neutral and every reduct of tu is either of the form t'u with t' a reduct of t, or of the form tu' with u' a reduct of u.

²⁴⁵ One can easily check that SN is a computability predicate.

Note also that a computability predicate is never empty: it contains every neutral term in normal form. In particular, it contains every variable.

²⁴⁸ We then get the following results (the proofs are given in Appendix A):

▶ Lemma 4. (a) For all terms *T* and substitutions σ , $\llbracket T \rrbracket_{\sigma} \in \mathbb{P}$.

- 250 (b) If T is typable, $T\sigma \in \mathcal{D}$ and $T \to T'$, then $[\![T]\!]_{\sigma} = [\![T']\!]_{\sigma}$.
- ²⁵¹ (c) If T is typable, $T\sigma \in \mathcal{D}$ and $\sigma \to \sigma'$, then $[\![T]\!]_{\sigma} = [\![T]\!]_{\sigma'}$.
- 252 (d) If $\forall x : A, B$ is typable and $\forall x : A\sigma, B\sigma \in \mathcal{D}$,
- then $\llbracket \forall x : A, B \rrbracket_{\sigma} = \Pi a \in \llbracket A \rrbracket_{\sigma} . \llbracket B \rrbracket_{[x \mapsto a, \sigma]}.$
- (e) If $\Delta \vdash U : s$, $\Gamma \vdash \gamma : \Delta$ and $U\gamma \sigma \in \mathcal{D}$, then $\llbracket U\gamma \rrbracket_{\sigma} = \llbracket U \rrbracket_{\gamma\sigma}$.

(f) Given $P \in \mathbb{P}$ and, for all $a \in P$, $Q(a) \in \mathbb{P}$ such that $Q(a') \subseteq Q(a)$ if $a \to a'$. Then, $\lambda x : A, b \in \Pi a \in P. Q(a)$ if $A \in SN$ and, for all $a \in P$, $b[x \mapsto a] \in Q(a)$.

²⁵⁷ We can finally prove that our model is adequate, that is, every term of type T belongs to ²⁵⁸ [T], if the typing map Θ itself is adequate. This reduces the termination of well-typed terms ²⁵⁹ to the computability of function symbols.

Lemma 5 (Adequacy). If Θ is adequate, $\Gamma \vdash t : T$ and $\sigma \models \Gamma$, then $t\sigma \in [\![T]\!]_{\sigma}$.

² Because we assume local confluence, every terminating term T has a unique normal form $T\downarrow$.

Proof. First note that, if $\Gamma \vdash t : T$, then either T = KIND or $\Gamma \vdash T : s$ [4, Lemma 28]. 261 Moreover, if $\Gamma \vdash a : A, A \downarrow B$ and $\Gamma \vdash B : s$ (the premises of the (conv) rule), then $\Gamma \vdash A : s$ 262 [4, Lemma 42] (because \rightarrow preserves typing). Hence, the relation \vdash is unchanged if one 263 adds the premise $\Gamma \vdash A : s$ in (conv), giving the rule (conv'). Similarly, we add the premise 264 $\Gamma \vdash \forall x : A, B : s$ in (app), giving the rule (app'). We now prove the lemma by induction on 265 $\Gamma \vdash t : T$ using (app') and (conv'): 266 (ax) It is immediate that $TYPE \in [KIND]_{\sigma} = \mathcal{D}$. 267 (var) By assumption on σ . 268 (weak) If $\sigma \models \Gamma, x : A$, then $\sigma \models \Gamma$. So, the result follows by induction hypothesis. 269 (prod) Is $(\forall x : A, B)\sigma$ in $[s]_{\sigma} = \mathcal{D}$? Wlog we can assume $x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma)$. So, 270 $(\forall x: A, B)\sigma = \forall x: A\sigma, B\sigma$. By induction hypothesis, $A\sigma \in [TYPE]_{\sigma} = \mathcal{D}$. Let now $a \in [TYPE]_{\sigma}$ 271

 $\mathcal{I}(A\sigma) \text{ and } \sigma' = [x \mapsto a, \sigma]. \text{ Note that } \mathcal{I}(A\sigma) = \llbracket A \rrbracket_{\sigma}. \text{ So, } \sigma' \models \Gamma, x : A \text{ and, by induction}$ hypothesis, $B\sigma' \in \llbracket s \rrbracket_{\sigma} = \mathcal{D}. \text{ Since } x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma), \text{ we have } B\sigma' = (B\sigma)[x \mapsto a].$ Therefore, $(\forall x : A, B)\sigma \in \llbracket s \rrbracket_{\sigma}.$

(abs) Is $(\lambda x : A, b)\sigma$ in $[\forall x : A, B]_{\sigma}$? Wlog we can assume that $x \notin \text{dom}(\sigma) \cup \text{FV}(\sigma)$. 275 So, $(\lambda x : A, b)\sigma = \lambda x : A\sigma, b\sigma$. By Lemma 4d, $[\![\forall x : A, B]\!]_{\sigma} = \Pi a \in [\![A]\!]_{\sigma}$. $[\![B]\!]_{[x \mapsto a, \sigma]}$. 276 By Lemma 4c, $\llbracket B \rrbracket_{[x \mapsto a,\sigma]}$ is an $\llbracket A \rrbracket_{\sigma}$ -indexed family of computability predicates such 277 that $\llbracket B \rrbracket_{[x \mapsto a',\sigma]} = \llbracket B \rrbracket_{[x \mapsto a,\sigma]}$ whenever $a \to a'$. Hence, by Lemma 4f, $\lambda x : A\sigma, b\sigma \in$ 278 $\llbracket \forall x : A, B \rrbracket_{\sigma}$ if $A\sigma \in SN$ and, for all $a \in \llbracket A \rrbracket_{\sigma}, (b\sigma)[x \mapsto a] \in \llbracket B \rrbracket_{\sigma'}$ where $\sigma' = [x \mapsto a]$ 279 $[a,\sigma]$. By induction hypothesis, $(\forall x: A, B)\sigma \in [s]_{\sigma} = \mathcal{D}$. Since $x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma)$, 280 $(\forall x: A, B)\sigma = \forall x: A\sigma, B\sigma \text{ and } (b\sigma)[x \mapsto a] = b\sigma'.$ Since $\mathcal{D} \subseteq SN$, we have $A\sigma \in SN$. 281 Moreover, since $\sigma' \models \Gamma, x : A$, we have $b\sigma' \in \llbracket B \rrbracket_{\sigma'}$ by induction hypothesis. 282

(app') Is $(ta)\sigma = (t\sigma)(a\sigma)$ in $\llbracket B[x \mapsto a] \rrbracket_{\sigma}$? By induction hypothesis, $t\sigma \in \llbracket \forall x : A, B \rrbracket_{\sigma}, a\sigma \in \llbracket A \rrbracket_{\sigma}$ and $(\forall x : A, B)\sigma \in \llbracket s \rrbracket = \mathcal{D}$. By Lemma 4d, $\llbracket \forall x : A, B \rrbracket_{\sigma} = \Pi \alpha \in \llbracket A \rrbracket_{\sigma}$. $\llbracket B \rrbracket_{[x \mapsto \alpha, \sigma]}$. Hence, $(t\sigma)(a\sigma) \in \llbracket B \rrbracket_{\sigma'}$ where $\sigma' = [x \mapsto a\sigma, \sigma]$. Wlog we can assume $x \notin \operatorname{dom}(\sigma) \cup FV(\sigma)$. So, $\sigma' = [x \mapsto a]\sigma$. Hence, by Lemma 4e, $\llbracket B \rrbracket_{\sigma'} = \llbracket B[x \mapsto a] \rrbracket_{\sigma}$.

(conv') By induction hypothesis, $a\sigma \in \llbracket A \rrbracket_{\sigma}$, $A\sigma \in \llbracket s \rrbracket_{\sigma} = \mathcal{D}$ and $B\sigma \in \llbracket s \rrbracket_{\sigma} = \mathcal{D}$. By Lemma 4b, $\llbracket A \rrbracket_{\sigma} = \llbracket B \rrbracket_{\sigma}$. So, $a\sigma \in \llbracket B \rrbracket_{\sigma}$.

(fun) By induction hypothesis, $\Theta_f \in [\![s_f]\!]_{\sigma} = \mathcal{D}$. Therefore, $f \in [\![\Theta_f]\!]_{\sigma} = [\![\Theta_f]\!]$ since Θ is adequate.

²⁹¹ **4** Dependency pairs theorem

305 306

Now, we prove that the adequacy of Θ can be reduced to the absence of infinite sequences of dependency pairs, as shown by Arts and Giesl for first-order rewriting [2].

▶ **Definition 6** (Dependency pairs). Let $f\vec{l} > g\vec{m}$ iff there is a rule $f\vec{l} \to r \in \mathcal{R}$, g is defined and $g\vec{m}$ is a subterm of r such that \vec{m} are all the arguments to which g is applied. The relation > is the set of dependency pairs.

Let $\tilde{>} = \rightarrow_{\arg}^* >_s$ be the relation on the set \mathbb{C} (Def. 2), where $f\vec{t} \rightarrow_{\arg} f\vec{u}$ iff $\vec{t} \rightarrow_{prod} \vec{u}$ (reduction in one argument), and $>_s$ is the closure by substitution and left-application of >: $ft_1 \dots t_p \tilde{>} gu_1 \dots u_q$ iff there are a dependency pair $fl_1 \dots l_i > gm_1 \dots m_j$ with $i \leq p$ and $j \leq q$ and a substitution σ such that, for all $k \leq i$, $t_k \rightarrow^* l_k \sigma$ and, for all $k \leq j$, $m_k \sigma = u_k$.

In our setting, we have to close $>_s$ by left-application because function symbols are curried. When a function symbol f is not fully applied wrt $\operatorname{ar}(\Theta_f)$, the missing arguments must be considered as potentially being anything. Indeed, the following rewriting system:

 $\texttt{app x y} \rightarrow \texttt{x y} \qquad \texttt{f x y} \rightarrow \texttt{app (f x) y}$

whose dependency pairs are f x y > app (f x) y and f x y > f x, does not terminate, but there is no way to construct an infinite sequence of dependency pairs without adding an argument to the right-hand side of the second dependency pair.

► Example 7. The rules of Example 1 have the following dependency pairs (the pairs whose
 left-hand side is headed by fil or fil_aux can be found in Appendix B):

```
312
    A :
                      El (arrow a b) > El a
313
   B:
                      El (arrow a b) > El b
314
    C:
                            (s p) + q > p + q
315
    D:
                  (cons _ x p l) q m > p + q
316
         app a
                  (cons_xpl) qm > app a plqm
317
   E:
    F:len_fil a f _ (cons _ x p l)
                                      > len_fil_aux (f x) a f p l
318
    G:len_fil a f _
                     (app _ p l q m) >
319
                                       (len_fil a f p l) + (len_fil a f q m)
320
    H:len_fil a f _ (app _ p l q m) > len_fil a f p l
321
    I:len_fil a f _ (app _ p l q m) > len_fil a f q m
322
          len_fil_aux true a f p l > len_fil a f p
    J:
                                                       1
323
          len_fil_aux false a f p l > len_fil a f
   K:
                                                     р
                                                       1
324
```

In [2], a sequence of dependency pairs interleaved with \rightarrow_{arg} steps is called a chain. Arts and Giesl proved that, in a first-order term algebra, $\rightarrow_{\mathcal{R}}$ terminates if and only if there are no infinite chains, that is, if and only if $\tilde{>}$ terminates. Moreover, in a first-order term algebra, $\tilde{>}$ terminates if and only if, for all f and \vec{t} , $f\vec{t}$ terminates wrt $\tilde{>}$ whenever \vec{t} terminates wrt \rightarrow_{330} . In our framework, this last condition is similar to saying that Θ is adequate.

³³¹ We now introduce the class of systems to which we will extend Arts and Giesl's theorem.

▶ Definition 8 (Well-structured system). Let \succeq be the smallest quasi-order on \mathbb{F} such that $f \succeq g$ if g occurs in Θ_f or if there is a rule $f\vec{l} \rightarrow r \in \mathcal{R}$ with g (defined or undefined) $f \succeq g$ if g occurs in Θ_f or if there is a rule $f\vec{l} \rightarrow r \in \mathcal{R}$ with g (defined or undefined) $f \succeq g$ if g occurs in Θ_f or if there is a rule $f\vec{l} \rightarrow r \in \mathcal{R}$ with g (defined or undefined) $f \succeq g$ if g occurs in Θ_f or if there is a rule $f\vec{l} \rightarrow r \in \mathcal{R}$ with g (defined or undefined) $f \succeq g$ is well-structured if:

336 (a) \succ is well-founded;

337 **(b)** for every rule $f\vec{l} \to r$, $|\vec{l}| \leq \operatorname{ar}(\Theta_f)$;

338 (c) for every dependency pair $f\vec{l} > g\vec{m}$, $|\vec{m}| \leq \operatorname{ar}(\Theta_q)$;

(d) every rule $f\vec{l} \to r$ is equipped with an environment $\Delta_{f\vec{l}\to r}$ such that, if $\Theta_f = \forall \vec{x} : \vec{T}, U$ and $\pi = [\vec{x} \mapsto \vec{l}]$, then $\Delta_{f\vec{l}\to r} \vdash_{f\vec{l}} r : U\pi$, where $\vdash_{f\vec{l}}$ is the restriction of \vdash defined in Fig. 2.

Condition (a) is always satisfied when \mathbb{F} is finite. Condition (b) ensures that a term of 341 the form $f\bar{t}$ is neutral whenever $|\bar{t}| = \operatorname{ar}(\Theta_f)$. Condition (c) ensures that > is included in $\tilde{>}$. 342 The relation $\vdash_{\vec{t}}$ corresponds to the notion of computability closure in [5], with the ordering 343 on function calls replaced by the dependency pair relation. It is similar to \vdash except that it 344 uses the variant of (conv) and (app) used in the proof of the adequacy lemma; (fun) is split 345 in the rules (const) for undefined symbols and (dp) for dependency pairs whose left-hand side 346 is fl; every type occurring in an object term or every type of a function symbol occurring in 347 a term is required to be typable by using symbols smaller than f only. 348

The environment $\Delta_{f\bar{l}\to r}$ can be inferred by DEDUKTI when one restricts rule left hand-sides to some well-behaved class of terms like algebraic terms or Miller patterns (in λ Prolog).

One can check that Example 1 is well-structured (the proof is given in Appendix B). Finally, we need matching to be compatible with computability, that is, if $f\vec{l} \rightarrow r \in \mathcal{R}$ and $\vec{l}\sigma$ are computable, then σ is computable, a condition called accessibility in [5]:

Figure 2 Restricted type systems $\vdash_{f\vec{l}}$ and $\vdash_{\prec f}$

$$\begin{array}{ll} \text{(ax)} & \frac{\Gamma \vdash_{\prec f} A : s \quad \Gamma \vdash_{f\vec{l}} b : B \quad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \vdash_{f\vec{l}} b : B} \\ \text{(var)} & \frac{\Gamma \vdash_{\prec f} A : s \quad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \vdash_{f\vec{l}} x : A} \\ \text{(prod)} & \frac{\Gamma \vdash_{\prec f} A : TYPE \quad \Gamma, x : A \vdash_{f\vec{l}} B : s}{\Gamma \vdash_{f\vec{l}} \forall x : A, B : s} \\ \text{(abs)} & \frac{\Gamma, x : A \vdash_{f\vec{l}} b : B \quad \Gamma \vdash_{\prec f} \forall x : A, B : s}{\Gamma \vdash_{f\vec{l}} \lambda x : A, b : \forall x : A, B} \\ \text{(app')} & \frac{\Gamma \vdash_{f\vec{l}} t : \forall x : A, B \quad \Gamma \vdash_{\forall f} a : A \quad \Gamma \vdash_{\prec f} \forall x : A, B : s}{\Gamma \vdash_{f\vec{l}} t a : B[x \mapsto a]} \\ \text{(conv')} & \frac{\Gamma \vdash_{f\vec{l}} g : s_g \quad \Gamma \vdash_{f\vec{l}} \gamma : \Sigma}{\Gamma \vdash_{f\vec{l}} g : g\vec{y} \gamma : V \gamma} \\ \text{(const)} & \frac{\vdash_{\prec f} \Theta_g : s_g \quad \Gamma \vdash_{f\vec{l}} \gamma : \Sigma}{\Gamma \vdash_{f\vec{l}} g : \Theta_g} \\ \text{(g undefined)} \end{array}$$

and $\vdash_{\prec f}$ is defined by the same rules as \vdash , except (fun) replaced by:

$$(\operatorname{fun}_{\prec f}) \quad \frac{\vdash_{\prec f} \Theta_g : s_g \quad g \prec f}{\vdash_{\prec f} g : \Theta_g}$$

▶ Definition 9 (Accessible system). A well-structured system \mathcal{R} is accessible if, for all substitutions σ and rules $f\vec{l} \rightarrow r$ with $\Theta_f = \forall \vec{x} : \vec{T}, U$ and $|\vec{x}| = |\vec{l}|$, we have $\sigma \models \Delta_{f\vec{l} \rightarrow r}$ whenever $\vdash \Theta_f : s_f, \Theta_f \in [\![s_f]\!]$ and $[\vec{x} \mapsto \vec{l}]\sigma \models \vec{x} : \vec{T}$.

This property is not always satisfied because the subterm relation does not preserve computability in general. Indeed, if C is an undefined type constant, then $\llbracket C \rrbracket = SN$. However, $\llbracket C \Rightarrow C \rrbracket \neq SN$ since $\omega = \lambda x : C, xx \in SN$ and $\omega \omega \notin SN$. Hence, if c is an undefined function symbol of type $\Theta_c = (C \Rightarrow C) \Rightarrow C$, then $c \omega \in \llbracket C \rrbracket$ but $\omega \notin \llbracket C \Rightarrow C \rrbracket$. We can now state the main lemma:

Lemma 10. Θ is adequate if $\tilde{>}$ terminates and \mathcal{R} is well-structured and accessible.

Proof. Since \mathcal{R} is well-structured, \succ is well-founded by condition (a). We prove that, for all $f \in \mathbb{F}$, $f \in \llbracket \Theta_f \rrbracket$, by induction on \succ . So, let $f \in \mathbb{F}$ with $\Theta_f = \forall \Gamma_f, U$ and $\Gamma_f = x_1 : T_1, \ldots, x_n : T_n$. By induction hypothesis, we have that, for all $g \prec f, g \in \llbracket \Theta_g \rrbracket$.

Since $\rightarrow_{\operatorname{arg}}$ and $\tilde{>}$ terminate on \mathbb{C} and $\rightarrow_{\operatorname{arg}} \tilde{>} \subseteq \tilde{>}$, we have that $\rightarrow_{\operatorname{arg}} \cup \tilde{>}$ terminates. We now prove that, for all $f\vec{t} \in \mathbb{C}$, we have $f\vec{t} \in \llbracket U \rrbracket_{\theta}$ where $\theta = [\vec{x} \mapsto \vec{t}]$, by a second induction on $\rightarrow_{\operatorname{arg}} \cup \tilde{>}$. By condition (b), $f\vec{t}$ is neutral. Hence, by definition of computability, it suffices to prove that, for all $u \in \rightarrow (f\vec{t}), u \in \llbracket U \rrbracket_{\theta}$. There are 2 cases:

 $u = f\vec{v}$ with $\vec{t} \rightarrow_{prod} \vec{v}$. Then, we can conclude by the first induction hypothesis.

There are $fl_1 \dots l_k \to r \in \mathcal{R}$ and σ such that $u = (r\sigma)t_{k+1} \dots t_n$ and, for all $i \in \{1, \dots, k\}$,

 $t_i = l_i \sigma$. Since $f\vec{t} \in \mathbb{C}$, we have $\pi \sigma \models \Gamma_f$. Since \mathcal{R} is accessible, we get that $\sigma \models \Delta_{f\vec{l} \to r}$.

By condition (d), we have $\Delta_{f\vec{l}\to r} \vdash_{f\vec{l}} r : V\pi$ where $V = \forall x_{k+1} : T_{k+1}, \dots \forall x_n : T_n, U$.

Now, we prove that, for all Γ , t and T, if $\Gamma \vdash_{f\vec{l}} t : T$ ($\Gamma \vdash_{\prec f} t : T$ resp.) and $\sigma \models \Gamma$, then $t\sigma \in [\![T]\!]_{\sigma}$, by a third induction on the structure of the derivation of $\Gamma \vdash_{f\vec{l}} t : T$

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- $(\Gamma \vdash_{\prec f} t : T \text{ resp.})$, as in the proof of Lemma 5 except for (fun) replaced by $(\operatorname{fun}_{\prec f})$ in one case, and (const) and (dp) in the other case.
- (fun $\prec f$) We have $g \in \llbracket \Theta_q \rrbracket$ by the first induction hypothesis on g.
- (const) Since g is undefined, it is neutral and normal. Therefore, it belongs to every computability predicate and in particular to $[\![\Theta_q]\!]_{\sigma}$.
- (dp) By the third induction hypothesis, $y_i \gamma \sigma \in [U_i \gamma]_{\sigma}$. By Lemma 4e, $[U_i \gamma]_{\sigma} = [U_i]_{\gamma \sigma}$.
 - So, $\gamma \sigma \models \Sigma$ and $g \vec{y} \gamma \sigma \in \mathbb{C}$. Now, by condition (c), $g \vec{y} \gamma \sigma \tilde{<} f \vec{l} \sigma$ since $g \vec{y} \gamma < f \vec{l}$. Therefore, by the second induction hypothesis, $g \vec{y} \gamma \sigma \in [\![V \gamma]\!]_{\sigma}$.
- So, $r\sigma \in \llbracket V\pi \rrbracket_{\sigma}$ and, by Lemma 4d, $u \in \llbracket U \rrbracket_{[x_n \mapsto t_n, \dots, x_{k+1} \mapsto t_{k+1}, \pi\sigma]} = \llbracket U \rrbracket_{\theta}$.

Note that the proof still works if one replaces the relation \succeq of Definition 8 by any well-founded quasi-order such that $f \succeq g$ whenever $f\vec{l} > g\vec{m}$. The quasi-order of Definition 8, defined syntactically, relieves the user of the burden of providing one and is sufficient in every practical case met by the authors. However it is possible to construct ad-hoc systems which require a quasi-order richer than the one presented here.

By combining the previous lemma and the Adequacy lemma (the identity substitution is computable), we get the main result of the paper:

³⁹² ► **Theorem 11.** The relation $\rightarrow = \rightarrow_{\beta} \cup \rightarrow_{\mathcal{R}}$ terminates on terms typable in $\lambda \Pi / \mathcal{R}$ if \rightarrow is ³⁹³ locally confluent and preserves typing, \mathcal{R} is well-structured and accessible, and $\tilde{>}$ terminates.

For the sake of completeness, we are now going to give sufficient conditions for accessibility and termination of $\tilde{>}$ to hold, but one could imagine many other criteria.

³⁹⁶ 5 Checking accessibility

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³⁹⁷ In this section, we give a simple condition to ensure accessibility and some hints on how to ³⁹⁸ modify the interpretation when this condition is not satisfied.

As seen with the definition of accessibility, the main problem is to deal with subterms of higher-order type. A simple condition is to require higher-order variables to be direct subterms of the left-hand side, a condition called plain function-passing (PFP) in [25], and satisfied by Example 1.

⁴⁰³ ► Definition 12 (PFP systems). A well-structured \mathcal{R} is PFP if, for all $f\vec{l} \to r \in \mathcal{R}$ with ⁴⁰⁴ $\Theta_f = \forall \vec{x} : \vec{T}, U \text{ and } |\vec{x}| = |\vec{l}|, \vec{l} \notin \mathbb{K} \cup \{\text{KIND}\} \text{ and, for all } y : T \in \Delta_{f\vec{l} \to r}, \text{ there is i such that}$ ⁴⁰⁵ $y = l_i \text{ and } T = T_i[\vec{x} \mapsto \vec{l}], \text{ or else } y \in \text{FV}(l_i) \text{ and } T = D\vec{t} \text{ with } D \text{ undefined and } |\vec{t}| = \text{ar}(D).$

406 ► Lemma 13. *PFP systems are accessible.*

⁴⁰⁷ **Proof.** Let $f\vec{l} \to r$ be a PFP rule with $\Theta_f = \forall \Gamma, U, \Gamma = \vec{x} : \vec{T}, \pi = [\vec{x} \mapsto \vec{l}]$. Following ⁴⁰⁸ Definition 9, assume that $\vdash \Theta_f : s_f, \Theta_f \in \mathcal{D}$ and $\pi\sigma \models \Gamma$. We have to prove that, for all ⁴⁰⁹ $(y:T) \in \Delta_{f\vec{l} \to r}, y\sigma \in [\![T]\!]_{\sigma}$.

410 Suppose $y = l_i$ and $T = T_i \pi$. Then, $y\sigma = l_i \sigma \in [\![T_i]\!]_{\pi\sigma}$. Since $\vdash \Theta_f : s_f, T_i \notin \mathbb{K} \cup \{\text{KIND}\}$.

⁴¹¹ Since $\Theta_f \in \mathcal{D}$ and $\pi\sigma \models \Gamma$, we have $T_i\pi\sigma \in \mathcal{D}$. So, $[\![T_i]\!]_{\pi\sigma} = \mathcal{I}(T_i\pi\sigma)$. Since $T_i \notin \mathbb{K} \cup \{\text{KIND}\}$ and $\vec{l} \notin \mathbb{K} \cup \{\text{KIND}\}, T_i\pi \notin \mathbb{K} \cup \{\text{KIND}\}$. Since $T_i\pi\sigma \in \mathcal{D}, [\![T_i\pi]\!]_{\sigma} = \mathcal{I}(T_i\pi\sigma)$. ⁴¹³ Thus, $y\sigma \in [\![T]\!]_{\sigma}$.

⁴¹⁴ Suppose $y \in FV(l_i)$ and T is of the form $C\vec{t}$ with $|\vec{t}| = ar(C)$. Then, $[T]_{\sigma} = SN$ and ⁴¹⁵ $y\sigma \in SN$ since $l_i\sigma \in [T_i]_{\sigma} \subseteq SN$.

⁴¹⁶ But many accessible systems are not PFP. They can be proved accessible by changing ⁴¹⁷ the interpretation of type constants (a complete development is left for future work).

```
Example 14 (Recursor on Brouwer ordinals).
418
419
      symbol Ord: TYPE
420
       symbol zero: Ord
                                  symbol suc: Ord⇒Ord
                                                                     symbol lim: (Nat⇒Ord)⇒Ord
421
422
      symbol ordrec: A \Rightarrow (Ord \Rightarrow A \Rightarrow A) \Rightarrow ((Nat \Rightarrow Ord) \Rightarrow (Nat \Rightarrow A) \Rightarrow Ord \Rightarrow A
423
         rule ordrec u v w zero
                                                \rightarrow u
424
         rule ordrec u v w (suc x) \rightarrow v x (ordrec u v w x)
425
         rule ordrec u v w (lim f) \rightarrow w f (\lambdan,ordrec u v w (f n))
439
```

The above example is not PFP because $f:Nat \Rightarrow Ord$ is not argument of ordrec. Yet, it is accessible if one takes for [Ord] the least fixpoint of the monotone function $F(S) = \{t \in SN \mid \text{if } t \to^* \lim f \text{ then } f \in [Nat]\} \Rightarrow S$, and if $t \to^* \operatorname{suc} u$ then $u \in S\}$ [5].

Similarly, the following encoding of the simply-typed λ -calculus is not PFP but can be proved accessible by taking

```
[[T c]] = \text{if } c \downarrow = \operatorname{arrow} a b \text{ then } \{t \in SN \mid \text{if } t \to^* \texttt{lam} f \text{ then } f \in [[T a]] \Rightarrow [[T b]] \} \text{ else } SN
```

```
Example 15 (Simply-typed \lambda-calculus).
```

```
^{435}_{436}symbol Sort : TYPEsymbol arrow : Sort \Rightarrow Sort \Rightarrow Sort^{437}_{438}symbol T : Sort \Rightarrow TYPE^{439}symbol lam : \forall a b, (T a \Rightarrow T b) \Rightarrow T (arrow a b)^{440}symbol app : \forall a b, T (arrow a b) \Rightarrow T a \Rightarrow T b^{412}_{412}rule app a b (lam _ _ f) x \rightarrow f x
```

6 Size-change termination

In this section, we give a sufficient condition for $\tilde{>}$ to terminate. For first-order rewriting, many techniques have been developed for that purpose. To cite just a few, see for instance [17, 14]. Many of them can probably be extended to $\lambda \Pi / \mathcal{R}$, either because the structure of terms in which they are expressed can be abstracted away, or because they can be extended to deal also with variable applications, λ -abstractions and β -reductions.

As an example, following Wahlstedt [37], we are going to use Lee, Jones and Ben-Amram's size-change termination criterion (SCT) [26]. It consists in following arguments along function calls and checking that, in every potential loop, one of them decreases. First introduced for first-order functional languages, it has then been extended to many other settings: untyped λ -calculus [21], a subset of OCAML [32], Martin-Löf's type theory [37], System F [27].

⁴⁵⁴ We first recall Hyvernat and Raffalli's matrix-based presentation of SCT [20]:

▶ Definition 16 (Size-change termination). Let \triangleright be the smallest transitive relation such that ft₁...t_n \triangleright t_i when $f \in \mathbb{F}$. The call graph $\mathcal{G}(\mathcal{R})$ associated to \mathcal{R} is the directed labeled graph on the defined symbols of \mathbb{F} such that there is an edge between f and g iff there is a dependency pair fl₁...l_p > gm₁...m_q. This edge is labeled with the matrix $(a_{i,j})_{i \leq \operatorname{ar}(\Theta_f), j \leq \operatorname{ar}(\Theta_g)}$ where: if $l_i \triangleright m_j$, then $a_{i,j} = -1$;

460 if
$$l_i = m_j$$
, then $a_{i,j} = 0$;

461 — otherwise $a_{i,j} = \infty$ (in particular if i > p or j > q).

462 \mathcal{R} is size-change terminating (SCT) if, in the transitive closure of $\mathcal{G}(\mathcal{R})$ (using the min-plus

semi-ring to multiply the matrices labeling the edges), all idempotent matrices labeling a loop
 have some -1 on the diagonal.

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Figure 3 Matrix of dependency pair C and call graph of the dependency pairs of Example 7

We add lines and columns of ∞ 's in matrices associated to dependency pairs containing partially applied symbols (cases i > p or j > q) because missing arguments cannot be compared with any other argument since they are arbitrary.

The matrix associated to the dependency pair C: (s p) + q > p + q and the call graph associated to the dependency pairs of Example 7 are depicted in Figure 3. The full list of matrices and the extensive call graph of Example 1 can be found in Appendix B.

↓ Lemma 17. $\tilde{>}$ terminates if \mathbb{F} is finite and \mathcal{R} is SCT.

Proof. Suppose that there is an infinite sequence $\chi = f_1 \vec{t_1} > f_2 \vec{t_2} > \dots$ Then, there is an 472 infinite path in the call graph going through nodes labeled by f_1, f_2, \ldots Since \mathbb{F} is finite, 473 there is a symbol g occurring infinitely often in this path. So, there is an infinite sequence 474 $g\vec{u}_1, g\vec{u}_2, \ldots$ extracted from χ . Hence, for every $i, j \in \mathbb{N}^*$, there is a matrix in the transitive 475 closure of the graph which labels the loops of g corresponding to the relation between \vec{u}_i and 476 \vec{u}_{i+j} . By Ramsey's theorem, there is an infinite sequence (ϕ_i) and a matrix M such that M 477 corresponds to all the transitions $g\vec{u}_{\phi_i}, g\vec{u}_{\phi_i}$ with $i \neq j$. M is idempotent, indeed $g\vec{u}_{\phi_i}, g\vec{u}_{\phi_{i+2}}$ 478 is labeled by M^2 by definition of the transitive closure and by M due to Ramsey's theorem, 479 so $M = M^2$. Since, by hypothesis, \mathcal{R} satisfies SCT, there is j such that $M_{j,j}$ is -1. So, for all $i, u_{\phi_i}^{(j)}(\to^* \rhd)^+ u_{\phi_{i+1}}^{(j)}$. Since $\rhd \to \subseteq \to \rhd$ and \to_{arg} is well-founded on \mathbb{C} , the existence of 480 481 an infinite sequence contradicts the fact that \triangleright is well-founded. 482

⁴⁸³ By combining all the previous results, we get:

⁴⁸⁴ ► **Theorem 18.** The relation $\rightarrow = \rightarrow_{\beta} \cup \rightarrow_{\mathcal{R}}$ terminates on terms typable in $\lambda \Pi / \mathcal{R}$ if \rightarrow is ⁴⁸⁵ locally confluent and preserves typing, F is finite and \mathcal{R} is well-structured, plain-function ⁴⁸⁶ passing and size-change terminating.

⁴⁸⁷ The rewriting system of Example 1 verifies all these conditions (proof in the appendix).

⁴⁸⁸ **7** Implementation and comparison with other criteria and tools

We implemented our criterion in a tool called SIZECHANGETOOL [12]. As far as we know, there are no other termination checker for $\lambda \Pi / \mathcal{R}$.

⁴⁹¹ If we restrict ourselves to simply-typed rewriting systems, then we can compare it with ⁴⁹² the termination checkers participating in the category "higher-order rewriting union beta" of ⁴⁹³ the termination competition: WANDA uses dependency pairs, polynomial interpretations,

HORPO and many transformation techniques [24]; SOL uses the General Schema [6] and other techniques. As these tools implement various techniques and SIZECHANGETOOL only one, it is difficult to compete with them. Still, there are examples that are solved by SIZECHANGETOOL and not by one of the other tools, demonstrating that these tools would benefit from implementing our new technique. For instance, the problem Hamana_Kikuchi_18/h17 is proved terminating by SIZECHANGETOOL but not by WANDA because of the rule:

rule map f (map g l) \rightarrow map (comp f g) l

And the problem Kop13/kop12thesis_ex7.23 is proved terminating by SIZECHANGETOOL but not by SOL because of the rules:³

505 506

```
rule f h x (s y) \rightarrow g (c x (h y)) y rule g x y \rightarrow f (\lambda_{-},s 0) x y
```

⁵⁰⁸ One could also imagine to translate a termination problem in $\lambda \Pi / \mathcal{R}$ into a simply-typed ⁵⁰⁹ termination problem. Indeed, the termination of $\lambda \Pi$ alone (without rewriting) can be reduced ⁵¹⁰ to the termination of the simply-typed λ -calculus [16]. This has been extended to $\lambda \Pi / \mathcal{R}$ when ⁵¹¹ there are no type-level rewrite rules like the ones defining E1 in Example 1 [22]. However, ⁵¹² this translation does not preserve termination as shown by the Example 15 which is not ⁵¹³ terminating if all the types $\mathbb{T}x$ are mapped to the same type constant.

In [30], Roux also uses dependency pairs for the termination of simply-typed higher-order rewriting systems, as well as a restricted form of dependent types where a type constant C is annotated by a pattern l representing the set of terms matching l. This extends to patterns the notion of indexed or sized types [18]. Then, for proving the absence of infinite chains, he uses simple projections [17], which can be seen as a particular case of SCT where strictly decreasing arguments are fixed (SCT can also handle permutations in arguments).

Finally, if we restrict ourselves to orthogonal systems, it is also possible to compare our technique to the ones implemented in the proof assistants COQ and AGDA. COQ essentially implements a higher-order version of primitive recursion. AGDA on the other hand uses SCT.

Because Example 1 uses matching on defined symbols, it is not orthogonal and can be written neither in COQ nor in AGDA. AGDA recently added the possibility of adding rewrite rules but this feature is highly experimental and comes with no guaranty. In particular, AGDA termination checker does not handle rewriting rules.

527 COQ cannot handle inductive-recursive definitions [11] nor function definitions with 528 permuted arguments in function calls while it is no problem for AGDA and us.

529 8 Conclusion and future work

We proved a general modularity result extending Arts and Giesl's theorem that a rewriting relation terminates if there are no infinite sequences of dependency pairs [2] from first-order rewriting to dependently-typed higher-order rewriting. Then, following [37], we showed how to use Lee, Jones and Ben-Amram's size-change termination criterion to prove the absence of such infinite sequences [26].

This extends Wahlstedt's work [37] from weak to strong normalization, and from orthogonal to locally confluent rewriting systems. This extends the first author's work [5] from orthogonal to locally confluent systems, and from systems having a decreasing argument in each recursive call to systems with non-increasing arguments in recursive calls. Finally, this

³ We renamed the function symbols for the sake of readability.

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⁵³⁹ also extends previous works on static dependency pairs [25] from simply-typed λ -calculus to ⁵⁴⁰ dependent types modulo rewriting.

To get this result, we assumed local confluence. However, one often uses termination to check (local) confluence. Fortunately, there are confluence criteria not based on termination. The most famous one is (weak) orthogonality, that is, when the system is left-linear and has no critical pairs (or only trivial ones) [35], as it is the case in functional programming languages. A more general one is when critical pairs are "development-closed" [36].

546 This work can be extended in various directions.

First, our tool is currently limited to PFP rules, that is, to rules where higher-order variables are direct subterms of the left-hand side. To have higher-order variables in deeper subterms like in Example 14, we need to define a more complex interpretation of types, following [5].

Second, to handle recursive calls in such systems, we also need to use an ordering more complex than the subterm ordering when computing the matrices labeling the SCT call graph. The ordering needed for handling Example 14 is the "structural ordering" of COQ and AGDA [9, 6]. Relations other than subterm have already been considered in SCT but in a first-order setting only [34].

⁵⁵⁶ But we may want to go further because the structural ordering is not enough to handle ⁵⁵⁷ the following system which is not accepted by AGDA:

Example 19 (Division). m/n computes $\lceil \frac{m}{n} \rceil$.

```
559
                                                           infix 1 "-" ≔ minus
     symbol minus: Nat⇒Nat⇒Nat
                                                     set
560
       rule 0 - n \rightarrow 0
                                  rule
                                              0
                                                 \rightarrow
                                                             rule (s m) - (s n)
561
                                         m
                                                     m
                                                                                           m
                                                                                                n
     symbol div: Nat⇒Nat⇒Nat
                                                     \texttt{set}
                                                          infix 1 "/" := div
562
        rule 0 / (s n) \rightarrow 0
                                    rule (s m) / (s n) \rightarrow s ((m - n) / (s n))
563
564
```

A solution to handle this system is to use arguments filterings (remove the second argument of -) or simple projections [17]. Another one is to extend the type system with size annotations as in AGDA and compute the SCT matrices by comparing the size of terms instead of their structure [1, 7]. In our example, the size of m - n is smaller than or equal to the size of m. One can deduce this by using user annotations like in AGDA, or by using heuristics [8].

Another interesting extension would be to handle function calls with locally size-increasing arguments like in the following example:

573 574 575

 $\begin{array}{cccc} \textbf{rule f } \textbf{x} \rightarrow \textbf{g} \ \textbf{(s x)} & \textbf{rule g (s (s x))} \rightarrow \textbf{f x} \end{array}$

where the number of s's strictly decreases between two calls to f although the first rule makes the number of s's increase. Hyvernat enriched SCT to handle such systems [19].

Acknowledgments. The authors thank the anonymous referees for their comments, which have improved the quality of this article.

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A Proofs of lemmas on the interpretation

666 A.1 Definition of the interpretation

667 ► Lemma 20. *F* is monotone wrt inclusion.

From Proof. We first prove that D is monotone. Let $I \subseteq J$ and $T \in D(I)$. We have to show that $T \in D(J)$. To this end, we have to prove (1) $T \in SN$ and (2) if $T \to^* (x : A)B$ then $A \in dom(J)$ and, for all $a \in J(A)$, $B[x \mapsto a] \in dom(J)$:

671 **1.** Since $T \in D(I)$, we have $T \in SN$.

2. Since $T \in D(I)$ and $T \to^* (x : A)B$, we have $A \in \text{dom}(I)$ and, for all $a \in I(A)$, $B[x \mapsto a] \in \text{dom}(I)$. Since $I \subseteq J$, we have $\text{dom}(I) \subseteq \text{dom}(J)$ and J(A) = I(A)since I and J are functional relations. Therefore, $A \in \text{dom}(J)$ and, for all $a \in I(A)$, $B[x \mapsto a] \in \text{dom}(J)$.

We now prove that F is monotone. Let $I \subseteq J$ and $T \in D(I)$. We have to show that F(I)(T) = F(J)(T). First, $T \in D(J)$ since D is monotone.

If $T \downarrow = (x : A)B$, then $F(I)(T) = \Pi a \in I(A)$. $I(B[x \mapsto a])$ and $F(J)(T) = \Pi a \in J(A)$. $J(B[x \mapsto a])$. Since $T \in D(I)$, we have $A \in \text{dom}(I)$ and, for all $a \in I(A)$, $B[x \mapsto a] \in \text{dom}(I)$. Since $\text{dom}(I) \subseteq \text{dom}(J)$, we have J(A) = I(A) and, for all $a \in I(A)$, $G_{681} \quad J(B[x \mapsto a]) = I(B[x \mapsto a])$. Therefore, F(I)(T) = F(J)(T).

Now, if $T \downarrow$ is not a product, then F(I)(T) = F(J)(T) = SN.

A.2 Computability predicates

- **684 Elemma 21.** \mathcal{D} is a computability predicate.
- 685 **Proof.** Note that $\mathcal{D} = D(\mathcal{I})$.
- 686 **1.** $\mathcal{D} \subseteq$ SN by definition of D.
- ⁶⁸⁷ 2. Let $T \in \mathcal{D}$ and T' such that $T \to T'$. We have $T' \in SN$ since $T \in SN$. Assume now that ⁶⁸⁸ $T' \to^* (x : A)B$. Then, $T \to^* (x : A)B$, $A \in \mathcal{D}$ and, for all $a \in \mathcal{I}(A)$, $B[x \mapsto a] \in \mathcal{D}$.
- 689 Therefore, $T' \in \mathcal{D}$.
- **3.** Let *T* be a neutral term such that $\rightarrow(T) \subseteq \mathcal{D}$. Since $\mathcal{D} \subseteq SN$, $T \in SN$. Assume now that $T \rightarrow^* (x : A)B$. Since *T* is neutral, there is $U \in \rightarrow(T)$ such that $U \rightarrow^* (x : A)B$.
- ⁶⁹² Therefore, $A \in \mathcal{D}$ and, for all $a \in \mathcal{I}(A)$, $B[x \mapsto a] \in \mathcal{D}$.
- ▶ Lemma 22. If $P \in \mathbb{P}$ and, for all $a \in P$, $Q(a) \in \mathbb{P}$, then $\Pi a \in P.Q(a) \in \mathbb{P}$.
- ⁶⁹⁴ **Proof.** Let $R = \Pi a \in P. Q(a)$.
- 1. Let $t \in R$. We have to prove that $t \in SN$. Let $x \in \mathbb{V}$. Since $P \in \mathbb{P}$, $x \in P$. So, $tx \in Q(x)$. Since $Q(x) \in \mathbb{P}$, $Q(x) \subseteq SN$. Therefore, $tx \in SN$, and $t \in SN$.
- ⁶⁹⁷ 2. Let $t \in R$ and t' such that $t \to t'$. We have to prove that $t' \in R$. Let $a \in P$. We have to ⁶⁹⁸ prove that $t'a \in Q(a)$. By definition, $ta \in Q(a)$ and $ta \to t'a$. Since $Q(a) \in \mathbb{P}$, $t'a \in Q(a)$.
- **3.** Let t be a neutral term such that $\rightarrow(t) \subseteq R$. We have to prove that $t \in R$. Hence, we take
- $a \in P$ and prove that $ta \in Q(a)$. Since $P \in \mathbb{P}$, we have $a \in SN$ and $\rightarrow^*(a) \subseteq P$. We now
- prove that, for all $b \in \to^*(a)$, $tb \in Q(a)$, by induction on \to . Since t is neutral, tb is neutral
- too and it suffices to prove that $\rightarrow(tb) \subseteq Q(a)$. Since t is neutral, $\rightarrow(tb) = \rightarrow(t)b \cup t \rightarrow(b)$.
- By induction hypothesis, $t \to (b) \subseteq Q(a)$. By assumption, $\to (t) \subseteq R$. So, $\to (t)a \subseteq Q(a)$.
- Since $Q(a) \in \mathbb{P}, \rightarrow(t)b \subseteq Q(a)$ too. Therefore, $ta \in Q(a)$ and $t \in R$.

▶ Lemma 23. For all $T \in D$, $\mathcal{I}(T)$ is a computability predicate.

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Proof. Since $\mathcal{F}_p(\mathbb{T}, \mathbb{P})$ is a chain-complete poset, it suffices to prove that $\mathcal{F}_p(\mathbb{T}, \mathbb{P})$ is closed by F. Assume that $I \in \mathcal{F}_p(\mathbb{T}, \mathbb{P})$. We have to prove that $F(I) \in \mathcal{F}_p(\mathbb{T}, \mathbb{P})$, that is, for all $T \in D(I), F(I)(T) \in \mathbb{P}$. There are two cases:

⁷⁰⁹ If $T \downarrow = (x : A)B$, then $F(I)(T) = \Pi a \in I(A)$. $I(B[x \mapsto a])$. By assumption, $I(A) \in \mathbb{P}$ and, ⁷¹⁰ for $a \in I(A)$, $I(B[x \mapsto a]) \in \mathbb{P}$. Hence, by Lemma 22, $F(I)(T) \in \mathbb{P}$.

711 • Otherwise, $F(I)(T) = SN \in \mathbb{P}$.

⁷¹² ► Lemma 4a. For all terms T and substitutions σ , $[T]_{\sigma} \in \mathbb{P}$.

Proof. By induction on T. If T = s, then $[\![T]\!]_{\sigma} = \mathcal{D} \in \mathbb{P}$ by Lemma 21. If $T = (x : A)K \in \mathbb{K}$, then $[\![T]\!]_{\sigma} = \Pi a \in [\![A]\!]_{\sigma}$. $[\![K]\!]_{[x \mapsto a,\sigma]}$. By induction hypothesis, $[\![A]\!]_{\sigma} \in \mathbb{P}$ and, for all $a \in [\![A]\!]_{\sigma}$, $[\![K]\!]_{[x \mapsto a,\sigma]} \in \mathbb{P}$. Hence, by Lemma 22, $[\![T]\!]_{\sigma} \in \mathbb{P}$. If $T \notin \mathbb{K} \cup \{\text{KIND}\}$ and $T\sigma \in \mathcal{D}$, then $[\![T]\!]_{\sigma} = \mathcal{I}(T\sigma) \in \mathbb{P}$ by Lemma 23. Otherwise, $[\![T]\!]_{\sigma} = \text{SN} \in \mathbb{P}$.

717 A.3 Invariance by reduction

⁷¹⁸ We now prove that the interpretation is invariant by reduction.

▶ Lemma 24. If $T \in \mathcal{D}$ and $T \to T'$, then $\mathcal{I}(T) = \mathcal{I}(T')$.

Proof. First note that $T' \in \mathcal{D}$ since $\mathcal{D} \in \mathbb{P}$. Hence, $\mathcal{I}(T')$ is well defined. Now, we have $T \in SN$ since $\mathcal{D} \subseteq SN$. So, $T' \in SN$ and, by local confluence and Newman's lemma, $T \downarrow = T' \downarrow$. If $T \downarrow = (x : A)B$ then $\mathcal{I}(T) = \Pi a \in \mathcal{I}(A)$. $\mathcal{I}(B[x \mapsto a]) = \mathcal{I}(T')$. Otherwise, $\mathcal{I}(T) = SN = \mathcal{I}(T')$.

⁷²⁴ ▶ Lemma 4b. If T is typable, $T\sigma \in D$ and $T \to T'$, then $[T]_{\sigma} = [T']_{\sigma}$.

Proof. By assumption, there are Γ and U such that $\Gamma \vdash T : U$. Since \rightarrow preserves typing, we also have $\Gamma \vdash T' : U$. So, $T \neq \text{KIND}$, and $T' \neq \text{KIND}$. Moreover, $T \in \mathbb{K}$ iff $T' \in \mathbb{K}$ since $\Gamma \vdash T : \text{KIND}$ iff $T \in \mathbb{K}$ and T is typable. In addition, we have $T'\sigma \in \mathcal{D}$ since $T\sigma \in \mathcal{D}$ and $\mathcal{D} \in \mathbb{P}$.

We now prove the result, with $T \to T'$ instead of $T \to T'$, by induction on T. If $T \notin \mathbb{K}$, then $T' \notin \mathbb{K}$ and, since $T\sigma, T'\sigma \in \mathcal{D}$, $[T]]_{\sigma} = \mathcal{I}(T\sigma) = \mathcal{I}(T'\sigma) = [T']]_{\sigma}$ by Lemma 24. If T = TYPE, then $[T]]_{\sigma} = \mathcal{D} = [T']]_{\sigma}$. Otherwise, T = (x : A)K and T' = (x : A')K'with $A \to A'$ and $K \to K'$. By inversion, we have $\Gamma \vdash A$: TYPE, $\Gamma \vdash A'$: TYPE, $\Gamma, x : A \vdash K$: KIND and $\Gamma, x : A' \vdash K'$: KIND. So, by induction hypothesis, $[A]]_{\sigma} = [A']]_{\sigma}$ and, for all $a \in [A]]_{\sigma}, [K]]_{\sigma'} = [[K']]_{\sigma'}$, where $\sigma' = [x \mapsto a, \sigma]$. Therefore, $[T]]_{\sigma} = [[T']]_{\sigma}$.

⁷³⁵ ► Lemma 4c. If *T* is typable, $T\sigma \in D$ and $\sigma \to \sigma'$, then $[T]_{\sigma} = [T]_{\sigma'}$.

⁷³⁶ **Proof.** By induction on T.

If $T \in \mathbb{S}$, then $\llbracket T \rrbracket_{\sigma} = \mathcal{D} = \llbracket T \rrbracket_{\sigma'}$.

⁷³⁸ If T = (x : A)K and $K \in \mathbb{K}$, then $\llbracket T \rrbracket_{\sigma} = \Pi a \in \llbracket A \rrbracket_{\sigma}$. $\llbracket K \rrbracket_{[x \mapsto a,\sigma]}$ and $\llbracket T \rrbracket_{\sigma'} = \Pi a \in \llbracket A \rrbracket_{\sigma'}$. ⁷³⁹ $\llbracket A \rrbracket_{\sigma'} . \llbracket K \rrbracket_{[x \mapsto a,\sigma']}$. By induction hypothesis, $\llbracket A \rrbracket_{\sigma} = \llbracket A \rrbracket_{\sigma'}$ and, for all $a \in \llbracket A \rrbracket_{\sigma}$, ⁷⁴⁰ $\llbracket K \rrbracket_{[x \mapsto a,\sigma]} = \llbracket K \rrbracket_{[x \mapsto a,\sigma']}$. Therefore, $\llbracket T \rrbracket_{\sigma} = \llbracket T \rrbracket_{\sigma'}$.

⁷⁴¹ If $T\sigma \in \mathcal{D}$, then $\llbracket T \rrbracket_{\sigma} = \mathcal{I}(T\sigma)$ and $\llbracket T \rrbracket_{\sigma'} = \mathcal{I}(T\sigma')$. Since $T\sigma \to^* T\sigma'$, by Lemma 4b, ⁷⁴² $\mathcal{I}(T\sigma) = \mathcal{I}(T\sigma')$.

⁷⁴³ Otherwise,
$$\llbracket T \rrbracket_{\sigma} = SN = \llbracket T \rrbracket_{\sigma'}$$
.

•

744 A.4 Adequacy of the interpretation

⁷⁴⁵ **Lemma 4d.** If (x : A)B is typable, $((x : A)B)\sigma \in \mathcal{D}$ and $x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma)$, then ⁷⁴⁶ $\llbracket (x : A)B \rrbracket_{\sigma} = \Pi a \in \llbracket A \rrbracket_{\sigma} \cdot \llbracket B \rrbracket_{[x \mapsto a, \sigma]}$.

Proof. If *B* is a kind, this is immediate. Otherwise, since $((x : A)B)\sigma \in \mathcal{D}$, $[[(x : A)B]]_{\sigma} = \mathcal{I}_{48} \quad \mathcal{I}(((x : A)B)\sigma)$. Since $x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma)$, we have $((x : A)B)\sigma = (x : A\sigma)B\sigma$. Since $x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma)$, we have $[(x : A)B]\sigma = (x : A\sigma)B\sigma$. Since $x \notin \operatorname{dom}(\sigma) \cup \operatorname{FV}(\sigma)$, we have $[[(x : A)B]]_{\sigma} = \Pi a \in \mathcal{I}(A\sigma \downarrow)$. $\mathcal{I}((B\sigma \downarrow)[x \mapsto a])$.

Since (x : A)B is typable, A is of type TYPE and $A \notin \mathbb{K} \cup \{\text{KIND}\}$. Hence, $[\![A]\!]_{\sigma} = \mathcal{I}(A\sigma)$ and, by Lemma 24, $\mathcal{I}(A\sigma) = \mathcal{I}(A\sigma\downarrow)$.

⁷⁵² Since (x : A)B is typable and not a kind, B is of type TYPE and $B \notin \mathbb{K} \cup \{\text{KIND}\}$. Hence, ⁷⁵³ $\llbracket B \rrbracket_{[x \mapsto a,\sigma]} = \mathcal{I}(B[x \mapsto a,\sigma])$. Since $x \notin \text{dom}(\sigma) \cup \text{FV}(\sigma)$, $B[x \mapsto a,\sigma] = (B\sigma)[x \mapsto a]$. Hence, ⁷⁵⁴ $\llbracket B \rrbracket_{[x \mapsto a,\sigma]} = \mathcal{I}((B\sigma)[x \mapsto a])$ and, by Lemma 24, $\mathcal{I}((B\sigma)[x \mapsto a]) = \mathcal{I}((B\sigma\downarrow)[x \mapsto a])$.

Therefore,
$$[\![(x:A)B]\!]_{\sigma} = \prod a \in [\![A]\!]_{\sigma}$$
. $[\![B]\!]_{[x\mapsto a,\sigma]}$

Note that, by iterating this lemma, we get that $v \in \llbracket \forall \vec{x} : \vec{T}, U \rrbracket$ iff, for all \vec{t} such that $\vec{x} \mapsto \vec{t} \models \vec{x} : \vec{T}, v\vec{t} \in \llbracket U \rrbracket_{|\vec{x} \mapsto \vec{t}|}.$

⁷⁵⁸ ► Lemma 4e. If $\Delta \vdash U : s, \Gamma \vdash \gamma : \Delta$ and $U\gamma \sigma \in \mathcal{D}$, then $\llbracket U\gamma \rrbracket_{\sigma} = \llbracket U \rrbracket_{\gamma\sigma}$.

⁷⁵⁹ **Proof.** We proceed by induction on U. Since $\Delta \vdash U : s$ and $\Gamma \vdash \gamma : \Delta$, we have $\Gamma \vdash U\gamma : s$.

- If s = TYPE, then $U, U\gamma \notin \mathbb{K} \cup \{\text{KIND}\}$ and $\llbracket U\gamma \rrbracket_{\sigma} = \mathcal{I}(U\gamma\sigma) = \llbracket U \rrbracket_{\gamma\sigma}$ since $U\gamma\sigma \in \mathcal{D}$.
- 761 Otherwise, s = KIND and $U \in \mathbb{K}$.
- ⁷⁶² If U = TYPE, then $\llbracket U \gamma \rrbracket_{\sigma} = \mathcal{D} = \llbracket U \rrbracket_{\gamma \sigma}$.

 $\begin{array}{ll} \text{763} & = & \text{Otherwise, } U = (x:A)K \text{ and, by Lemma 4d, } \llbracket U\gamma \rrbracket_{\sigma} = \Pi a \in \llbracket A\gamma \rrbracket_{\sigma}. \llbracket K\gamma \rrbracket_{[x\mapsto a,\sigma]} \text{ and} \\ \llbracket U \rrbracket_{\gamma\sigma} = \Pi a \in \llbracket A \rrbracket_{\gamma\sigma}. \llbracket K \rrbracket_{[x\mapsto a,\gamma\sigma]}. \text{ By induction hypothesis, } \llbracket A\gamma \rrbracket_{\sigma} = \llbracket A \rrbracket_{\gamma\sigma} \text{ and, for} \\ \text{all } a \in \llbracket A\gamma \rrbracket_{\sigma}, \llbracket K\gamma \rrbracket_{[x\mapsto a,\sigma]} = \llbracket K \rrbracket_{\gamma[x\mapsto a,\sigma]}. \text{ Wlog we can assume } x \notin \text{dom}(\gamma) \cup \text{FV}(\gamma). \\ \text{So, } \llbracket K \rrbracket_{\gamma[x\mapsto a,\sigma]} = \llbracket K \rrbracket_{[x\mapsto a,\gamma\sigma]}. \end{array}$

⁷⁶⁷ ► Lemma 4f. Let P be a computability predicate and Q a P-indexed family of computability ⁷⁶⁸ predicates such that $Q(a') \subseteq Q(a)$ whenever $a \to a'$. Then, $\lambda x : A.b \in \Pi a \in P. Q(a)$ whenever ⁷⁶⁹ $A \in SN$ and, for all $a \in P$, $b[x \mapsto a] \in Q(a)$.

Proof. Let $a_0 \in P$. Since $P \in \mathbb{P}$, we have $a_0 \in SN$ and $x \in P$. Since $Q(x) \in \mathbb{P}$ and $b = b[x \mapsto x] \in Q(x)$, we have $b \in SN$. Let $a \in \to^* (a_0)$. We can prove that $(\lambda x : A, b)a \in Q(a_0)$ by induction on (A, b, a) ordered by $(\to, \to, \to)_{\forall}$.,Since $Q(a_0) \in \mathbb{P}$ and $(\lambda x : A, b)a$ is neutral, it suffices to prove that $\to ((\lambda x : A, b)a) \subseteq Q(a_0)$. If the reduction takes place in A, b or a, we can conclude by induction hypothesis. Otherwise, $(\lambda x : A, b)a \to b[x \mapsto a] \in Q(a)$ by assumption. Since $a_0 \to^* a$ and $Q(a') \subseteq Q(a)$ whenever $a \to a'$, we have $b[x \mapsto a] \in Q(a_0)$.

B Termination proof of Example 1

778

⁷⁷⁷ Here is the comprehensive list of dependency pairs in the example:

```
A :
                       El (arrow a b) > El a
779
    B:
                       El
                          (arrow a b) > El b
780
                            (s p) + q > p + q
    C:
781
         app a _ (cons _ x p l) q m > p + q
    D:
782
         app a _ (cons _ x p l) q m > app a p l q m
783
    E:
    F:len_fil a f _ (cons _ x p l) > len_fil_aux (f x) a f p l
784
785
    G:len_fil a f _ (app _ p l q m) >
```

```
(len_fil a f p l) + (len_fil a f q m)
786
   H:len_fil a f _ (app _ p l q m) > len_fil a f p l
787
    I:len_fil a f _ (app
                           p l q m) > len_fil a f
                                                   q
                                                     m
788
                         afpl > len_fil af
    J:
         len_fil_aux true
                                                   р
                                                     1
789
   K:
         len_fil_aux false a f p l > len_fil a f p
                                                     1
790
         fil a f _ (cons _ x p l)
                                    > fil_aux (f x) a f x p l
   L:
791
   M:
         fil a f _ (app _ p l q m) >
792
                             app a (len_fil a f p l) (fil a f p l)
793
                                   (len_fil a f q m) (fil a f q m)
794
         fil a f _ (app _
                           plqm) > len_fil a f p l
   N:
795
         fil a f _ (app
                        _plqm) > filafpl
   0:
796
                                                   q m
   P:
         fil a f _ (app _ p l q m) > len_fil a f
797
   Q:
         fil a f _
                    (app
                        _plqm) > filafqm
798
   R:
            fil_aux true
                          afxpl>len_filaf
                                                   p 1
799
   S:
            fil_aux true
                          afxpl>filafpl
800
            fil_aux false a f x p l > fil a f p l
   T:
801
802
```

The whole callgraph is depicted below. The letter associated to each matrix corresponds to the dependency pair presented above and in example 7, except for TC 's which comes from the computation of the transitive closure and labels dotted edges.



806

⁸⁰⁷ The argument **a** is omitted everywhere on the matrices presented below:

$$\begin{array}{ll} A, B = (-1), \ C = \begin{pmatrix} -1 & \infty \\ \infty & 0 \end{pmatrix}, \ D = \begin{pmatrix} \infty & \infty \\ -\infty & 0 \\ \infty & \infty \end{pmatrix}, \ E = \begin{pmatrix} \infty & 0 & \infty & \infty \\ -\infty & 0 & 0 \\ \infty & \infty & 0 \end{pmatrix}, \ F = \begin{pmatrix} \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & -1 & -1 \end{pmatrix}, \ J = K = \begin{pmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & 0 & 0 \end{pmatrix}, \\ G = \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix}, \ H = I = N = 0 = P = \mathbb{Q} = \begin{pmatrix} 0 & \infty & \infty \\ \infty & \infty & \infty \\ \infty & -1 & -1 \end{pmatrix}, \ L = \begin{pmatrix} \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & -1 & -1 & -1 \end{pmatrix}, \ M = \begin{pmatrix} \infty & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & \infty \end{pmatrix}, \\ R = S = T = \begin{pmatrix} 0 & \infty & \infty \\ 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & 0 & 0 \end{pmatrix}.$$

811 Which leads to the matrices labeling a loop in the transitive closure:

It would be useless to compute matrices labeling edges which are not in a strongly connected component of the call-graph (like $S \times R$), but it is necessary to compute all the products which could label a loop, especially to verify that all loop-labeling matrices are idempotent, which is indeed the case here.

We now check that this system is well-structured. For each rule $f\vec{l} \to r$, we take the environment $\Delta_{f\vec{l}\to r}$ made of all the variables of r with the following types: a:Set, b:Set, p:N, q:N, x:El a, l:L a p, m:L a q, f:El a \Rightarrow B.

821	The precedence infered for this example is the smallest containing:						
822	comparisons linked to the typing of symbols:						
	Set \preceq	arrow	Set,L,0	\preceq	nil		
823	Set \preceq	E1	Set,El, \mathbb{N},\mathbb{L},s	\preceq	cons		
	\mathbb{B} \preceq	true	Set, $\mathbb{N},\mathbb{L},+$	\preceq	app		
	\mathbb{B} \preceq	false	$\texttt{Set},\texttt{El},\mathbb{B},\mathbb{N},\mathbb{L}$	\preceq	len_fil		
	\mathbb{N} \preceq	0	$\mathbb{B}, \mathtt{Set}, \mathtt{El}, \mathbb{N}, \mathbb{L}$	\preceq	len_fil_aux		
	\mathbb{N} \preceq	S	Set,El, \mathbb{B} , \mathbb{N} , \mathbb{L} ,len_fil	\preceq	fil		
	\mathbb{N} \preceq	+ I	B,Set,El,N,L,len_fil_aux	\preceq	fil_aux		
	Set, \mathbb{N} \preceq	L					
824	and comparisons related	d to calls	3:				
	S	≚ +	s,len	_fil	\preceq len_fil_aux		
825	cons,+	\preceq app	<pre>nil,fil_aux,app,len</pre>	_fil	\preceq fil		
	0,len_fil_aux,+	\preceq len	_fil fil,cons,len	_fil	≺ fil_aux		

This precedence can be sum up in the following diagram, where symbols in the same box are equivalent:



828