



HAL
open science

Kalman 1960: The birth of modern system theory

Pierre Bernhard, Marc Deschamps

► **To cite this version:**

Pierre Bernhard, Marc Deschamps. Kalman 1960: The birth of modern system theory. *Mathematical Population Studies*, 2019, 26 (3), pp.123-145. 10.1080/08898480.2018.1553393 . hal-01940560

HAL Id: hal-01940560

<https://inria.hal.science/hal-01940560>

Submitted on 30 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1

Kalman 1960:

2

the birth of modern system theory

3

Pierre Bernhard

Biocore, Université Côte d'Azur-Institut national

de recherche en informatique et en automatique (Inria), France

Corresponding author: Pierre.Bernhard@inria.fr

and Marc Deschamps

Crese EA3190, Université de Bourgogne Franche-Comté,

Centre national de la recherche scientifique (Cnrs) Beta and Gredeg,

Ofce-SiencesPo, France

4

September 2018

5

Abstract

6 Rudolph E. Kalman is mainly known for the Kalman filter, first
7 published in 1960. In this year, he published two equally important
8 contributions, one about linear state space system theory and the other
9 about linear quadratic optimal control theory. These three domains
10 are intertwined in the later theory of linear quadratic Gaussian control.
11 An extended version of linear quadratic optimal control is put into
12 practice in an example of cooperation in population ecology.

13 *Keywords:* Kalman filter, linear quadratic Gaussian control, linear system,
14 state space, cooperation

15 **1 Introduction**

16 Rudolf Emil Kalman passed away on the 2nd of July 2016. His name will
17 remain tied to the *Kalman filter*. The purpose of the Kalman filter is to
18 estimate time-varying variables from noisy and incomplete measurements,
19 using a (noisy) model of the underlying dynamics. Kalman objected that his
20 filter belonged to “estimation theory,” arguing that this term had come to
21 name a body of non-science. He later also denied that the filter was based

22 on probabilities; as much as he refused to reduce it to a mere algorithm,
23 because, for him, the filter was to be acknowledged as a theory on its own.

24 Because the filter allows one to estimate a variable which is not directly
25 measured, it has sometimes been publicized by computer scientists as a “soft-
26 ware sensor”. The filter and its extensions are now used in many domains,
27 be they industrial and technological, or in the health, social, biological, or
28 earth sciences. It is also implemented in global positioning systems (GPS).

29 In the same year 1960 when Kalman published his article about the filter
30 (Kalman, 1960b), he also published a major article on linear quadratic opti-
31 mal control (Kalman, 1960a), which he had earlier presented at a conference
32 in 1959, and an also major article on system theory (Kalman, 1960c), com-
33 pleted by Kalman (1962, 1963). These contributions were ground-breaking
34 for control theoreticians. They transformed control theory and linear system
35 theory, and their influence went beyond the mere domain of filtering and
36 prediction. Kalman, in this year 1960, also published important articles on
37 the stability of linear dynamical systems (Kalman and Bertram, 1960a, b).

38 Bolstered by the advent of the digital computer, the theories presented in
39 (Kalman, 1960a, b) were extensively put into use in the “Automatic Synthesis
40 Program” involved in the Apollo lunar landing program, which started also

41 in 1960. A new theory has seldom been adopted so quickly by practitioners
42 and for such significant a project. This had been the case, though, of the
43 Wiener filter, which was conceived in 1940 for the anti-aircraft radar, and
44 had been embargoed until 1949 for its military sensitivity.

45 We shall present here Kalman's contributions in the early 1960's, with a
46 focus on control theory and with a toy application in population ecology.

47 **2 System theory**

48 **2.1 A paradigmatic change**

49 **2.1.1 State of the theory before 1960**

50 Before 1960, system and control theories were confined to linear time-invariant
51 (LTI) systems, mostly so-called "monovariable" ones, which are characterized
52 by scalar signals.

53 A (linear) dynamical system is a device which is excited by a time-varying
54 signal, called the *input*, to produce a time-varying signal, called the *output*.
55 For mathematicians, a linear system amounts to a linear operator, trans-
56 forming a time function into another one. Because this excitation happens
57 in real time, the transformation must satisfy the principle of *causality* (the

58 current output does not depend on future inputs). Moreover, description
59 and analysis based on the theory available at that time require linearity,
60 time-invariance, and some other properties about behavior at infinity.

61 Mathematicians had developed an elegant and powerful way for handling
62 such transformations, through the use of the (quite esoteric) *Laplace trans-*
63 *form*, which leads to the representation of a system through a “transfer func-
64 tion,” which is a ratio of polynomials of a complex variable. An important
65 point is that the transfer function is easily obtained from a linear differential
66 equation governing the system.

67 The transfer function has the property of transforming a cascade of sys-
68 tems (whereby the output of a system is the input of the next one) into
69 simple products of their transfer functions. This allows the analysis of *feed-*
70 *back* systems, where the output is re-introduced as a component of the input
71 of the same system. This loop is necessary to servomechanisms. The concept
72 of transfer function also led to the Wiener filter, which, at that epoch, was
73 the standard tool in signal processing.

74 The transfer function and the Wiener filter were intimately tied to the
75 *frequency response* of the system, that is, its behavior if excited by sinusoidal
76 signals of various frequencies.

77 A clever trick of Wiener filtering was to consider a noisy signal to be
78 “filtered” (net its noise) as the output of a linear system excited by a noise
79 with adequate statistical properties. Kalman used this move in the Kalman
80 filter, advising one of the authors (Pierre Bernhard): “Take the Kalman filter,
81 which, as everybody knows, was invented by Wiener”.

82 **2.1.2 Innovations in the year 1960**

83 Kalman represented the transformation of inputs into outputs by the media-
84 tion of an internal *state* of the system, consisting of a vector of real variables,
85 whose temporal variations are governed by a first-order differential equation
86 in continuous time or by a first-order difference equation in discrete time.
87 This is why this description of the system is called *internal*, the classical
88 one then being called *external*. In that representation, the input acts on
89 the dynamics of the state, and the state instantaneously produces an out-
90 put. Because the state is a vector and all relations are linear, linear algebra
91 is appropriate. This mathematical machinery allows the consideration of
92 vector-valued inputs and outputs. Moreover, if the matrices defining the sys-
93 tem vary over time, the system is no longer time invariant. Part of realization
94 theory, which lies at the heart of linear system theory, not only concerns lin-

95 ear time-invariant systems, but also can be extended to systems that are not
96 invariant over time, including filtering and optimal control.

97 The concept of the state of a system was imprecisely known as “*a set*
98 *of numbers from which the entire future behavior of the plant may be de-*
99 *termined provided that the future inputs of the plant are known*” (Kalman
100 and Koepcke, 1958: 1821); for example positions and velocities in a mechan-
101 ical system, intensities in inductors and charges of capacitors in an electrical
102 circuit. Kalman (1960c) mentioned that it is the *smallest* such set. This
103 property was present in the (earlier) concept of Nerode equivalence of formal
104 languages and used in the theory of automata (Kalman et al., 1969). The
105 intimate link of the concept of state with first-order differential equations
106 was now acknowledged, but often confined to the derivation of the transfer
107 function of the linear time-invariant system. The direct use of differential
108 (or difference) equations in optimization had become more common in the
109 late fifties under the influence of Bellman’s Dynamic Programming (Bellman,
110 1957). Kalman’s bold move was to implement this use in the very definition
111 of a linear system and to investigate its properties.

112 The internal description avoids using the Laplace transform, with the
113 consequence that systems no longer need to be time invariant. Signals now

114 amount to time functions, which justifies the term *time domain*, in contrast
115 to the term *frequency domain*. The tools developed by Kalman and after
116 him are treated with the calculus of variations for optimal control, and with
117 Markov processes for filtering.

118 The advent of the digital computer and of direct digital control led
119 Kalman and others to develop a discrete-time theory, originally as a the-
120 ory of *sampled data systems*, which consists of considering continuous-time
121 systems only at discrete instants. The parallel was natural and elegant in
122 the new theory. Kolmogorov (1941), independently of Wiener, had earlier
123 pioneered a discrete-time version of the Wiener filter.

124 The transfer function of a time-invariant system in internal form results
125 from a simple algebraic formula. Conversely, finding the internal representa-
126 tion of a system given in external form is a deep question, which involves a
127 thorough analysis of a linear system in internal form. This is the object of
128 realization theory.

129 **2.2 Realization theory**

130 Kalman's contribution to system theory began with his article "On the
131 general theory of control systems" (Kalman, 1960c), but the founding publi-

132 cations, which we shall follow here, are Kalman (1962, 1963).

133 **Definition 1** *A realization of a linear (or affine) input-output transforma-*
134 *tion is a representation in internal form as expressed in Eq. (1) and (2), or*
135 *(3) and (4) below.*

Let $x \in \mathbb{R}^n$ be the state of the system (n is called the dimension of the realization), $u \in \mathbb{R}^m$ the input, or control, and $y \in \mathbb{R}^p$ the output. The interesting cases are $m \leq n$ and $p \leq n$. We use Newton's notation for time derivatives: $\dot{x} = dx/dt$. A continuous-time system is of the form

$$\dot{x}(t) = Fx(t) + Gu(t), \quad (1)$$

$$y(t) = Hx(t), \quad (2)$$

and in discrete time

$$x(t+1) = Fx(t) + Gu(t), \quad (3)$$

$$y(t) = Hx(t). \quad (4)$$

136 Kalman argued against the possibility of augmenting the output equation
137 with a term $+Ju(t)$.

138 For the sake of completeness, we mention that if all three matrices F , G ,
139 and H are constant, then the system is linear time-invariant, and its transfer

140 function is

$$\mathcal{H}(s) = H(sI - F)^{-1}G. \quad (5)$$

141 Cramer's rule (Birkhoff and Mac Lane, 1967) implies that $\mathcal{H}(s)$ is a matrix
142 of (strictly proper) rational fractions of s , with the characteristic polynomial
143 of F as their common denominator, and thus the poles of $\mathcal{H}(s)$ are the
144 eigenvalues of F .

Even without referring to the transfer function, the transformation from input to output induced by these equations is unchanged when changing the basis in the state space, or equivalently if we use the state $\xi = Tx$, where T is an invertible matrix. The continuous-time system becomes

$$\dot{\xi} = TFT^{-1}\xi + TGu, \quad (6)$$

$$y = HT^{-1}\xi. \quad (7)$$

145 Hence changing (H, F, G) into (HT^{-1}, TFT^{-1}, TG) leaves the system un-
146 changed, with the same transfer function if the system is time-invariant.
147 This points out that the same input-output system may correspond to a new
148 representation, which is then not unique. The lack of uniqueness also occurs
149 with a higher dimensional vector, say z , made of x and a vector ξ of arbitrary

150 dimension:

$$\begin{aligned} z &= \begin{pmatrix} x \\ \xi \end{pmatrix}, \\ \dot{z} &= \begin{pmatrix} F & 0 \\ A & B \end{pmatrix} z + \begin{pmatrix} G \\ C \end{pmatrix} u, \\ y &= (H \ 0)z. \end{aligned} \tag{8}$$

151 A , B , and C are arbitrary matrices. They play no role, because ξ does
152 not influence y , neither directly nor through x . Therefore, realizations of
153 different dimensions may represent the same input-output system. In the
154 case of Eq. (8), it is trivial but if a change of the basis such as the previous
155 one mixes x and ξ , then the dimension in excess may be more difficult to
156 detect. Moreover, other cases are possible.

157 The solution of this problem involves *controllability* and *observability*. A
158 state is controllable if there exists a control function $u(\cdot)$ that drives the
159 system from this state to the origin. The system is said to be *completely*
160 *controllable* if every state is controllable.

161 For the theory to encompass both continuous- and discrete-time systems,
162 we now use the later concept of *reachability*. A state is *reachable* if there
163 exists a control function that drives the system from the origin to that state.
164 A system is *completely reachable* if every state is reachable. The two concepts

165 are equivalent for continuous-time systems, but not for discrete-time ones,
166 unless the matrices $F(t)$ are invertible for all t . Consequently, a system is
167 completely reachable if the application $u([t_0, t_1]) \mapsto x(t_1)$, which is linear if
168 $x(t_0) = 0$, is onto (that is, surjective) for some $t_1 > t_0$.

169 A state $x_0 \neq 0$ is *unobservable* if the output of the “free” system, corre-
170 sponding to $u(\cdot) = 0$, initialized at x_0 , is $y([t_0, t_1]) = 0$ for any $t_1 \geq t_0$. A
171 system is *completely observable* if no state is unobservable. Consequently, the
172 system is completely observable if the application $x(t_0) \mapsto y([t_0, t_1])$, which
173 is linear if $u(\cdot) = 0$, is one to one (that is, injective) for some $t_1 > t_0$.

174 Kalman (1960a) also gave efficient criteria to examine these properties.
175 In the case of time-invariant systems, the Kalman criteria are expressed in
176 terms of the ranks of composite matrices:

Theorem 1

$$(F \ G) \text{ completely reachable} \Leftrightarrow \text{rank}(G \ FG \ F^2G \ \dots \ F^{n-1}G) = n, \quad (9)$$

$$(H \ F) \text{ completely observable} \Leftrightarrow \text{rank} \begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{pmatrix} = n \quad (10)$$

177 Kalman (1960a) also gave simple criteria for systems that are not time-
178 invariant. They are less algebraic, more analytic, but they share the property
179 called *duality* (we use prime for transposed), defined as:

$$(F \ G) \text{ completely reachable} \Leftrightarrow (G' \ F') \text{ completely observable} \quad (11)$$

180

$$(H \ F) \text{ completely observable} \Leftrightarrow (F' \ H') \text{ completely reachable.} \quad (12)$$

181 We shall see that this duality concerns optimal control and filtering. Kalman
182 (1960c) analyzed the duality between the Wiener filter and the linear quadratic
183 regulator. He pointed out that it is also related to the known duality between
184 the differential equations $\dot{x} = Fx$ and $\dot{p} = -F'p$ or the difference equations
185 $x(t+1) = Fx(t)$ and $p(t) = F'p(t+1)$, which leave the inner product $p'x$
186 invariant.

187 But the whole extent of duality in the linear quadratic Gaussian theory,
188 which we present below, remains difficult to explain; the more so that it
189 mysteriously extends into modern \mathcal{H}_∞ -optimal control (Başar and Bernhard,
190 1995).

191 We use these concepts in realization theory with the formal definition:

192 **Definition 2** *A completely reachable and completely observable realization*
193 *is called canonical.*

194 The main theorem is:

195 **Theorem 2** *A realization is minimal (has a state space of minimum dimen-*
196 *sion) if and only if it is canonical. It is unique up to a change of the basis*
197 *in the state space.*

Kalman (1962) further showed that the state space of any linear system in internal form, not necessarily time-invariant (that is, a system such as Eq. (1) and (2) with matrices H , F , and G depending continuously on time) can be decomposed canonically as the direct sum of four subspaces —which can vary over time if the system is not time-invariant— as represented in the left-hand side diagram of Figure 1, or more classically, in block diagram, as in Kalman (1963), and reproduced in all textbooks since then. A is the sub-

space of reachable but unobservable states; B the subspace of reachable and observable states; C the subspace of unreachable and unobservable states; D the subspace of unreachable but observable states. Using a basis adapted to

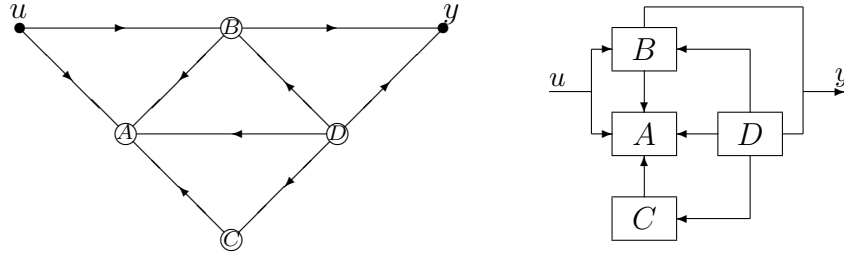


Figure 1: The canonical decomposition of a linear system in internal form.

Source: Kalman (1962).

that decomposition yields a canonical decomposition of the system matrices,

which takes the form:

$$F = \begin{pmatrix} F_{AA} & F_{AB} & F_{AC} & F_{AD} \\ 0 & F_{BB} & 0 & F_{BD} \\ 0 & 0 & F_{CC} & F_{CD} \\ 0 & 0 & 0 & F_{DD} \end{pmatrix}, \quad G = \begin{pmatrix} G_A \\ G_B \\ 0 \\ 0 \end{pmatrix}, \quad (13)$$

$$H = (\quad 0 \quad H_B \quad 0 \quad H_D). \quad (14)$$

198 The subspaces B , C , and D are not uniquely defined, contrary to the decom-
 199 position of the matrices of the system (H, F, G) , up to changes of the basis
 200 within each of the four subspaces.

201 **2.3 Compensator design by pole placement**

202 The Kalman filter was the first *observer*, also called “observing system” in
203 Kalman (1960c), who proposed either an “optimal observer” or an “optimal
204 observing system” in terms of the total number of time steps necessary to
205 retrieve the state in a discrete-time system. The term “observer” was coined
206 by Luenberger (1964), who extended the concept in a less explicit and more
207 complicated form, ignoring the stability argument we use below in paragraph
208 4.1.2. The equations of the Kalman filter are, in discrete time as it appeared
209 originally in Kalman (1960b):

$$\hat{x}(t+1) = F\hat{x}(t) + Gu(t) + K(y(t) - H\hat{x}(t)), \quad (15)$$

210 or in continuous time (Kalman and Bucy, 1961):

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + K(y(t) - H\hat{x}(t)), \quad (16)$$

211 providing an estimate $\hat{x}(t)$ of the state, optimal in the sense that it minimizes
212 the expected squared L^2 norm of the *error signal* $\tilde{x}(t) = x(t) - \hat{x}(t)$. The
213 natural idea, proposed by Kalman (1960c) for monovariable discrete-time
214 systems, is to associate such an observer with a control law

$$u(t) = -C\hat{x}(t). \quad (17)$$

215 We now examine the choice of the gains K and C .

216 With a change of variables from (x, \hat{x}) to (x, \tilde{x}) , one can prove the *prin-*
217 *ciple of separation of the dynamics*:

218 **Theorem 3 (Separation of dynamics)** *The set of the eigenvalues of the*
219 *dynamic matrix of the closed loop observer-controller is the union of the*
220 *eigenvalues of the “controller” $F - GC$ and those of the “observer” $F - KH$.*

221 Wonham (1967) extended the *pole shifting theorem* from single-input sys-
222 tems (Luenberger, 1964) to multi-input ones (early alternative proofs are in
223 Heymann (1968) and Davison (1968)):

224 **Theorem 4** *If the pair (F, G) is completely reachable, then, given any monic*
225 *polynomial $p(z)$ of degree n , there exists a matrix C such that the character-*
226 *istic polynomial of $F - GC$ is p . Dually, if the pair (H, F) is completely*
227 *observable, the characteristic polynomial of $F - KH$ can be assigned to any*
228 *desired monic polynomial of degree n by the choice of K .*

229 We deduce an argument of pure system theory in favor of the proposed
230 control structure, and a means of choosing C and K (see paragraph 4.1.2
231 below).

232 Moreover, in the discrete-time case, consequently also in the sampled-data

233 problem of any digital control, the observer behaves as a one-step predictor,
234 because the system estimates $\hat{x}(t+1)$ for $x(t+1)$ with the data $y(\tau)$, $\tau \leq t$.
235 The control algorithm then has one time step at its disposal to compute the
236 control $u(t+1) = -C\hat{x}(t+1)$, with the knowledge of the gain C , which has
237 been computed off line.

238 **3 Linear quadratic Gaussian (LQG) theory**

239 **3.1 Linear quadratic (LQ) optimal control**

240 The topic covered here is partially addressed in Kalman (1960c), but the
241 authoritative article is Kalman (1960a), whose notation we adopt for the
242 most part.

243 In his introduction, Kalman (1960a: 102) stated: “This problem dates
244 back, in its modern form, to Wiener and Hall at about 1943.” He also cited
245 Newton Jr et al. (1957), as representing the state of the art.

246 Therefore, in its infinite horizon (optimal regulator) form, Kalman’s lin-
247 ear quadratic optimal control theory was not new. However, the solution
248 provided by the theory then available (Newton Jr et al., 1957) was in terms
249 of spectral factorization, which is comparable to the Wiener Filter and did

250 not easily lead to efficient algorithms, particularly so for “multivariable” prob-
251 lems. It could not be extended to finite-horizon problems either. Actually,
252 one finite-horizon non-homogeneous scalar-control linear-quadratic optimiza-
253 tion problem is solved in Merriam III (1959), with the correct Riccati equa-
254 tion and the linear equations for the non-homogeneous terms, although the
255 latter are difficult to recognize.

256 **3.1.1 Finite-horizon problem**

257 The theory of linear-quadratic optimal control began with the investigation
258 of a finite-horizon optimal control problem, which is then not time-invariant.
259 This problem involves quadratic forms. We express it as: for any positive
260 definite or semi-definite $\ell \times \ell$ matrix M and z a ℓ -vector,

$$\langle z, Mz \rangle = z'Mz = \|z\|_M^2. \quad (18)$$

261 The problem is (presentation in a more general form in Kalman (1960a)):

262 **Linear quadratic optimal control problem** Given the system of Eq. (1)
263 and (2), with all system matrices (possibly piecewise) continuous functions of
264 time, and $x(t_0) = x_0$, and given the symmetric $n \times n$ matrix $A \geq 0$, and the
265 symmetric (piecewise) continuous $n \times n$ matrix function $Q(t) \geq 0$ and $m \times m$
266 matrix function $R(t) > 0$, find the control law, if it exists, that minimizes

267 the performance index

$$V(x_0, t_0, t_1; u(\cdot)) = \|x(t_1)\|_A^2 + \int_{t_0}^{t_1} (\|y(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2) dt. \quad (19)$$

268 While Kalman and Koepcke (1958) and Kalman (1960c) explicitly use Bell-
269 man (1957) and discrete dynamic programming, which was a strong incentive
270 to use the state-space representation, Kalman (1960a) for the continuous-
271 time problem used the theory of Carathéodory (1935). Conversely, Bellman
272 (1957) did not refer to Carathéodory. Yet, continuous-time dynamic pro-
273 gramming is mostly a re-discovery of Carathéodory's theory, with Bellman's
274 *return function* playing the same role as Carathéodory's *principal function*.
275 Define a symmetric matrix function $P(t)$ as the solution, if it exists, of the
276 matrix Riccati equation (where all matrices are time-dependent)

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH, \quad P(t_1) = A. \quad (20)$$

277 (By Cauchy's theorem, there exists a solution on some open time interval
278 (t_2, t_1) . However, there is no guarantee a priori that a solution exists over
279 the time interval $[t_0, t_1]$, because the solution might diverge to infinity before
280 reaching t_0 by decreasing values.)

281 The full theorem is:

282 **Theorem 5**

283 1. *The Riccati Eq. (20) has a solution $P(t) \geq 0$ over $[t_0, t_1]$ for every*
284 *$t_0 < t_1$.*

285 2. *The solution of the linear quadratic optimal control problem is given in*
286 *state feedback form by*

$$u(t) = -C(t)x(t), \quad C(t) = R(t)^{-1}G'(t)P(t), \quad (21)$$

287 3. *and the optimal value of the performance index is*

$$V^0(x_0, t_0, t_1) = \|x_0\|_{P(t_0)}^2. \quad (22)$$

288 We emphasize that the Riccati Eq. (20) and the optimal feedback gain in
289 Eq. (21) are the duals of the Riccati equation and gain associated with the
290 Kalman filter (see section 3.2)

291 **3.1.2 Optimal regulator (infinite-horizon) problem**

292 For the sake of simplicity, we give here only the linear time-invariant version,
293 which is the only one that the previous theory could handle. Kalman (1960a)
294 however gave also the solution for a non time-invariant problem.

295 **Optimal regulator problem** Given the time-invariant linear system Eq. (1)
296 and (2) with initial state $x(0) = x_0$, and a non-negative definite $p \times p$ matrix

297 Q and a positive definite $m \times m$ matrix R , find the control, if it exists, that
 298 minimizes the performance index

$$V(x_0; u(\cdot)) = \int_0^\infty (\|y(t)\|_Q^2 + \|u(t)\|_R^2) dt. \quad (23)$$

299 This study requires the introduction of both controllability and observability.
 300 Indeed, in his introduction, Kalman states that “the principal contribution
 301 of the paper lies in the introduction and exploitation of the concepts of
 302 *controllability and observability.*”

303 Retrospectively, he could also have quoted the Riccati Eq. (20). We use
 304 here its algebraic version, where all matrices are now constant:

$$PF + F'P - PGR^{-1}G'P + H'QH = 0. \quad (24)$$

305 The full theorem is:

306 **Theorem 6**

307 1. If the pair (F, G) is completely controllable, then

308 (a) the solution $P(t)$ of the Riccati Eq. (20) has a limit \bar{P} as $t \rightarrow -\infty$,
 309 which solves the algebraic Riccati Eq. (24),

310 (b) the solution of the optimal regulator problem in state feedback form
 311 is

$$u(t) = -Cx(t), \quad C = R^{-1}G'\bar{P}, \quad (25)$$

312 (c) and the optimal value of the performance index is

$$V^0(x_0) = \|x_0\|_{\bar{P}}^2 \quad (26)$$

313 (it is actually sufficient that the pair (F, G) is stabilizable, that is,

314 $\exists D : F - GD$ stable);

315 2. If furthermore the pair (H, F) is completely observable and $Q > 0$, \bar{P} is

316 positive definite and the system governed by Eq. (25) is asymptotically

317 stable (for the stability result, it is sufficient that $Q \geq 0$ and $(Q^{1/2}H, F)$

318 is detectable, that is, $\exists L : F - LQ^{1/2}H$ stable).

319 The duality we pointed out in the finite-horizon problem holds here, making

320 the optimal regulator dual to the stationary Kalman filter, that is, to a

321 realization of the Wiener filter.

322 3.1.3 Discrete-time case

323 Kalman (1960b, c), building on Kalman and Koepcke (1958: Appendix),

324 which dealt with sampled data control (which, as we mentioned before, con-

325 sists of considering continuous-time systems only at discrete instants), of a

326 continuous-time system provides the equivalent discrete-time results. The

327 system is made of Eq. (3) and (4), and the performance index is:

$$V(x_0, t_0, t_1; \{u(\cdot)\}) = \|x(t_1)\|_A^2 + \sum_{t=t_0}^{t_1-1} (\|y(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2). \quad (27)$$

328 The Riccati differential equation is replaced by the so-called discrete Riccati
329 equation (the system matrices may all depend on time) or its “algebraic”
330 version (with all system matrices constant), where $P(t) = P(t+1) = \bar{P}$:

$$\begin{aligned} P(t) &= F'P(t+1)F - F'P(t+1)G(G'P(t+1)G + R)^{-1}G'P(t+1)F + H'QH, \\ P(t_1) &= A. \end{aligned} \quad (28)$$

331 The optimal feedback control is

$$u(t) = -C(t)x(t) \quad \text{and} \quad C(t) = (G'P(t+1)G + R)^{-1}G'P(t+1)F. \quad (29)$$

332 Both the finite- and the infinite-horizon results follow exactly as for the
333 continuous-time case, with the same controllability and observability con-
334 ditions.

335 **3.2 The Kalman Filter**

336 For the sake of completeness, and to stress duality, we briefly review the
337 Kalman filter (Kalman, 1960b; Kalman and Bucy, 1961). Existence and
338 stability properties for both the finite and the infinite horizon cases derive
339 directly from those of the dual linear-quadratic control problem.

340 **3.2.1 Discrete time**

We start with the discrete-time problem, after Kalman (1960b) (where there is neither added noise in the measurement equation nor control). We consider a discrete-time linear system excited by *white noise* and a known control $u(\cdot)$:

$$x(t+1) = F(t)x(t) + G(t)u(t) + D(t)v(t), \quad x(t_0) = x_0, \quad (30)$$

$$y(t) = H(t)x(t) + w(t), \quad (31)$$

341 where $(v(t), w(t))$ is a Gaussian random variable with zero mean and known
 342 covariance, independent of all the $(v(\tau), w(\tau))$ for $\tau \neq t$. In the simplest
 343 case, $v(t)$ is also independent of $w(t)$, but this is not necessary to the theory.
 344 A possible non-zero cross-correlation between $v(t)$ and $w(t)$ is dual to the
 345 presence of a cross term $x'Su$ in the quadratic performance index of linear-
 346 quadratic control. The noise is characterized by its covariance matrix (with
 347 $\delta_{t,\tau}$ the Kronecker symbol):

$$\mathbb{E} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} (v'(\tau) \ w'(\tau)) = \begin{pmatrix} V(t) & 0 \\ 0 & W(t) \end{pmatrix} \delta_{t,\tau}. \quad (32)$$

348 The initial state is also a Gaussian random variable of known distribution:

$$\mathbb{E}(x(t_0)) = \hat{x}_0, \quad \mathbb{E}(x(t_0) - \hat{x}_0)(x(t_0) - \hat{x}_0)' = \Sigma_0. \quad (33)$$

349 The problem is to compute the conditional mathematical expectation:

$$\hat{x}(t) = \mathbb{E}(x(t)|y(\tau), \tau < t). \quad (34)$$

350 The solution is of the form of Eq. (15) initialized at $\hat{x}(t_0) = \hat{x}_0$, where the gain

351 K is given through the error covariance matrix $\Sigma(t) = \mathbb{E}(x(t) - \hat{x}(t))(x(t) -$

352 $\hat{x}(t))'$, which is solution of the discrete Riccati Eq. (35), which is dual of

353 Eq. (28), and by the formula in Eq. (36), which is dual of Eq. (29):

$$\Sigma(t+1) = F\Sigma(t)F' - F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}H\Sigma(t)F' + DVD', \quad (35)$$

354

$$K(t) = F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}. \quad (36)$$

355 The time-invariant infinite-horizon case is the internal form of the Kolmogorov

356 filter.

357 **Continuous-time** Given a continuous-time system in internal form excited

358 by “white noises” (quotes because such a thing does not exist rigorously; it

359 is an “engineering” presentation) involved both in the dynamics and in mea-

360 surement, as in the previous subsection on discrete time, but in continuous

361 time, with

$$\mathbb{E} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} (v'(\tau) \ w'(\tau)) = \begin{pmatrix} V(t) & 0 \\ 0 & W(t) \end{pmatrix} \delta(t - \tau) \quad (37)$$

362 (we use the Dirac δ as in Kalman and Bucy (1961), but probabilists nowa-
 363 days write the system differently, with a different ontology, in terms of dif-
 364 fusions, Itô calculus, and without “white noise”), the conditional expectation
 365 in Eq. (34) we are looking for is the solution of the continuous-time observer
 366 of Eq. (16) with the dual formulas from linear-quadratic control:

$$\dot{\Sigma} = F\Sigma + \Sigma F' - \Sigma H'W^{-1}H\Sigma + DVD', \quad \Sigma(t_0) = \Sigma_0. \quad (38)$$

367 and

$$K(t) = \Sigma(t)H'W^{-1}. \quad (39)$$

368 The time-invariant infinite-horizon case coincides with the internal represen-
 369 tation of the Wiener filter, which, like the Kolmogorov filter, was given in
 370 external form (Wiener, 1949).

371 **3.3 The separation theorem**

372 The control laws in Eq. (21), (25), or (29) are based on the exact measure-
 373 ment of the state $x(t)$. However, the underlying assumption of this theory
 374 is that only the output $y(t)$ is measured. The natural idea, then, is to as-
 375 sociate a Kalman filter with the optimal linear-quadratic control law in an
 376 “optimum” observer-controller $u(t) = -C\hat{x}(t)$. This idea was proposed by

377 Kalman (1960a, c), who suggested that the duality principle legitimizes the
378 association of the Kalman filter estimate with the optimal linear-quadratic
379 gain. He was then anticipating the separation theorem. We now relate the
380 results published in the year 1960 with the results that were to come later.

381 Optimality is a consequence of the *separation theorem*. This theorem was
382 first proved by Joseph and Tou (1961) in discrete time, with no observation
383 noise, as in Kalman (1960b), and, dually, no control cost, for a non-quadratic
384 performance index. It was proved in continuous time by Wonham (1968). An
385 early proof, more specific to the linear quadratic Gaussian case, was also due
386 to Faurre (1968). “Certainty equivalence” results, in some particular cases
387 with perfect state information and without formulation in terms of system
388 theory, had appeared earlier in the economic literature (Simon, 1956; Theil,
389 1957).

390 The continuous-time problem is much more difficult, because part of the
391 problem is to state the problem precisely. This involves continuous-time
392 Brownian motions, Itô calculus, filtrations, and measurability. As Kalman in
393 a conference on applications of the Kalman filter in hydrogeology, hydraulics,
394 and water resources (1978) put it: “There are three types of filters: (i) those
395 which keep tea leaves from falling into the tea cup, (ii) those we are talking

396 about today, (iii) those which are so fancy that only topologists use them.”
397 We will not attempt to present this statement in modern terms, but rather
398 stick to the form devised for engineers in the early sixties. Our purpose is
399 to set firm grounds to devise a feed-back dynamic compensator (paragraph
400 4.1.3 below.)

401 **Linear quadratic Gaussian (LQG) stochastic optimal control prob-**
402 **lem**

403 Given a linear system in internal form with additive Gaussian white ran-
404 dom disturbances in the dynamics and output equations, find a control law
405 $u(\cdot)$, if it exists, where $u(t)$ depends only on past outputs $y(\tau)$, for $\tau < t$, that
406 minimizes the mathematical expectation of a quadratic performance index
407 among all such control laws.

408 The answer is the *separation and certainty equivalence theorem*, holding
409 true for discrete time and continuous time, finite-horizon problems, and sta-
410 tionary infinite-horizon problems:

411 **Theorem 7** *The solution of the linear quadratic Gaussian stochastic opti-*
412 *mal control problem exists and is obtained by replacing the state $x(t)$ by the*
413 *Kalman filter estimate $\hat{x}(t)$ in the linear quadratic deterministic optimal con-*
414 *trol state feedback law. (Positive terms however must be added to Eq. (22)*

415 *for the optimal criterion value.)*

416 **4 Applications**

417 **4.1 Compensator design in engineering**

418 **4.1.1 Linear and linearized control systems**

419 Already in the “pre-Kalmanian” era, the linear time-invariant theory was used
420 in various physical systems. Some had reasonable linear physical models,
421 which were time-invariant when considered at the steady state. However,
422 most industrial systems such as transportation or energy systems, demand
423 a nonlinear model. The engineering practice, then, is to define a desired or
424 *nominal* output trajectory, and take the *error signal* as the output of the
425 control system, that is, the difference between the actual and the desired
426 outputs. The objective of the control system is then to keep this error signal
427 close to zero thanks to a *dynamic compensator*, which is a dynamic system
428 whose input is the measured error signal, and the output the control input
429 of the to-be-controlled system (that is, a feedback system, as understood by
430 Wiener.)

431 In order to achieve this goal, one builds a linear model as the linearization

432 of the nonlinear model for small deviations around the nominal trajectory.
433 This can be done either on the basis of an analytic nonlinear model linearized
434 by a first-order expansion, or of experiments using further parts of the theory
435 (such as the consideration of cross-correlations between input and output
436 pseudo-random small deviations).

437 Because the model is an approximation of the real physical system, the
438 variables in the model are approximations of the physical variables. This is
439 the “paradox of linear control theory;” the error stays close to zero because
440 the control is efficient; the control is efficient because the system is well
441 approximated by the model; the system is well approximated by the model
442 because the linear approximation is good; the linear approximation is good
443 because the error stays close to zero. The errors, which are approximated
444 by the variables appearing in the control model, are then to be kept close to
445 zero, in spite of disturbances in the dynamics, measurement errors, lack of
446 direct measurement of some key variables, not to mention modeling errors
447 and biases. These errors and biases differ from “noises” —here not a well-
448 defined concept—, and were at the inception of “robust control” theories, and
449 for our purpose, \mathcal{H}_∞ -optimal control by Zames (1981).

450 4.1.2 Observer-controller: algebraic procedure

451 Keeping a steady-state variable close to zero is achieved by forcing the system
452 to be sufficiently stable. In that process, the poles of the transfer function,
453 that is, the eigenvalues of the internal description of the overall system, are
454 key information. The reason is that, for continuous-time systems, the real
455 parts of these eigenvalues, which must be negative to insure stability, reflect
456 the degree of stability. The imaginary parts measure the oscillatory behavior
457 of the response of the system. (In discrete time, their modulus must be less
458 than one to insure stability.)

459 As we mentioned in subsection 2.1.1, engineers had developed advanced
460 tools beyond the mere inspection of the poles of the transfer function. These
461 poles however remain of paramount importance, which justifies the use of
462 the theorems on the separation of dynamics and on pole shifting, as well
463 as the fact that the practitioner chooses the poles of the observer and the
464 controller, such that these poles are independent of one another (this is the
465 principle of separation of the dynamics). A rule of thumb is that the observer
466 must be an order of magnitude faster than the controller. Trials and errors
467 with the localization of the poles using simulation models (either linearized
468 or nonlinear if available) allow one to construct an efficient control device.

469 4.1.3 Quadratic synthesis

470 Advancement of modeling science, which was largely due to digital comput-
471 ers, allowed the treatment of an increasingly high dimension of the models,
472 to a point that simple methods based on the location of poles were no longer
473 practical. For example, the rigid body dynamics of a landing airplane are de-
474 scribed by a 12th-order system. Adding engines and control surface dynamics
475 and flexible modes in modern airliners has pushed the order way beyond the
476 12th one.

477 Moreover, some problems, such as automatic landing of an airplane or
478 control of an industrial baking cycle by heating and cooling an oven, intrin-
479 sically have their time horizon finite and are not linear time invariant, with
480 emphasis sometimes on terminal error control. These problems are beyond
481 the reach of algebraic methods.

482 Engineers may have a fairly precise idea of the origin and magnitude of
483 the disturbances occurring in the dynamics and in the sources of error during
484 measurements, let alone in biases inherent in the modeling. This provides a
485 sound basis for using the Kalman filter to compute an observer's gain.

486 As for the controller, the control gain is computed through a quadratic
487 performance index and the Kalman optimal gain. It is tuned by searching

488 for satisfactory weighting matrices of the performance index by trials and
489 errors. This operation is made easy and efficient if we understand the gain
490 as the value minimizing this performance index. Then, one knows how the
491 different state and control variables will respond to a change in the weighting
492 matrices. This process, known as *quadratic synthesis*, is what made the new
493 theory so popular among engineers, to the point of being the main tool used
494 in designing the control systems for the Apollo mission.

495 **4.2 Population ecology**

496 The use of Kalman filtering in estimating a population, such as in a census,
497 is nothing new. In the line of our presentation, we provide now an example
498 of the use of linear quadratic optimal control in a rudimentary model of
499 cooperation within an animal population. We show how the individual quest
500 of self interest may sustain a collaborative behavior. We use this model to
501 present several extensions of the linear quadratic theory, mainly:

- 502 • the problem comprises non-homogeneous terms,
- 503 • there is a time-discounted criterion,
- 504 • several players are involved, rather than a single optimizing one, and

505 there may exist a Nash equilibrium.

506 A large finite population of animals live in an area where individuals have
507 two actions at their disposal: improve the foraging environment or forage.
508 We have a colony of beavers constructing a dam in mind, for example. The
509 food intake individuals obtain for a given foraging effort is an increasing
510 function of environmental quality, and a decreasing one of total foraging
511 effort, which depletes the resource. The quality of the environment decays
512 exponentially when left unattended, but improves when animals invest in it,
513 with a decreasing marginal effect. It may also be slightly degraded by the
514 foraging effort.

515 We investigate the behavior at the Nash equilibrium among “selfish” in-
516 dividuals either over a season $[0, T]$ or over lifetime, with an exponentially
517 distributed random time of death. Many models of evolutionary dynamics
518 converge toward a Nash equilibrium, mainly so when, as here, this equilib-
519 rium is unique (Vincent and Brown, 2005; Sandholm, 2010: chap. 6).

520 We therefore introduce the scalar variables

- 521 • the quality x of the environment,

- 522 • the effort U_i produced by individual i to improve the (common) envi-

- 523 ronment, and $u_i = \sqrt{U_i}$ the decreasing marginal effect of effort,
- 524 • the foraging effort v_i of individual i ,
- 525 • the efficiency factor K of any individual's foraging effort, in terms of
- 526 energy intake,
- 527 and the (positive) parameters
- 528 • the population size n ,
- 529 • the natural rate f of decay of the environment quality,
- 530 • the coefficient g of total effort $\sum_k u_k$ and h of total effort $\sum_k v_k$ in the
- 531 dynamics governing environmental quality,
- 532 • the cost q of one unit of foraging effort and the cost r of one unit of
- 533 effort devoted to improve the environment per time unit, in terms of
- 534 energy consumption,
- 535 • the coefficients a and b in the affine function K (see below), and $\alpha =$
- 536 $a - q$ assumed positive,
- 537 • the coefficient ρ of the exponential mortality rate.

538 The model is:

$$\dot{x} = -fx + g \sum_k u_k - h \sum_k v_k, \quad (40)$$

539

$$K = a + x - b \sum_k v_k, \quad (41)$$

540 Each individual (each beaver) seeks to maximize its performance index,
541 which has the the same expression for the seasonal or for the lifetime opti-
542 mization. In the first case, ρ may (optionally) be taken equal to zero (if the
543 possibility of death during the season may be ignored); in the second case,
544 $T = \infty$ and the stationary theory applies:

$$J_i = \int_0^T e^{-\rho t} (Kv_i - qv_i - ru_i^2) dt. \quad (42)$$

545 We analyze this problem in the appendix. We deal with the non homo-
546 geneous character by introducing a non-homogeneous Value function, with
547 exponential discounting by introducing the same exponential term in the
548 Value function, and with the many-player problem by looking for a Nash
549 equilibrium, which we obtain by solving an optimization problem, where
550 each individual assumes that all other players use the same optimal strategy.

The result is that cooperation is sustained. We introduce two coefficients P and p (variable in the finite-horizon case, fixed in the infinite-horizon case),

and find that the equilibrium strategies are:

$$u^* = \frac{g}{r}(Px + p), \quad (43)$$

$$v^* = \frac{1}{(n+1)b}((1 - 2hP)x - 2hp + \alpha) \quad (44)$$

551 For suitable values of the parameters, and in particular if n is large enough,
552 Eq. (44) yields positive controls, with positive coefficients P and $(1 - 2hP)$,
553 as well as stable dynamics and positive values for the quality x .

554 5 Carrying on

555 Kalman's contributions in the years 1960 and 1961 were a powerful stimulus
556 for system theory and control research. Many researchers followed suit, both
557 to get further theoretical advances (such as those mentioned in sections 3.3
558 and 4.2.2), and to develop algorithms implementing those theoretical results.

559 Algebraic system theory attracted many researchers such as Wonham
560 (1967) and remained Kalman's main research area. It benefited from ad-
561 vanced algebraic tools and ideas coming from the theory of automata (Kalman,
562 1965; Kalman et al., 1969; Kalman, 1972)). The linear quadratic theory of
563 control and observation was deeply renewed by the theory of \mathcal{H}_∞ -optimal con-
564 trol, pioneered by Zames (1981) in the classical "external" frequency-domain

565 description, and later formulated in terms of time domain *à la* Kalman (Başar
566 and Bernhard, 1995), with a minimax probability-free treatment of uncer-
567 tainties, where the same duality shows up, in a more complex setup, and a
568 bit mysteriously.

569 The basic theory quickly appeared in all textbooks of engineering control,
570 and later in textbooks of many other domains (Weber, 2011). The algorithms
571 were coded into publicly available software packages such as Matlab or Scilab,
572 and they have been used in a wide range of application domains, well beyond
573 the industrial and transportation systems of the early times, encompassing
574 all branches of engineering as well as natural and bio-medical sciences.

575 Linear plus quadratic performance indices were used in the study of
576 animal cooperation (Brown and Vincent, 2008). Yet, the *dynamic* linear-
577 quadratic optimization theory is mostly absent from the literature on popu-
578 lation dynamics. Most models used are nonlinear (Clark and Mangel, 2000;
579 Pastor, 2008), such as the logistic growth model, but, as mentioned in para-
580 graph 4.1.1, this is also true in other domains where the linear-quadratic opti-
581 mization theory has had many applications. We have sketched here a possible
582 use of the linear-quadratic optimal control theory in population ecology.

583 References

584 Başar, T. and Bernhard, P. (1995). *H[∞]-Optimal Control and Related Mini-*
585 *max Design Problems: a Differential Games approach*. Boston: Birkhäuser,
586 second edition.

587 Bellman, R. E. (1957). *Dynamic Programming*. Princeton: Princeton Uni-
588 versity Press.

589 Birkhoff and Mac Lane (1967). *Linear Algebra*. London: AMS Chelsea Pub-
590 lishing.

591 Brown, J. S. and Vincent, T. L. (2008). Evolution of cooperation with shared
592 costs and benefits. *Proceedings of the Royal Society B*, 275: 1985–1994.

593 Carathéodory, C. (1935). *Variations-Rechnung und Partielle Differential-*
594 *Gleichungen erster Ordnung*. Leipzig: Tubner. Translated into
595 Carathéodory, C. (1999). *Calculus of Variations and Partial Differen-*
596 *tial Equations of the First Order (3rd edition)*. New York: Chelsea
597 Publishing Company.

598 Clark, C. W. and Mangel, M. (2000). *Dynamic State Variable Models in Ecol-*

- 599 *ogy, Methods and Applications*. Oxford Series in Ecology and Evolution.
600 New York, USA: Clarendon Press.
- 601 Davison, E. J. (1968). Comments on “On pole assignment in multi-input
602 controllable linear systems.” *IEEE Transactions on Automatic Control*,
603 *AC-13*: 747–748.
- 604 Faurre, P. (1968). Commande optimale stochastique et principe de séparation
605 de l’estimation et de la commande. *Revue du CETHEDEC*, *16*: 129–135.
- 606 Heymann, M. (1968). Comments on On pole assignment in multi-input con-
607 trollable linear systems. *IEEE Transactions on Automatic Control*, *AC-13*:
608 748–749.
- 609 Joseph, P. D. and Tou, J. T. (1961). On linear control theory. *Transactions*
610 *of the A.I.E.E. II, Applications and Industry*, *80*: 193–196.
- 611 Kalman, R. E. (1960a). Contributions to the theory of optimal control. *Bo-*
612 *letin de la Sociedad Matematica Mexicana. Proceedings of the symposium*
613 *on ordinary differential equations, Mexico City 1959*, *5*: 102–119.
- 614 Kalman, R. E. (1960b). A new approach to linear filtering and prediction

- 615 problems. *Transactions of the ASME - Journal of Basic Engineering*, 82:
616 35–45.
- 617 Kalman, R. E. (1960c). On the general theory of control systems. *Proceedings*
618 *of the First International Congress of the IFAC*, 1, 481–491.
- 619 Kalman, R. E. (1962). Canonical structure of linear dynamical systems. *Pro-*
620 *ceedings of the National Academy of Sciences of the United States of Amer-*
621 *ica*, 48: 596–600. Communicated by S. Lefschetz, January 23, 1961.
- 622 Kalman, R. E. (1963). Mathematical description of linear dynamical systems.
623 *SIAM Journal on Control*, 1: 152–192.
- 624 Kalman, R. E. (1965). Algebraic structure of linear dynamical systems, I—
625 The module of Σ . *Proceedings of the National Academy of Sciences of the*
626 *USA*, 54: 1503–1508.
- 627 Kalman, R. E. (1972). Kronecker invariants and feedback. In Leonard Weiss
628 (ed.), *Proceedings of the NRL-MRC Conference on Ordinary Differential*
629 *Equations, Washington D.C., June 14–23 1971*, 459–471. New York, Lon-
630 don: Academic Press.
- 631 Kalman, R. E. and Bertram, J. E. (1960a). Control system analysis and

- 632 design by the second method of Lyapunov. I—Continuous-time systems.
633 *Journal of Basic Engineering, 82 D*: 371–393.
- 634 Kalman, R. E. and Bertram, J. E. (1960b). Control system analysis and de-
635 sign by the second method of Lyapunov. II—Discrete-time systems. *Journal*
636 *of Basic Engineering, 82 D*: 394–400.
- 637 Kalman, R. E. and Bucy, R. S. (1961). New results in linear filtering and pre-
638 diction theory. *Transactions of the ASME - Journal of Basic Engineering,*
639 *1983*: 95–107.
- 640 Kalman, R. E., Falb, P. L., and Arbib, M. A. (1969). *Topics in Mathematical*
641 *System Theory*. New York: McGraw-Hill.
- 642 Kalman, R. E. and Koepcke, R. W. (1958). Optimal synthesis of linear sam-
643 pling control systems using generalized performance indexes. *Transactions*
644 *of the ASME, 80*: 1820–1826.
- 645 Kolmogorov, Andrei (1941). Interpolation and Extrapolation. *Bulletin of the*
646 *USSR Academy of Sciences, (Doklady Akademii Nauk USSR) Mathematics*
647 *5*: 3-14. (In Russian.)

- 648 Luenberger, D. G. (1964). Observing the state of a linear system. *IEEE*
649 *Transactions on Military Electronics*, 8: 74–80.
- 650 Mageirou, E. F. (1976). Values and Strategies for Infinite Time Linear
651 Quadratic Games, *IEEE Transactions on Automatic Control*, AC-21, 547–
652 550.
- 653 Merriam III, C. W. (1959). A class of optimum control systems. *Journal of*
654 *the Franklin Institute*, 267: 267–281.
- 655 Newton Jr, G. C., Gould, L. A., and Kaiser, J. F. (1957). *Analytical Design*
656 *of Linear Feedback Controls*. NY: John Wiley & Sons.
- 657 Pastor, J. (2008). *Mathematical Ecology*. NY: Wiley-Blackwell.
- 658 Roos, C. F. (1925). Mathematical theory of competition. *American Journal*
659 *of Mathematics*, 46: 163–175.
- 660 Sandholm, W. H. (2010). *Population Games and Evolutionary Dynamics*.
661 Cambridge (MA): The MIT Press.
- 662 Simon, H. A. (1956). Programming under uncertainty with a quadratic cri-
663 terion function. *Econometrica*, 24: 74–81.

- 664 Theil, H. (1957). A note on certainty equivalence in dynamic planning.
665 *Econometrica*, 25: 346–349.
- 666 Vincent, T. L. and Brown, J. S. (2005). *Evolutionary Game Theory, Natural*
667 *Selection, and Darwinian Dynamics*. Cambridge: Cambridge University
668 Press.
- 669 Weber, T. A. (2011). *Optimal Control with Applications in Economics*. Cam-
670 bridge (MA): The MIT Press.
- 671 Wiener, N. (1949). *Extrapolation, Interpolation and Smoothing of Stationary*
672 *Time Series*. New York: John Wiley and Sons.
- 673 Willems, J. C. (1971). Least squares stationary optimal control and the alge-
674 braic Riccati equation. *IEEE Transactions on Automatic Control*, AC 16:
675 621–634.
- 676 Wonham, W. M. (1967). On pole assignment in multi-input controllable lin-
677 ear systems. *IEEE Transactions on Automatic Control*, AC-12: 660–665.
- 678 Wonham, W. M. (1968). On the separation theorem of stochastic control.
679 *SIAM Journal on Control*, 6: 312–326.

680 Zames, G. (1981). Feedback and optimal sensitivity: model reference trans-
681 formations, multiplicative seminorms, and approximate inverses. *IEEE*
682 *Transactions on Automatic Control, AC-26*: 301–320.

683 A A model in population ecology

684 A.1 The Nash equilibrium

685 A.1.1 The Hamilton-Jacobi-Carathéodory-Isaacs-Bellman (HJCIB) 686 equation

687 We assumed that all individuals are identical. Therefore they share a unique
688 Bellman (or rather Isaacs Value) function, which we seek of the form $V(t, x) =$
689 $\exp(-\rho t)W(t, x)$, with W a non-homogeneous quadratic function of x :

$$W(t, x) = P(t)x^2 + 2p(t)x + \pi(t), \quad (45)$$

690 Thus, the HJCIB equation is

$$-\dot{W} + \rho W = \max_{u_i, v_i} \left\{ 2(Px + p) \left(-fx(t) + g \sum_{k=1}^n u_k(t) - h \sum_{k=1}^n v_k \right) \right. \\ \left. + v_i (\alpha + x(t) - b \sum_{k=1}^n v_k(t)) - ru_i^2 \right\}. \quad (46)$$

We derive the equations for the maximizing strategies u^* and v^* as

$$ru^* = g(Px + p), \quad (47)$$

$$(n + 1)bv^* = (1 - 2hP)x + \alpha - 2hp. \quad (48)$$

691 Putting these expressions back into the HJCIB Eq. (46) and equating the
692 coefficients of the same powers of x , we obtain

$$\begin{aligned} -\dot{P} &= \left((2n - 1)\frac{g^2}{r} + \frac{4n^2h^2}{(n+1)^2b} \right) P^2 - 2\left(f + \frac{\rho}{2} + \frac{(n^2+1)h}{(n+1)^2b} \right) P + \frac{1}{(n+1)^2b}, \\ -\dot{p} &= \left(-f - \rho + (2n - 1)\frac{g^2}{r} P + \frac{4n^2h^2P - (n^2+1)h}{(n+1)^2b} \right) p + \frac{1 - (n^2+1)hP}{(n+1)^2b} \alpha, \\ -\dot{\pi} &= -\rho\pi + \left((2n - 1)\frac{g^2}{r} - \frac{2h\alpha}{b} \right) + \frac{(\alpha + 2nhp)^2}{(n+1)^2b}. \end{aligned} \quad (49)$$

693 These differential equations are to be integrated backward (numerically) from
694 the terminal conditions (at T), which are equal to zero. The equation for P
695 is decoupled from those for p and for π . Once P is computed, the equation
696 for p is linear, and $\exp(-\rho t)\pi(t)$ is obtained as an integral.

697 A.1.2 Analysis in finite horizon

698 The equation for P is a Riccati equation $\dot{P} = -\Pi(P)$, where Π is a second-
699 degree polynomial with two positive roots if these roots are real. The dis-
700 criminant of Π is always positive if

$$f + \frac{\rho}{2} \geq \left(\frac{(2n - 1)g^2}{(n + 1)^2br} \right)^{\frac{1}{2}}, \quad (50)$$

701 (the right-hand side goes to zero as n increases) and positive otherwise if h
702 is larger than a limit value that decreases as n increases. At terminal time
703 T , P reaches 0 with a negative slope; as $t \rightarrow -\infty$, $P(t)$ converges toward
704 the smallest (real) root \bar{P} of Π if it exists. It diverges to $+\infty$ if not. (The
705 possibility of P diverging, contrary to what is stated in Theorem 6, is due
706 to a cross term xv in the criterion.) Therefore $P(t)$ remains positive, and
707 smaller than \bar{P} when it exists. \bar{P} goes to zero as $1/n^2$ when n increases to
708 infinity.

709 For h larger than a limit h^* (which goes to zero as n goes to ∞), the
710 quantity

$$\Pi \left(\frac{1}{(n^2 + 1)h} \right) = \frac{(2n - 1)g^2}{(n^2 + 1)^2 r} \times \frac{1}{h^2} - 2 \frac{f + \frac{e}{2}}{n^2 + 1} \times \frac{1}{h} - \frac{1}{(n + 1)^2 b} \quad (51)$$

711 is negative, ensuring that a) Π has real roots, and b) $1 - (n^2 + 1)hP$ and a
712 fortiori $1 - 2hP$ are positive for all t . The equation for $-\dot{p}$ in Eq. (49) ensures
713 that p is also positive for all t and smaller than a limit value \bar{p} , which goes
714 to zero as $1/n$ when $n \rightarrow \infty$. Hence, for n large enough, if x is positive, so
715 are u^* and v^* .

716 Finally, the dynamics of environment quality under the equilibrium strate-

gies are:

$$\dot{x} = \left(-f - \frac{nh}{(n+1)b} + \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b} \right) P \right) x + \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b} \right) p - \frac{nh\alpha}{(n+1)b}. \quad (52)$$

Therefore, x remains positive provided that α is smaller than a limit value, which is easy to find. (Actually, if $\alpha < 0$ but $x + \alpha > 0$, although $p < 0$, we still have $u^* > 0$.)

A.1.3 Analysis in infinite horizon

In infinite horizon, if the algebraic Riccati Eq. (24) has real roots, then all coefficients P , p , and π are constant and equal to their limit values given by the algebraic equations obtained by setting the derivatives to zero. Closed form solutions are easy to write and useful for an efficient numerical implementation, but otherwise of little interest.

Existence of the positive limit value \bar{p} requires that $1 - (n^2 + 1)h\bar{P} > 0$, which is a condition we have already examined, and

$$f + \rho + \frac{(n^2 + 1)h}{(n + 1)^2b} > \left((2n - 1)\frac{g^2}{r} + \frac{4n^2h^2}{(n + 1)^2b} \right) \bar{P}, \quad (53)$$

again a condition which is satisfied, provided n is large.

730 The dynamics of environment quality are stable provided that

$$f + \frac{nh}{(n+1)b} > \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b} \right) \bar{P}, \quad (54)$$

731 again a condition which is satisfied provided that n is large enough. It also
732 ensures that $e^{-\rho t}x^2$ is asymptotically stable, and that this stability is pre-
733 served if some of the players play $u_i = 0$. Both are necessary conditions for
734 the Nash equilibrium in infinite horizon to exist (Mageirou, 1976).

735 **A.2 Time domain versus frequency domain**

736 With this elementary example, our main objective was to show

- 737 1. that Kalman's standard linear-quadratic theory can be extended in
738 various directions, and
- 739 2. that some dynamic population ecology models may lead to such prob-
740 lems.

741 The infinite-horizon problem in a one-player version with $h = 0$ could be
742 treated with the pre-Kalmanian theory of Newton Jr et al. (1957). In that
743 case indeed, the v_i have no dynamic effect and may be chosen so as to max-
744 imize the integrand. Thus the problem becomes monovariable. The game

745 version was more problematic, although Roos (1925) obtained the Cournot-
746 Nash equilibrium of a linear quadratic differential game by the standard
747 variational methods of the calculus of variations. The case with $h \neq 0$ was
748 solvable in principle, provided that one performs a spectral factorization for
749 the two-input multivariable system. The simplest way to achieve this task is
750 with the Riccati Eq. (24), revived by Kalman (Willems, 1971). The finite-
751 horizon seasonal problems were out of reach for this theory.