Admissible generalizations of examples as rules (2018)
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Admissible generalizations of examples as rules

Philippe Besnard\textsuperscript{1} — Thomas Guyet\textsuperscript{3} — Véronique Masson\textsuperscript{2,3}

\textsuperscript{1} CNRS-IRIT \quad \textsuperscript{2} Université Rennes-1 \quad \textsuperscript{3} IRISA/LACODAM

Machine Learning and Explainability – 8 October 2018
And now . . .

Introduction

Formalizing rule learning
- General formalism
- Notations

Admissibility for generalization
- Formalizing admissibility
- Classes of choice functions

Application to an analysis of CN2

Conclusion
Attribute-value rule learning

<table>
<thead>
<tr>
<th>(C)</th>
<th>(A_1)</th>
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</table>

- Task: induce rules to predict the value of the class attribute (C)
- Rules extracted by Algorithm CN2

\[
\pi_1^{CN2} : A_5 = \text{Downtown} \Rightarrow C = \text{expensive}
\]
\[
\pi_2^{CN2} : A_2 < 2.50 \land A_4 = \text{Toulouse} \Rightarrow C = \text{low-priced}
\]
\[
\pi_3^{CN2} : A_1 > 36.00 \land A_3 = D \Rightarrow C = \text{low-priced}
\]
Interpretability of rules and rulesets

- The logical structure of a rule can be easily interpreted by users

  \[ \text{IF } \text{conditions} \text{ THEN class-label} \]

- Rule learning algorithms generate rules according to implicit or explicit principles\(^1\)
  - are the generated rules the *interpretable* ones?
  - would it be possible to have different rulesets?
  - why a ruleset would be better than another one from the interpretability point of view?

⇒ We need ways to **analyze the interpretability of the outputs of rule learning algorithms**

---

\(^1\)principles mainly based on statistical properties!
Analyzing the interpretability of rules

Analyzing the interpretativeness of ruleset

- Objective criteria on ruleset syntax [CZV13, BS15]
  - size of the rule (number of attributes)
  - size of the ruleset
- Intuitiveness of rules through the effects of cognitive biases [KBF18]

⇒ Our approach formalizes rule learning and some expected properties on rules to shed light on properties of some extracted ruleset
Analyzing the interpretability of rules

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⇒ Our approach formalizes rule learning and some expected properties on rules to shed light on properties of some extracted ruleset

In this talk

- We present the formalisation of rule learning, and we focus on the generalization of examples as a rule
- We introduce the notion of admissible rule that attempts to capture an intuitive generalization of the examples
- We develop the example of numerical attributes
Impact of examples generalization on rule interpretability

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- Rules extracted by Algorithm CN2

\[ \pi^{CN2}_1 : A_5 = \text{Downtown} \Rightarrow C = \text{expensive} \]
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Impact of examples generalization on rule interpretability

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- **Rules extracted by Algorithm CN2**

\[ \pi_{1}^{CN2} : A_5 = \text{Downtown} \Rightarrow C = \text{expensive} \]
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Toward the notion of admissibility

- Rote learning of a rule
  \[ A_1 = \{75\} \Rightarrow C = \text{low-priced} \]

- Most generalizing rule
  \[ A_1 = [32 : 75] \Rightarrow C = \text{low-priced} \]

- Would the following rule be better?
  \[ A_1 = [32 : 40] \cup [70 : 75] \Rightarrow C = \text{low-priced} \]
Toward the notion of admissibility

- **Rote learning of a rule**
  \[ A_1 = \{75\} \Rightarrow C = \text{low-priced} \]

- **Most generalizing rule**
  \[ A_1 = [32 : 75] \Rightarrow C = \text{low-priced} \]

- **Would the following rule be better?**
  \[ A_1 = [32 : 40] \cup [70 : 75] \Rightarrow C = \text{low-priced} \]

⇒ this is the question of admissibility!
Toward the notion of admissibility

Overall, this is the question of admissibility!

The notion of admissibility has to capture an intuitive notion of generalization . . .
And now . . .

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At a glance

Rule learning is formalized by two main functions

- \( \phi \): selects possible subsets of data
- \( f \): generalizes examples as a rule (LearnOneRule process [Mit82])

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The attribute-value model

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</table>

Rows: items $x_1, x_2, \ldots, x_m$

Columns: attributes $A_1, A_2, \ldots, A_n$

$\forall i, j \ a_{j,i} \in \text{Rng} \ A_i$

$\text{Rng} \ A_i$ denotes the set of possible values for attribute $A_i$
Subsets of data to generalize

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<th></th>
<th>$A_1$</th>
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<tr>
<td>item 1</td>
<td>$a_{1,1}$</td>
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<td>item 2</td>
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<td>$a_{3,9}$</td>
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<td>item 4</td>
<td>$a_{4,1}$</td>
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<td>$a_{4,3}$</td>
<td>$a_{4,4}$</td>
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<td>item 5</td>
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<td>item 7</td>
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<td>$a_{7,9}$</td>
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<tr>
<td>item 8</td>
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<td>$a_{9,6}$</td>
<td>$a_{9,7}$</td>
<td>$a_{9,8}$</td>
<td>$a_{9,9}$</td>
</tr>
</tbody>
</table>

$\Rightarrow$ "Square" = selection of rows and columns in the data
A rule $\pi$ expresses constraints (for a generic item $x$) which lead to conclusion $C(x)$ (class which the item belongs to)

$$\pi : A_1(x) \in \nu_1^\pi \land \cdots \land A_n(x) \in \nu_n^\pi \rightarrow C(x) \in \nu_0^\pi$$

(*)

where

$$\left\{ \begin{array}{l} \nu_i^\pi \subseteq \text{Rng } A_i \text{ pour } i = 1, \ldots, n, \\ \nu_0^\pi \subseteq \text{Rng } C. \end{array} \right. $$

attributes $i = 1, \ldots, n$ without constraints are such that $\nu_i^\pi = \text{Rng } A_i$. 
And now . . .

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Conclusion
Eliciting a rule

- $S$ being a square is supposed to capture a rule $\pi$ requires that every item of $S$ satisfies $\pi$.
- Generalisation does not capture the statistical representativeness of dataset, but only elicits a rule generalizing all its items.

$$f \downarrow \quad A_0 = 2 \land A_4 = \text{Toulouse} \Rightarrow C = \text{low-priced}$$

$$f \downarrow \quad A_0 \in [2, 4] \Rightarrow C \in \{\text{low-priced}, \text{expensive}\}$$
Eliciting a rule (\( f \) function)

For every attribute \( A_i \), \( S_i \) is the set of values of \( A_i \) in items of \( S \)

<table>
<thead>
<tr>
<th>((A_0)) Price</th>
<th>((A_1)) Area</th>
<th>((A_2)) Rooms</th>
</tr>
</thead>
<tbody>
<tr>
<td>low-priced</td>
<td>70</td>
<td>2</td>
</tr>
<tr>
<td>low-priced</td>
<td>75</td>
<td>4</td>
</tr>
<tr>
<td>low-priced</td>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>low-priced</td>
<td>40</td>
<td>2</td>
</tr>
</tbody>
</table>

\( S_0 = \{\text{low-priced}\} \)
\( S_1 = \{32, 40, 70, 75\} \)
\( S_2 = \{2, 4\} \)
Eliciting a rule \((f\ function)\)

| \(A_0\) \((A_1)\) \((A_2)\) |
| --- | --- | --- |
| **Price** | **Area** | **Rooms** |
| 1 | low-priced | 70 | 2 |
| 2 | low-priced | 75 | 4 |
| 4 | low-priced | 32 | 2 |
| 7 | low-priced | 40 | 2 |

\(S_0 = \{\text{low-priced}\}\)
\(S_1 = \{32, 40, 70, 75\}\)
\(S_2 = \{2, 4\}\)

\(\hat{S}_0\) \(\hat{S}_1\) \(\hat{S}_2\)

- For every attribute \(A_i\), \(S_i\) is the set of values of \(A_i\) in items of \(S\).
- Each superset of \(S_i\) is, theoretically speaking, a generalization of \(S_i\).
Eliciting a rule ($f$ function)

<table>
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<th>(A₀) Price</th>
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<td>2</td>
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- For every attribute $A_i$, $S_i$ is the set of values of $A_i$ in items of $S$
- Each superset of $S_i$ is, theoretically speaking, a generalization of $S_i$
- The generalisation process thus consists in selecting one of these supersets:

**$f$ choice function** that is given as input a collection of supersets of $S_i$ and picks one

We are looking for an appropriate $\hat{\cdot}$ for ($\S$) i.e.

$$A_1(x) \in \hat{S}_1 \land \cdots \land A_n(x) \in \hat{S}_n \rightarrow C(x) \in \hat{S}_0$$  \hspace{1cm} (§)
Eliciting a rule \((f\) function\)

<table>
<thead>
<tr>
<th>((A_0))</th>
<th>((A_1))</th>
<th>((A_2))</th>
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Each superset of \(S_i\) is, theoretically speaking, a generalization of \(S_i\)

The generalisation process thus consists in selecting one of these supersets:

\[
f\text{ choice function that is given as input a collection of supersets of } S_i \text{ and picks one}
\]

We are looking for an appropriate \(\hat{\cdot}\) for (§) i.e.

\[
A_1(x) \in \hat{S}_1 \land \cdots \land A_n(x) \in \hat{S}_n \rightarrow C(x) \in \hat{S}_0 \tag{§}
\]

Generalization of \(S_i\): \(\hat{S}_i = f(\{Y | S_i \subseteq Y \subseteq \text{Rng } A_i\})\)
Notion of admissibility: propositions

Generalization of $S_i$: $\hat{S}_i = f(\{Y \mid S_i \subseteq Y \subseteq \text{Rng } A_i\})$

What collection $\mathcal{X} = \{\hat{S}_i \mid S_i \subseteq \text{Rng } A_i\}$ would do?

What choice function(s) can in practice capture these expected algebraic properties?
Notion of admissibility: propositions

Generalization of $S_i$: $\widehat{S}_i = f(\{Y \mid S_i \subseteq Y \subseteq \text{Rng } A_i\})$

What collection $\mathcal{X} = \{\widehat{S}_i \mid S_i \subseteq \text{Rng } A_i\}$ would do?

(i) $\text{Rng } A_i \in \mathcal{X}$

(ii) if $X$ and $Y$ are in $\mathcal{X}$ then so $X \cap Y$.

- $\mathcal{X}$ is a closure system upon $\text{Rng } A_i$.
- $\widehat{\cdot}$ is an operation enjoying weaker properties than closure operators; alternatives looked at:
  - pre-closure operator
  - capping operator

What choice function(s) can in practice capture these expected algebraic properties?
Notion of admissibility: propositions

Generalization of $S_i$: $\hat{S}_i = f(\{Y \mid S_i \subseteq Y \subseteq \text{Rng } A_i\})$

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  • pre-closure operator
  • capping operator

What choice function(s) can in practice capture these expected algebraic properties?

• Proposal for some classes of choice functions generating specific types of operators
• Concrete examples of such functions for numerical rules
Weakening closure operators

- List of Kuratowski’s axioms [Kur14] (closure system):

\[
\begin{align*}
\hat{\emptyset} &= \emptyset \\
S &\subseteq \hat{S} \subseteq \text{Rng } A_i \\
\hat{S} &= \hat{S} \\
S \cup S' \subseteq \hat{S} \cup \hat{S}' \quad \text{(pre-closure)}
\end{align*}
\]

- Actually, we downgrade Kuratowski’s axioms as follows

\[
\begin{align*}
\hat{S} &\subseteq \hat{S}' \text{ whenever } S \subseteq S' \quad \text{(closure)} \\
\hat{S} &= \hat{S}' \text{ whenever } S \subseteq S' \subseteq \hat{S} \quad \text{(cumulation)} \\
S \cup S' &\subseteq \hat{S} \text{ whenever } S' \subseteq \hat{S} \quad \text{(capping)}
\end{align*}
\]

Lemma: Kuratowksi \implies closure \implies cumulation \implies capping
Class of choice functions satisfying pre-closure

**Theorem.** Given a set $Z$, let $f : 2^{2^Z} \to 2^Z$ be a function st for every upward closed $\mathcal{X} \subseteq 2^Z$ and every $\mathcal{Y} \subseteq 2^Z$:

1. $f(2^Z) = \emptyset$
2. $f(\mathcal{X}) \in \mathcal{X}$
3. $f(\mathcal{X} \cap \mathcal{Y}) = f(\mathcal{X}) \cup f(\mathcal{Y})$

whenever $\bigcup \min(\mathcal{X} \cap \mathcal{Y}) = \bigcup \min \mathcal{X} \cup \bigcup \min \mathcal{Y}$

Then, $\hat{\cdot} : 2^Z \to 2^Z$ as defined by

$$\hat{\mathcal{X}} \overset{\text{def}}{=} f(\{ Y \mid \mathcal{X} \subseteq Y \subseteq Z \})$$

is a pre-closure operator upon $Z$.

**Intuition:** $Z$ is $\text{Rng } A_i$

$\mathcal{X}$ (and $\mathcal{Y}$, too) is a collection of intervals over $\text{Rng } A_i$

moreover, $\mathcal{X}$ is a collection containing all super-intervals of an interval belonging to the collection
Class of choice functions satisfying pre-closure

**Theorem.** Given a set $Z$, let $f : 2^2^Z \to 2^Z$ be a function st for every upward closed $\mathcal{X} \subseteq 2^Z$ and every $\mathcal{Y} \subseteq 2^Z$:

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is a pre-closure operator upon $Z$.

**Numerical attributes:** principle of single point ($u$) interpolation

$$A_i(x) \in [u - r : u + r] \to C(x) = c.$$
Class of choice functions satisfying capping

**Theorem.** Given a set $Z$, let $f : 2^2^Z \to 2^Z$ be a function st for every $\mathcal{X} \subseteq 2^Z$ such that $\bigcap \mathcal{X} \in \mathcal{X}$ and for every $\mathcal{Y} \subseteq 2^Z$

1. $f(\mathcal{X}) \in \mathcal{X}$

2. if $\mathcal{Y} \subseteq \mathcal{X}$ and $\exists W \in \mathcal{Y}$, $W \subseteq f(\mathcal{X})$ then $f(\mathcal{Y}) \subseteq f(\mathcal{X})$

Then, $\hat{\cdot} : 2^Z \to 2^Z$ as defined by

$$\hat{\mathcal{X}} \overset{\text{def}}{=} f(\{Y \mid X \subseteq Y \subseteq Z\})$$

is a capping operator upon $Z$.

**Intuition:** $Z$ is $\text{Rng } A_i$

$\mathcal{X}$ (and $\mathcal{Y}$, too) is a collection of intervals over $\text{Rng } A_i$

moreover, $\mathcal{X}$ is a collection whose intersection is itself a member of the collection
Class of choice functions satisfying capping

**Theorem.** Given a set $Z$, let $f : 2^Z \rightarrow 2^Z$ be a function st for every $\mathcal{X} \subseteq 2^Z$ such that $\bigcap \mathcal{X} \in \mathcal{X}$ and for every $\mathcal{Y} \subseteq 2^Z$

1. $f(\mathcal{X}) \in \mathcal{X}$
2. if $\mathcal{Y} \subseteq \mathcal{X}$ and $\exists W \in \mathcal{Y}, W \subseteq f(\mathcal{X})$ then $f(\mathcal{Y}) \subseteq f(\mathcal{X})$

Then, $\hat{\cdot} : 2^Z \rightarrow 2^Z$ as defined by

$$\hat{\mathcal{X}} \overset{\text{def}}{=} f(\{Y \mid X \subseteq Y \subseteq Z\})$$

is a capping operator upon $Z$.

**Numerical attributes:** principle of pairwise point interpolation
And now . . .

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Illustrations of the behaviour of CN2

Generation of synthetic data:

▶ Data with 2 dimensions: a numerical attribute and a symbolic class attribute
▶ Data with two classes (green and blue)

Objective:

▶ Illustrate the behaviour of the rule learning algorithm in terms of characteristics of generalisation of examples
  ▶ based on a pre-closure (interpolation over single points)
  ▶ based on a capping (interpolation over pairs of points)

Using Algorithm CN2 [CN89]
Separating interesting intervals

Comparison of rules obtained out of uniform distributions vs. normal distributions

- distance between two successive values are small wrt the range of the attribute
- mixing normal distributions causes disparate average distances (pairwise distance between examples)
- the second dataset can be viewed as a super set of the first dataset (add of examples in between examples)

Figure: Distributions of the data for the classes blue and green.
Separating interesting intervals

Figure: Distributions of the data for the classes blue and green.

Expected rules assuming capping or pre-closure for each class:

- **topmost dataset:**
  - \( v \in [-\infty: 15] \Rightarrow A_0 = \text{blue} \)
  - \( v \in [10: +\infty] \Rightarrow A_0 = \text{green} \)

- **bottom dataset:**
  - \( v \in [-\infty: 15] \Rightarrow A_0 = \text{blue} \)
  - \( v \in [0: +\infty] \Rightarrow A_0 = \text{green} \)
Separating interesting intervals

Figure: Distributions of the data for the classes blue and green.

Rules learned by CN2, topmost dataset:
- \( v \in [\infty : 10.03] \Rightarrow A_0 = \text{blue} \)
- \( v \in [12.73 : 14.83] \Rightarrow A_0 = \text{blue} \)
- \( v \in [10.65 : 12.81] \Rightarrow A_0 = \text{green} \)
- \( v \in [15.01 : +\infty] \Rightarrow A_0 = \text{green} \)

Rules learned by CN2, bottom dataset:
- \( v \in [\infty : 0.96] \Rightarrow A_0 = \text{blue} \)
- \( v \in [0.97 : 2.57] \Rightarrow A_0 = \text{blue} \)
- \( v \in [3.09 : 10.04] \Rightarrow A_0 = \text{blue} \)
- \( v \in [3.50 : 7.18] \Rightarrow A_0 = \text{green} \)
- \( v \in [11.55 : 13.14] \Rightarrow A_0 = \text{green} \)
- \( v \in [13.15 : +\infty] \Rightarrow A_0 = \text{green} \)
Does density influence the choice of boundaries?

Comparison of rules obtained out of well-separated uniform distributions, for two similar situations

- topmost dataset: same number of examples in both classes
- bottom dataset: the blue class is under-represented as compared to the green class

Figure: Distributions of the data for the classes blue and green.
Does density influence the choice of boundaries?

Observed behaviour:

- no difference between the extracted rules for either dataset
- CN2 systematically chooses the boundary to be the middle of the limits in between the two classes

Figure: Distributions of the data for the classes blue and green.
Does density influence the choice of boundaries?

Figure: Distributions of the data for the classes blue and green.

Behaviour from *capping*:
- adding extra examples can alter boundaries
  - to be insensitive to density of examples corresponds to a *cumulation* operator.

⇒ to be insensitive to density of examples corresponds to a *cumulation* operator.
And now . . .

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Conclusion (1)

- The logical structure of rules makes them easy to read but ...
- The interpretability of rules learned from examples requires, in particular, to take care of the way examples are generalized
  - Example of numerical attributes, but also symbolic attributes with structures (e.g. orders)
- Qualifying the interpretable nature of rule learning outputs is challenging

What can our approach do for rule interpretability

- Our work contributes by giving a way to do such analysis
  - A proposal of a general framework for rule learning
  - A topological study of admissible generalisations of examples
Conclusion (2)

Formalisation of rule learning

- $\phi$: selects possible subsets of data
- $\widehat{\cdot}$: elicits the rule

→ offer a framework for analysing rule learning algorithms

<table>
<thead>
<tr>
<th>(C) Price</th>
<th>(A1) Area</th>
<th>(A2) #Rooms</th>
<th>(A3) Energy</th>
<th>(A4) Town</th>
<th>(A5) District</th>
<th>(A6) Exposure</th>
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</thead>
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<tr>
<td>low-priced</td>
<td>70</td>
<td>2</td>
<td>D</td>
<td>Toulouse</td>
<td>Minimes</td>
<td></td>
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<tr>
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<td>4</td>
<td>D</td>
<td>Toulouse</td>
<td>Rangueil</td>
<td>Downtown</td>
</tr>
<tr>
<td>expensive</td>
<td>65</td>
<td>3</td>
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<td>Downtown</td>
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</table>
Admissible generalisation of examples

- Admissible generalisations resulting of a choice among the supersets of the examples
- Proposed topological property of the choice: closure-like operators (pre-closure, capping)
- Definition of classes of choice functions
  - Proposal of concrete choice functions upon numerical attributes
  - Can be generalized to symbolic attributes, including attributes with structure (e.g. total order)
Perspectives

- Long term objective: study the characteristics of extracted rulesets
  - Comparing the set of rules extracted by machine learning
- Need a formalism to represent a set of rules
  - A formalism that enables to represent
    - rules actually extracted by machine learning algorithms (e.g., Ripper, CN2, etc)
    - rules selected using a selection criteria (interestingness measures, etc)
- Formalize essential notions of rule learning
- The formalism will be a way to reason about the machine learning algorithms


