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16<sup>ème</sup> Journées de l'Hydrodynamique  
November 2018

## Domain decomposition methods for linearized Boussinesq type equations

J.G Caldas Steintraesser <sup>1,3</sup>    G. Kemlin <sup>2</sup>    A. Rousseau <sup>1,3</sup>

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<sup>2</sup>Inria Chile, Avenida Apoquindo 2827, Las Condes, Santiago, Chile

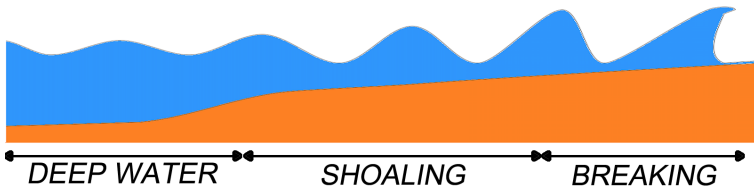
<sup>3</sup>MERIC, Avenida Apoquindo 2827, Las Condes, Santiago, Chile

# 1

## Introduction

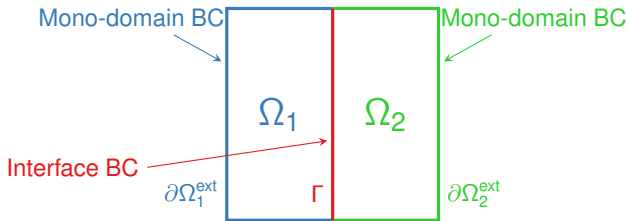
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- Domain Decomposition Methods (DDMs)
  - “Divide and conquer”;
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# Introduction

- Wave propagation in nearshore area: Boussinesq-type equations;
- Domain Decomposition Methods (DDMs)
  - “Divide and conquer”;
  - Main issue: interface boundary conditions (IBCs);
  - Future objective: couple different wave propagation models;
- Transparent Boundary Conditions (TBCs)
  - Numerical resolution in finite computational domain;
  - Optimal interface conditions for DDMs;
  - Cannot be computed exactly.

# Objectives

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- Derive discrete TBCs for the linearized equations;
- Test their efficiency as TBCs;
- Test their efficiency as IBCs in a DDM;
- Test their efficiency for the full nonlinear problem.

# The governing equations

- Full equations [Roeber and Cheung, 2012]:

$$\begin{cases} \eta_t + \nabla \cdot [(h + \eta) \mathbf{u}] + \nabla \cdot \left[ \left( \frac{z^2}{2} - \frac{h^2}{6} \right) h \nabla (\nabla \cdot \mathbf{u}) + \left( z + \frac{h}{2} \right) h \nabla (\nabla \cdot (h\mathbf{u})) \right] = 0, \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \eta + z \left[ \frac{z}{2} \nabla (\nabla \cdot \mathbf{u}_t) + \nabla (\nabla \cdot (h\mathbf{u}_t)) \right] + \boldsymbol{\tau} = 0. \end{cases}$$

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- Simplified framework:
  - 1D;
  - Flat bottom;
  - Linearization around  $(\bar{\eta}, \bar{\mathbf{u}}) = (0, 0)$ ;
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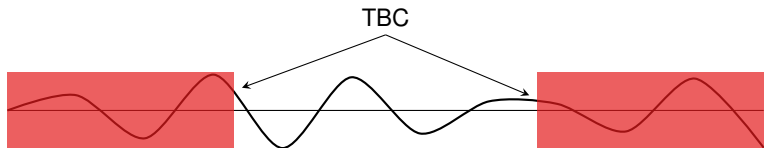
- Simplified framework:
  - 1D;
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  - Bottom shear stress  $\boldsymbol{\tau} = 0$ .
- Simplified equations:

$$\begin{cases} \eta_t + h_0 u_x + \tilde{h} u_{xxx} = 0, \\ u_t + g \eta_x + \bar{h} u_{xxt} = 0, \end{cases} \quad \implies \quad u_{tt} + \bar{h} u_{xxtt} - g h_0 u_{xx} - g \tilde{h} u_{xxxx} = 0$$

# 2

## Transparent Boundary Condi- tions

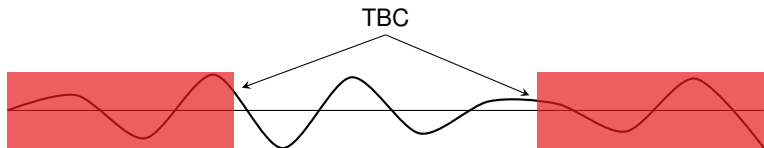
# Transparent Boundary Conditions (TBCs)



- Problems are well posed in an infinite domain, but the numerical resolution must be done in a finite computational domain;

$$(A) \begin{cases} \mathcal{L}(u) = 0, & \text{in } \mathbb{R} \\ u \rightarrow 0, & x \rightarrow \pm\infty \end{cases} \quad (B) \begin{cases} \mathcal{L}(u) = 0, & \text{in } ]0, L[ \\ \text{TBCs}, & \text{in } x = 0, L \end{cases}$$

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- TBCs assure that the solution of (A), restricted to  $[0, L]$ , coincides with the solution of (B).



# Difficulties

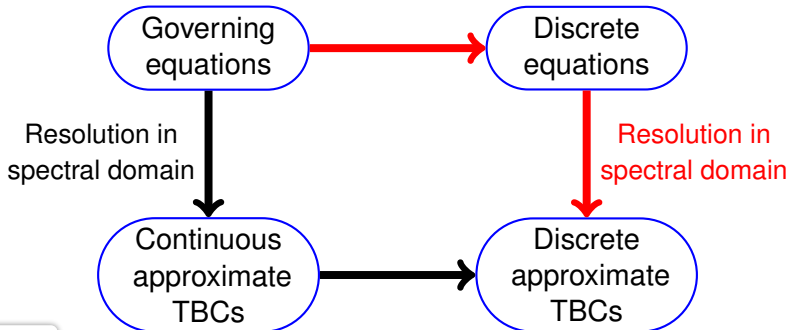
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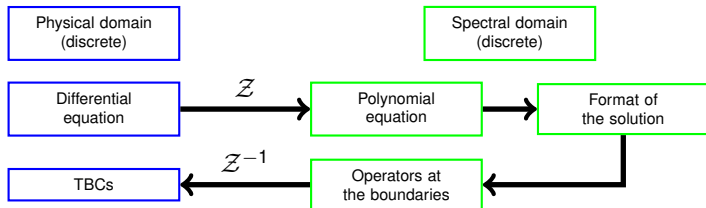
- In general, TBCs
  - Are non local in time and space;
  - Cannot be computed exactly, both analytically and numerically;
- Therefore, one must find approximate TBCs;
- Two usual approaches:



# Derivation of TBCs for the discrete equations

Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

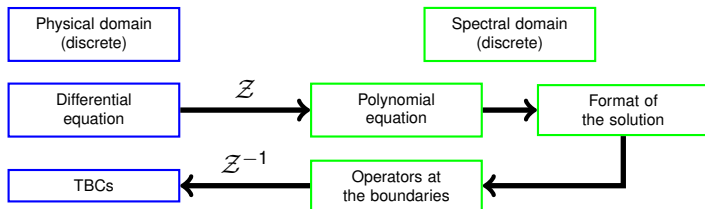
- Discretize the governing equations;



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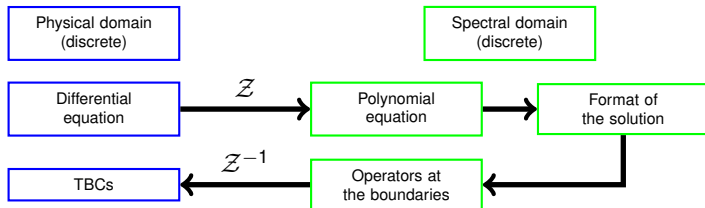
- Discretize the governing equations;
- Use the  $\mathcal{Z}$ -transform in the complementary set  $\mathbb{R} \setminus [0, L]$ ;



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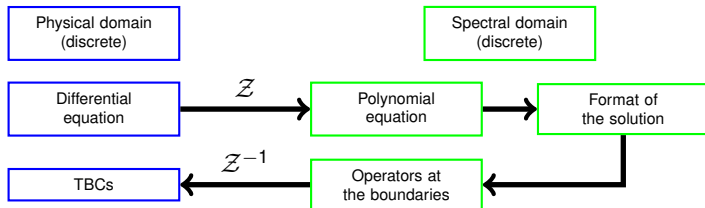
- Discretize the governing equations;
- Use the  $\mathcal{Z}$ -transform in the complementary set  $\mathbb{R} \setminus [0, L]$ ;
- Solve the problem in the  $\mathcal{Z}$ -space and find the conditions fulfilled by the solutions at the two boundaries;



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- Solve the problem in the  $\mathcal{Z}$ -space and find the conditions fulfilled by the solutions at the two boundaries;
- Use the inverse  $\mathcal{Z}$ -transform to find the TBCs.

$$\text{Left: } \begin{cases} \Gamma_1^l(\{u^n\}) := u_0^n - (Y_5 * u_1)^n + (Y_7 * u_2)^n = 0, \\ \Gamma_2^l(\{u^n\}) := u_0^n - (Y_6 * u_2)^n + 2(Y_9 * u_3)^n - (Y_8 * u_4)^n = 0, \end{cases}$$

where  $(Y_i^n)_{n \in \mathbb{N}}$  depend on the equation and  $(Y_i * u_j)^n = \sum_{m=0}^n Y_i^m u_j^{n-m}$ .



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Methodology [Arnold and Ehrhardt, 2001, Besse et al., 2015, Kazakova and Noble, 2017]:

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$$\text{Right: } \begin{cases} \Gamma_1^r(\{u^n\}) := u_j^n - (Y_1 * u_{j-1})^n + (Y_3 * u_{j-2})^n = 0, \\ \Gamma_2^r(\{u^n\}) := u_j^n - 2(Y_1 * u_{j-1})^n + (Y_2 * u_{j-2})^n - (Y_4 * u_{j-4})^n = 0, \end{cases}$$

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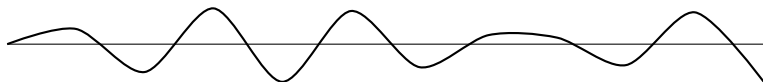
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- These operators are non local in time.

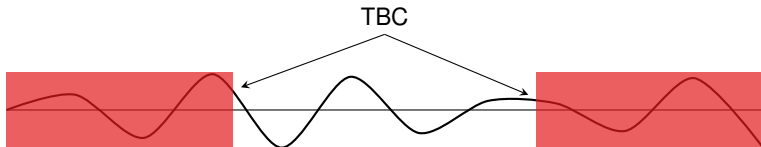
# Numerical tests

- Reference solution: computed in a bigger domain.



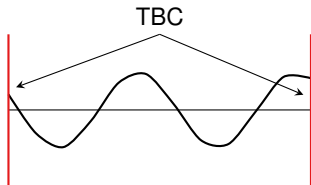
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## Numerical tests

- Reference solution: computed in a bigger domain.

| $t$  | $e^n$               |
|------|---------------------|
| 0.25 | $1.5 \cdot 10^{-9}$ |
| 0.5  | $9.5 \cdot 10^{-7}$ |
| 0.75 | $5.2 \cdot 10^{-6}$ |
| 1.0  | $8.4 \cdot 10^{-6}$ |

$$e_T = 4.5 \cdot 10^{-6}$$

$$e^n = \frac{\|u_{\text{ref}}(\cdot, t^n) - u_{\text{num}}(\cdot, t^n)\|_{L^2}}{\|u_{\text{ref}}(\cdot, t^n)\|_{L^2}},$$

$$e_T = \sqrt{\delta t \times \sum_{n=1}^{n_{\text{max}}} (e^n)^2}.$$

# 3

## Domain Decomposition Method

# Domain Decomposition Methods (DDMs)

- Idea: divide the domain in two or more subdomains (possibly overlapping) and solve the problem in each one of them.



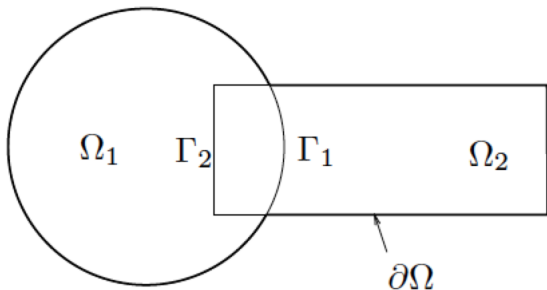
$$\Omega = \Omega_1 \cup \Omega_2, \quad \partial\Omega_1^{\text{int}} = \partial\Omega_1 \cap \Omega_2, \quad \partial\Omega_2^{\text{int}} = \partial\Omega_2 \cap \Omega_1$$

- Main difficulty: definition of the interface boundary conditions (IBCs) on  $\partial\Omega_1^{\text{int}}$  and  $\partial\Omega_2^{\text{int}}$  for the communication between the subdomains.



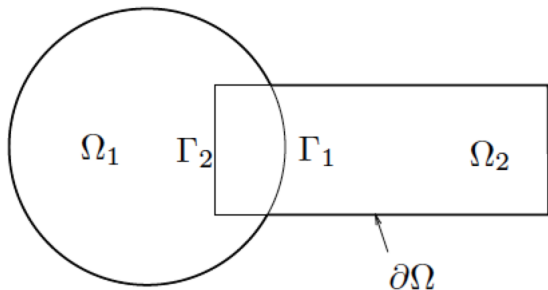
# The Schwarz method

- The first and one of the most important types of DDM [Gander, 2008];



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- The first and one of the most important types of DDM [Gander, 2008];
- Iterative method;



# The Schwarz method (1870)

- Equations solved in each subdomain (in parallel or additive form), for updating the solution to time  $t_{n+1}$ :

$$\begin{cases} \mathcal{L}(u_1^{n+1,k+1}) = 0, & \text{in } \Omega_1 \\ u_1^{n+1,k+1} = g, & \text{on } \partial\Omega_1^{\text{ext}}, \\ \mathcal{B}_1(u_1^{n+1,k+1}) = \mathcal{B}_1(u_2^{n+1,k}), & \text{on } \partial\Omega_1^{\text{int}} \end{cases}$$

$$\begin{cases} \mathcal{L}(u_2^{n+1,k+1}) = 0, & \text{in } \Omega_2 \\ u_2^{n+1,k+1} = g, & \text{on } \partial\Omega_2^{\text{ext}}, \\ \mathcal{B}_2(u_2^{n+1,k+1}) = \mathcal{B}_2(u_1^{n+1,k}), & \text{on } \partial\Omega_2^{\text{int}} \end{cases}$$

# The Schwarz method

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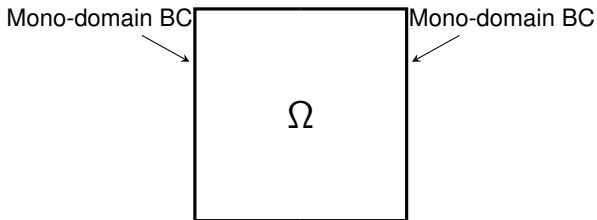
# The Schwarz method

- Optimal IBCs? **TBCs** [Japhet and Nataf, 2001]
  - Convergence to the solution of the monodomain;
  - Fastest convergence
- We implement the discrete TBCs as IBCs.



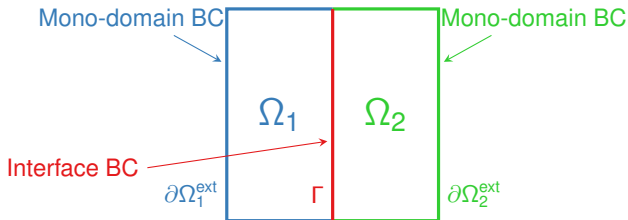
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  - DDM using Dirichlet IBCs (original Schwarz method):

$$\begin{cases} u_1^{n+1,k+1} = u_2^{n+1,k}, & \text{on } \partial\Omega_1^{\text{int}} \\ u_2^{n+1,k+1} = u_1^{n+1,k}, & \text{on } \partial\Omega_2^{\text{int}} \end{cases}$$

# Numerical results

- Reference solution: numerical solution of the problem solved in the monodomain;
- Objective: compare the number of iterations for convergence to the reference solution in two cases:
  - DDM using the discrete TBCs as IBCs:

$$\begin{cases} \Gamma_1^r \left( \{u_1^{n+1}\}^{k+1} \right) = \Gamma_1^r \left( \{u_2^{n+1}\}^k \right) \\ \Gamma_2^r \left( \{u_1^{n+1}\}^{k+1} \right) = \Gamma_2^r \left( \{u_2^{n+1}\}^k \right) \end{cases} \quad \text{on } \partial\Omega_1^{\text{int}}$$

$$\begin{cases} \Gamma_1^l \left( \{u_2^{n+1}\}^{k+1} \right) = \Gamma_1^l \left( \{u_1^{n+1}\}^k \right) \\ \Gamma_2^l \left( \{u_2^{n+1}\}^{k+1} \right) = \Gamma_2^l \left( \{u_1^{n+1}\}^k \right) \end{cases} \quad \text{on } \partial\Omega_2^{\text{int}}$$

(these IBCs are **local in time**).

## Numerical results

| $t$  | TBC | Dirichlet |
|------|-----|-----------|
| 0.25 | 2   | 604       |
| 0.5  | 2   | 612       |
| 0.75 | 2   | 546       |
| 1.0  | 2   | 556       |

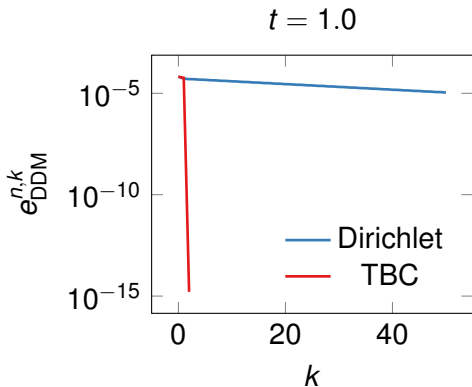


Table: Convergence of the additive Schwarz method, for a precision of  $\varepsilon = 10^{-15}$ , for an overlap size of 5.

# 4

## Towards TBCs for nonlinear problems

# The Serre equations

- Objective: use TBCs in the numerical resolution of nonlinear problems;



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- Serre equations [Cienfuegos and Carter, 2011]:

$$\begin{cases} h_t + (hu)_x = 0, \\ u_t + uu_x + gh_x - \frac{1}{3h} \left( h^3 \left( u_{xt} + uu_{xx} - (u_x)^2 \right) \right)_x = 0. \end{cases} \quad (1)$$

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- Analogous procedure lead to similar discrete TBCs as obtained for Boussinesq equations, differing only by the coefficients of the TBCs;

# Numerical simulations

- Use discrete TBCs:
  - for simulating nonlinear problem;
  - as IBCs in a DDM.

# Numerical simulations

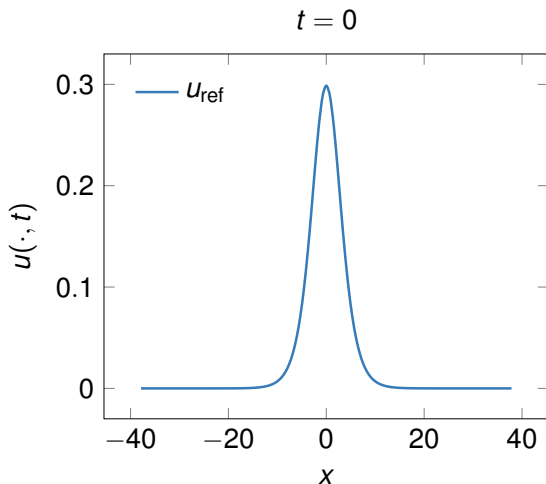
- Use discrete TBCs:
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- Compare results for nonlinear Serre equations using
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  - Discrete TBCs coefficients obtained for linearized Serre equations;
  - Discrete TBCs coefficients obtained for linearized Boussinesq equations.
- Two testcases:
  - Gaussian wave;
  - Solitary wave;

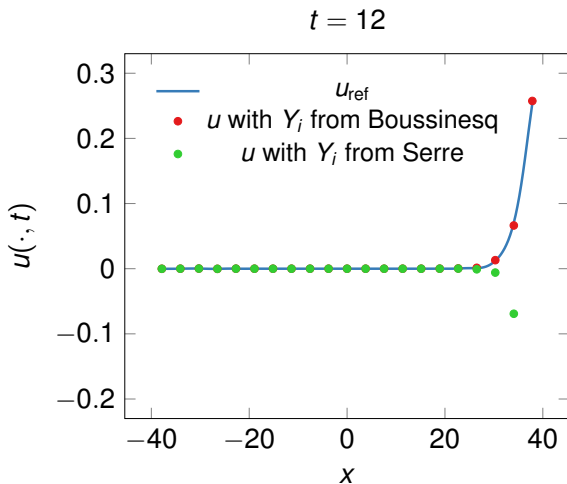
# Numerical results - TBCs - Solitary wave

Initial solution:



# Numerical results - TBCs - Solitary wave

TBCs with different coefficients:



## Numerical results - DDM

| Case            | Dirichlet | TBC   |            |
|-----------------|-----------|-------|------------|
|                 |           | Serre | Boussinesq |
| <b>Gaussian</b> | 120 ~ 160 | 18    | 9          |
| <b>Solitary</b> | 11        | 8     | 8          |

Table: Convergence of the additive Schwarz method, for a precision of  $\varepsilon = 10^{-15}$ , for an overlap size of 5.

**Question:** Coefficients from linearized Boussinesq equations seem to be more efficient than the coefficients from linearized Serre equations.



# 5

## Conclusion

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- Application to the numerical resolution of nonlinear problems shall be investigated in more details.

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  - as TBCs;
  - as IBCs in a DDM;
- Application to the numerical resolution of nonlinear problems shall be investigated in more details.
- More detailed paper under review [Caldas Steinstraesser et al., 2018].

# References I

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