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► **To cite this version:**

Cedric Herzet, Mamadou Diallo, Patrick Héas. Beyond Petrov-Galerkin projection by using “multi-space” priors. MoRePas 2018 - Model Reduction for Parametrized Systems IV, Apr 2018, Nantes, France. hal-01937876

**HAL Id: hal-01937876**

**<https://inria.hal.science/hal-01937876>**

Submitted on 4 Jul 2019

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# Beyond Petrov-Galerkin projection by using « multi-space » priors

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Joint work with M. Diallo & P. Héas

# The target problem...

Find  $\mathbf{h}^* \in \mathcal{H}$  such that  $a(\mathbf{h}^*, \mathbf{h}) = b(\mathbf{h}) \quad \forall \mathbf{h} \in \mathcal{H}$

where  $\mathcal{H}$  is a Hilbert space ( $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ )  
 $a : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  is a bilinear operator  
 $b : \mathcal{H} \rightarrow \mathbb{R}$  is a linear operator

... and its Petrov-Galerkin approximation

Find  $\hat{\mathbf{h}}_{\text{PG}} \in V_n$  such that  $a(\hat{\mathbf{h}}_{\text{PG}}, \mathbf{h}) = b(\mathbf{h}) \quad \forall \mathbf{h} \in Z_n$

where  $V_n \subset \mathcal{H}$ ,  $Z_n \subset \mathcal{H}$  are  $n$ -dimensional subspaces

The precision of Petrov-Galerkin can be quantified by an « instance optimal property »

$$\left\| \mathbf{h}^* - \hat{\mathbf{h}}_{\text{PG}} \right\| \leq C(V_n, Z_n) \text{dist}(\mathbf{h}^*, V_n),$$

Standard methods constructing  $V_n$  often return a set of subspaces and their « widths »

Standard outputs of methods constructing  $V_n$ :

$$V_0 \subset V_1 \subset \dots \subset V_n, \quad \dim(V_k) = k$$

such that

$$\text{dist}(\mathbf{h}^*, V_k) \leq \hat{\epsilon}_k, \quad k = 0 \dots n.$$

*E.g.*, « reduced basis » methods

# The Petrov-Galerkin projection discards most of the available information

Standard outputs of methods constructing  $V_n$ :

$$\mathbf{X} \subset \mathbf{X} \subset \dots \subset V_n, \quad \dim(V_k) = k$$

such that

$$\text{dist}(\mathbf{h}^*, \mathbf{X}) \leq \hat{\epsilon}_k, \quad k = 0 \dots n.$$

*E.g.*, « reduced basis » methods



Can we use this information to improve the projection process?



The Petrov-Galerkin projection can be reformulated as a variational problem

$$\hat{\mathbf{h}}_{\text{PG}} = \arg \min_{\mathbf{h} \in V_n} \sum_{j=1}^n (b_j - \langle \mathbf{a}_j, \mathbf{h} \rangle)^2$$

where  $\text{span} \left( \{\mathbf{z}_j\}_{j=1}^n \right) = Z_n$

$\mathbf{a}_j$  is the Riesz's representer of  $a(\cdot, \mathbf{z}_j)$

$b_j = b(\mathbf{z}_j)$

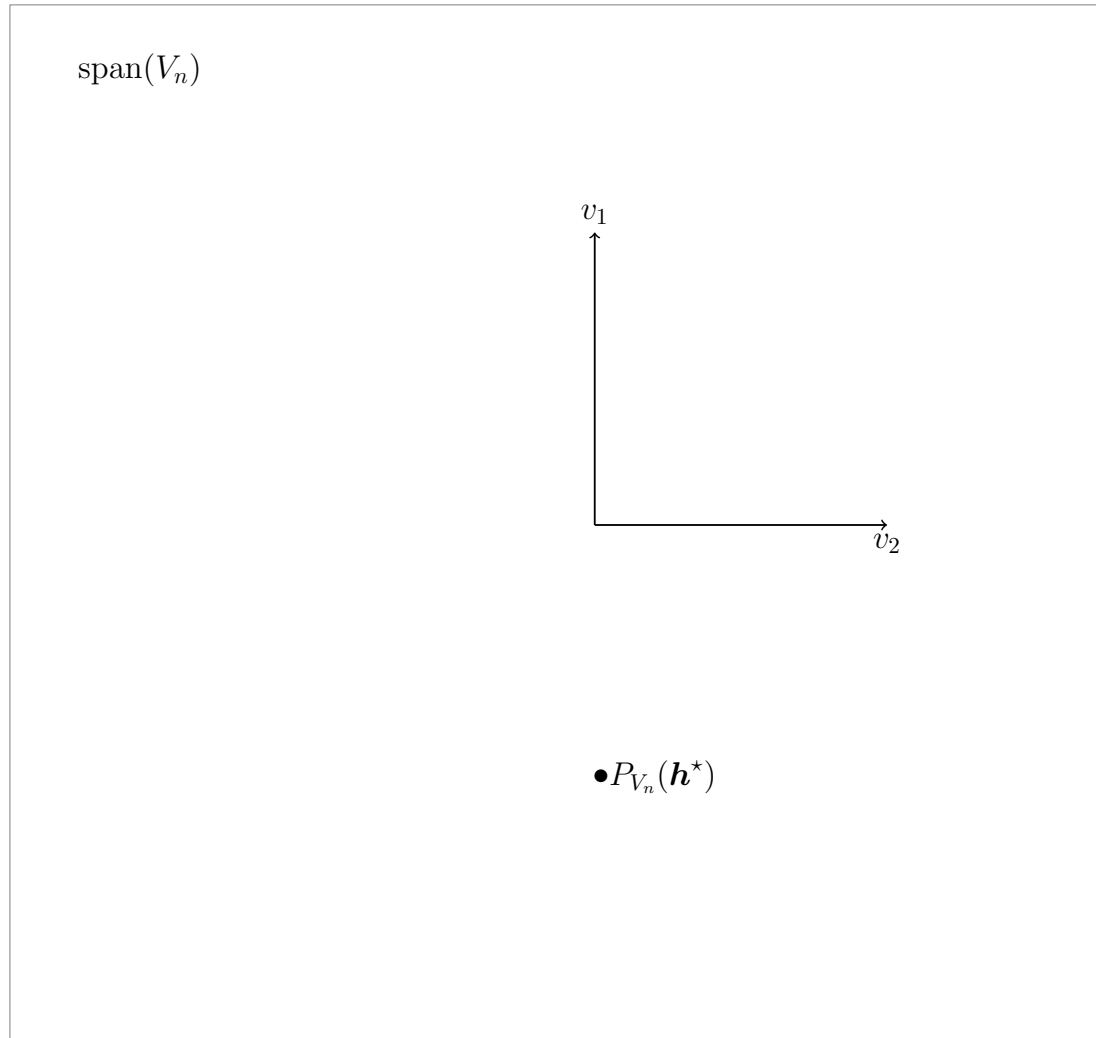
The proposed « multi-space » decoder adds new constraints to the variational problem

$$\hat{\mathbf{h}}_{\text{MS}} = \arg \min_{\mathbf{h} \in V_n} \sum_{j=1}^n (b_j - \langle \mathbf{a}_j, \mathbf{h} \rangle)^2,$$

subject to  $\text{dist}(\mathbf{h}, V_k) \leq \hat{\epsilon}_k, \quad k = 0 \dots n - 1.$

See [*Binev et al., SIAM JUQ 17*] for a related work.

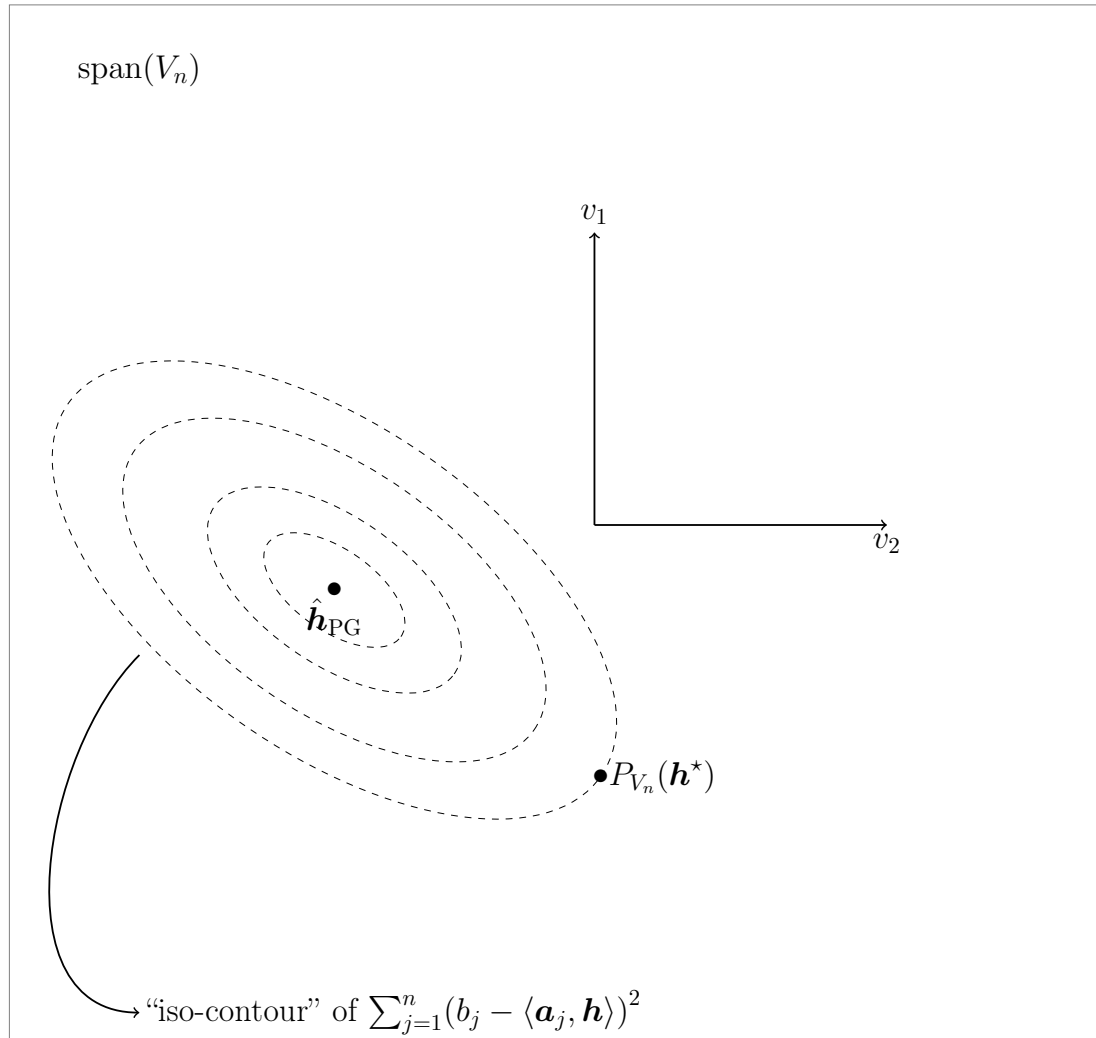
# A graphical representation of the problem



$$n = 2$$

$$V_k = \text{span} \left( \{\mathbf{v}_i\}_{i=1}^k \right)$$

# A graphical representation of the problem

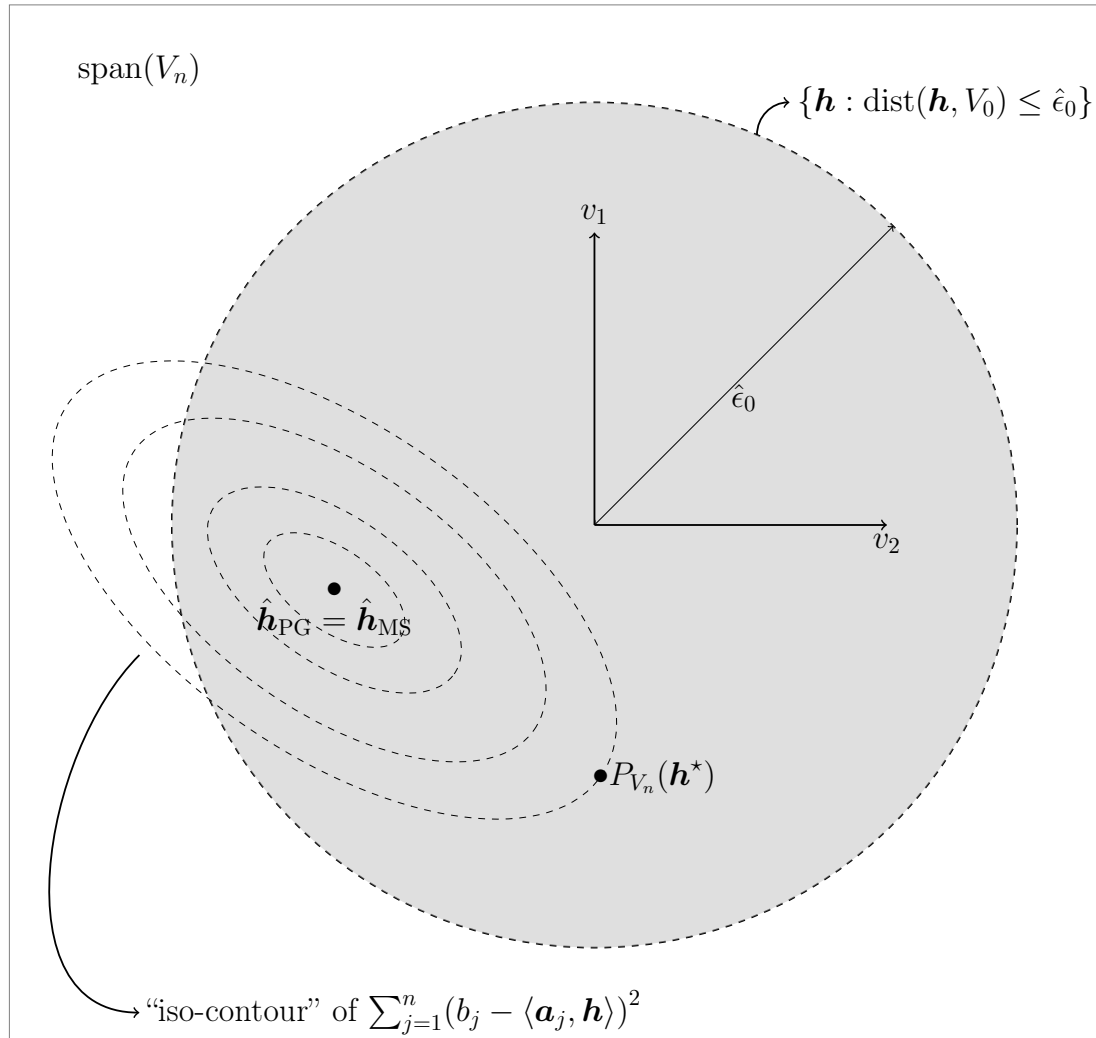


$$n = 2$$

$$V_k = \text{span} \left( \{ \mathbf{v}_i \}_{i=1}^k \right)$$

The shape of the iso-contours depends on  $\mathbf{G} = [\langle \mathbf{a}_i, \mathbf{v}_j \rangle]_{ij}$

# A graphical representation of the problem

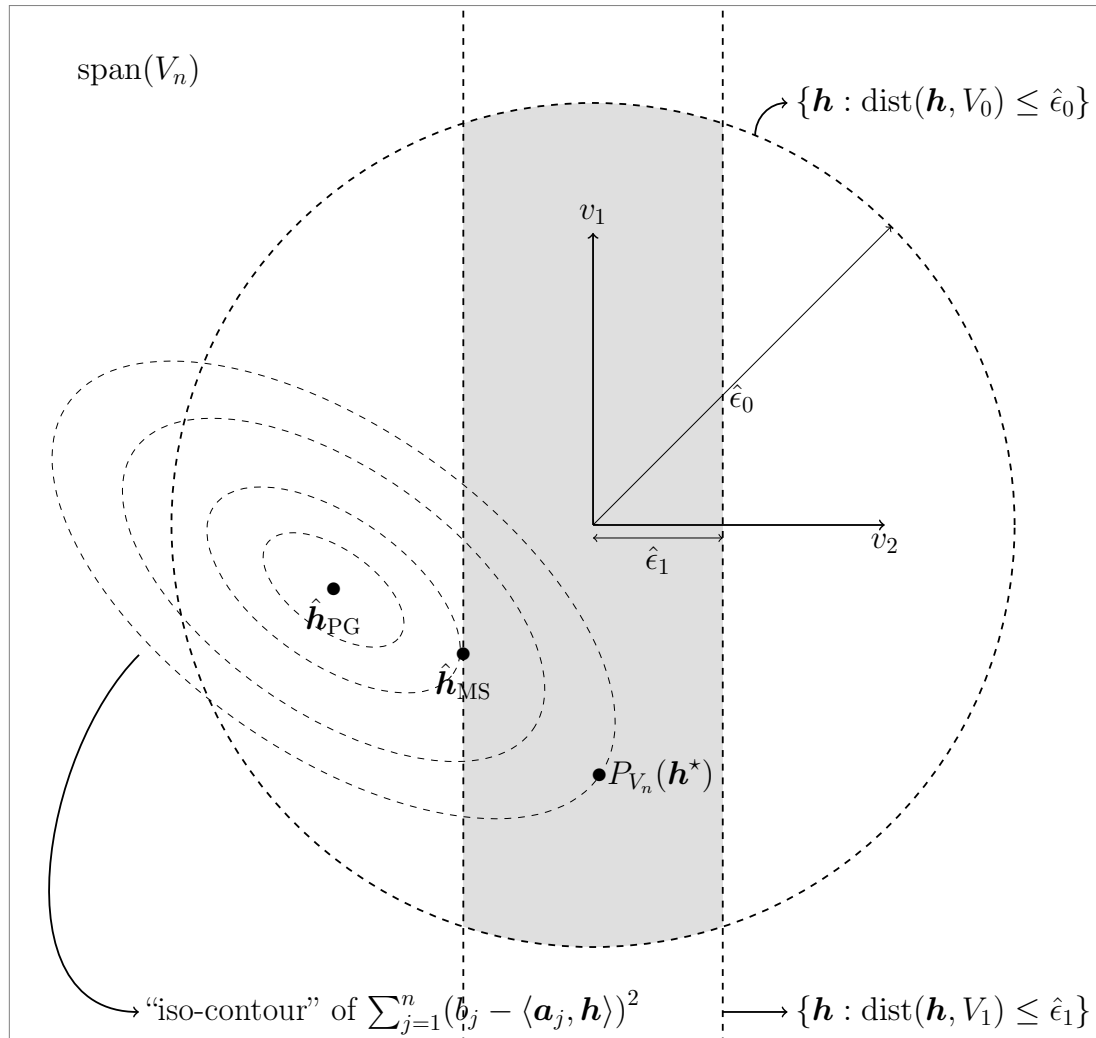


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# A graphical representation of the problem



$$n = 2$$

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The shape of the iso-contours depends on  $\mathbf{G} = [\langle \mathbf{a}_i, \mathbf{v}_j \rangle]_{ij}$

The feasibility region depends on  $\{V_k\}_{k=1}^n$  and  $\{\hat{\epsilon}_k\}_{k=1}^n$



Can we give some guarantee on the performance of the « multi-space » decoder?

# Instance optimality properties

- *Petrov-Galerkin*:  $\left\| \mathbf{h}^* - \hat{\mathbf{h}}_{\text{PG}} \right\| \leq C(\mathbf{G}) \text{dist}(\mathbf{h}^*, V_n),$

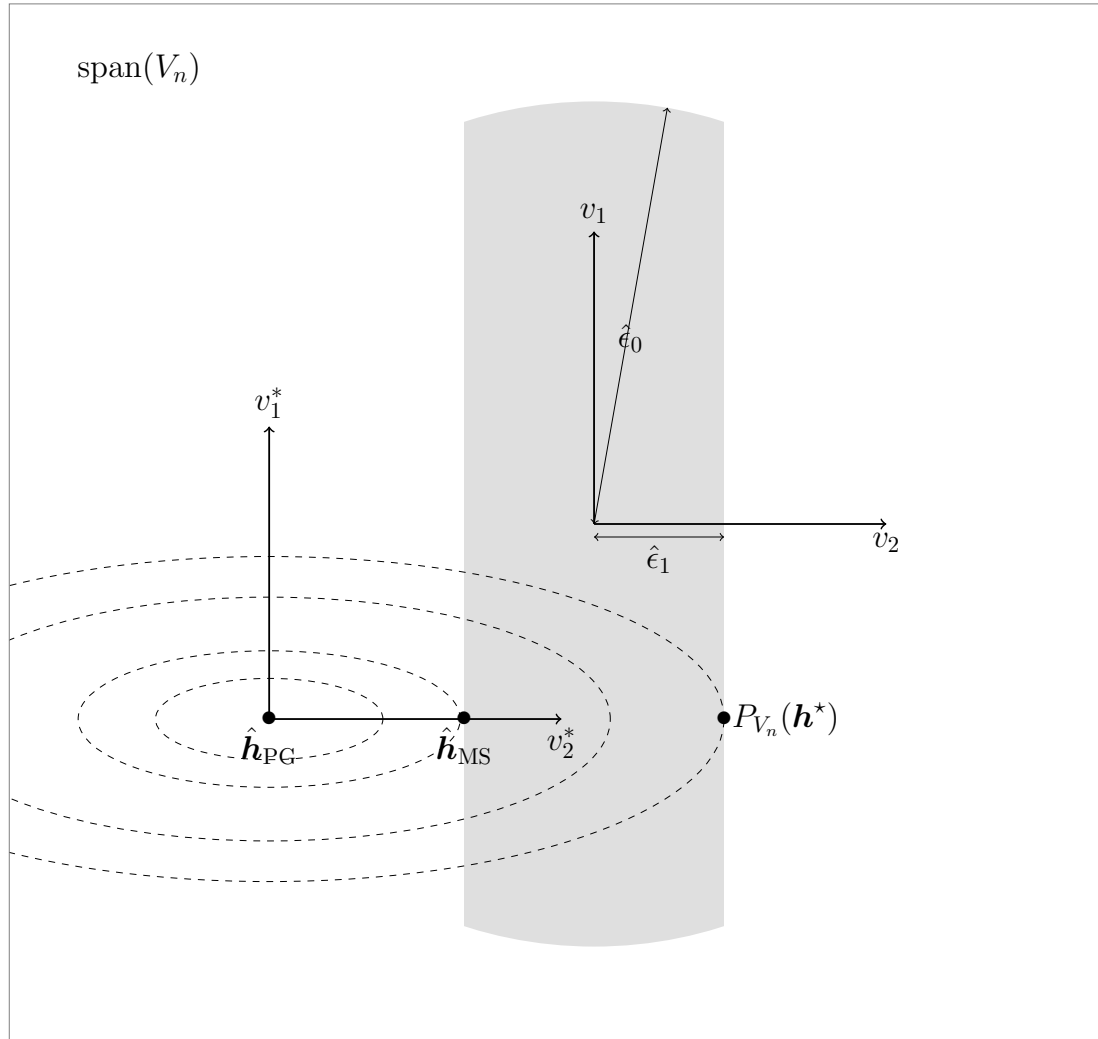
- « *Multi-space* » decoder:

$$\left\| \mathbf{h}^* - \hat{\mathbf{h}}_{\text{MS}} \right\| \leq \left( \sum_{j=\ell+1}^n \delta_j^2 + \rho \delta_\ell^2 + (\text{dist}(\mathbf{h}^*, V_n))^2 \right)^{\frac{1}{2}}$$

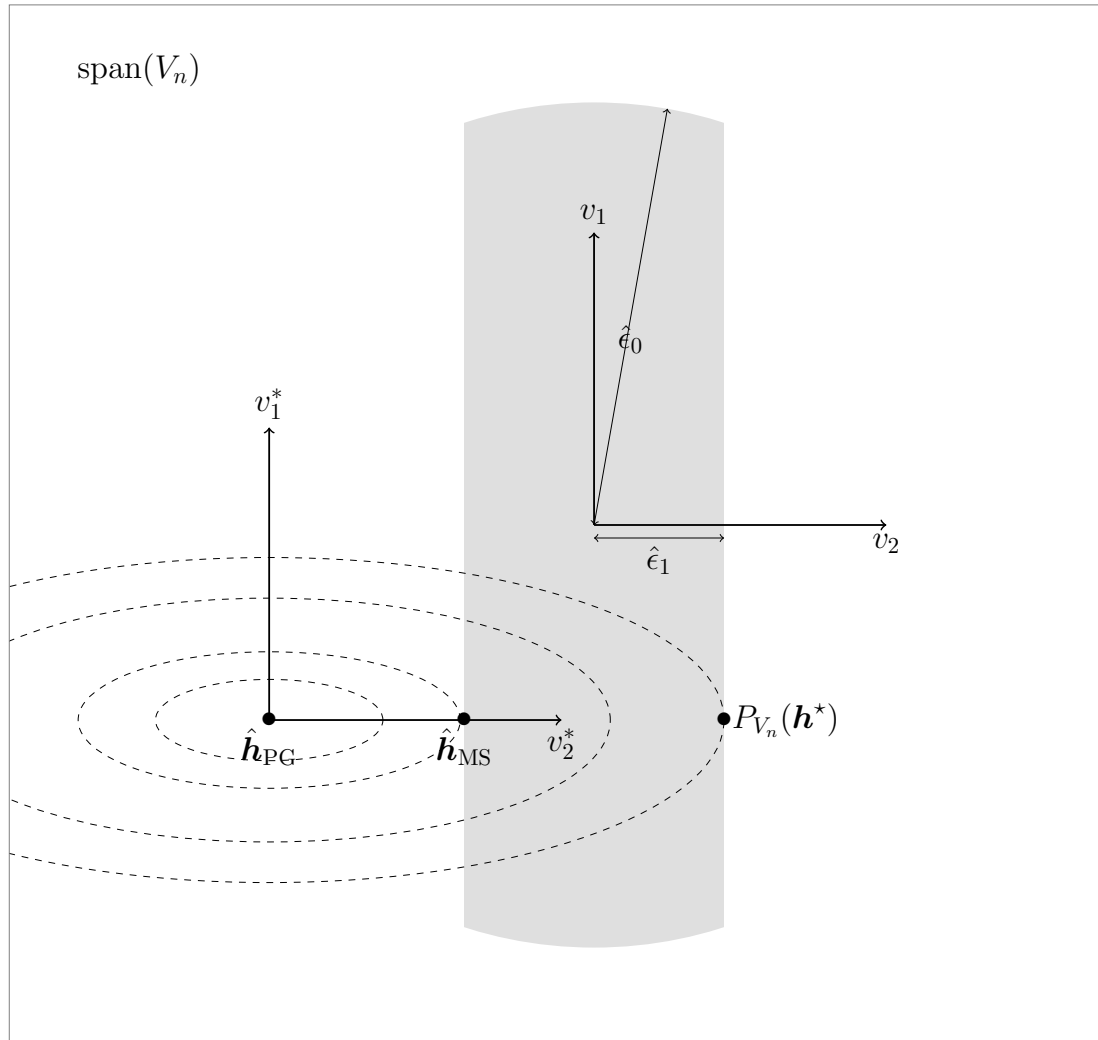
where  $\ell$  and  $\delta_j$ 's are “easily-computable” quantities only depending on  $\mathbf{G} = [\langle \mathbf{a}_i, \mathbf{v}_j \rangle]_{i,j}$ ,  $\{\hat{\epsilon}_k\}_{k=1}^{n-1}$  and  $\{\text{dist}(\mathbf{h}^*, V_k)\}_{k=1}^n$



# Particularization of the instance optimality bound to some examples



# Particularization of the instance optimality bound to some examples



$$\hat{\epsilon}_j = \begin{cases} 1 & j = 0 \dots n-3, \\ \epsilon^{\frac{1}{2}} & j = n-2, n-1, \\ \epsilon & j = n, \end{cases}$$

$$\text{dist}(\mathbf{h}^*, V_k) = \hat{\epsilon}_j$$

$\sigma_j = \text{sing. values of } \mathbf{G}$

$$= \begin{cases} 1 & j = 1 \dots n-3, \\ \epsilon^{\frac{1}{2}} & j = n-2, n-1, \\ \epsilon & j = n. \end{cases}$$

$$\left\| \mathbf{h}^* - \hat{\mathbf{h}}_{\text{PG}} \right\| \leq 1$$

$$\left\| \mathbf{h}^* - \hat{\mathbf{h}}_{\text{MS}} \right\| \leq 3\epsilon^{\frac{1}{2}}$$



*Quid* of the computational complexity?

PG projection can be carried out efficiently with a complexity  $\mathcal{O}(n^2)$  per iteration

$$\hat{\mathbf{h}}_{\text{PG}} = \arg \min_{\mathbf{h} \in V_n} \sum_{j=1}^n (b_j - \langle \mathbf{a}_j, \mathbf{h} \rangle)^2$$

« Least square » problem: can be solved efficiently via gradient-based methods with a complexity  $\mathcal{O}(n^2)$  per iteration.

Our decoder can also be implemented with a complexity  $\mathcal{O}(n^2)$  per iteration

Our problem can be rewritten as:

$$\hat{\mathbf{h}}_{\text{MS}} = \arg \min_{\mathbf{h} \in V_n} \sum_{j=1}^n (b_j - \langle \mathbf{a}_j, \mathbf{h} \rangle)^2,$$

subject to  $\|P_{V_k}^\perp(\mathbf{h})\| \leq \hat{\epsilon}_k, \quad k = 0 \dots n - 1.$

We use the « Alternating Directions Method of Multipliers » to solve this convex problem with a complexity  $\mathcal{O}(n^2)$  per iteration

# Some results

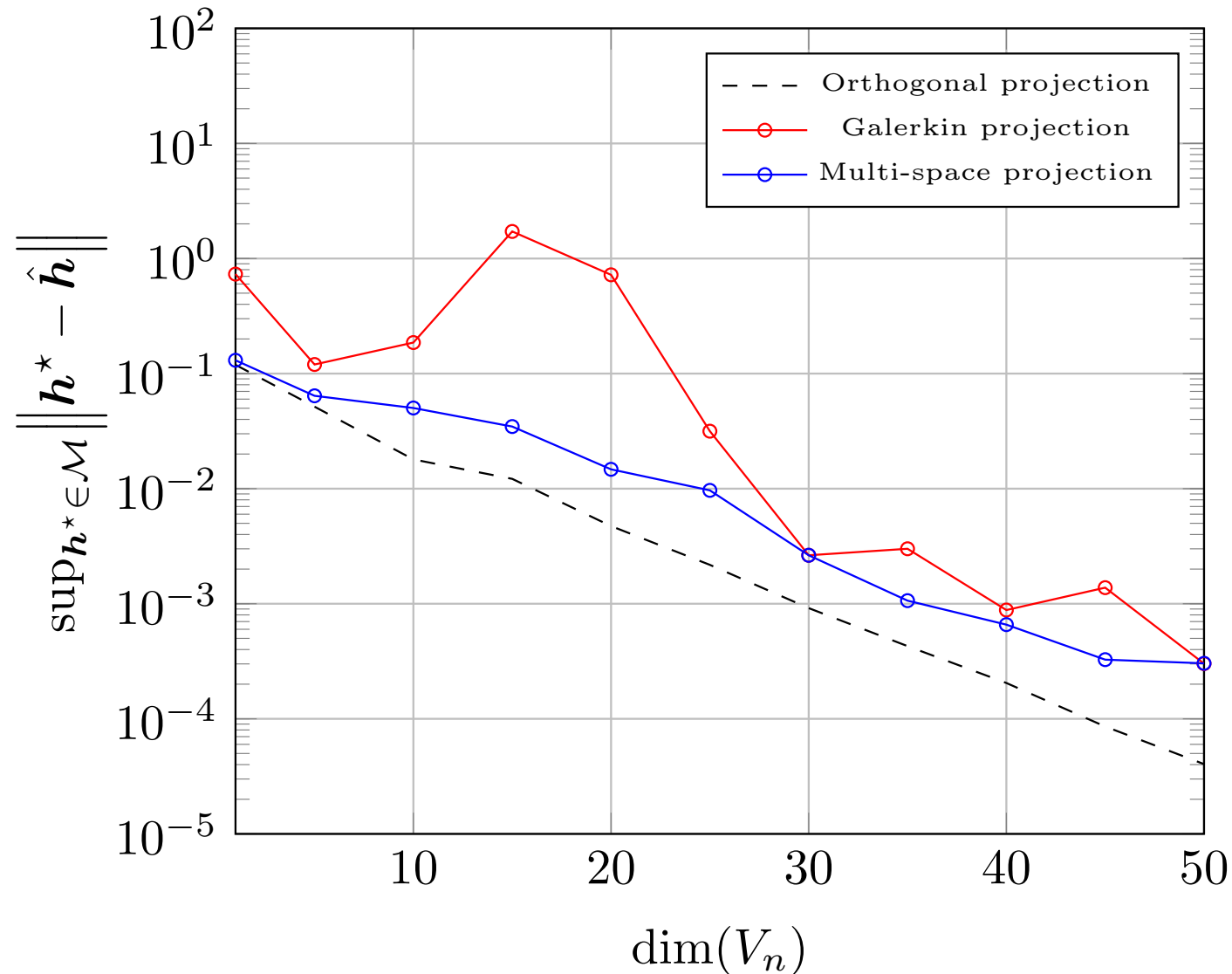
We consider a parametric mass transfert problem

$$\begin{aligned}\mu_1 \Delta \mathbf{h} + \mathbf{b}(\mu_2) \cdot \nabla \mathbf{h} &= \mathbf{s} && \text{in } \Omega \\ \mu_1 \nabla \mathbf{h} \cdot \mathbf{n} &= 0 && \text{in } \partial\Omega\end{aligned}$$

where

$$\begin{aligned}\mathbf{b}(\mu_2) &= [\cos(\mu_2) \sin(\mu_2)]^T \\ \mathbf{s} &= \exp\left(-\frac{\|\mathbf{x} - \mathbf{m}\|_2^2}{2\sigma^2}\right) \\ \mu_1 &= [0.03, 0.05] \\ \mu_2 &= [0, 2\pi]\end{aligned}$$

# The multi-space decoder improves the worst-case approximation error over PG





# Our contributions

- We exploit additional information in the Petrov-Galerkin projection
- We derive an instance optimal guarantee for our « multi-space » decoder
- We propose an efficient implementation for the « multi-space » decoder

Thank you for your attention.