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# A Polynomial-Division-Based Algorithm for Computing Linear Recurrence Relations 

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#### Abstract

Sparse polynomial interpolation, sparse linear system solving or modular rational reconstruction are fundamental problems in Computer Algebra. They come down to computing linear recurrence relations of a sequence with the Berlekamp-Massey algorithm. Likewise, sparse multivariate polynomial interpolation and multidimensional cyclic code decoding require guessing linear recurrence relations of a multivariate sequence.

Several algorithms solve this problem. The so-called Berlekamp-Massey-Sakata algorithm (1988) uses polynomial additions and shifts by a monomial. The Scalar-FGLM algorithm (2015) relies on linear algebra operations on a multi-Hankel matrix, a multivariate generalization of a Hankel matrix. The Artinian Gorenstein border basis algorithm (2017) uses a Gram-Schmidt process.

We propose a new algorithm for computing the Gröbner basis of the ideal of relations of a sequence based solely on multivariate polynomial arithmetic. This algorithm allows us to both revisit the Berlekamp-Massey-Sakata algorithm through the use of polynomial divisions and to completely revise the Scalar-FGLM algorithm without linear algebra operations.

A key observation in the design of this algorithm is to work on the mirror of the truncated generating series allowing us to use polynomial arithmetic modulo a monomial ideal. It appears to have some similarities with Padé approximants of this mirror polynomial.

As an addition from the paper published at the ISSAC conferance, we give an adaptive variant of this algorithm taking into account the shape of the final Gröbner basis gradually as it is discovered. The main advantage of this algorithm is that its complexity in terms of operations and sequence queries only depends on the output Gröbner basis.

All these algorithms have been implemented in Maple and we report on our comparisons.


Keywords: Gröbner bases; linear recursive sequences; Berlekamp-Massey-Sakata; extended Euclidean algorithm; Padé approximants

[^0]
## 1. Introduction

The Berlekamp-Massey algorithm (BM), introduced by Berlekamp in 1968 Berlekamp (1968) and Massey in 1969 Massey (1969) is a fundamental algorithm in Coding Theory, Bose and Ray-Chaudhuri (1960); Hocquenghem (1959), and Computer Algebra. It allows one to perform efficiently sparse polynomial interpolation, sparse linear system solving or modular rational reconstruction.

In 1988, Sakata extended the BM algorithm to dimension $n$. This algorithm, known as the Berlekamp-Massey-Sakata algorithm (BMS), can be used to compute a Gröbner basis of the zero-dimensional ideal of the relations satisfied by a sequence, Sakata (1988, 1990, 2009). Analogously to dimension 1 , the BMS algorithm allows one to decode cyclic code in dimension $n>1$, an extension of Reed-Solomon's codes. Furthermore, the latest versions of the Sparse-FGLM algorithm rely heavily on the efficiency of the BMS algorithm to compute the change of ordering of a Gröbner basis, Faugère and Mou (2011); Faugère and Mou (2017).

### 1.1. Related Work

In dimension 1 , it is well known that the BM algorithm can be seen in a matrix form requiring to solve a linear Hankel system of size $D$, the order of the recurrence, see Jonckheere and Ma (1989), or the Levinson-Durbin method, Levinson (1947); Wiener (1964). If we let $\mathrm{M}(D)$ be a cost function for multiplying two polynomials of degree $D$, for instance $\mathrm{M}(D) \in O(D \log D \log \log D)$, Cantor and Kaltofen (1991); Cooley and Tukey (1965), then solving a linear Hankel system of size $D$ comes down to performing a truncated extended Euclidean algorithm called on two polynomials of degree $D$, Blackburn (1997); Brent et al. (1980); Dornstetter (1987). More precisely, it can be done in $O(\mathrm{M}(D) \log D)$ operations.

In Berthomieu et al. (2015, 2017), the authors present the Scalar-FGLM algorithm, extending the matrix version of the BM algorithm for multidimensional sequences. It consists in computing the relations of the sequence through the computation of a maximal full-rank matrix of a multi-Hankel matrix, a multivariate generalization of a Hankel matrix. Then, it returns the minimal Gröbner basis $\mathcal{G}$ of the ideal of relations satisfied by the sequence. These notions are recalled in Section 2. If we denote by $S$ the staircase defined by $\mathcal{G}$ and $T$ the input set of monomials containing $S \cup \mathcal{G}$, then the complexity of the Scalar-FGLM algorithm is $O\left((\# T)^{\omega}\right)$, where $2 \leq \omega \leq 3$ is the linear algebra exponent. However, we do not know how to exploit the multi-Hankel structure to improve this complexity.

The Artinian Gorenstein border bases algorithm (AGbb) was presented in Mourrain (2017) for computing a border basis $\mathcal{B}$ of the ideal of relations. It extends the algorithm of Berthomieu et al. (2015) using polynomial arithmetic allowing it to reach the better complexity $O((\# S+\# \mathcal{B})$. $\# S \cdot \# T)$ with the above notation.

Another viewpoint is that computing linear recurrence relations can be seen as computing Padé approximants of a truncation of the generating series $\sum_{i_{1}, \ldots, i_{n} \geq 0} w_{i_{1}, \ldots, i_{n}} x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}$. In Fitzpatrick and Flynn (1992), the authors extend the extended Euclidean algorithm for computing multivariate Padé approximants. Given a polynomial $P$ and an ideal $B$, find polynomials $F$ and $C$ such that $P=\frac{F}{C} \bmod B$, where the leading monomials of $F$ and $C$ satisfy some constraints.

It is also worth noticing that we now know that both the BMS and the Scalar-FGLM algorithms are not equivalent, see Berthomieu and Faugère (2017), i.e. it is not possible to tweak one algorithm to mimic the behavior of the other. However, if the input sequence is linear recurrent and sufficiently many sequence terms are visited, then both algorithms compute a Gröbner basis of the zero-dimensional ideal of relations.

### 1.2. Contributions

In all the paper, we assume that the input sets of the Scalar-FGLM algorithm are the sets of all the monomials less than a given monomial. In order to improve the complexity of the algorithm, we will use polynomial arithmetic in all the operations. Even though they are not equivalent, this reduces the gap between the BMS and the Scalar-FGLM algorithms and provides a unified presentation.

In Section 3, we present the BM, the BMS and the Scalar-FGLM algorithms in a unified polynomial viewpoint. Using the mirror of the truncated generating series is a key ingredient letting us perform the computations modulo a specific monomial ideal $B$ : a vector in the kernel of a multi-Hankel matrix is a polynomial $C$ such that

$$
\begin{equation*}
\operatorname{Lм}(F)=\operatorname{Lм}(P C \bmod B)<t_{C}, \tag{1}
\end{equation*}
$$

where $P$ is the mirror of the truncated generating series, Lм denotes the leading monomial and $t_{C}$ is a monomial associated to $C$.

One interpretation of this is the computation of multivariate Padé approximants $\frac{F}{C}$ of $P$ modulo $B$ with different constraints than in Fitzpatrick and Flynn (1992) since we require that $\mathrm{Lm}(C)$ is in a given set of terms and $\operatorname{Lm}(F)$ satisfies equation (1).

This polynomial point of view allows us to design the Polynomial Scalar-FGLM algorithm (Algorithm 4.18) in Section 4 based on multivariate polynomial divisions. It computes polynomials whose product with $P$ modulo $B$ must satisfy equation (1). If they do not, by polynomial divisions, we make new ones until finding minimal polynomials satisfying this constraint. It is worth noticing that in dimension 1, we recover the truncated extended Euclidean algorithm applied to the mirror polynomial of the generated series of the input sequence, truncated in degree $D$, and $x^{D+1}$. All the examples are available on Berthomieu and Faugère (2018).

Our main result is Theorem 21, a simplified version of which is
Theorem 1. Let $\mathbf{w}$ be a sequence, < be a total degree monomial ordering and a be a monomial. Let us assume that the Gröbner basis $\mathcal{G}$ of the ideal of relations of $\mathbf{w}$ for $<$ and its staircase $S$ satisfy $a \geq \max (S \cup \operatorname{LM}(\mathcal{G}))$ and for all $g \leq a, s=\max _{\sigma \leq a}\{\sigma, \sigma g \leq a\}$, we have $\max (S) \leq s$. Then, the Polynomial Scalar-FGLM algorithm terminates and computes a Gröbner basis of the ideal of relations of $\mathbf{w}$ for $<$ in $O(\# S(\# S+\# \mathcal{G}) \#\{\sigma, \sigma \leq a\})$ operations in the base field.

In applications such as the Sparse-FGLM one, sequence queries are costly, Faugère and Mou (2017). In Berthomieu et al. (2015), an adaptive variant of the Scalar-FGLM algorithm was designed aiming to minimize the number of sequence queries to recover the relations.

In Section 5 , we show how we can transform the Adaptive Scalar-FGLM algorithm of Berthomieu et al. (2015) into an algorithm using polynomial arithmetic. This algorithm is output sensitive and probabilistic, like the Adaptive Scalar-FGLM algorithm is. However, its main advantage is that its complexity only depends on the sizes of the computed staircase and Gröbner basis.

Theorem 2 (see Theorem 24). Let $\mathbf{w}$ be a sequence, < be a total degree monomial ordering.
Let us assume that calling the Adaptive Polynomial Scalar-FGLM algorithm on $\mathbf{w}$ and $\prec$ yields the Gröbner basis $\mathcal{G}$ and its staircase $S$.

Then, the Adaptive Polynomial Scalar-FGLM algorithm performs at most $O\left((\# S+\# \mathcal{G})^{2} \# 2 S\right)$ operations in the base field and \#2(S $\cup \mathrm{LM}(\mathcal{G}))$ table queries to recover $\mathcal{G}$, where for a set $T$, $2 T=\left\{t t^{\prime}, t, t^{\prime} \in T\right\}$.

Finally, in Section 6, we compare the Polynomial Scalar-FGLM algorithm with our implementations of the BMS, the Scalar-FGLM and the AGbb algorithms. Our algorithm performs always fewer arithmetic operations than the others starting from a certain size. Even for an example family favorable towards the BMS algorithm, our algorithm performs better.

Although we have compared the numbers of arithmetic operations, it would be beneficial to have an efficient implementation. This would be the first step into designing a similar efficient algorithm for computing linear recurrence relations with polynomial coefficients, extending the Beckermann-Labahn algorithm Beckermann and Labahn (1994) for computing multivariate Hermite-Padé approximants.

Amongst the changes from the ISSAC version of the paper, Berthomieu and Faugère (2018), the main addition is a full description of the Adaptive Polynomial Scalar-FGLM algorithm, an adaptive variant of the Polynomial Scalar-FGLM algorithm using polynomial divisions as well, in Section 5.2 Generically, one could expect to make one relation with leading monomial $m x_{i}^{2}$ through a division of polynomials related to relations with leading monomials $m$ and $m x_{i}$. Yet, the naive approach given in (Berthomieu and Faugère, 2018, Section 5) could not do so as it does not perform any division. The Adaptive Polynomial Scalar-FGLM algorithm visits the monomials in the same order as the Adaptive Scalar-FGLM algorithm to recover the relations and replaces any linear algebra computations by polynomial ones, see Berthomieu et al. (2015). We also give the complexity of this algorithm in terms both of the number of operations and the number of sequence queries.

Furthermore, one of the main obstructions to the design of this adaptive variant is that at each step, some polynomials are updated. This update process adds terms supposed to be small with respect to the ordering. Yet, their leading terms were not stable during this update process.

Lastly, in Section 3. we now more clearly define the condition for a relation to be valid (see Proposition 13) in the execution of the Polynomial Scalar-FGLM algorithm.

## 2. Notation

We give a brief description of classical notation used in the paper.

### 2.1. Sequences and relations

For $n \geq 1$, we let $\mathbf{i}=\left(i_{1}, \ldots, i_{n}\right) \in \mathbb{N}^{n}$ and for $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$, we write $\mathbf{x}^{\mathbf{i}}=x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}$.
Definition 3. Let $\mathbb{K}$ be a field, $\mathcal{K} \subseteq \mathbb{N}^{n}$ be finite and $f=\sum_{\mathbf{k} \in \mathcal{K}} \gamma_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{K}[\mathbf{x}]$. We let $[f]_{\mathbf{w}}$, or $[f]$, be the linear combination $\sum_{\mathbf{k} \in \mathcal{K}} \gamma_{\mathbf{k}} w_{\mathbf{k}}$. If for all $\mathbf{i} \in \mathbb{N}^{n},\left[\mathbf{x}^{\mathbf{i}} f\right]=0$, then we say that $f$ is the polynomial of the relation induced by $\gamma=\left(\gamma_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{K}} \in \mathbb{K}^{\# K}$.

The main benefit of the [] notation resides in the immediate fact that for all index $\mathbf{i}$, its shift by $\mathbf{x}^{\mathbf{i}}$ is $\left[\mathbf{x}^{\mathbf{i}} f\right]=\sum_{\mathbf{k} \in \mathcal{K}} \gamma_{\mathbf{k}} w_{\mathbf{k}+\mathbf{i}}$.

Example 4. Let $\mathbf{b}=\left(\binom{i}{j}\right)_{(i, j) \in \mathbb{N}^{2}}$ be the sequence of the binomial coefficients. Then, $x y-y-1$ is the polynomial of Pascal's rule:

$$
\forall(i, j) \in \mathbb{N}^{2},\left[x^{i} y^{j}(x y-y-1)\right]=\mathbf{b}_{i+1, j+1}-\mathbf{b}_{i, j+1}-\mathbf{b}_{i, j}=0 .
$$

Definition 5 (Fitzpatrick and Norton (1990); Sakata (1988)). Let $\mathbf{w}=\left(w_{i}\right)_{i \in \mathbb{N} n}$ be an n-dimensional sequence with coefficients in $\mathbb{K}$. The sequence $\mathbf{w}$ is linear recurrent if from a nonzero finite number of initial terms $\left\{w_{\mathbf{i}}, \mathbf{i} \in S\right\}$, and a finite number of relations, without any contradiction, one can compute any term of the sequence.

Equivalently, $\mathbf{w}$ is linear recurrent if $\{f, \forall m \in \mathbb{K}[\mathbf{x}],[m f]=0\}$, its ideal of relations, is zero-dimensional.

As the input parameters of the algorithms are the first terms of a sequence, a table shall denote a finite subset of terms of a sequence.

### 2.2. Gröbner bases

Let $\mathcal{T}=\left\{\mathbf{x}^{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^{n}\right\}$ be the set of all monomials in $\mathbb{K}[\mathbf{x}]$. A monomial ordering $<$ on $\mathbb{K}[\mathbf{x}]$ is an order relation satisfying the following three classical properties:

1. for all $m \in \mathcal{T}, 1 \leq m$;
2. for all $m, m^{\prime}, s \in \mathcal{T}, m<m^{\prime} \Rightarrow m s<m^{\prime} s$;
3. every subset of $\mathcal{T}$ has a least element for $<$.

For a monomial ordering $<$ on $\mathbb{K}[\mathbf{x}]$, the leading monomial of $f$, denoted $\operatorname{LM}(f)$, is the greatest monomial in the support of $f$ for $<$. The leading coefficient of $f$, denoted $\operatorname{Lc}(f)$, is the nonzero coefficient of $\operatorname{Lm}(f)$. The leading term of $f, \operatorname{Lt}(f)$, is defined as $\operatorname{ct}(f)=\operatorname{Lc}(f) \operatorname{Lm}(f)$. For an ideal $I$, we denote $\operatorname{Lm}(I)=\{\operatorname{Lm}(f), f \in I\}$.

We recall briefly the definition of a Gröbner basis and a staircase.
Definition 6. Let I be a nonzero ideal of $\mathbb{K}[\mathbf{x}]$ and let $<$ be a monomial ordering. A set $\mathcal{G} \subseteq I$ is $a$ Gröbner basis of I if for all $f \in I$, there exists $g \in \mathcal{G}$ such that $\operatorname{Lm}(g) \mid \operatorname{LM}(f)$.

A Gröbner basis $\mathcal{G}$ of $I$ is minimal if for any $g \in \mathcal{G},\langle\mathcal{G} \backslash\{g\}\rangle \neq I$.
Furthermore, $\mathcal{G}$ is reduced if for any $g, g^{\prime} \in \mathcal{G}, g \neq g^{\prime}$ and any monomial $m \in \operatorname{supp} g^{\prime}$, $\mathrm{LM}(g) \nmid m$.

The staircase of $\mathcal{G}$ is defined as $S=\operatorname{Staircase}(\mathcal{G})=\{s \in \mathcal{T}, \forall g \in \mathcal{G}, \operatorname{LM}(g) \nmid s\}$. It is also the canonical basis of $\mathbb{K}[\mathbf{x}] / I$.

Gröbner basis theory allows us to choose any monomial ordering, among which, we mainly use the
$\operatorname{LEx}\left(x_{n}<\cdots<x_{1}\right)$ ordering which satisfies $\mathbf{x}^{\mathbf{i}}<\mathbf{x}^{\mathbf{i}^{\prime}}$ if and only if there exists $k, 1 \leq k \leq n$ such that for all $\ell<k, i_{\ell}=i_{\ell}^{\prime}$ and $i_{k}<i_{k}^{\prime}$, see (Cox et al. 2015, Chapter 2, Definition 3);
$\operatorname{DRL}\left(x_{n}<\cdots<x_{1}\right)$ ordering which satisfies $\mathbf{x}^{\mathbf{i}}<\mathbf{x}^{\mathbf{i}^{\prime}}$ if and only if $i_{1}+\cdots+i_{n}<i_{1}^{\prime}+\cdots+i_{n}^{\prime}$ or $i_{1}+\cdots+i_{n}=i_{1}^{\prime}+\cdots+i_{n}^{\prime}$ and there exists $k, 2 \leq k \leq n$ such that for all $\ell>k, i_{\ell}=i_{\ell}^{\prime}$ and $i_{k}>i_{k}^{\prime}$, see (Cox et al. 2015. Chapter 2, Definition 6).
However, in the BMS algorithm, we need to be able to enumerate all the monomials up to a bound monomial. This forces the user to take an ordering $<$ such that for all $M \in \mathcal{T}$, the set $\mathcal{T}_{\leq a}=\{m \leq a, m \in \mathcal{T}\}$ is finite. Such an ordering $<\operatorname{makes}\left(\mathbb{N}^{n},<\right)$ isomorphic to $(\mathbb{N},<)$. Hence, for a monomial $m$, it makes sense to speak about the previous (resp. next) monomial $m^{-}$(resp. $m^{+}$) for $<$. The DRL ordering is an example for an ordering on which every term other than 1 has an immediate predecessor.

This request excludes for instance the LEX ordering, and more generally any elimination ordering. In other words, only weighted degree ordering, or weight ordering, should be used.

Now that a monomial ordering is defined, we can say that a relation given by a polynomial $f \in \mathbb{K}[\mathbf{x}]$ fails when shifted by $s$ if for all monomials $\sigma<s,[\sigma f]=0$ but $[s f] \neq 0$, see also Sakata (1990, 2009).

### 2.3. Multi-Hankel matrices

A matrix $H \in \mathbb{K}^{m \times n}$ is Hankel, if there exists a sequence $\mathbf{w}=\left(w_{i}\right)_{i \in \mathbb{N}}$ such that for all $\left(i, i^{\prime}\right) \in\{1, \ldots, m\} \times\{1, \ldots, n\}$, the coefficient $h_{i, i^{\prime}}$ lying on the $i$ th row and $i^{\prime}$ th column of $H$ satisfies $h_{i, i^{\prime}}=w_{i+i^{\prime}}$.

In a multivariate setting, we can extend this notion to multi-Hankel matrices. For two sets of monomials $U$ and $T$, we let $H_{U, T}$ be the multi-Hankel matrix with rows (resp. columns) indexed with $U$ (resp. $T$ ) so that the coefficient of $H_{U, T}$ lying on the row labeled with $\mathbf{x}^{\mathbf{i}} \in U$ and column labeled with $\mathbf{x}^{\mathbf{i}^{\prime}} \in T$ is $w_{\mathbf{i + i ^ { \prime }}}$.
Example 7. Let $\mathbf{w}=\left(w_{i, j, k}\right)_{(i, j, k) \in \mathbb{N}^{3}}$ be a sequence.

1. For $U=\left\{1, z, z^{2}, y, y z, y z^{2}\right\}=\left\{1, z, z^{2}\right\} \cup y\left\{1, z, z^{2}\right\}$ and $T=\left\{1, z, y, y z, y^{2}, y^{2} z\right\}=\{1, z\} \cup$ $y\{1, z\} \cup y^{2}\{1, z\}$, ordered for $\operatorname{LEx}(z<y<x)$,

$$
\left.H_{U, T}=\begin{array}{c} 
\\
1 \\
z \\
z^{2} \\
y \\
y z \\
y z^{2}
\end{array} \begin{array}{cc|cc|cc}
1 & z & y & y z & y^{2} & y^{2} z \\
w_{0,0,0} & w_{0,0,1} & w_{0,1,0} & w_{0,1,1} & w_{0,2,0} & w_{0,2,1} \\
w_{0,0,1} & w_{0,0,2} & w_{0,1,1} & w_{0,1,2} & w_{0,2,1} & w_{0,2,2} \\
w_{0,0,2} & w_{0,0,3} & w_{0,1,2} & w_{0,1,3} & w_{0,2,2} & w_{0,2,3} \\
\hline w_{0,1,0} & w_{0,1,1} & w_{0,2,0} & w_{0,2,1} & w_{0,3,0} & w_{0,3,1} \\
w_{0,1,1} & w_{0,1,2} & w_{0,2,1} & w_{0,2,2} & w_{0,3,1} & w_{0,3,2} \\
w_{0,1,2} & w_{0,1,3} & w_{0,2,2} & w_{0,2,3} & w_{0,3,2} & w_{0,3,3}
\end{array}\right)
$$

is a $2 \times 3$-block-Hankel matrix with $3 \times 2$-Hankel blocks.
2. For $\tilde{U}=U \cup x U \cup x^{2} U \cup x^{3} U$ and $\tilde{T}=T \cup x T \cup x^{2} T \cup x^{3} T \cup x^{4} T$, also ordered for $\operatorname{LEx}(z<y<x)$,

$H_{\tilde{U}, \tilde{T}}=$|  |
| :---: |
| $U$ |
| ${ }_{x}$ |
| $x^{2} U$ |
| $x^{3} U$ |\(\left(\begin{array}{c|c|c|c|c}T \& x^{2} T \& x^{2} T \& \& x^{3} T <br>

x^{4} T <br>
H_{U, T} \& H_{U, x T} \& H_{U, x^{2} T} \& H_{U, x^{3} T} \& H_{U, x^{4} T} <br>
\hline H_{x U, T} \& H_{x U, x T} \& H_{x U, x^{2} T} \& H_{x U, x^{3} T} \& H_{x U, x^{4} T} <br>
\hline H_{x^{2} U, T} \& H_{x^{2} U, x T} \& H_{x^{2} U, x^{2} T} \& H_{x^{2} U, x^{3} T} \& H_{x^{2} U, x^{4} T} <br>
\hline H_{x^{3} U, T} \& H_{x^{3} U, x T} \& H_{x^{3} U, x^{2} T} \& H_{x^{3} U, x^{3} T} \& H_{x^{3} U, x^{4} T}\end{array}\right)\),
where $H_{x^{i} U, x^{\prime} T}=H_{x^{i+i^{\prime}} U, T}$ for any $i, i^{\prime}$ since $H_{x^{i} U, x^{i^{\prime}} T}$ is the same matrix as $H_{U, T}$ where each coefficient $w_{0, j, k}$ has been replaced by $w_{i+i^{\prime}, j, k}$. Therefore, $H_{\tilde{U}, \tilde{T}}$ is a $4 \times 5$-block-Hankel matrix with $6 \times 6$-multi-Hankel blocks like $H_{U, T}$.
3. For $T=\left\{1, y, x, y^{2}, x y, x^{2}\right\}$, ordered for $\operatorname{DRL}(z<y<x), H_{T, T}$ is a multi-Hankel matrix whose structure is less clear. It can be considered as a block-Hankel matrix with blocks of different sizes, noticing that $T=\{1\} \cup y\left\{1, \frac{x}{y}\right\} \cup y^{2}\left\{1, \frac{x}{y}, \frac{x^{2}}{y^{2}}\right\}$,

$$
\left.H_{T, T}=\begin{array}{c}
1 \\
\\
1 \\
y \\
x \\
y^{2} \\
x y \\
x^{2}
\end{array} \begin{array}{c|cc|ccc} 
\\
w_{0,0,0} & w_{0,1,0} & w_{1,0,0} & y^{2} & x y & x_{0,2,0} \\
w_{0,1,0} & w_{1,1,0} & w_{2,0,0} \\
w_{1,0,0} & w_{1,1,0} & w_{1,1,0} & w_{2,0,0} & w_{0,3,0} & w_{1,2,0} \\
w_{1,2,0} & w_{2,1,0} \\
w_{0,2,0} & w_{0,3,0} & w_{1,2,0} & w_{3,0,0} \\
w_{1,1,0} & w_{1,2,0} & w_{2,1,0} & w_{1,3,0} & w_{2,2,0} \\
w_{2,0,0} & w_{2,3,0} & w_{2,2,0} & w_{3,3,0} \\
w_{2,0,0} & w_{2,2,0} & w_{3,1,0} & w_{4,0,0}
\end{array}\right) .
$$

4. For $U=\left\{1, z, y, x, z^{2}, y z, x z, y^{2}, x y, x^{2}\right\}=\{1\} \cup z\left\{1, \frac{y}{z}, \frac{x}{z}\right\} \cup z^{2}\left\{1, \frac{y}{z}, \frac{x}{z}, \frac{y^{2}}{z^{2}}, \frac{x y}{z^{2}}, \frac{x^{2}}{z^{2}}\right\}$, also ordered for $\operatorname{DRL}(z<y<x)$, the matrix $H_{U, U}$ can be seen as a block-matrix like $H_{T, T}$ except each block is a multi-Hankel matrix in two variables. In fact, the bottom-right block would be the same as $H_{T, T}$ where each coefficient $w_{i, j, 0}$ is replaced by $w_{i, j, 4-i-j}$.

### 2.4. Polynomials associated to multi-Hankel matrices

For two sets of terms $T$ and $U$, we let $T+U$ denote their Minkowsky sum, i.e. $T+U=$ $\{t u, t \in T, u \in U\}$, and $2 T=T+T$.

For a set of terms $T$, we let $M=\operatorname{LCM}(T)$. We let $P_{T}$ be the mirror polynomial of the truncated generating series of a sequence $\mathbf{w}$, i.e.

$$
P_{T}=\sum_{t \in T}[t] \frac{M}{t} .
$$

Example 8. Let $\mathbf{w}=\left(w_{i, j, k}\right)_{(i, j, k) \in \mathbb{N}^{3}}$ be a sequence and $T=\left\{1, z, y, x, z^{2}, y z\right\}$, then $M=x y z^{2}$ and

$$
\begin{aligned}
P_{T} & =[1] x y z^{2}+[z] x y z+[y] x z^{2}+[x] y z^{2}+\left[z^{2}\right] x y+[y z] x z \\
& =w_{0,0,0} x y z^{2}+w_{0,0,1} x y z+w_{0,1,0} x z^{2}+w_{1,0,0} y z^{2}+w_{0,0,2} x y+w_{0,1,1} x z
\end{aligned}
$$

In this paper, we will mostly deal with polynomials $P_{T+U}$ as there is a strong connection between $H_{U, T}$ and $P_{T+U}$.

Finally, letting $M=\operatorname{LCM}(T+U)=x_{1}^{D_{1}} \cdots x_{n}^{D_{n}}$ and $B$ be the monomial ideal $\left(x_{1}^{D_{1}+1}, \ldots, x_{n}^{D_{n}+1}\right)$, we will use pairs of multivariate polynomials $R_{m}=\left[F_{m}, C_{m}\right]$ where $\operatorname{Lm}\left(C_{m}\right)=m$ and $F_{m}=$ $P_{T+U} C_{m} \bmod B$.

## 3. From matrices to polynomials

Before detailing the unified polynomial viewpoint, we recall the linear algebra viewpoint of the BM, the BMS and the Scalar-FGLM algorithms.

### 3.1. The BM algorithm

Let $\mathbf{w}=\left(w_{i}\right)_{i \in \mathbb{N}}$ be a one-dimensional table. Classically, when calling the BM algorithm, one does not know in advance the order of the output relation. Therefore, from a matrix viewpoint, one wants to compute the greatest collection of vectors

$$
\left(\begin{array}{c}
\gamma_{1} \\
\vdots \\
\gamma_{x^{d-1}} \\
1 \\
0 \\
0 \\
\vdots \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
\gamma_{1} \\
\vdots \\
\gamma_{x^{d-1}} \\
1 \\
0 \\
\vdots \\
0
\end{array}\right), \ldots,\left(\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
\gamma_{1} \\
\vdots \\
\gamma_{x^{d-1}} \\
1
\end{array}\right)
$$

in the kernel of $H_{\left\{11,\left\{1, \ldots, x^{D}\right\}\right.}=1\left(\begin{array}{ccc}1 & \cdots & x^{D} \\ w_{0} & \cdots & w_{D}\end{array}\right)$, that is $\gamma_{1}, \ldots, \gamma_{x^{d-1}}$ such that the relation $\left[C_{x^{d}}\right]=w_{d}+\sum_{k=0}^{d-1} \gamma_{x^{k}} w_{k}$ and its shifts, $\left[x C_{x^{d}}\right], \ldots,\left[x^{D-d} C_{x^{d}}\right]$, are all 0 . Equivalently, we look for the least $d$ such that $H_{\mathcal{T}_{\leq x} D-d}, \mathcal{T}_{\leq x^{d}}\left(\begin{array}{c}\gamma_{1} \\ \vdots \\ \gamma_{x} d^{-1} \\ 1\end{array}\right)=0$.

This Hankel matrix-vector product can be extended into

$$
\left(\begin{array}{cccc}
w_{0} & \cdots & w_{d-1} & w_{d}  \tag{2}\\
w_{1} & \cdots & w_{d} & w_{d+1} \\
\vdots & & \vdots & \vdots \\
w_{D-d} & \cdots & w_{D-1} & w_{D} \\
w_{D-d+1} & \cdots & w_{D} & 0 \\
\vdots & . & . \cdot & \vdots \\
w_{D} & 0 & \cdots & 0
\end{array}\right)\left(\begin{array}{c}
\gamma_{1} \\
\vdots \\
\gamma_{x^{d-1}} \\
1
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
f_{x^{d-1}} \\
\vdots \\
f_{1}
\end{array}\right)
$$

representing the product of polynomials $P_{\mathcal{T}_{\leq x} D}=\sum_{i=0}^{D} w_{i} x^{D-i}$ and $C_{x^{d}}=x^{d}+\sum_{k=0}^{d-1} \gamma_{x^{k}} x^{k}$ modulo $B=x^{D+1}$. The requirement for $C_{x^{d}}$ to encode a valid relation is now that $\operatorname{Lm}\left(F_{x^{d}}\right)<x^{d}$ with $F_{x^{d}}=P_{\mathcal{T}_{\leq x^{D}}} C_{x^{d}} \bmod B$.

This viewpoint gave rise to the following version of the BM algorithm: Start with $R_{B}=$ $\left[F_{B}, C_{B}\right]=[B, 0]$ and $B=x^{D+1}$, and $R_{1}=\left[F_{1}, C_{1}\right]=\left[P_{\mathcal{T}_{\leq x D}}, 1\right]$. Compute $Q$ the quotient of the Euclidean division of $F_{B}=B$ by $F_{1}$ and then compute $R_{\mathrm{LM}(Q)}=R_{B}-Q R_{1}=\left[F_{B}-Q F_{1}, C_{B}-\right.$ $\left.Q C_{1}\right]=\left[F_{\mathrm{LM}(Q)}, C_{\mathrm{LM}(Q)}\right]$. Repeat with $R_{1}$ and $R_{\mathrm{LM}(Q)}$ until reaching a pair $R_{x^{d}}=\left[P_{\mathcal{T}_{\leq x^{D}}} C_{x^{d}} \bmod \right.$ $\left.B, C_{x^{d}}\right]=\left[F_{x^{d}}, C_{x^{d}}\right]$ with $\operatorname{Lm}\left(C_{x^{d}}\right)=x^{d}$ and $\operatorname{Lm}\left(F_{x^{d}}\right)<x^{d}$. This is in fact the extended Euclidean algorithm called on $B=x^{D+1}$ and $F_{1}$ without any computation of the Bézout's cofactors of $x^{D+1}$.

Example 9. Let us consider the Fibonacci table $\mathbf{F}=\left(F_{i}\right)_{i \in \mathbb{N}}$ with $F_{0}=F_{1}=1$ and assume $D=5$. On the one hand, although the kernel of

$$
H_{\{1\},\left\{1, x, x^{2}, x^{3}, x^{4}, x^{5}\right\}}=1 \quad\left(\begin{array}{cccccc}
1 & x & x^{2} & x^{3} & x^{4} & x^{5} \\
1 & 1 & 2 & 3 & 5 & 8
\end{array}\right)
$$

has dimension 5 and $\left(\begin{array}{c}-1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right)$ is in this kernel, it corresponds to $[x-1]=0$, its shifted vectors $\left(\begin{array}{c}0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{c}0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{c}0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0\end{array}\right),\left(\begin{array}{c}0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1\end{array}\right)$ are not, as they correspond to $\left[x^{i}(x-1)\right] \neq 0$, for $1 \leq i \leq 4$. However, these vectors

$$
\left(\begin{array}{c}
-1 \\
-1 \\
1 \\
0 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
-1 \\
-1 \\
1 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
-1 \\
-1 \\
1 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
0 \\
-1 \\
-1 \\
1
\end{array}\right)
$$

are in the kernel and form the greatest collection of shifted vectors as such. They correspond to $\left[x^{i}\left(x^{2}-x-1\right)\right]=0$, for $0 \leq i \leq 3$. Finally, we have

$$
\left(\begin{array}{lll}
1 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 5 \\
3 & 5 & 8 \\
5 & 8 & 0 \\
8 & 0 & 0
\end{array}\right) \underset{8}{\left(\begin{array}{c}
-1 \\
-1 \\
1
\end{array}\right)}=\underset{\left(\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
-13 \\
-8
\end{array}\right), ~}{( }
$$

where the gray zeroes (0) are due to the matrix extension and not the sequence itself.
On the other hand, $B=x^{6}, R_{B}=[B, 0]$ and $R_{1}=\left[x^{5}+x^{4}+2 x^{3}+3 x^{2}+5 x+8,1\right]$. As we can see $R_{1}=\left[F_{1}, C_{1}\right]$ with $\operatorname{LM}\left(C_{1}\right)=1$ and $\operatorname{LM}\left(F_{1}\right)=x^{5} \geq 1$.

The first step of the extended Euclidean algorithm yields $R_{x}=\left[x^{4}+x^{3}+2 x^{2}+3 x-8, x-1\right]=$ $\left[F_{x}, C_{x}\right]$ with $\operatorname{LM}\left(C_{x}\right)=x$ and $\operatorname{LM}\left(F_{x}\right)=x^{4} \geq x$.

Then, the second step yields $R_{x^{2}}=\left[-13 x-8, x^{2}-x-1\right]=\left[F_{x^{2}}, C_{x^{2}}\right]$ with $\operatorname{Lm}\left(C_{x^{2}}\right)=x^{2}$ and $\operatorname{Lm}\left(F_{x^{2}}\right)=x<x^{2}$ so $C_{x^{2}}$ is a valid relation. We return $C_{x^{2}}$.

Remark 10. The BM algorithm always returns a relation. If no pair $R_{x^{b}}=\left[F_{x^{b}}, C_{x^{b}}\right]$ satisfies the requirements, then it will return a pair $R_{x^{d}}$ with $\operatorname{Lg}\left(C_{x^{d}}\right)>x^{D}$. From a matrix viewpoint, it returns an element of the kernel of the empty matrix $H_{0, \mathcal{T}_{\leq 1}}$.

### 3.2. Multidimensional extension

For a multidimensional table $\mathbf{w}=\left(w_{\mathbf{i}}\right)_{i \in \mathbb{N}^{n}}$, the BMS algorithm extends the BM algorithm by computing vectors in the kernel of a multi-Hankel matrix

$$
\left.H_{\{1\}, T_{\leq a}}=1 \begin{array}{cccc}
1 & \cdots & a^{-} & a \\
{[1]} & \cdots & {\left[a^{-}\right]} & {[a]}
\end{array}\right)
$$

corresponding to having relations $\left[C_{g}\right]=0$, with $\operatorname{Lm}\left(C_{g}\right)=g$ minimal for $\mid$ and for all $t$ such that $t g \leq a,\left[t C_{g}\right]=0$ as well. This also comes down to finding the least (for the partial order |) monomials $g_{1}, \ldots, g_{r} \leq a$ such that dim $\operatorname{ker} H_{\mathcal{T}_{\leq s_{k}}} \mathcal{T}_{\leq s_{k}}>0$ with $s_{k}$ the greatest monomial such that $s_{k} g_{k} \leq a$ for all $k, 1 \leq k \leq r$. Then, each multi-Hankel matrix-vector product can be extended further as in equation 2, taking the multi-Hankel matrix $H_{\mathcal{T}_{\leq a}, \mathcal{T}_{\Sigma_{s k}}}$ and setting to zero any sequence term $[t u]$ with $t u \notin \mathcal{T}_{\leq a}$.

$$
\left(\begin{array}{cccc}
{[1]} & \cdots & {\left[g_{k}^{-}\right]} & g_{k}  \tag{3}\\
{\left[1^{+}\right]} & \cdots & {\left[1^{+} g_{k}^{-}\right]} & {\left[1^{+} g_{k}\right]} \\
\vdots & & \vdots & \vdots \\
{\left[s_{k}\right]} & \cdots & {\left[s_{k} g_{k}^{-}\right]} & {\left[s_{k} g_{k}\right]} \\
{\left[s_{k}^{+}\right]} & \cdots & {\left[s_{k}^{+} g_{k}^{-}\right]} & 0 \\
\vdots & & . & \vdots \\
{[a]} & 0 & \cdots & 0
\end{array}\right)\left(\begin{array}{c}
\gamma_{1} \\
\vdots \\
\gamma_{g_{k}^{-}} \\
1
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
f_{M / s_{k}^{+}} \\
\vdots \\
f_{M / a}
\end{array}\right),
$$

where $M=\operatorname{LCM}\left(\mathcal{T}_{\leq a}\right)=x_{1}^{D_{1}} \cdots x_{n}^{D_{n}}$. It then represents the product of polynomials $P_{\mathcal{T}_{\leq a}}=$ $\sum_{t \leq a}[t] \frac{M}{t}$ and $C_{g_{k}}=g_{k}+\sum_{t<g_{k}} \gamma_{t} t$ modulo $B=\left(x_{1}^{D_{1}+1}, \ldots, x_{n}^{D_{n}+1}\right)$. The requirement for $C_{g_{k}}$ to encode a valid relation is now that $\operatorname{Lm}\left(F_{g_{k}}\right)<\frac{M}{s_{k}}$ with $F_{g_{k}}=P_{\mathcal{T}_{\leq a}} C_{g_{k}} \bmod B$.

Let us notice that $\left[s_{k}^{+} g_{k}^{-}\right.$] can also be a 0 if $s_{k}^{+} g_{k}^{-}>a$ and that, more generally, the gray zeroes need not be diagonally aligned like they are in the univariate case. This is illustrated by the following example.

Example 11. Let us consider the binomial table $\mathbf{b}=\left(\begin{array}{l}\left.\binom{i}{j}\right)_{(i, j) \in \mathbb{N}^{2}} \text { with } \operatorname{DRL}(y<x) \text { and assume }\end{array}\right.$ $a=x^{2} y$. The kernel of

$$
H_{\{1\}, \mathcal{T}_{\leq x^{2} y}}=1 \quad\left(\begin{array}{ccccccccc}
1 & y & x & y^{2} & x y & x^{2} & y^{3} & x y^{2} & x^{2} y \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2
\end{array}\right)
$$

has clearly dimension 8. Two vectors in the kernel, $\left(\begin{array}{l}0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right)$ and $\left(\begin{array}{c}-1 \\ \alpha \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right)$,
independant relation $[y]=[x+\alpha y-1]=0$. However, not all their shifts belong to the kernel. For the former, $\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right)$, corresponding to $\left[y^{2}\right]=0$, belongs to the kernel but $\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0\end{array}\right)$ does not, as $[x y] \neq 0$. For the latter, $\left(\begin{array}{c}0 \\ -1 \\ 0 \\ \alpha \\ 1 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right)$ does not belong to the kernel, as $\left[x y+\alpha y^{2}-y\right] \neq 0$, whatever $\alpha$ is.

Therefore, the vectors in the kernel that we seek must correspond to relations $\left[C_{g}\right]=0$ with $g \in\left\{y^{2}, x y, x^{2}, \ldots, x^{2} y\right\}$.

In fact, as $\left[m y^{2}\right]=0$ for $1 \leq m \leq x$, the vectors $\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0\end{array}\right)$, are in the kernel and fulfill the requirements. We can indeed notice that
where the gray zeroes (0) are due to the matrix extension and not the binomial sequence itself. From the polynomial point of view,

$$
\begin{aligned}
& F_{y^{2}}=P_{\mathcal{T}_{\leq x^{2} y}} C_{y^{2}} \bmod B \\
&=\left(x^{2} y^{3}+x y^{3}+x y^{2}+y^{3}+2 y^{2}\right) y^{2} \bmod \left(x^{3}, y^{4}\right) \\
&=0 . \\
& \text { tors }\left(\begin{array}{c}
-(1+\alpha) \\
\beta \\
\alpha \\
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
-(1+\alpha) \\
0 \\
\beta \\
\alpha \\
0 \\
0 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
-(1+\alpha) \\
0 \\
\beta \\
\alpha \\
0 \\
0 \\
1
\end{array}\right), \text { are in the kernel for } \alpha=0 \text { and } \beta=-1,
\end{aligned}
$$

Likewise, the vectors as they correspond to $[m(x y+\alpha x+\beta y-(1+\alpha))]=[m(x y-y-1)]=0$, for $1 \leq m \leq x$.

## Furthermore,

$$
\begin{gathered}
\\
1 \\
y \\
x \\
y^{2} \\
x y \\
x^{2} \\
y^{3} \\
x y^{2} \\
x^{2} y
\end{gathered}\left(\begin{array}{ccccc}
1 & y & x & y^{2} & x y \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
-1 \\
-1 \\
0 \\
0 \\
1
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
-1 \\
-3 \\
0 \\
0 \\
-2
\end{array}\right),
$$

From the polynomial point of view, $F_{x y}=P_{\mathcal{T}_{\leq x^{2} y}}(x y-y-1) \bmod \left(x^{3}, y^{4}\right)=-x y^{2}-3 y^{3}-2 y^{2}$. Finally, the vectors $\left(\begin{array}{c}-(1+\alpha) \\ \beta \\ \alpha \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{c}0 \\ -(1+\alpha) \\ 0 \\ \beta \\ \alpha \\ 0 \\ 0 \\ 0 \\ 1\end{array}\right)$ are in the kernel for $\alpha=-2$ and $\beta=0$, as they correspond to $\left[m\left(x^{2}+\alpha x+\beta y-(1+\alpha)\right)\right]=\left[m\left(x^{2}-2 x+1\right)\right]=0$, for $1 \leq m \leq y$. Furthermore,

| 1 |
| :---: |
| 1 |
| $y$ |
| $x$ |
| $y^{2}$ |
| $x y$ |
| $x^{2}$ |
| $y^{3}$ |
| $x y^{2}$ |
| $x^{2} y$ |\(\left(\begin{array}{llllll}1 \& y \& x \& y^{2} \& x y \& x^{2} <br>

1 \& 0 \& 1 \& 0 \& 1 \& 1 <br>
0 \& 0 \& 1 \& 0 \& 0 \& 2 <br>
1 \& 1 \& 1 \& 0 \& 2 \& 0 <br>
0 \& 0 \& 0 \& 0 \& 0 \& 0 <br>
1 \& 0 \& 2 \& 0 \& 0 \& 0 <br>
1 \& 2 \& 0 \& 0 \& 0 \& 0 <br>
0 \& 0 \& 0 \& 0 \& 0 \& 0 <br>
0 \& 0 \& 0 \& 0 \& 0 \& 0 <br>
2 \& 0 \& 0 \& 0 \& 0 \& 0\end{array}\right)\left($$
\begin{array}{c}1 \\
0 \\
-2 \\
0 \\
0 \\
1\end{array}
$$\right)=\left($$
\begin{array}{c}0 \\
0 \\
-1 \\
0 \\
-3 \\
1 \\
0 \\
0 \\
2\end{array}
$$\right)\).

From the polynomial point of view, $F_{x^{2}}=P_{\mathcal{T}_{\leq x^{2} y}}\left(x^{2}-2 x+1\right) \bmod \left(x^{3}, y^{4}\right)=-x y^{3}-3 x y^{2}+$ $y^{3}+2 y^{2}$.

The BMS algorithm will return the three relations $C_{y^{2}}=y^{2}, C_{x y}=x y-y-1$ and $C_{x^{2}}=$ $x^{2}-2 x+1$.

Remark 12. As for the BM algorithm, the BMS algorithm will always return a relation $C_{g}$ with $\mathrm{Lm}\left(C_{g}\right)=g$ a pure power in each variable. Therefore, it can return $C_{g}$ with $g>a$, corresponding to a vector in the kernel of the empty matrix $H_{0, \mathcal{T}_{\leq g}}$.

The Scalar-FGLM algorithm corresponds to computing vectors in the kernel of a more general multi-Hankel matrix $H_{U, T}$, with $T$ and $U$ two ordered sets of monomials, corresponding also to relations $\left[C_{g}\right]=0$ with $\operatorname{Lm}\left(C_{g}\right)=g$, such that for all $t \in \mathcal{T}$, if $t g \in T$, then $\left[t C_{g}\right]=0$, i.e. the vectors corresponding to $t C_{g}$ are also in the kernel of $H_{U, T}$. We now consider that both sets of terms $T$ and $U$ satisfy $T=\mathcal{T}_{\leq a}$ and $U=\mathcal{T}_{\leq b}$. This allows us to encompass both the BMS algorithm and the Scalar-FGLM algorithm.

The multi-Hankel matrix-vector product

$$
\begin{gather*}
1  \tag{4}\\
\vdots \\
b^{-} \\
b
\end{gather*}\left(\begin{array}{cccc}
1 & \cdots & a^{-} & a \\
{[1]} & \cdots & {\left[a^{-}\right]} & {[a]} \\
\vdots & & \vdots & \vdots \\
{\left[b^{-}\right]} & \cdots & {\left[a^{-} b^{-}\right]} & {\left[a b^{-}\right]} \\
{[b]} & \cdots & {\left[a^{-} b\right]} & {[a b]}
\end{array}\right)\left(\begin{array}{c}
\gamma_{1} \\
\vdots \\
\gamma_{g^{-}} \\
1 \\
0 \\
\vdots \\
0
\end{array}\right)=\left(\begin{array}{l}
0 \\
\vdots \\
0 \\
0
\end{array}\right)
$$

can be extended in the same way as in equations (2) and (3) with rows up to monomial $a b$ and any table term $[t u]$ set to zero whenever $t u>a b$. This extension corresponds to the product of $P_{T+U}=\sum_{\tau \in(T+U)}[\tau] \frac{M}{\tau}$, where $M=x_{1}^{D_{1}} \cdots x_{n}^{D_{n}}=\operatorname{LCM}(T+U)$, and $C_{g}=g+\sum_{t<g} \gamma_{t} t$, with $g \leq a$, modulo $B=\left(x_{1}^{D_{1}+1}, \ldots, x_{n}^{D_{n}+1}\right)$. In that case, equation (4) is equivalent to asking that the coefficients of monomials $\frac{M}{u}$ of $F_{g}=P_{T+U} C_{g} \bmod B$ are all zero for $u \leq b$, i.e. $\operatorname{LM}\left(F_{g}\right)<\frac{M}{b}$. Thus, we are aiming for a Padé approximant of $P_{T+U}$ of the form $\frac{F_{g}}{C_{g}}$ with $\operatorname{LM}\left(F_{g}\right)<\frac{M}{b}$ and $\operatorname{lm}\left(C_{g}\right)=g$, see also Fitzpatrick and Flynn (1992).

The Scalar-FGLM algorithm requires to compute the kernel of $H_{\mathcal{T}_{\leq b}, \mathcal{T}_{\leq a}}$ as relations $\left[C_{g}\right]=0$ such that $\left[t C_{g}\right]=0$, for any $t \in \mathcal{T}$ such that $t g \in T$. Denoting $s$ the greatest monomial such that $s g \leq a$, this comes down to requiring that $C_{g}$ represents a kernel vector of the matrix $H_{\mathcal{T}_{\leq b}+\mathcal{T}_{\leq s}, \mathcal{T}_{\leq g}}$. As the set $\mathcal{T}_{\leq s}+\mathcal{T}_{\leq b}$ is stable by division, some monomials are missing compared to the set $\mathcal{T}_{\leq b s}$. Then, we can extend the matrix $H_{\mathcal{T}_{\leq b}+\mathcal{T}_{\leq s}, \mathcal{T}_{\leq g}}$ by adding all the rows with label in $\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}$ except those with monomials divisible by a monomial in $\mathcal{T}_{\leq b s} \backslash\left(\mathcal{T}_{\leq b}+\mathcal{T}_{\leq s}\right)$. Doing so, the extended matrix-vector product does not encode the polynomial product $F_{g}=P_{T+U} C_{g} \bmod B$ any more, as some monomials are missing. Specifically, monomials $\frac{M}{m}$ where $m$ is a multiple of a monomial in $\mathcal{T}_{\leq b s} \backslash\left(\mathcal{T}_{\leq b}+\mathcal{T}_{\leq s}\right)$.

Therefore, it comes down to finding |-minimal pairs $R_{g}=\left[F_{g}, C_{g}\right]=\left[P_{\mathcal{T}_{\leq b}, \mathcal{T}_{\leq a}} C_{g} \bmod B, C_{g}\right]$ with $g=\operatorname{Lm}\left(C_{g}\right) \leq a$ and a condition on $F_{g}$ so that $C_{g}$ represents a vector in the kernel of $H_{U_{s}, \tau_{\leq g}}$ with $U_{s}$ as large as possible. It is clear that this matrix should only have table terms [ $\tau$ ] that also appear in $H_{\mathcal{T}_{\leq b}, \mathcal{T}_{\leq a}}$, i.e. $\tau \in(T+U)$ and that $s$ should be the greatest monomial as such. Hence, we have the following proposition.

Proposition 13. Let $T=\mathcal{T}_{\leq a}$ and $U=\mathcal{T}_{\leq b}$ be finite sets of monomial terms in $\mathbb{K}[\mathbf{x}]$, let $M=$ $\operatorname{LCM}(T+U)=x_{1}^{D_{1}} \cdots x_{n}^{D_{n}}$ and $B=\left(x_{1}^{D_{1}+1}, \ldots, x_{n}^{D_{n}+1}\right)$.

Let $C_{g}$ be a polynomial with support in $T$ and with leading monomial $g$ and let $s$ be the greatest monomial such that $s g \leq a$.

Let $\mathcal{G}_{s}$ be a minimal set of monomials generating the sets of monomials less than $b s$ but not in $\mathcal{T}_{\leq s}+\mathcal{T}_{\leq b}$, i.e. $\mathcal{G}_{s}$ is a Gröbner basis of $\mathcal{T}_{\leq b s} \backslash\left(\mathcal{T}_{\leq s}+\mathcal{T}_{\leq b}\right)$.

From $F_{g}=P_{T+U} C_{g} \bmod B=\sum_{\tau \in(T+U)} f_{g, \tau} \frac{M}{\tau}$, we let $\tilde{F}_{g}$ be the polynomial obtained from $F_{g}$ by setting to zero all the coefficients of monomials $\frac{M}{m u}$ of $F_{g}$, with $m u \mid M$ and $u \in \mathcal{G}_{s}$. That is, $\tilde{F}_{g}=\sum_{\tau \in(T+U)} \tilde{f}_{g, \tau} \frac{M}{\tau}$, where NormalForm $\left(\sum_{\tau \in(T+U)} f_{g, \tau} \tau, \mathcal{G}_{s}\right)=\sum_{\tau \in(T+U \tilde{U}} \tilde{f}_{g, \tau} \tau$.

Then, $\left[u \sigma C_{g}\right]=0$ for all $u \in U$ and all $\sigma \in \mathcal{T}_{\leq s}$ if, and only if, $\operatorname{LM}\left(\tilde{F}_{g}\right)<\frac{M}{b s}$.
Example 14. We still consider the binomial table $\mathbf{b}$ with $\operatorname{DRL}(y<x)$. We let $T=U=\mathcal{T}_{\leq x^{2} y}$ so
that

$$
H_{U, T}=\begin{gathered}
\\
1 \\
y \\
x \\
y^{2} \\
x y \\
x^{2} \\
y^{3} \\
x y^{2} \\
x^{2} y
\end{gathered}\left(\begin{array}{ccccccccc}
1 & y & x & y^{2} & x y & x^{2} & y^{3} & x y^{2} & x^{2} y \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 2 & 1 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 3 & 0 & 0 & 3 \\
1 & 2 & 1 & 1 & 3 & 1 & 0 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 1 \\
2 & 1 & 3 & 0 & 3 & 4 & 0 & 1 & 6
\end{array}\right)
$$

The computation of the kernel of this matrix yields the vectors $\left(\begin{array}{c}-1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{c}0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0\end{array}\right),\left(\begin{array}{c}0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1\end{array}\right)$ corresponding to $[u(x y-y-1)]=[u y(x y-y-1)]=[u x(x y-y-1)]=0$ for all $u \in U$ and the vector 000000 corresponding to $\left[u y^{3}\right]=0$ for all $u \in U$.

For $g=x y$, we thus have $s=x$ and the extended matrix-vector product

|  | $1 \begin{array}{lllll} & y & x & y^{2} & x y\end{array}$ |  |
| :---: | :---: | :---: |
| 1 | $\left(\begin{array}{lllll}1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0\end{array}\right)$ | $(0)$ |
| $y$ | $\begin{array}{lllll}0 & 0 & 1 & 0 & 0\end{array}$ | 0 |
| $x$ | $\begin{array}{lllll}1 & 1 & 1 & 0 & 2\end{array}$ | 0 |
| $y^{2}$ | $0 \begin{array}{lllll}0 & 0 & 0 & 0 & 0\end{array}$ | 0 |
| xy | $\begin{array}{lllll}1 & 0 & 2 & 0 & 1\end{array}$ | 0 |
| $x^{2}$ | $\begin{array}{lllll}1 & 2 & 1 & 1 & 3\end{array}$ | 0 |
| $y^{3}$ | 0 | 0 |
| $x y^{2}$ | $\begin{array}{lllll}0 & 0 & 1 & 0 & 0\end{array}$ | 0 |
| $x^{2} y$ | $\begin{array}{lllll}2 & 1 & 3 & 0 & 3\end{array}$ | (-1) 0 |
| $x^{3}$ | $\begin{array}{lllll}1 & 3 & 1 & 3 & 4\end{array}$ | $\binom{-1}{-1} \quad 0$ |
| $y^{4}$ | $\begin{array}{lllll}0 & 0 & 0 & 0 & 0\end{array}$ | $\binom{-1}{0}=0$ |
| $x y^{3}$ | 0 | 0 |
| $x^{2} y^{2}$ | $1 \begin{array}{lllll}1 & 0 & 3 & 0 & 1\end{array}$ | $\binom{0}{1} \quad 0$ |
| $x^{3} y$ | $\begin{array}{lllll}3 & 3 & 4 & 1 & 6\end{array}$ | (1) 0 |
| $y^{5}$ | 0 | 0 |
| $x y^{4}$ | 0 | 0 |
| $x^{2} y^{3}$ | $\begin{array}{lllll}0 & 0 & 1 & 0 & 0\end{array}$ | 0 |
| $x^{3} y^{2}$ | $\begin{array}{lllll}3 & 1 & 6 & 0 & 0\end{array}$ | -4 |
| $y^{6}$ | $0 \begin{array}{lllll}0 & 0 & 0 & 0 & 0\end{array}$ | 0 |
| $x y^{5}$ | $\begin{array}{lllll}0 & 0 & 0 & 0 & 0\end{array}$ | 0 |
| $x^{2} y^{4}$ | $\begin{array}{llllll}0 & 0 & 0 & 0 & 0\end{array}$ | 0 |
| $x^{3} y^{3}$ | $\left(\begin{array}{lllll}1 & 0 & 0 & 0 & 0\end{array}\right)$ | (-1) |

Since $\mathcal{T}_{\leq x}+\mathcal{T}_{\leq x^{2} y}=\mathcal{T}_{\leq x^{3} y} \backslash\left\{x^{4}\right\}$, we did not add any row with label multiple of $x^{4}$.
On the other hand, the polynomial $F_{x y}=P_{T+U}(x y-y-1) \bmod \left(x^{5}, y^{7}\right)=-5 y^{6}-4 x y^{4}-$ $10 y^{5}-x y^{3}-6 y^{4}$ and thus $\tilde{F}_{x y}=-4 x y^{4}-x y^{3}=\sum_{\tau \in(T+U)} \tilde{f}_{g, \tau} \frac{M}{\tau}$ is obtained from $F_{x y}$ by removing any monomial $\frac{x^{4} y^{6}}{m x^{4}}$. The vector $\left(\begin{array}{c}-1 \\ -1 \\ 0 \\ 0 \\ 1\end{array}\right)$ being in the kernel of $H_{\mathcal{T}_{\leq x}+\mathcal{T}_{\leq x^{2} y}, \mathcal{T}_{\leq x y}}$ is then equivalent to asking that $\operatorname{LM}\left(\tilde{F}_{x y}\right)<x y^{5}=\frac{x^{4} y^{6}}{x^{3} y}=\frac{M}{b s}$.

For $g=y^{3}, s=1$, we have

The polynomial $F_{y^{3}}=P_{T+U} y^{3} \bmod \left(x^{5}, y^{7}\right)=x y^{6}$ and thus $\tilde{F}_{y^{3}}=y^{3}$. The vector $\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1\end{array}\right)$ being in the kernel of $H_{\mathcal{T}_{\leq x^{2},}, \mathcal{T}_{\leq y^{3}}}$ is then equivalent to asking that $\operatorname{LM}\left(\tilde{F}_{y^{3}}\right)<x^{2} y^{5}=\frac{x^{4} y^{6}}{x^{2} y}$.

## 4. A division-based algorithm

The goal is now to design an algorithm based on polynomial division to determine all the $C_{g}$ for |-minimal $g$ such that $\operatorname{Lm}\left(\tilde{F}_{g}\right)$ is small enough, where $F_{g}=P_{T+U} C_{g} \bmod B$ and $\tilde{F}_{g}$ is the higher part of $F_{g}$ obtained as in Proposition 13 .

We start with two sets of terms $T=\mathcal{T}_{\leq a}$ and $U=\mathcal{T}_{\leq b}$ so that $M=x_{1}^{D_{1}} \cdots x_{n}^{D_{n}}=\operatorname{LCM}(T+U)$. We initialize $B=\left(B_{1}, \ldots, B_{n}\right)=\left(x_{1}^{D_{1}+1}, \ldots, x_{n}^{D_{n}+1}\right), R_{B_{1}}=\left[B_{1}, 0\right], \ldots, R_{B_{n}}=\left[B_{n}, 0\right]$ and $R_{1}=$ [ $\left.P_{T+U}, 1\right]$.

For any $R_{g}=\left[F_{g}, C_{g}\right]=\left[P_{T+U} C_{g} \bmod B, C_{g}\right]$, by Proposition 13, $C_{g}$ is a valid relation if $g \in T$ and $\operatorname{Lm}\left(\tilde{F}_{g}\right)<\frac{M}{b s}$ with $s=\max \{\sigma, \sigma g \leq a\}$. To go along with the fact that the BMS algorithm always returns a relation $C_{g}$ with $g=\operatorname{Lm}\left(C_{g}\right)$ a pure power of a variable, if $g \notin T$, then $C_{g}$ will automatically be considered a valid relation as well.

From a failing relation $C_{m}$, we get that $m$ is in the staircase of the Gröbner basis of relations. Thus, each time a built relation is not valid, we update the staircase of the ideal of relations. At each step, we know the staircase $S$ and equivalently the set $\mathcal{H}=\min _{\mid}\{h \in \mathcal{T} \backslash S\}$ which are the leading terms of the candidate relations.

For $h \in \mathcal{H}$, we now want to build $R_{h}$ with the least $\operatorname{Lm}\left(F_{h}\right)$.
Instruction 15. Pick a failing pair $R_{m}=\left[F_{m}, C_{m}\right]$ with $h=q m$.

1. if there exists another failing pair $R_{m^{\prime}}=\left[F_{m^{\prime}}, C_{m^{\prime}}\right]$ such that $\operatorname{LM}\left(F_{m^{\prime}}\right)=q \operatorname{LM}\left(F_{m}\right)$, then compute $R_{h}$ as the normal form of $R_{m^{\prime}}$ wrt. the list $\left[R_{m}, R_{B_{1}}, \ldots, R_{B_{n}}, R_{t_{1}}, \ldots, R_{t_{n}}\right]$ where $C_{t_{1}}, \ldots, C_{t_{r}}$ are failing relations and $\operatorname{Lm}\left(F_{t_{1}}\right)>\cdots>\operatorname{Lm}\left(F_{t_{r}}\right)$. To do so, compute first $Q_{m}, Q_{B_{1}}, \ldots, Q_{B_{n}}, Q_{t_{1}}, \ldots, Q_{t_{r}}$ the quotients of the division of $F_{m^{\prime}}$ by the list of polynomials $\left[F_{m}, B_{1}, \ldots, B_{n}, F_{t_{1}}, \ldots, F_{t_{r}}\right]$ and then return $R_{h}=R_{m^{\prime}}-Q_{m} R_{m}-Q_{B_{1}} R_{B_{1}}-\cdots-Q_{B_{n}} R_{B_{n}}-$ $Q_{t_{1}} R_{t_{1}}-\cdots-Q_{t_{r}} R_{t_{r}}$.
2. otherwise, compute $R_{h}$ as the normal form of $q R_{m}$ wrt. to the list $\left[R_{B_{1}}, \ldots, R_{B_{n}}, R_{t_{1}}, \ldots, R_{t_{n}}\right]$.

Remark 16. If $q \operatorname{Lm}\left(F_{m}\right)$ is in the ideal spanned by $B$, then case 2 of Instruction 15 is equivalent to computing the normal form of $\left[q \operatorname{LM}\left(F_{m}\right), 0\right]$ wrt. $\left[R_{m}, R_{B_{1}}, \ldots, R_{B_{n}}, R_{t_{1}}, \ldots, R_{t_{r}}\right]$. In fact, at the start, $R_{1}=\left[P_{T+U}, 1\right]$ fails when shifted by a monomial $s=x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}$ and we have to make new pairs $R_{x_{1}^{i_{1}+1}}, \ldots, R_{x_{n}^{i_{n}+1}}$. Since $\operatorname{LM}\left(P_{T+U}\right)=\frac{M}{s}$, then these pairs can be computed as the normal forms of $\left[x_{k}^{i_{k}+1} \frac{M}{s}, 0\right]=\left[B_{k}, 0\right] M^{\prime}$, with $M^{\prime} \in \mathcal{T}$, wrt. the ordered list $\left[R_{1}, R_{B_{1}}, \ldots, R_{B_{n}}\right]$. In dimension 1, this comes down to reducing $\left[x_{1}^{D_{1}+1}, 0\right]=\left[B_{1}, 0\right]=R_{B_{1}}$ wrt. $\left[R_{1}, R_{B_{1}}\right]$, and thus $R_{1}$ only. This is indeed the first step of the extended Euclidean algorithm called on $B_{1}$ and $F_{1}$ as described in Section 3.1
 be the $\operatorname{DRL}(y<x)$ monomial ordering and $T=\mathcal{T}_{\leq x^{3}}$ and $U=\{1\}$ be sets of terms. We have $a=x^{3}$, $b=1, T=T+U$ and $M=\operatorname{LCM}(T)=x^{3} y^{3}$ so that $P_{T}=x^{3} y^{3}+x^{2} y^{3}+x^{2} y^{2}+x y^{3}+2 x y^{2}+y^{3}$, $R_{1}=\left[P_{T}, 1\right]=\left[F_{1}, C_{1}\right]$ and $R_{B_{1}}=\left[x^{4}, 0\right], R_{B_{2}}=\left[y^{4}, 0\right]$.

- $m=1$, thus $s=x^{3}$ and as $\operatorname{Lm}\left(F_{1}\right)=\operatorname{Lm}\left(P_{T}\right)=x^{3} y^{3}=\frac{M}{1}$, then the relation $C_{1}$ fails when shifting by 1 so that 1 is in the staircase. Thus $\mathcal{H}=\{y, x\}$. We create $R_{y}$ by computing the normal form of $\left[y \operatorname{LM}\left(F_{1}\right), 0\right]=\left[x^{3} y^{4}, 0\right]$ wrt. $\left[R_{1}, R_{B_{1}}, R_{B_{2}}\right]$ and get $R_{y}=\left[F_{y}, C_{y}\right]=$ $\left[x^{2} y^{3}+2 x y^{3}, y\right]$. Likewise $R_{x}=\left[F_{x}, C_{x}\right]=\left[x^{3} y^{2}+x^{2} y^{2}-2 x y^{2}-y^{3}, x-1\right]$.
- $m=y$, thus $s=x^{2}$ and as $\operatorname{LM}\left(F_{y}\right)=x^{2} y^{3}=\frac{M}{x}$, then the relation $C_{y}$ fails when shifting by $x$ so that $y$ is in the staircase. Thus $\mathcal{H}=\left\{x, y^{\frac{x}{2}}\right\}$. We create
$-R_{y^{2}}=\left[0, y^{2}\right]$ by computing the normal form of $\left[y \mathrm{LM}\left(F_{y}\right), 0\right]=\left[x^{2} y^{4}, 0\right]$ wrt. $\left[R_{y}, R_{B_{1}}, R_{B_{2}}, R_{1}, R_{x}\right]$.
- $m=x$, thus $s=x^{2}$ and as $\operatorname{Lm}\left(F_{x}\right)=x^{3} y^{2}=\frac{M}{y}$, then the relation $C_{x}$ fails when shifting by $y$ so that $x$ is in the staircase. Thus $\mathcal{H}=\left\{y^{2}, x y, x^{2}\right\}$. We create
$-R_{x y}=\left[-x^{2} y^{2}-3 x y^{3}-2 x y^{2}-y^{3}, x y-y-1\right]$ by computing the normal form of $R_{1}$ wrt. [ $\left.R_{y}, R_{B_{1}}, R_{B_{2}}, R_{x}\right]$;
$-R_{x^{2}}=\left[-3 x^{2} y^{2}-x y^{3}+2 x y^{2}+y^{3}, x^{2}-2 x+1\right]$ by computing the normal form of $\left[x \operatorname{LM}\left(F_{x}\right), 0\right]=\left[x^{4} y^{2}, 0\right]$ wrt. $\left[R_{x}, R_{B_{1}}, R_{B_{2}}, R_{1}, R_{y}\right]$.
- $m=y^{2}$, thus $s=x$ and as $F_{y^{2}}=0$, then the relation is necessarily valid.
- $m=x y$, thus $s=x$ and as $\operatorname{LM}\left(F_{x y}\right)=x^{2} y^{2}=\frac{M}{x y}$, then the relation $C_{x y}$ is valid.
- $m=x^{2}$, thus $s=x$ and, likewise, as $\operatorname{LM}\left(F_{x^{2}}\right)=x^{2} y^{2}=\frac{M}{x y}$, then the relation $C_{x^{2}}$ is valid.

We return $C_{y^{2}}=y^{2}, C_{x y}=x y-y-1$ and $C_{x^{2}}=x^{2}-2 x+1$.
The algorithm is the following Algorithm 4.18. It uses functions
$\operatorname{NormalForm}\left(R_{m},\left[R_{B_{1}}, \ldots, R_{B_{n}}, R_{t_{1}}, \ldots, R_{t_{r}}\right]\right)$, for computing the normal form of $\left[F_{m}, C_{m}\right]$ wrt. to the list $R_{B_{1}}, \ldots, R_{B_{n}}, R_{t_{1}}, \ldots, R_{t_{r}}$ with $\operatorname{LM}\left(F_{t_{1}}\right)>\cdots>\operatorname{LM}\left(F_{t_{r}}\right)$;
Stabilize $(S)$, for computing the true staircase containing $S$, i.e. all the divisors of terms in $S$;
$\operatorname{Border}(S)$, for computing the least terms for $\mid$ outside of $S$.

Algorithm 4.18: Polynomial Scalar-FGLM
Input: A table $\mathbf{w}=\left(w_{\mathbf{i}}\right)_{i \in \mathbb{N}^{n}}$ with coefficients in $\mathbb{K}$, a monomial ordering $<$ and two monomials $a$ and $b$.
Output: A set $G$ of relations generating the ideal of relations.
$T:=\{t \in \mathcal{T}, t \leq a\}, U:=\{u \in \mathcal{T}, u \leq b\}$.
$M:=\operatorname{LCM}(T) \operatorname{LCM}(U)$.
For $i$ from 1 to $n$ do $R_{B_{i}}:=\left[x_{i}^{1+\operatorname{deg}_{x_{i}} M}, 0\right]$.
// pairs on the edge
$P:=\sum_{\tau \in(T+U)}[\tau] \frac{M}{\tau}$.
// the mirror of the truncated generating series
$R:=\{[P, 1]\} . \quad / /$ set of pairs $\left[F_{m}, C_{m}\right]=\left[P \cdot C_{m} \bmod B, C_{m}\right]$ to be tested
$R^{\prime}:=\emptyset$.
$G:=\emptyset, S:=\emptyset$.

> // set of failing pairs

While $R \neq \emptyset$ do

$$
R_{m}=\left[F_{m}, C_{m}\right]:=\text { first element of } R \text { and remove it from } R .
$$

If $m \notin T$ or $\operatorname{LM}\left(\tilde{F}_{m}\right)$ is as in Proposition 13 then // good relation

$$
G:=G \cup\left\{C_{m}\right\}
$$

Else
// bad relation
$R^{\prime}:=R^{\prime} \cup\left\{R_{m}\right\}$.
For all $r \in R$ do $\quad / /$ reduce next pairs with it
$r:=\operatorname{NormalForm}\left(r,\left[R_{B_{1}}, \ldots, R_{B_{r}}, R_{m}\right]\right)$.
$S:=$ Stabilize $(S \cup\{m\})$.
$H:=\operatorname{Border}(S)$.
For all $h \in H$ do
// compute new pairs
If there is no relation $C_{h} \in G$ or no pair $R_{h} \in R$ then
Make a new pair $R_{h}=\left[F_{h}, C_{h}\right]$ following Instruction 15 and add it to $R$.

Return $G$.

Remark 19. Like the BMS algorithm, this algorithm creates new potential relations by making polynomial combinations offailing relations. As a consequence, at each step of the main loop, the potential relations, i.e. elements of $R$, are not necessarily interreduced. Either we can interreduce the final Gröbner basis before returning it at the last line of the algorithm, or when $C_{g}$ is added to the set $G$ we can update all the current relations by removing multiples of $\left[F_{g}, C_{g}\right]$ and likewise, reduce by $\left[F_{g}, C_{g}\right]$, any subsequent pair $\left[F_{m}, C_{m}\right]$.
Example 20 (See Berthomieu and Faugère (2018)). We give the trace of the Polynomial ScalarFGLM algorithm with the slight modification above called on the table $\mathbf{w}=((2 i+1)+(2 j-$ 1) $\left.(-1)^{i+j}\right)_{(i, j) \in \mathbb{N}^{2}}$, the stopping monomials 1 and $y^{5}$ and the monomial ordering $\operatorname{DRL}(y<x)$.

We set $T:=\mathcal{T}_{\leq y^{5}}, U:=\{1\}, M:=x^{4} y^{5}$ and $P=4 x^{3} y^{5}+4 x^{4} y^{3}+4 x^{3} y^{4}+4 x^{2} y^{5}-4 x^{4} y^{2}+$ $4 x^{2} y^{4}+8 x y^{5}+8 x^{4} y+8 x^{3} y^{2}+8 x^{2} y^{3}+8 x y^{4}+8 y^{5}-8 x^{4}, R_{B_{1}}:=\left[x^{5}, 0\right], R_{B_{2}}:=\left[y^{6}, 0\right]$, $R=[[P, 1]]$.

Pair $R_{1}=\left[F_{1}, C_{1}\right]=[P, 1], R:=\emptyset$ and since $1 \in T$ but $\operatorname{Lm}\left(F_{1}\right)=x^{3} y^{5} \geq \frac{M}{s}=x^{4}$, then

- $R^{\prime}:=\left\{R_{1}\right\}, S:=\{1, x\}$ and $H:=\left\{y, x^{2}\right\}$.
- We make new pairs added to $R$ :
- $R_{y}=\left[F_{y}, C_{y}\right]:=\operatorname{NormalForm}\left(\left[y \operatorname{Lm}\left(F_{1}\right), 0\right],\left[R_{1}, R_{B_{1}}, R_{B_{2}}\right]\right)$ which can be normalized into $R_{y}$, $=\left[4 x^{4} y^{4}-\cdots, y-1\right]$;
$-R_{x^{2}}=\left[F_{x^{2}}, C_{x^{2}}\right]:=\operatorname{NormalForm}\left(\left[x^{2} \operatorname{LM}\left(F_{1}\right), 0\right],\left[R_{1}, R_{B_{1}}, R_{B_{2}}\right]\right)$ which can be normalized into $R_{x^{2}}=\left[4 x^{4} y^{3}-\cdots, x^{2}-x-1\right]$.
Pair $R_{y}=\left[F_{y}, C_{y}\right], R:=\left\{R_{x^{2}}\right\}$ and since $y \in T$ but $\operatorname{LM}\left(F_{y}\right)=x^{4} y^{4} \geq \frac{M}{s}=x^{4} y$, then
- $R^{\prime}:=\left\{R_{1}, R_{y}\right\}, S:=\{1, y, x\}$ and $H:=\left\{y^{2}, x y, x^{2}\right\}$.
- We make new pairs added to $R$ :
- As $y \operatorname{LM}\left(F_{y}\right)=x^{4} y^{5} \notin\left\langle x^{5}, y^{6}\right\rangle$ and $\operatorname{LM}\left(F_{1}\right) \neq y \operatorname{LM}\left(F_{y}\right)$, we can only set $R_{y^{2}}=$ [ $\left.F_{y^{2}}, C_{y^{2}}\right]:=\operatorname{NormalForm}\left(y R_{y},\left[R_{B_{1}}, R_{B_{2}}, R_{1}, R_{y}\right]\right)$ which can be normalized into $R_{y^{2}}=\left[-4 x^{4} y^{3}-\cdots, y^{2}-x+2 y-1\right]$;
$-R_{x y}=\left[F_{x y}, C_{x y}\right]:=\operatorname{NormalForm}\left(\left[x \operatorname{LM}\left(F_{y}\right), 0\right],\left[R_{y}, R_{B_{1}}, R_{B_{2}}, R_{1}\right]\right)$ which can be normalized into $R_{x y}=\left[4 x^{4} y^{2}-\cdots, x y-x+y-1\right]$.
- Nothing is done for $x^{2}$ since $R_{x^{2}}$ already exists.

Pair $R_{y^{2}}=\left[F_{y^{2}}, C_{y^{2}}\right], R:=\left\{R_{x y}, R_{x^{2}}\right\}$ and since $y^{2} \in T$ but $\operatorname{LM}\left(F_{y^{2}}\right)=x^{4} y^{3} \geq \frac{M}{s}=x^{4} y^{2}$, then

- As $\operatorname{Lm}\left(F_{x^{2}}\right) \geq \operatorname{LM}\left(F_{y^{2}}\right)$, we reduce it and obtain $R_{x^{2}}:=\left[-8 x^{2} y^{4}-\cdots, x^{2}+y^{2}-2 x+\right.$ $2 y-2]$.
- $R^{\prime}:=\left\{R_{1}, R_{y}, R_{y^{2}}\right\}, S:=\left\{1, y, x, y^{2}\right\}$ and $H:=\left\{x y, x^{2}, y^{3}\right\}$.
- We make new pairs added to $R$ :
- $R_{x y}$ and $R_{x^{2}}$ already exist so we do nothing for them.
- Since $\operatorname{Lm}\left(F_{y}\right)=y \operatorname{LM}\left(F_{y^{2}}\right)$, we can set $R_{y^{3}}=\left[F_{y^{3}}, C_{y^{3}}\right]:=\operatorname{NormalForm}\left(R_{y},\left[R_{y^{2}}, R_{B_{1}}, R_{B_{2}}, R_{y}, R_{1}\right]\right)$ which can be normalized into $R_{y^{3}}=\left[4 x^{3} y^{4}-\cdots, y^{3}-x y+y^{2}+x-2 y\right]$.

Pair $R_{x y}=\left[F_{x y}, C_{x y}\right], R:=\left\{R_{x^{2}}, R_{y^{3}}\right\}$ and since $x y \in T$ and $\operatorname{LM}\left(F_{x y}\right)=x^{4} y^{2}<\frac{M}{s}=x^{2} y^{5}$, then

- $G:=\{x y-x+y-1\}$.
- As $C_{y^{3}}=y^{3}-x y+y^{2}+x-2 y$ has a term in $x y$, we update $R_{y^{3}}:=R_{y^{3}}+R_{x y}=$ $\left[4 x^{3} y^{4}-\cdots, y^{3}+y^{2}-y-1\right]$.
Pair $R_{x^{2}}=\left[F_{x^{2}}, C_{x^{2}}\right], R:=\left\{R_{y^{3}}\right\}$ and since $x^{2} \in T$ and $\operatorname{LM}\left(F_{x^{2}}\right)=x^{2} y^{4}<\frac{M}{s}=x^{2} y^{5}$, then
- $G:=\left\{x y-x+y-1, x^{2}+y^{2}-2 x+2 y-2\right\}$.

Pair $R_{y^{3}}=\left[F_{y^{3}}, C_{y^{3}}\right], R:=\emptyset$ and since $y^{3} \in T$ and $\operatorname{Lm}\left(F_{y^{3}}\right)=x^{3} y^{4}<\frac{M}{s}=x^{4} y^{3}$, then

- $G:=\left\{x y-x+y-1, x^{2}+y^{2}-2 x+2 y-2, y^{3}+y^{2}-y-1\right\}$.

We return $G$.
Theorem 21. Let a table $\mathbf{w}$, a monomial ordering < and two monomials $a$ and $b$ be the input of the Polynomial Scalar-FGLM algorithm. Let us assume that the Gröbner basis $\mathcal{G}$ of the ideal of relations of $\mathbf{w}$ for $<$ and its staircase $S$ satisfy $a \geq \max (S \cup \operatorname{Lm}(\mathcal{G}))$ and for all $g \leq a$, $s=\max \{\sigma \in \mathcal{T}, \sigma g \leq a\}$, we have $\max (S) \leq s$.

Then, the Polynomial Scalar-FGLM algorithm terminates and computes a Gröbner basis of the ideal of relations of $\mathbf{w}$ for $<$ in $O\left(\# S(\# S+\# G) \#\left(\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}\right)\right)$ operations in the base field.

Proof. The proof is mainly based on the termination and validity of the BMS algorithm. For any monomial $m \in \mathcal{T}_{\leq a}$, we denote by $C_{m}^{\star}$ the last (and therefore one with the largest fail) relation made by the BMS algorithm starting with $m$, if there is any.

Starting with $R_{1}=\left[F_{1}, C_{1}\right]=\left[P_{\left.\mathcal{T}_{\leq a}+\tau_{\leq b}, 1\right], \operatorname{LM}\left(F_{1}\right) \text { yields exactly the fail of relation } C_{1}=C_{1}^{\star} .}^{\text {and }}\right.$ so that, as in the BMS algorithm, we know the leading monomials of the potential next relations.

Let us assume now that for any monomial $\mu<h$, the pair $R_{\mu}=\left[F_{\mu}, C_{\mu}\right]$ made by the Polynomial Scalar-FGLM algorithm is equivalent to $C_{\mu}^{\star}$, that is either both $C_{\mu}$ and $C_{\mu}^{\star}$ fail when shifting by exactly the same monomial or they both succeed on $\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}$.

Since $C_{\mu}$ and $C_{\mu}^{\star}$ are equivalent, the current discovered staircase by the BMS and the Polynomial Scalar-FGLM algorithms are the same. Thus either $h$ is a leading monomial of a relation to be built by both algorithms or it is not. Without loss of generality, we can assume it is. There exists a monomial $m$ such that $m \mid h$ and $R_{m}=\left[F_{m}, C_{m}\right]$ and $C_{m}^{\star}$ have been made. In the BMS algorithm, the relation $C_{h}^{\star}$ is obtained as $\frac{h}{m} C_{m}^{\star}-\sum_{\mu<h} q_{\mu}^{\star} C_{\mu}^{\star}$ while in the Polynomial Scalar-FGLM algorithm, $C_{h}$ is made as $\frac{h}{m} C_{m}-\sum_{\mu<h} q_{\mu} C_{\mu}$. In each computation, $q_{\mu}^{\star}$ and $q_{\mu}$ are chosen so that $C_{m}^{\star}$ and $C_{m}$ have the largest fail (or equivalently $\tilde{F}_{m}$ has the least leading monomial), hence $C_{m}^{\star}$ and $C_{m}$ are equivalent. For $h \in S$, the potential relation $C_{h}$ made by the algorithm must fail when shifted by a monomial in $S$. Thus, there exist $\sigma_{1}, \sigma_{2}$ such that $\sigma_{1} \sigma_{2} \in S, \sigma_{1} h \leq a, \sigma_{2} \leq b$ and the column labeled with $\sigma_{1} h$ of the matrix $H_{\mathcal{T}_{\leq b}, \mathcal{T}_{\leq a}}$ is independent from the previous ones. For $g \in \operatorname{lm}(\mathcal{G})$, by Section 3.2, the relation $C_{g}$ has been tested shifted by all the monomials in $\mathcal{T}_{\leq s}+\mathcal{T}_{\leq b}$, with $s=\max \{\sigma \in \mathcal{T}, \sigma g \leq a\}$. The theorem hypothesis is exactly that the full staircase is included in the set of tested shifts, hence we can ensure that $C_{g}$ corresponds to a kernel vector of $H_{S, S \cup\{g\}}$ with the last coordinate equal to 1 .

Concerning the complexity of the algorithm. Since $\mathcal{T}_{\leq a}$ and $\mathcal{T}_{\leq b}$ are stable by division, so is $\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}$. Let us recall that the support of $P_{\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}}$ is $\left\{\frac{M}{\tau}, \tau \in\left(\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}\right)\right\}, M=\operatorname{LCM}\left(\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}\right)$. Since each $F_{m}$ satisfies $F_{m}=P_{\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}} C_{m} \bmod B$, then the monomials in the support of $F_{m}$ are multiples of the monomials in the support of $P_{\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}}$ and thus are included in the support of $P_{\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}}$. Each pair $R_{m}=\left[F_{m}, C_{m}\right]$ for $m \in S \cup \operatorname{LM}(\mathcal{G})$ must be reduced by all the previous ones lying in the staircase in at most \#S \# $\left(\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}\right)$ operations. Reducing the relations to obtain a minimal Gröbner basis can be done in $O\left(\# S \# \mathcal{G} \#\left(\mathcal{T}_{\leq a}+\mathcal{T}_{\leq b}\right)\right)$ operations, hence this part is not the bottleneck of the algorithm.

## 5. An adaptive variant

In general, the BMS, the Scalar-FGLM and thus the Polynomial Scalar-FGLM algorithms are efficient when the staircase of the Gröbner basis of the ideal of relations is closer to a simplex than to a line. Indeed, for the $\operatorname{DRL}\left(x_{n}<\cdots<x_{1}\right)$ ordering, whether the Gröbner basis is $\left\{x_{n}, \ldots, x_{2}, x_{1}^{d}\right\}$ or all the monomials of degree $d$ : $\left\{x_{n}^{d}, x_{n-1} x_{n}^{d-1}, \ldots, x_{1} x_{n}^{d-1}, \ldots, x_{1}^{d}\right\}$, the BMS algorithm requires to visit all the monomials up to $x_{1}^{2 d-1}$ making the algorithm costly in the former case compared to the size of the staircase.

The Adaptive Scalar-FGLM algorithm Berthomieu et al. (2015) was designed to take into account the shape of the Gröbner basis gradually as it is discovered. The idea is to start from $S$, the empty set, and a list of monomial to visit in increasing order. At step $m$, the first monomial of that list, if $H_{S \cup\{m\}, S \cup\{m\}}$ has a greater rank than $H_{S, S}$, then $m$ is added to $S$ and removed from the list. Otherwise, a relation $C_{m}$ is found and any multiple of $m$ is removed from the list. For instance, in the line staircase with Gröbner basis $\left\{x_{n}, \ldots, x_{2}, x_{1}^{d}\right\}$, it computes the rank of matrices $H_{\{1\},\{1\}}, H_{\left\{1, x_{n}\right\},\left\{1, x_{n}\right\}}, \ldots, H_{\left\{1, x_{2}\right\},\left\{1, x_{2}\right\}}, H_{\left\{1, x_{1}\right\},\left\{1, x_{1}\right\}}, H_{\left\{1, x_{1}, x_{1}^{2}\right\},\left\{1, x_{1}, x_{1}^{2}\right\}}, \ldots, H_{\left\{1, x_{1}, \ldots, x_{1}^{d}\right\},\left\{1, x_{1}, \ldots, x_{1}^{d}\right\}}$. It thus requires merely $2(n+d)-1$ table terms instead of $\binom{n+2 d-1}{n}$.

The main issue with the Polynomial Scalar-FGLM ${ }^{n}$ algorithm in this context is that it is based on polynomials from matrices with columns set $\mathcal{T}_{\leq a}$ and rows set $\mathcal{T}_{\leq b}$.

### 5.1. A naive approach

The idea of the adaptive variant of the Polynomial Scalar-FGLM algorithm is to replace the linear algebra arithmetic with matrices $H_{S \cup\{m\}, S \cup\{m\}}$ by polynomial arithmetic like in Sections 2.4 and 3.2. At each step, we have the polynomial $P_{2 S}$, a monomial ideal $B_{2 S}$, determined as in Section 3.2 and pairs $R_{2 S, t}=\left[F_{2 S, t}, C_{t}\right]=\left[P_{2 S} C_{t} \bmod B_{2 S}, C_{t}\right]$. At the next step, we compute $P_{2(S \cup\{m\})}$ from $P_{2 S}$ by shifting it by $\frac{\operatorname{Lcm}(S \cup\{m\})^{2}}{\operatorname{LCM}(S)^{2}}$ and adding the missing terms, update $B_{2 S}$ into $B_{2(S \cup\{m\})}$ and likewise update each $R_{2(S \cup\{m\}), t}$ by shifting $F_{2 S, t}$ and adding the missing terms to make $F_{2(S \cup\{m\}), t}$. Then, $R_{2(S \cup\{m\}), m}$ is initialized as $m R_{2(S \cup\{m\}), 1}$ and then reduced, as in Section4, by $R_{2(S \cup\{m\}), B_{1}}, \ldots, R_{2(S \cup\{m\}), B_{n}}$, where $B_{1}, \ldots, B_{n} \in B_{2 S}$.

When a relation $C_{g}$ is found, any multiple of $g$ is removed from the set of potential monomials to add to $S$. Moreover, we can further reduce a future relation $R_{2(S \cup\{m\}), m}=\left[F_{2(S \cup\{m\}), m}, C_{m}\right]$ with any pair $\left[\frac{M}{\mu g}, 0\right], m \geq \mu g$, like we reduce it with $R_{2(S \cup\{m\}), B_{1}}, \ldots, R_{2(S \cup\{m\}), B_{n}}$.

Although this naive approch is based on polynomial arithmetic, it does not use polynomial division like the Polynomial Scalar-FGLM does.

### 5.2. A division-based adaptive variant

In Berthomieu and Faugère (2018), we did not go further in the design of an adaptive variant of the Polynomial Scalar-FGLM. In particular, that version could not initialize a pair $R_{2(S \cup\{m\}), m}$ as the quotient of two pairs of polynomials. In the following part of this section, we show how to initialize a new pair as the quotient of two previously computed pairs of polynomials.

The algorithm uses some new procedures that are not needed in Polynomial Scalar-FGLM. First NormalFormRightSide behaves like NormalForm except the normal form algorithm is applied between the right sides of all the pairs, and then passed on the left sides. By reducing pairs with pairs $R_{2 S^{\prime}, t}, t \in G$, this allows us to ensure that the support of $C_{m}$ is in $S^{\prime}=S \cup\{m\}$.

Procedure NormalFormHigherPart behaves like NormalForm, except only the higher part $\tilde{F}_{2 S^{\prime}, t}$ of a polynomial $F_{2 S^{\prime}, t}$ is used. For $t=B_{i}, \tilde{F}_{2 S^{\prime}, t}=F_{2 S^{\prime}, t}=B_{i}$, otherwise $\tilde{F}_{2 S^{\prime}, t}$ is obtained from $F_{2 S^{\prime}, t}$ by only removing any monomial dividing $\frac{\operatorname{cmm}\left(S^{\prime}\right)^{2}}{g}$ with $g \in \operatorname{LM}(\mathcal{G})$. Then

## Algorithm 5.22: Adaptive Polynomial Scalar-FGLM

Input: A table $\mathbf{w}=\left(w_{\mathbf{i}}\right)_{i \in \mathbb{N}^{n}}$ with coefficients in $\mathbb{K}$, a monomial ordering $<$.
Output: A set $G$ of relations generating the ideal of relations.
$L:=\{1\}$.
$G:=\emptyset, S:=\emptyset . \quad / /$ the future Gröbner basis and staircase
$B:=\left\{B_{1}, \ldots, B_{n}\right\}=\left\{x_{1}^{0}, \ldots, x_{n}^{0}\right\}$.
$R_{B_{1}}:=\left[B_{1}, 0\right], \ldots, R_{B_{n}}:=\left[B_{n}, 0\right]$.
While $L \neq \emptyset$ do
$m:=$ first element of $L$ and remove it from $L$.
$S^{\prime}:=S \cup\{m\}$.
Update $B$ and all pairs $R_{2 S, t}$ into $R_{2 S^{\prime}, t}$ for $t \in S \cup G \cup B$.
If $m=1$ then $R_{1}:=\left[w_{0, \ldots,}, 1\right]$.
Else if $m=\mu x_{i}^{2}$ with $\mu, \mu x_{i} \in S$ then
$R_{2 S^{\prime}, m}:=$ NormalFormHigherPart $\left(R_{2 S^{\prime}, \mu},\left[R_{2 S^{\prime}, \mu x_{i}}, R_{2 S^{\prime}, B_{1}}, \ldots, R_{2 S^{\prime}, B_{n}}\right]\right)$.
Else
$/ / m=\mu x_{i}$ with $\mu \in S$
$R_{2 S^{\prime}, m}:=\operatorname{NormalForm}\left(x_{i} R_{2 S^{\prime}, \mu},\left[R_{2 S^{\prime}, B_{1}}, \ldots, R_{2 S^{\prime}, B_{n}}\right]\right)$.
$R_{2 S^{\prime}, m}:=\operatorname{NormalFormRightSide}\left(R_{2 S^{\prime}, m},\left[R_{2 S^{\prime}, g 1}, \ldots, R_{\left.2 S^{\prime}, g_{r}\right]}\right]\right)$.
$/ / g_{1}, \ldots, g_{r} \in G$
$R_{2 S^{\prime}, m}:=\operatorname{NormalForm}\left(R_{2 S^{\prime}, m},\left[R_{2 S^{\prime}, B_{1}}, \ldots, R_{2 S^{\prime}, B_{n}}\right]\right)$.
$R_{2 S^{\prime}, m}:=$ NormalFormHigherPart $\left(R_{2 S^{\prime}, m},\left[R_{2 S^{\prime}, s_{1}}, \ldots, R_{2 S^{\prime}, s_{q}}\right]\right)$ / // $s_{1}, \ldots, s_{q} \in S$
$R_{2 S^{\prime}, m}=\left[F_{2 S^{\prime}, m}, C_{m}\right]$.
If $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, m}\right)<\frac{\operatorname{LcM}\left(S^{\prime}\right)^{2}}{m}$ then
// Relation succeeds
Update $R_{2 S^{\prime}, m}$ into $R_{2 S, m}$.
$G:=G \cup\left\{R_{2 S, m}\right\}$.
Remove multiples of $m$ in $L$.
Else
// Relation fails
Delete every pair $R_{2 S, t}$ for $t \in S \cup G \cup B$.
$S:=S^{\prime}$.
$L:=L \cup\left\{x_{1} m, \ldots, x_{n} m\right\}$, remove any multiples of $\operatorname{Lм}(G)$ and sort it by increasing order.
return $G$.

NormalFormHigherPart $\left(R_{2 S^{\prime}, m},\left[R_{2 S^{\prime}, s}, \ldots, R_{2 S^{\prime}, s_{q}}\right]\right)$ computes the normal form of $\tilde{F}_{2 S^{\prime}, m}$ with respect to $\left[\tilde{F}_{2 S^{\prime}, s_{1}}, \ldots, \tilde{F}_{2 S^{\prime}, s_{q}}\right]$ and the corresponding quotients $Q_{1}, \ldots, Q_{q}$. It then returns $R_{2 S^{\prime}, m}$ $Q_{1} R_{2 s^{\prime}, s_{1}}-\cdots-Q_{q} R_{2 S^{\prime}, s_{q}}$.

For the correctness of the algorithm, we need to prove the following lemma.
Lemma 23. Let $S$ be the current computed staircase and let $\mathcal{G}$ be the current computed Gröbner basis. Let $m$ be the least monomial not in $S$ and not divisible by $\operatorname{Lm}(\mathcal{G})$ and let $S^{\prime}=S \cup\{m\}$. Then, for all $t, \operatorname{Lm}\left(\tilde{F}_{2 S^{\prime}, t}\right)=\frac{\operatorname{com}\left(S^{\prime}\right)^{2}}{\operatorname{cov}\left(S^{2}\right)^{2}} \operatorname{Lm}\left(\tilde{F}_{2 S, t}\right)$ and $\operatorname{Lm}\left(\tilde{F}_{2 S^{\prime}, m}\right) \leq \frac{\operatorname{com}\left(S^{\prime}\right)^{2}}{m}$. Furthermore, this weak inequality is an equality if, and only if, $C_{m}$ is not a valid relation.

Proof. If $\mathcal{G}=\emptyset$, then the result is clear as $\tilde{F}_{2 S^{\prime}, m}=F_{2 S^{\prime}, m}$. Otherwise, the leading monomial of $\tilde{F}_{2 S^{\prime}, m}$ is less than $\frac{\operatorname{ccc}\left(S^{\prime}\right)}{t}$ with $t \in S$ and cannot be a divisor of $\frac{\operatorname{ccc}\left(S^{\prime}\right)}{g}$ with $g \in \operatorname{Lm}(\mathcal{G})$. Hence it must be at most $\frac{\operatorname{ccm}\left(S^{\prime}\right)}{m}$.

Furthermore, for $S=\left\{1, \ldots, s_{q}\right\}$, the coefficient of $\frac{\operatorname{LcM}\left(S^{\prime}\right)}{m}$ can be read as the last coeffcient of this product

$$
\begin{gathered}
1 \\
\vdots \\
s_{q} \\
m
\end{gathered}\left(\begin{array}{cccc}
1 & \cdots & s_{q} & m \\
{[1]} & \cdots & {\left[s_{q}\right]} & {[m]} \\
\vdots & & \vdots & \vdots \\
{\left[s_{q}\right]} & \cdots & {\left[s_{q}^{2}\right]} & {\left[m s_{q}\right]} \\
{[m]} & \cdots & {\left[m s_{q}\right]} & {\left[m^{2}\right]}
\end{array}\right)\left(\begin{array}{c}
\alpha_{1} \\
\vdots \\
\alpha_{s_{q}} \\
1
\end{array}\right)=\left(\begin{array}{c}
0 \\
\vdots \\
0 \\
f_{2 s^{\prime}, m}
\end{array}\right) .
$$

Hence, the relation is valid if, and only if, this coefficient is zero.
The leading monomial of the higher part of $F_{2 s^{\prime}, m}$ tells us whether $C_{m}$ is a valid relation or not.

Theorem 24. Assuming the Adaptive Polynomial Scalar-FGLM called on table $\mathbf{w}$ returns a Gröbner basis $\mathcal{G}$ with staircase $S$, then the algorithm does not need more that $\# 2(S \cup \operatorname{lm}(\mathcal{G}))$ table queries to recover $\mathcal{G}$.

Furthermore, it performs at most $O\left((\# S+\# \mathcal{G})^{2} \# 2 S\right)$ operations to recover $\mathcal{G}$.
Proof. Since $\tilde{F}_{2 S, t}$ is obtained from $F_{2 S, t}$ by removing all the monomials dividing $\frac{\operatorname{cms}(S)^{2}}{g}, g \in$ Lм $(\mathcal{G})$, then $\tilde{F}_{2 S, t}$ is actually the polynomial obtained from the product of the extended matrix $H_{T, S}$ and the vector representing $C_{t}$, where $T=2 S \backslash\{g \tau, g \in \operatorname{Lm}(\mathcal{G}), \tau \in \mathcal{T}\}$. Hence, updating from $\tilde{F}_{2 S, t}$ to $\tilde{F}_{2 S^{\prime}, t}$ is equivalent to updating $H_{T, S}$ to $H_{T^{\prime}, S^{\prime}}$, with $T^{\prime}=2 S^{\prime} \backslash\{g \tau, g \in \operatorname{Lm}(\mathcal{G}), \tau \in$ $\mathcal{T}$ \} in this matrix-vector product. Yet, the first nonzero coefficient of this product remains the same through this updating process.

At step $m, F_{2 S^{\prime}, t}=P_{2 S^{\prime}} C_{t} \bmod B_{2 S^{\prime}}$ for any $t \in S \cup \operatorname{lm}(\mathcal{G})$. As in the end of the step, $m$ is either found to be a member of the final $S$ or the final $\operatorname{Lm}(\mathcal{G})$, then $S^{\prime} \subseteq(S \cup \operatorname{LM}(\mathcal{G}))$ and thus $2 S^{\prime} \subseteq 2(S \cup \mathrm{Lм}(\mathcal{G}))$.

At step $m$, a pair $R_{2 S, t}=\left[F_{2 S, t}, C_{t}\right]$ is updated into $R_{2(S \cup\lfloor m\}, t}$ by adding terms with support in $2(S \cup\{m\})$, so in the last step, all the polynomials have support in a set of size \# 2 $(S \cup\{g\})$ where $g \in \operatorname{Lm}(\mathcal{G})$.

Furthermore, for each monomial $t \in S \cup \operatorname{Lm}(\mathcal{G})$, the pair $R_{2 S^{\prime}, t}$ is reduced by all the previous ones lying in the staircase or the Gröbner basis in at most $(\# S+\# \mathcal{G}) \# 2 S$ operations.

Example 25. We consider the following sequence $\mathbf{w}=\left(w_{i, j}\right)_{(i, j) \in \mathbb{N}^{2}}$

$$
\mathbf{w}=\left(\begin{array}{cccccccc}
6 & 9 & 5 & 1 & 10 & -6 & -9 & \cdots \\
3 & 12 & 2 & 4 & 7 & -3 & -12 & \cdots \\
6 & 9 & 5 & 1 & 10 & -6 & -9 & \cdots \\
3 & 12 & 2 & 4 & 7 & -3 & -12 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

For a pair $R_{2 S, t}=\left[F_{2 S, t}, C_{t}\right]$, we let $\tilde{R}_{2 S, t}=\left[\tilde{F}_{2 S, t}, C_{t}\right]$.
We start with $L=\{1\}, G=S=\emptyset, B=(1,1)$ and $R_{2 S, B_{1}}=[1,0], R_{2 S, B_{2}}=[1,0]$.

1. We set $m=1, S^{\prime}=\{1\}, R_{2 S^{\prime}, B_{1}}=[1,0], R_{2 S^{\prime}, B_{2}}=[y, 0]$ and initialize $R_{2 S^{\prime}, 1}=[6,1]$.

As $\tilde{F}_{2 S^{\prime}, 1}=F_{2 S^{\prime}, 1}=1$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, 1}\right)=1=\frac{\operatorname{ccm}\left(S^{\prime}\right)^{2}}{1}$ the relation fails.
$L$ is updated to $\{y, x\}$.
2. We set $m=y, S^{\prime}=\{1, y\}, R_{2 S^{\prime}, B_{1}}=[x, 0], R_{2 S^{\prime}, B_{2}}=\left[y^{3}, 0\right]$ and $R_{2 S^{\prime}, 1}=\left[6 y^{2}+9 y+5,1\right]=$ $\tilde{R}_{2 S^{\prime}, 1}$.
We initialize $R_{2 S^{\prime}, y}=\left[9 y^{2}+5 y, y\right]$ using NormalForm on $y R_{2 S^{\prime}, 1}$ and then reduce it to $\left[-\frac{17}{2} y-\frac{15}{2}, y-\frac{3}{2}\right]$.
As $\tilde{F}_{2 S^{\prime}, y}=F_{2 S^{\prime}, y}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, y}\right)=\frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{y}$ the relation fails.
$L$ is updated to $\left\{x, y^{2}\right\}$.
3. We set $m=x, S^{\prime}=\{1, y, x\}, R_{2 S^{\prime}, B_{1}}=\left[x^{3}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{3}, 0\right]$,

- $R_{2 S^{\prime}, 1}=\left[6 x^{2} y^{2}+9 x^{2} y+3 x y^{2}+5 x^{2}+12 x y+6 y^{2}, 1\right]=\tilde{R}_{2 S^{\prime}, 1}$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y+\frac{15}{2} x y^{2}-\frac{15}{2} x^{2}-18 x y-9 y^{2}, y-\frac{3}{2}\right]=\tilde{R}_{2 S^{\prime}, y}$.

We initialize $R_{2 S^{\prime}, x}=\left[3 x^{2} y^{2}+12 x^{2} y+6 x y^{2}, x\right]$ using NormalForm on $x R_{2 S^{\prime}, 1}$ and then reduce it to $\left[\frac{189}{17} x y^{2}-\frac{155}{17} x^{2}-\frac{372}{17} x y-\frac{186}{17} y^{2}, x+\frac{15}{17} y-\frac{31}{17}\right]$.
As $\tilde{F}_{2 S^{\prime}, x}=F_{2 S^{\prime}, x}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, x}\right)=\frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{x}$ the relation fails.
$L$ is updated to $\left\{y^{2}, x y, x^{2}\right\}$.
4. We set $m=y^{2}, S^{\prime}=\left\{1, y, x, y^{2}\right\}, R_{2 S^{\prime}, B_{1}}=\left[x^{3}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{5}, 0\right]$,

- $R_{2 S^{\prime}, 1}=\left[6 x^{2} y^{4}+\cdots, 1\right]=\tilde{R}_{2 S^{\prime}, 1}$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{3}+\cdots, y-\frac{3}{2}\right]=\tilde{R}_{2 S^{\prime}, y}$,
- $R_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{4}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right]=\tilde{R}_{2 S^{\prime}, x}$.

We initialize $R_{2 S^{\prime}, y^{2}}=\left[-\frac{106}{17} x y^{4}+\cdots, y^{2}-\frac{13}{17} y+\frac{16}{51}\right]$ using NormalFormHigherPart on $R_{2 S^{\prime}, 1}$ and $R_{2 S^{\prime}, y}$ and then reduce it to $\left[\frac{1381}{189} x^{2} y^{2}+\cdots, y^{2}+\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right]$.
As $\tilde{F}_{2 S^{\prime}, y^{2}}=F_{2 S^{\prime}, y^{2}}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, y^{2}}\right)=\frac{\operatorname{LCM}\left(S^{\prime}\right)^{2}}{y^{2}}$ the relation fails.
$L$ is updated to $\left\{x y, x^{2}, y^{3}\right\}$.
5. We set $m=x y, S^{\prime}=\left\{1, y, x, y^{2}, x y\right\}, R_{2 S^{\prime}, B_{1}}=\left[x^{3}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{5}, 0\right]$,

- $R_{2 S^{\prime}, 1}=\left[6 x^{2} y^{4}+\cdots, 1\right]=\tilde{R}_{2 S^{\prime}, 1}$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{3}+\cdots, y-\frac{3}{2}\right]=\tilde{R}_{2 S^{\prime}, y}$,
- $R_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{4}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right]=\tilde{R}_{2 S^{\prime}, x}$,
- $R_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{2}+\cdots, y^{2}+\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right]=\tilde{R}_{2 S^{\prime}, y^{2}}$.

We initialize $R_{2 S^{\prime}, x y}=\left[-\frac{15}{2} x^{2} y^{4}+\cdots, x y-\frac{3}{2} x\right]$ using NormalForm on $x R_{2 S^{\prime}, y}$ and then reduce it to $\left[-15 y^{4}-7 x^{2} y-x y^{2}-14 y^{3}-10 x^{2}-4 x y-5 y^{2}, x y+x-y-1\right]$.
As $\tilde{F}_{2 S^{\prime}, x y}=F_{2 S^{\prime}, x y}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, x y}\right)=y^{4}<\frac{\operatorname{L\operatorname {cm}(S^{\prime })^{2}}}{x y}$ the relation succeeds!
We update $R_{2 S, x y}=\left[-9 x y^{4}+\cdots, x y+x-y-1\right]$ and put it in $G$.
6. We set $m=x^{2}, S^{\prime}=\left\{1, y, x, y^{2}, x^{2}\right\}, R_{2 S^{\prime}, B_{1}}=\left[x^{5}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{5}, 0\right]$

- $R_{2 S^{\prime}, 1}=\left[6 x^{4} y^{4}+\cdots, 1\right], \tilde{R}_{2 S^{\prime}, 1}=\left[6 x^{4} y^{4}+\cdots, 1\right]$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{4} y^{3}-+\cdots, y-\frac{3}{2}\right], \tilde{R}_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{4} y^{3}-+\cdots, y-\frac{3}{2}\right]$,
- $R_{2 S^{\prime}, x}=\left[\frac{189}{17} x^{3} y^{4}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right], \tilde{R}_{2 S^{\prime}, x}=\left[\frac{189}{17} x^{3} y^{4}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right]$,
- $R_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{4} y^{2}+\cdots, y^{2}+\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right], \tilde{R}_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{4} y^{2}+\cdots, y^{2}+\right.$ $\left.\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right]$,
- $R_{2 S^{\prime}, x y}=\left[-4 x^{4} y^{2}+\cdots, x y+x-y-1\right]$.

We initialize $R_{2 S^{\prime}, x^{2}}=\left[-\frac{945}{34} x^{4} y^{3}-x^{4} y^{3}+\cdots, x^{2}+\frac{15}{17} x y-\frac{45}{34} x+\frac{15}{34} y-\frac{47}{17}\right]$ using NormalFormHigherPart on $\tilde{R}_{2 S^{\prime}, 1}$ and $\tilde{R}_{2 S^{\prime}, x}$ and notice that $C_{x^{2}}$ has not its support in $S^{\prime}$.
We then reduce it to $\left[-12 x^{3} y^{3}+\cdots, x^{2}-1\right]$ by computing first the reduction of the right part by $x y+x-y-1$ and then by computing the reduction of the left parts without taking into account any monomials $\frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{x y t}=\frac{x^{3} y^{3}}{t}$.
As $\tilde{F}_{2 S^{\prime}, x^{2}}=-x^{4} y^{3}+\cdots \neq F_{2 S^{\prime}, x^{2}}$ and $\operatorname{Lm}\left(\tilde{F}_{2 S^{\prime}, x^{2}}\right)=x^{4} y^{3}<\frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{x^{2}}$ the relation succeeds!
We update $R_{2 S, x^{2}}=\left[-9 x^{2} y^{3}+\cdots, x^{2}-1\right]$ and put it in $G$.
7. We set $m=y^{3}, S^{\prime}=\left\{1, y, x, y^{2}, y^{3}\right\}, R_{2 S^{\prime}, B_{1}}=\left[x^{3}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{7}, 0\right]$,

- $R_{2 S^{\prime}, 1}=\left[6 x^{2} y^{6}+\cdots, 1\right], \tilde{R}_{2 S^{\prime}, 1}=\left[6 x^{2} y^{6}+\cdots, 1\right]$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{5}-+\cdots, y-\frac{3}{2}\right], \tilde{R}_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{5}-+\cdots, y-\frac{3}{2}\right]$,
- $R_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{6}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right], \tilde{R}_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{6}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right]$,
- $R_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{4}+\cdots, y^{2}+\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right], \tilde{R}_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{4}+\cdots, y^{2}+\right.$ $\left.\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right]$,
- $R_{2 S^{\prime}, x y}=\left[-9 x y^{6}+\cdots, x y+x-y-1\right]$,
- $R_{2 S^{\prime}, x^{2}}=\left[-9 x^{2} y^{5}+\cdots, x^{2}-1\right]$.

We initialize $R_{2 S^{\prime}, y^{3}}=\left[-\frac{4204}{1381} x y^{5}+\cdots, y^{3}+\frac{1354}{1381} y^{2}-\frac{607}{1381} x-\frac{190}{1381} y-\frac{770}{1381}\right]$ using NormalFormHigherPart on $R_{2 S^{\prime}, y}$ and $R_{2 S^{\prime}, y^{2}}$.
We then reduce it to $\left[-\frac{4204}{1381} x y^{5}-\frac{4620}{1381} y^{6}-\frac{25651}{1381} x^{2} y^{3}+\cdots, y^{3}+\frac{1354}{1381} y^{2}-\frac{607}{1381} x-\frac{190}{1381} y-\frac{770}{1381}\right]$ by computing the reduction of the left parts without taking into account any monomials $\frac{\operatorname{Lcc}\left(S^{\prime}\right)^{2}}{x y t}=\frac{x y^{5}}{t}, \frac{\operatorname{cmm}\left(S^{\prime}\right)^{2}}{x^{2} t}=\frac{y^{6}}{t}$.
As $\tilde{F}_{2 S^{\prime}, y^{3}}=-\frac{25651}{1381} x^{2} y^{3}+\cdots \neq F_{2 S^{\prime}, y^{3}}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, y^{3}}\right)=x^{2} y^{3}=\frac{\operatorname{LcM}\left(S^{\prime}\right)^{2}}{y^{3}}$ the relation fails!
$L$ is updated to $\left\{y^{4}\right\}$.
8. We set $m=y^{4}, S^{\prime}=\left\{1, y, x, y^{2}, y^{3}, y^{4}\right\}, R_{2 S^{\prime}, B_{1}}=\left[x^{3}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{9}, 0\right]$,

- $R_{2 S^{\prime}, 1}=\left[6 x^{2} y^{8}+\cdots, 1\right], \tilde{R}_{2 S^{\prime}, 1}=\left[6 x^{2} y^{8}+\cdots, 1\right]$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{7}-+\cdots, y-\frac{3}{2}\right], \tilde{R}_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{7}-+\cdots, y-\frac{3}{2}\right]$,
- $R_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{8}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right], \tilde{R}_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{8}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right]$,
- $R_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{6}+\cdots, y^{2}+\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right], \tilde{R}_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{6}+\cdots, y^{2}+\right.$ $\left.\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right]$,
- $R_{2 S^{\prime}, y^{3}}=\left[\frac{5463}{1381} x y^{7}-\frac{4620}{1381} y^{8}-\frac{25651}{1381} x^{2} y^{5}+\cdots, y^{3}+\frac{1354}{1381} y^{2}-\frac{607}{1381} x-\frac{190}{1381} y-\frac{770}{1381}\right]$, $\tilde{R}_{2 S^{\prime}, y^{3}}=\left[-\frac{25651}{1381} x^{2} y^{5}+\cdots, y^{3}+\frac{1354}{1381} y^{2}-\frac{607}{1381} x-\frac{190}{1381} y-\frac{770}{1381}\right]$,
- $R_{2 S^{\prime}, x y}=\left[-9 x y^{8}+\cdots, x y+x-y-1\right]$,
- $R_{2 S^{\prime}, x^{2}}=\left[-9 x^{2} y^{7}+\cdots, x^{2}-1\right]$.

We initialize $R_{2 S^{\prime}, y^{4}}$ using NormalFormHigherPart on $R_{2 S^{\prime}, y^{2}}$ and $R_{2 S^{\prime}, y^{3}}$ and then reduce it to $\left[-\frac{571416}{25651} x y^{7}+\cdots+\frac{504287}{25651} x^{2} y^{4}+\cdots, y^{4}-\frac{29900}{25651} y^{3}+\frac{7703}{25651} y^{2}+\frac{72041}{25651} x-\frac{35598}{25651} y-\frac{26811}{25651}\right]$ by computing the reduction of the left parts without taking into account any monomials $\frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{x y t}=\frac{x y^{7}}{t}, \frac{\operatorname{LCM}\left(S^{\prime}\right)^{2}}{x^{2} t}=\frac{y^{8}}{t}$.
As $\tilde{F}_{2 S^{\prime}, y^{4}}=\frac{504287}{25651} x^{2} y^{4}+\cdots \neq F_{2 S^{\prime}, y^{3}}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, y^{4}}\right)=x^{2} y^{4}=\frac{\operatorname{LcM}\left(S^{\prime}\right)^{2}}{y^{4}}$ the relation fails! $L$ is updated to $\left\{y^{5}\right\}$.
9. We set $m=y^{5}, S^{\prime}=\left\{1, y, x, y^{2}, y^{3}, y^{4}, y^{5}\right\}, R_{2 S^{\prime}, B_{1}}=\left[x^{3}, 0\right], R_{2 S^{\prime}, B_{2}}=\left[y^{11}, 0\right]$,

- $R_{2 S^{\prime}, 1}=\left[6 x^{2} y^{10}+\cdots, 1\right], \tilde{R}_{2 S^{\prime}, 1}=\left[6 x^{2} y^{10}+\cdots, 1\right]$,
- $R_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{9}-+\cdots, y-\frac{3}{2}\right], \tilde{R}_{2 S^{\prime}, y}=\left[-\frac{17}{2} x^{2} y^{9}-+\cdots, y-\frac{3}{2}\right]$,
- $R_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{10}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right], \tilde{R}_{2 S^{\prime}, x}=\left[\frac{189}{17} x y^{10}+\cdots, x+\frac{15}{17} y-\frac{31}{17}\right]$,
- $R_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{8}+\cdots, y^{2}+\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right], \tilde{R}_{2 S^{\prime}, y^{2}}=\left[\frac{1381}{189} x^{2} y^{8}+\cdots, y^{2}+\right.$ $\left.\frac{106}{189} x-\frac{17}{63} y-\frac{134}{189}\right]$,
- $R_{2 S^{\prime}, y^{3}}=\left[\frac{5463}{1381} x y^{9}-\frac{4620}{1381} y^{10}-\frac{25651}{1381} x^{2} y^{7}+\cdots, y^{3}+\frac{1354}{1381} y^{2}-\frac{607}{1381} x-\frac{190}{1381} y-\frac{770}{1381}\right]$, $\tilde{R}_{2 S^{\prime}, y^{3}}=\left[-\frac{25651}{1381} x^{2} y^{7}+\cdots, y^{3}+\frac{1354}{1381} y^{2}-\frac{607}{1381} x-\frac{190}{1381} y-\frac{770}{1381}\right]$,
- $R_{2 S^{\prime}, y^{4}}=\left[-\frac{648369}{25651} x y^{9}+\cdots+\frac{504287}{25651} x^{2} y^{6}+\cdots, y^{4}-\frac{29900}{25651} y^{3}+\frac{7703}{25651} y^{2}+\frac{72041}{25651} x-\frac{35598}{25651} y-\right.$ $\left.\frac{26811}{25651}\right], \tilde{R}_{2 S^{\prime}, y^{4}}=\left[\frac{504287}{25651} x^{2} y^{6}+\cdots, y^{4}-\frac{29900}{25651} y^{3}+\frac{7703}{25651} y^{2}+\frac{72041}{25651} x-\frac{35598}{25651} y-\frac{26811}{25651}\right]$,
- $R_{2 S^{\prime}, x y}=\left[-9 x y^{10}+\cdots, x y+x-y-1\right]$,
- $R_{2 S^{\prime}, x^{2}}=\left[-9 x^{2} y^{9}+\cdots, x^{2}-1\right]$.

We initialize $R_{2 S^{\prime}, y^{5}}$ using NormalFormHigherPart on $R_{2 S^{\prime}, y^{3}}$ and $R_{2 S^{\prime}, y^{4}}$ and then reduce it to $\left[12 x y^{9}+6 y^{10}+2 x y^{8}+4 x y^{7}+7 x y^{6}-9 x^{2} y^{4}-3 x y^{5}-5 x^{2} y^{3}-x^{2} y^{2}-10 x^{2} y+6 x^{2}, y^{5}+1\right]$ by computing the reduction of the left parts without taking into account any monomials $\frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{x y t}=\frac{x y^{9}}{t}, \frac{\operatorname{LcM}\left(S^{\prime}\right)^{2}}{x^{2} t}=\frac{y^{10}}{t}$.
As $\tilde{F}_{2 S^{\prime}, y^{5}}=-9 x^{2} y^{4}+\cdots \neq F_{2 S^{\prime}, y^{3}}$ and $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, y^{5}}\right)=x^{2} y^{4}<\frac{\operatorname{LCM}\left(S^{\prime}\right)^{2}}{y^{5}}$ the relation suceeds! We update $R_{2 S, y^{5}}=\left[3 x y^{8}+\cdots, y^{5}+1\right]$ and put it in $G$.

The algorithms returns $G$, in particular the second part of each pair: $\left[x y+x-y-1, x^{2}-1, y^{5}+1\right]$.
Remark 26. At step $y^{4}$, updating $R_{2 S, y^{3}}$ into $R_{2 S^{\prime}, y^{3}}$ makes the leading monomial of $F_{2 S^{\prime}, y^{3}}$, $\frac{5463}{1381} \frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{x y}$, totally different from this of $F_{2 S, y^{3}},-\frac{4204}{1381} \frac{\operatorname{Lcm}(S)^{2}}{x y}$. However, this is not the case when considering the leading monomials of $\tilde{F}_{2 S^{\prime}, y^{3}}$ and $\tilde{F}_{2 S, y^{3}}$, as $\operatorname{LM}\left(\tilde{F}_{2 S^{\prime}, y^{3}}\right)=-\frac{25651}{1381} \frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{y^{3}}=$ $\operatorname{LM}\left(\tilde{F}_{2 S, y^{3}}\right) \frac{\operatorname{Lcm}\left(S^{\prime}\right)^{2}}{\operatorname{LCM}(S)^{2}}$.

## 6. Experiments

In this section, we report on the number of arithmetic operations done by the different algorithms for computing the Gröbner basis of the ideal of relations of some table families. They are counted using naive multiplications. Three families in dimension 2 (Figure 1) and dimension 3 (Figure 2) are tested. For each of them we use the $\operatorname{DRL}(z<y<x)$ ordering and denote by $S$ the staircase and $\operatorname{lm}(\mathcal{G})$ the set of the leading terms of the Gröbner basis of relations.

Rectangle tables: $\operatorname{LM}(\mathcal{G})=\left\{y^{[d / 2]}, x^{d}\right\}$ in dimension 2 and $\operatorname{LM}(\mathcal{G})=\left\{z^{[d / 3]}, y^{[d / 2]}, x^{d}\right\}$ dimension 3 . This case is the best for the size of the Gröbner basis compared to the size of the staircase.

L-shape tables: $\operatorname{Lm}(\mathcal{G})=\left\{x y, y^{d}, x^{d}\right\}$ in dimension 2 and $\operatorname{Lm}(\mathcal{G})=\left\{y z, x z, x y, z^{d}, y^{d}, x^{d}\right\}$ in dimension 3. This case is the worst for the number of table queries compared to the sizes of the staircase and the Gröbner basis.

Simplex tables: $\operatorname{LM}(\mathcal{G})=\left\{y^{d}, x y^{d-1}, \ldots, x^{d}\right\}$ in dimension 2 and $\operatorname{LM}(\mathcal{G})=\left\{z^{d}, y z^{d-1}, x z^{d-1}, \ldots, y^{d}\right.$, $\left.x y^{d-1}, \ldots, x^{d}\right\}$ in dimension 3, i.e. all the monomials of degree $d$. This case is the best for the number of table queries and the worst for the size of the Gröbner basis, both compared to the size of the staircase.

Let $a=\max (S \cup \operatorname{Lm}(\mathcal{G}))$. Generically, a relation $C_{m}$ fails when shifted by $m$. From (BrasAmorós and O'Sullivan, 2006, Prop. 10), we know that the BMS algorithm recover all the relations when called up to $\operatorname{monomial} \max (S) \max (S \cup \operatorname{Lm}(\mathcal{G}))$. Yet, if $\max (\operatorname{Lm}(\mathcal{G}))>\max (S)$, then for $g \in \operatorname{Lm}(\mathcal{G})$, the relation $C_{g}$ is not necessarily shifted by $g$, so we called it with $a^{2}$. The ScalarFGLM algorithm was called on $U=T=\mathcal{T}_{\leq a}$. The Polynomial Scalar-FGLM algorithm was called on $U=\{1\}, T=\mathcal{T}_{\leq a^{2}}$ and $U=T=\mathcal{T}_{\leq a}$ and we report the lower number of operations. The needed input set of monomials for the Artinian Gorenstein border bases algorithm (AGbb) of Mourrain (2017) to recover the ideal of relations was higher than expected in some situations. It explains the big overhead in Figure 1 in dimension 2:

Rectangle tables: The border basis is recovered whenever the algorithm visits all the monomials of degree at most $2(d+\lfloor d / 2\rfloor+\lceil d / 3\rceil-1)$.

L-shape tables: The correct border basis is not recovered. In dimension 2, we have to visit all the monomials of degree $5 d-10$ to close the staircase. This yields relations of degree higher than expected, though. For instance, with $d=8$, we obtained $\operatorname{Lm}(\mathcal{B})=$ $\left\{x y^{i}, x^{i} y, y^{13}, x^{13}, 1 \leq i \leq 12\right\}$.

Simplex tables: The correct border basis is not recovered. In dimension 2, we have to visit all the monomials of degree around $5 d$ or $6 d$ to close the staircase. This also yields relations of degree higher than expected. For instance, with $d=8$, we obtained $\operatorname{lm}(\mathcal{B})=$ $\left\{x^{i} y^{18-i}, 0 \leq i \leq 18\right\}$.

The Polynomial Scalar-FGLM algorithm performs fewer arithmetic operations than the others, for large $d$. More precisely, its number of operations appears to be linear in $(\# S)^{2}=$ $O(\# S(\# S+\# c G))$ in fixed dimension.

Simplex tables: While it seems the Scalar-FGLM algorithm is the fastest in Figure 2, we can expect that it will not be the case in higher degrees, like in Figure 1 This would confirm the observed speedup in dimension 2 to dimension 3 .



L-shape tables: Although the obtained speedups are not negligible, the adaptive variant should allow us to perform even fewer operations. See Section 5

Rectangle tables: While this family has the best behavior for the BMS algorithm, the Polynomial Scalar-FGLM algorithm has an even greater speedup than in the Simplex case.

In the following Figures 3 and 4 , we compare the number of operations performed by the Polynomial Scalar-FGLM, the Adaptive Scalar-FGLM and the Adaptive Polynomial ScalarFGLM algorithms on these three families. First, as we can expect, the Adaptive Polynomial Scalar-FGLM algorithms always performs fewer operations than the Polynomial Scalar-FGLM algorithm. Even when giving the sharper sets of monomials $\mathcal{T}_{\leq a}$ and $\mathcal{T}_{\leq b}$ for the Polynomial Scalar-FGLM algorithm. Furthermore, like in the non-adaptive case, the polynomial arithmetic seems to be faster than the linear algebra ones. Though, it would be interesting to compare with a less naive implementation of the Adaptive Scalar-FGLM algorithm.

That being said, in three variables, the Adaptive Scalar-FGLM algorithm is the better one for the L-shape family in small size, until the Adaptive Polynomial Scalar-FGLM becomes the better one for bigger sizes. As the crossing happens later in 3D than in 2D, it is possible than the more variables, the bigger the degrees need to be for the Adaptive Polynomial Scalar-FGLM algorithm to be better than the Adaptive Scalar-FGLM algorithm, for this family. This makes the Adaptive Scalar-FGLM algorithm still a suited algorithm in this context.


Figure 3: Number of arithmetic operations, for the adaptive variants, compared with the Polynomial Scalar-FGLM algorithm (2D)

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Figure 4: Number of arithmetic operations, for the adaptive variants, compared with the Polynomial Scalar-FGLM algorithm (3D)

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