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# An adaptive $hp$ -refinement strategy with inexact solvers and computable guaranteed bound on the error reduction factor\*

Patrik Daniel<sup>†‡</sup>    Alexandre Ern<sup>†‡</sup>    Martin Vohralík<sup>†‡</sup>

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## Abstract

In this work we extend our recently proposed adaptive refinement strategy for  $hp$ -finite element approximations of elliptic problems by taking into account an inexact algebraic solver. Namely, on each level of refinement and on each iteration of an (arbitrary) iterative algebraic solver, we compute guaranteed a posteriori error bounds on the algebraic and the total errors in energy norm. For the algebraic error upper bound, we crucially exploit the nested hierarchy of  $hp$ -finite element spaces created throughout the adaptive algorithm, whereas the rest of the components of the total error upper and lower bounds are computed using the finest space only. These error bounds allow us to formulate adaptive stopping criteria for the algebraic solver ensuring that the algebraic error does not significantly contribute to the total error. Next, we use the total error bound to mark mesh vertices for refinement via Dörfler’s bulk-chasing criterion. On patches associated with marked vertices only, we solve two separate primal finite element problems with homogeneous Dirichlet (Neumann) boundary conditions, which serve to decide between  $h$ -,  $p$ -, or  $hp$ -refinement. Altogether, we show that these ingredients lead to a computable guaranteed bound on the ratio of the total errors of the inexact approximations between successive refinements (the error reduction factor), when the stopping criteria are satisfied. Finally, in a series of numerical experiments, we investigate the practicality of the proposed adaptive solver, the accuracy of our bound on the reduction factor, and show that exponential convergence rates are also achieved even in the presence of an inexact algebraic solver.

**Key words:** elliptic problem, finite element method, a posteriori error estimate, equilibrated flux,  $hp$ -adaptivity, error reduction, algebraic error

## 1 Introduction

The adaptive finite element method (AFEM), developed back in the 1980s [1, 2, 3, 4], is still one of the fundamental and widely used numerical methods for solving the boundary value problems arising in physics or engineering sciences. In short, it can be described as a numerical method which automatically, in an iterative fashion, adapts the employed finite element space until a sufficiently accurate approximation of the solution is obtained. For an overview and further insight, we refer the reader to work of Nochetto, Siebert and Veiser [5], and the references therein.

In the vast majority of the publications, the resulting linear systems are assumed to be solved *exactly*. However, in practical applications, including large scale numerical computations, the exact solve is not feasible in most cases; it may actually be greatly advantageous to employ an *inexact* (iterative) algebraic solver. The incorporation of an inexact algebraic solver, as an alternative to the use of (sparse) direct solvers, within the AFEM framework and its rigorous analysis is rather an exception. It has been addressed by Stevenson in [6, 7], Arioli *et al.* [8, 9] for linear elliptic problems and by Holst, Szypowski and Zhu [10],

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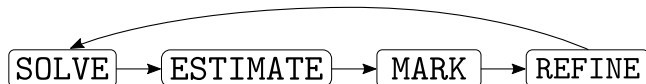
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Carstensen *et al.* [11] and Gantner *et al.* [12] for nonlinear elliptic problems, all in the context of the  $h$ -AFEM. We also mention the work of Becker, Johnson and Rannacher [13], where the authors deal with the issue of settling an objective stopping criterion for a multigrid iterative solver, and the work [14] where the authors devise a posteriori stopping criteria for inexact Newton methods and iterative linear solvers in the context of diffusion PDEs.

The main goal of this manuscript is to extend the recently proposed  $hp$ -adaptive refinement strategy with *computable guaranteed bound on the error reduction factor* [15] to approximate elliptic problems by taking into account an inexact algebraic iterative solver inside the adaptive loop. In this work, we consider as a model problem the Poisson equation with homogeneous Dirichlet boundary conditions. Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a polytopal domain (open, bounded and connected set) with a Lipschitz boundary  $\partial\Omega$ . The model problem in its weak form reads as follows: Seek  $u \in H_0^1(\Omega)$ , such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (1.1)$$

where  $H_0^1(\Omega)$  denotes the Sobolev space of all functions in  $L^2(\Omega)$  which have all their first-order weak derivatives in  $L^2(\Omega)$  and a vanishing trace on  $\partial\Omega$ , and  $(\cdot, \cdot)$  stands for the  $L^2(\Omega)$  or  $[L^2(\Omega)]^d$  inner product. We employ the conforming  $hp$ -finite element method to discretize the model problem (1.1) on a matching (no hanging nodes) simplicial mesh. The well-established paradigm of the adaptive iterative procedures, used in [15] as well, comprises at each step the four independent, but concatenated, modules, see Scheme 1. The



**Scheme 1:** Paradigm of an adaptive loop with exact algebraic solver.

module SOLVE, as already mentioned, usually stands for rather unrealistic *exact* (up to machine precision) solution of the underlying, possibly very large and/or ill-conditioned, linear algebraic problem. Thus, we opt to replace the module SOLVE in Scheme 1 by the module ONE\_SOLVER\_STEP coupled directly together with the module ESTIMATE in an adaptive sub-loop. This is conceptually described in Scheme 2.



**Scheme 2:** Paradigm of an adaptive loop employing an inexact algebraic solver.

The ONE\_SOLVER\_STEP in Scheme 2 stands for performing only one (or a certain small number of) iteration(s) of the iterative solver to the resulting algebraic system. The obtained inexact solution is then immediately analyzed within the ESTIMATE module which now distinguishes the *algebraic error* and the *total error*. The interplay between the modules ONE\_SOLVER\_STEP and ESTIMATE, which is indicated by the forward and backward arrows between them in Scheme 2, corresponds to the progressive improvement of the current approximate solution by performing additional iteration(s) of the algebraic solver within module ONE\_SOLVER\_STEP with the immediate calls of module ESTIMATE. Our present choice of the module ESTIMATE amounts to a natural extension of the developments of algebraic a posteriori error bounds via a *multilevel approach* by Papež *et al.* [16] to the present setting with variable polynomial degree. The adaptive sub-loop is piloted by a tailored *adaptive stopping criterion*, namely, we stop at the moment when we are sure that the algebraic error lies below the total error.

The remaining two modules extend the workflow of the strategy proposed in [15]. The module MARK refers to applying a bulk-chasing criterion inspired by the well-known Dörfler's marking [17]; we mark mesh vertices and not simplices since we observe a smoother performance in practice and since we later work with some vertex-based auxiliary quantities. The module REFINE, including our  $hp$ -decision criterion, then proceeds in three steps. First, we solve two local finite element problems on each patch of simplices attached to a mesh vertex marked for refinement, with either the mesh refined or the polynomial degree increased. These conforming residual liftings allow us, in particular, to estimate the effect of applying  $h$ - or  $p$ -refinement, and

lead to a partition of the set of marked vertices into two disjoint subsets, one collecting the mesh vertices flagged for  $h$ -refinement and the other collecting the mesh vertices flagged for  $p$ -refinement. The second step of the module **REFINE** uses these two subsets to flag the simplices for  $h$ -,  $p$ -, or  $hp$ -refinement. Finally, the third step of the module **REFINE** uses the above sets of flagged simplices to build the next simplicial mesh and the next polynomial-degree distribution.

We are particularly interested in recovering the computable guaranteed bound on the error reduction factor introduced in [15] also in the inexact setting described in Scheme 2, and for this reason our bounds are derived using various equilibrated flux reconstructions in the spirit of [14]. One of the main contributions of the present work is to show that using a properly designed stopping criterion for the algebraic solver, see Section 4.3, it is possible, at the end of each iteration of the inexact adaptive loop given in Scheme 2, to compute a real number  $C_{\text{red}} \in [0, 1]$  such that

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\text{red}} \|\nabla(u - u_{\ell})\|, \quad (1.2)$$

where  $u$  is the unknown weak solution of (1.1),  $u_{\ell}$  and  $u_{\ell+1}$  are its discrete inexact approximations on step  $\ell$ , and  $\ell + 1$  respectively, of the adaptive loop of Scheme 2. Note that in (1.2), the inexact solution  $u_{\ell}$  on the  $\ell$ -th iteration is at our disposal while the weak solution  $u$  and the next level's inexact solution  $u_{\ell+1}$  are unknown. The number  $C_{\text{red}}$  is fully computable, giving a guaranteed upper bound on the ratio of the total errors of the *inexact approximations* between two successive refinements.

The rest of this paper is organized as follows: in Section 2 we specify the discrete setting and some useful notation, in particular, concerning the inexact finite element approximation. In Section 3, we introduce the theoretical background of the a posteriori error bounds computed later within our **ESTIMATE** module. The overall description of all the modules of the proposed inexact  $hp$ -adaptive algorithm follows in Section 4. The result on a computable guaranteed bound on the reduction factor in the inexact setting is given in Section 5. Section 6 illustrates our theoretical findings and applicability of the proposed strategy with numerical experiments carried out on two-dimensional test cases. Finally, conclusions are drawn in Section 7.

## 2 Setting and notation

While using the adaptive loop of Scheme 2 a sequence of discrete finite element spaces  $\{V_{\ell}\}_{\ell \geq 0}$ , with  $\ell \geq 0$  the iteration counter, is generated. We enforce the  $H_0^1$ -conformity  $V_{\ell} \subset H_0^1(\Omega)$  for all  $\ell \geq 0$  and make the following nestedness assumption:

$$V_{\ell} \subset V_{\ell+1}, \quad \forall \ell \geq 0. \quad (2.1)$$

Each space  $V_{\ell}$  is built up on the pair  $(\mathcal{T}_{\ell}, \mathbf{p}_{\ell})$ , where  $\mathcal{T}_{\ell}$  denotes a matching simplicial mesh of the computational domain  $\Omega$ , i.e. a finite collection of (closed) non-overlapping simplices  $K \in \mathcal{T}_{\ell}$  covering  $\bar{\Omega}$  exactly and such that the intersection of two different simplices is either empty, a common vertex, a common edge, or a common face, and where the polynomial-degree distribution vector  $\mathbf{p}_{\ell} := \{p_{\ell, K}\}_{K \in \mathcal{T}_{\ell}}$  assigns a degree  $p_{\ell, K} \in \mathbb{N}_{\geq 1}$  to each simplex  $K \in \mathcal{T}_{\ell}$ . The conforming finite element space  $V_{\ell}$  is then defined as

$$V_{\ell} := \mathbb{P}_{\mathbf{p}_{\ell}}(\mathcal{T}_{\ell}) \cap H_0^1(\Omega), \quad \forall \ell \geq 0, \quad (2.2)$$

where  $\mathbb{P}_{\mathbf{p}_{\ell}}(\mathcal{T}_{\ell})$  denotes the space of piece-wise polynomials of total degree at most  $p_{\ell, K}$  on each simplex  $K \in \mathcal{T}_{\ell}$ . In other words, any function  $v_{\ell} \in V_{\ell}$  satisfies  $v_{\ell} \in H_0^1(\Omega)$  and  $v_{\ell}|_K \in \mathbb{P}_{p_{\ell, K}}(K)$  for all  $K \in \mathcal{T}_{\ell}$ , where  $\mathbb{P}_p(K)$  stands for the space of all polynomials of total degree at most  $p$  on the simplex  $K$ . Let us denote by  $N_{\ell}$  the dimension of the  $\ell$ -th level space  $V_{\ell}$ .

The initial coarse mesh and the initial polynomial-degree distribution  $(\mathcal{T}_0, \mathbf{p}_0)$  are assumed to be given. The purpose of each step  $\ell \geq 0$  of the adaptive loop of Scheme 2 is to produce the next pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$ . The nestedness property (2.1) gives us two restrictions on the meshes and polynomial-degree distributions defining the spaces  $V_{\ell}$ : (i) the sequence of meshes  $\{\mathcal{T}_{\ell}\}_{\ell \geq 0}$  needs to be *hierarchically nested*, i.e., for all  $\ell \geq 1$  the mesh  $\mathcal{T}_{\ell}$  is a refinement of  $\mathcal{T}_{\ell-1}$  such that for all  $K \in \mathcal{T}_{\ell}$ , there is a unique simplex  $\tilde{K} \in \mathcal{T}_{\ell-1}$ , called the parent of  $K$ , satisfying  $K \subseteq \tilde{K}$ ; (ii) The local polynomial degree is *locally increasing*, i.e., for all  $\ell \geq 1$  and all  $K \in \mathcal{T}_{\ell}$ ,  $p_{\ell, K} \geq p_{\ell-1, \tilde{K}}$ , where  $\tilde{K} \in \mathcal{T}_{\ell-1}$  is the parent of  $K$ . Moreover, we assume the following standard shape-regularity property: There exists a constant  $\kappa_{\mathcal{T}} > 0$  such that  $\max_{K \in \mathcal{T}_{\ell}} h_K / \rho_K \leq \kappa_{\mathcal{T}}$  for all  $\ell \geq 0$ , where  $h_K$  is the diameter of  $K$  and  $\rho_K$  is the diameter of the largest ball inscribed in  $K$ .

Let us now introduce some additional useful notation. We denote by  $\mathcal{V}_\ell$  the set of vertices of  $\mathcal{T}_\ell$  decomposed into interior vertices  $\mathcal{V}_\ell^{\text{int}}$  and vertices on the boundary  $\mathcal{V}_\ell^{\text{ext}}$ . For each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ ,  $\ell \geq 0$ , the so-called hat function  $\psi_\ell^{\mathbf{a}}$  is the continuous, piecewise affine function that takes the value 1 at the vertex  $\mathbf{a}$  and the value 0 at all the other vertices of  $\mathcal{V}_\ell$ ; the function  $\psi_\ell^{\mathbf{a}}$  is in  $V_\ell$  for all  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$ . Furthermore, we consider the simplex patch  $\mathcal{T}_\ell^{\mathbf{a}} \subset \mathcal{T}_\ell$  which is the collection of the simplices sharing the vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , with  $\omega_\ell^{\mathbf{a}}$  the corresponding open subdomain coinciding with the support of  $\psi_\ell^{\mathbf{a}}$ . Finally, for each simplex  $K \in \mathcal{T}_\ell$ ,  $\mathcal{V}_K$  denotes the set of vertices of  $K$ .

The Galerkin finite element method constructs an approximation of the weak solution  $u$  of (1.1) by solving the problem: Find  $u_\ell^{\text{ex}} \in V_\ell$  such that

$$(\nabla u_\ell^{\text{ex}}, \nabla v_\ell) = (f, v_\ell) \quad \forall v_\ell \in V_\ell. \quad (2.3)$$

The problem (2.3) is equivalent to solving the system of linear algebraic equations

$$\mathbb{A}_\ell \mathbf{U}_\ell^{\text{ex}} = \mathbf{F}_\ell, \quad (2.4)$$

where we employed  $\psi_\ell^n$ ,  $1 \leq n \leq N_\ell$ , the basis of the  $\ell$ -th level space  $V_\ell$  such that  $u_\ell^{\text{ex}} := \sum_{n=1}^{N_\ell} (\mathbf{U}_\ell^{\text{ex}})_n \psi_\ell^n$ . Hence,  $(\mathbb{A}_\ell)_{mn} := (\nabla \psi_\ell^m, \nabla \psi_\ell^n)$  is the symmetric positive-definite stiffness matrix and  $(\mathbf{F}_\ell)_m := (f, \psi_\ell^m)$  is the corresponding right-hand side vector.

However, in this work we do not assume that the algebraic system (2.4) is solved exactly (for  $\ell \geq 1$ ). Let us denote by  $\mathbf{U}_\ell \in \mathbb{R}^{N_\ell}$  an *arbitrary approximation* to the exact solution  $\mathbf{U}_\ell^{\text{ex}}$  of system (2.4), corresponding to a continuous piecewise polynomial  $u_\ell = \sum_{n=1}^{N_\ell} (\mathbf{U}_\ell)_n \psi_\ell^n \in V_\ell$ . The *algebraic residual vector*  $\mathbf{R}_\ell$  associated with  $\mathbf{U}_\ell$  is given by

$$\mathbf{R}_\ell := \mathbf{F}_\ell - \mathbb{A}_\ell \mathbf{U}_\ell. \quad (2.5)$$

Moreover, we introduce its functional representation  $\mathbf{r}_\ell \in \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell)$ ,  $\mathbf{r}_\ell|_{\partial\Omega} = 0$ , i.e. a *discontinuous* polynomial of total degree at most  $p_{\ell,K}$  on each  $K \in \mathcal{T}_\ell$  vanishing on the boundary  $\partial\Omega$  and satisfying

$$(\mathbf{r}_\ell, \psi_\ell^n) = (\mathbf{R}_\ell)_n \quad 1 \leq n \leq N_\ell. \quad (2.6)$$

Following Papež *et al.* [18, Section 5.1], we define  $\mathbf{r}_\ell$  in an elementwise manner by prescribing  $\mathbf{r}_\ell|_K \in \mathbb{P}_{p_{\ell,K}}(K)$ ,  $\mathbf{r}_\ell|_{\partial K \cap \partial\Omega} = 0$  for each simplex  $K \in \mathcal{T}_\ell$  individually such that

$$(\mathbf{r}_\ell, \psi_\ell^n)_K = \frac{(\mathbf{R}_\ell)_n}{N_\ell^n} \quad \text{for each } \psi_\ell^n \text{ non-vanishing on } K, \quad (2.7)$$

where  $N_\ell^n$  denotes the number of elements forming the support of the basis function  $\psi_\ell^n$ . Note that the property (2.6) together with the definition of the algebraic system (2.4) yield the functional equivalent of algebraic relation (2.5)

$$(\mathbf{r}_\ell, v_\ell) = (f, v_\ell) - (\nabla u_\ell, \nabla v_\ell) \quad v_\ell \in V_\ell. \quad (2.8)$$

### 3 Guaranteed total and algebraic a posteriori error bounds

Let the iteration number  $\ell \geq 0$  and an arbitrary approximate solution  $u_\ell \in V_\ell$  be fixed. In this section we derive the a posteriori error bounds based on equilibrated flux reconstructions by local problems, see e.g. [19, 20, 21, 22, 23], adapted to the present setting of conforming  $hp$ -finite elements. To be more precise, we will follow the concepts from the works of Jiránek *et al.* [24], Ern and Vohralík [14], Rey *et al.* [25] and Papež *et al.* [16] in order to distinguish in the guaranteed upper bound  $\eta(u_\ell, \mathcal{T}_\ell)$  on the *total energy error*  $\|\nabla(u - u_\ell)\|$  two different contributions: one serving as the guaranteed upper bound on the *algebraic error*  $\|\nabla(u_\ell^{\text{ex}} - u_\ell)\|$ , and the rest which corresponds to the *discretization error*  $\|\nabla(u - u_\ell^{\text{ex}})\|$ . Finally, for the total energy error; we also need to construct a guaranteed lower bound, so that a reliable confidence interval for the true value of  $\|\nabla(u - u_\ell)\|$  is at our disposal.

The two main ingredients for the error estimators bounding from above the total and the algebraic error in energy norm are an  $\mathbf{H}(\text{div}, \Omega)$ -conforming *total flux reconstruction* and an  $\mathbf{H}(\text{div}, \Omega)$ -conforming *algebraic error flux reconstruction*:

**Definition 3.1** (Total flux reconstruction  $\sigma_{\ell,\text{tot}}$ ). *We call total flux reconstruction any function  $\sigma_{\ell,\text{tot}}$  constructed from the approximate solution  $u_\ell$  satisfying*

$$\sigma_{\ell,\text{tot}} \in \mathbf{H}(\text{div}, \Omega), \quad (3.1a)$$

$$(\nabla \cdot \sigma_{\ell,\text{tot}}, q_\ell)_K = (f, q_\ell)_K \quad \forall K \in \mathcal{T}_\ell, \quad \forall q_\ell \in \mathbb{P}_{p_\ell, K}(K). \quad (3.1b)$$

**Definition 3.2** (Algebraic error flux reconstruction  $\sigma_{\ell,\text{alg}}$ ). *We call algebraic error flux reconstruction any function  $\sigma_{\ell,\text{alg}}$  constructed from  $\tau_\ell$  defined in (2.7), which satisfies*

$$\sigma_{\ell,\text{alg}} \in \mathbf{H}(\text{div}, \Omega), \quad (3.2a)$$

$$(\nabla \cdot \sigma_{\ell,\text{alg}}, q_\ell)_K = (\tau_\ell, q_\ell)_K \quad \forall K \in \mathcal{T}_\ell, \quad \forall q_\ell \in \mathbb{P}_{p_\ell, K}(K). \quad (3.2b)$$

Moreover, as we uncover in Theorem 3.3, there exists a natural decomposition of the total flux reconstruction  $\sigma_{\ell,\text{tot}}$  from Definition 3.1 in the form

$$\sigma_{\ell,\text{tot}} := \sigma_{\ell,\text{alg}} + \sigma_{\ell,\text{dis}}, \quad (3.3)$$

with  $\sigma_{\ell,\text{alg}}$  of Definition 3.2 and  $\sigma_{\ell,\text{dis}} \in \mathbf{H}(\text{div}, \Omega)$ , the *discretization flux reconstruction*, for which (3.2b) and (3.1b) yield

$$(\nabla \cdot \sigma_{\ell,\text{dis}}, q_\ell)_K = (f - \tau_\ell, q_\ell)_K \quad \forall K \in \mathcal{T}_\ell, \quad \forall q_\ell \in \mathbb{P}_{p_\ell, K}(K). \quad (3.4)$$

Note that unlike in the work [16], here the properties (3.1b), (3.2b), and (3.4) are imposed on the divergences of the flux reconstructions  $\sigma_{\ell,\text{tot}}$ ,  $\sigma_{\ell,\text{alg}}$ , and  $\sigma_{\ell,\text{dis}}$  only in a weak sense.

**Theorem 3.3** (Guaranteed upper bound on total and algebraic errors). *Let  $u \in H_0^1(\Omega)$  be the weak solution of the problem (1.1) and  $u_\ell^{\text{ex}} \in V_\ell$  be its exact finite element approximation given by (2.3). Let  $u_\ell \in V_\ell$  be arbitrary. Furthermore, let  $\sigma_{\ell,\text{tot}}$ ,  $\sigma_{\ell,\text{alg}}$  be given by Definitions 3.1 and 3.2, respectively, and  $\sigma_{\ell,\text{dis}} \in \mathbf{H}(\text{div}, \Omega)$  by (3.3). Then the following upper bound on the energy norm of the total error holds true:*

$$\|\nabla(u - u_\ell)\| \leq \eta(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_K^2(u_\ell) \right\}^{\frac{1}{2}}, \quad (3.5a)$$

$$\eta_K(u_\ell) := \underbrace{\|\nabla u_\ell + \sigma_{\ell,\text{dis}}\|_K}_{\eta_{\text{dis}, K}(u_\ell)} + \underbrace{\|\sigma_{\ell,\text{alg}}\|_K}_{\eta_{\text{alg}, K}(u_\ell)} + \underbrace{\frac{h_K}{\pi} \|f - \nabla \cdot \sigma_{\ell,\text{tot}}\|_K}_{\eta_{\text{osc}, K}(u_\ell)}, \quad (3.5b)$$

and we have the upper bound on the energy norm of the algebraic error

$$\|\nabla(u_\ell^{\text{ex}} - u_\ell)\| \leq \eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_{\text{alg}, K}^2(u_\ell) \right\}^{\frac{1}{2}}. \quad (3.6)$$

*Proof.* The proof follows the proofs of equivalent statements in [21, 22, 16] in a straightforward way. Let us remark that it is sufficient to enforce weakly the equilibration property (3.2b), as opposed to [16] where it is enforced strongly, to prove the algebraic error upper bound (3.6).  $\square$

As discussed in, e.g. [21, Remark 3.6], the term  $\eta_{\text{osc}, K}(u_\ell)$  represents, for all  $K \in \mathcal{T}_\ell$ , a local oscillation in the source datum  $f$  that, under suitable smoothness assumptions, converges to zero two orders faster than the error. The detailed description of the actual construction of the algebraic error flux reconstruction  $\sigma_{\ell,\text{alg}}$  and the discretization flux reconstruction  $\sigma_{\ell,\text{dis}}$ , yielding the total flux reconstruction  $\sigma_{\ell,\text{tot}}$ , is given in Section 4.2.

Following [18, Theorem 2], the key ingredient for bounding the total energy error from below is:

**Definition 3.4** (Total residual lifting  $\rho_{\ell,\text{tot}}$ ). *For each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , let  $V_\ell^{\mathbf{a}}$  be a finite-dimensional subspace of  $H_*^1(\omega_\ell^{\mathbf{a}})$*

$$H_*^1(\omega_\ell^{\mathbf{a}}) := \{v \in H^1(\omega_\ell^{\mathbf{a}}), (v, 1)_{\omega_\ell^{\mathbf{a}}} = 0\}, \quad \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}, \quad (3.7a)$$

$$H_*^1(\omega_\ell^{\mathbf{a}}) := \{v \in H^1(\omega_\ell^{\mathbf{a}}), v = 0 \text{ on } \partial\omega_\ell^{\mathbf{a}} \cap \partial\Omega\}, \quad \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}. \quad (3.7b)$$

The total residual lifting is constructed as  $\rho_{\ell,\text{tot}} := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \psi_\ell^{\mathbf{a}} \rho_{\ell,\text{tot}}^{\mathbf{a}} \in H_0^1(\Omega)$ , where each vertex contribution solves the local primal finite element problem

$$(\nabla \rho_{\ell,\text{tot}}^{\mathbf{a}}, \nabla v_\ell)_{\omega_\ell^{\mathbf{a}}} = (f, \psi_\ell^{\mathbf{a}} v_\ell)_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla(\psi_\ell^{\mathbf{a}} v_\ell))_{\omega_\ell^{\mathbf{a}}} \quad \forall v_\ell \in V_\ell^{\mathbf{a}}.$$

**Theorem 3.5** (Guaranteed lower bound on the total error). *Let  $u \in H_0^1(\Omega)$  be the weak solution of the problem (1.1). Let  $\rho_{\ell,\text{tot}}$  be associated with the approximate solution  $u_\ell$  as in Definition 3.4. Then, the following holds true:*

$$\|\nabla(u - u_\ell)\| \geq \frac{\sum_{\mathbf{a} \in \mathcal{V}_\ell} \|\nabla \rho_{\ell,\text{tot}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}^2}{\|\nabla \rho_{\ell,\text{tot}}\|} =: \mu(u_\ell). \quad (3.8)$$

## 4 The inexact $hp$ -adaptive algorithm

In this section we present the modules `ONE_SOLVER_STEP`, `ESTIMATE`, `MARK`, and `REFINE` of the adaptive loop described in Scheme 2. We recall that  $\ell \geq 0$  denotes the iteration number.

### 4.1 The module `ONE_SOLVER_STEP`

First, let the current iteration number be  $\ell = 0$ . Note that at this stage the nested sequence of spaces characterized by (2.1) and (2.2) contains only the initial  $H_0^1$ -conforming finite element space  $V_0$ . This represents a special case where the module `ONE_SOLVER_STEP` takes as input only the space  $V_0$  and sets the output function  $u_0 \in V_0$  directly to be the exact solution of (2.3); here the corresponding (still small) linear algebraic problem (2.4) is considered to be solved exactly.

Otherwise, for the iteration  $\ell \geq 1$ , the module `ONE_SOLVER_STEP` may and in most cases will be called several times due to the coupling with the `ESTIMATE` module (cf. Scheme 2). Let  $P_{\ell-1}^\ell : V_{\ell-1} \rightarrow V_\ell$  be a canonical prolongation operator between the consecutive nested finite element spaces. Before the initial call of `ONE_SOLVER_STEP` at iteration  $\ell$ , we initialize the  $\ell$ -th level approximation  $u_\ell \in V_\ell$  of the exact finite element solution  $u_\ell^{\text{ex}}$ , by setting  $u_\ell := P_{\ell-1}^\ell u_{\ell-1}$ . This corresponds to setting the initial guess for the algebraic solver. The module `ONE_SOLVER_STEP` for  $\ell \geq 1$  takes as input not only the space  $V_\ell$ , but also the current approximation  $u_\ell$  which is in turn improved and returned as the output of the module. Here, by the improvement of  $u_\ell$ , we mean applying one or a given small number of steps of the given iterative algebraic solver to the system (2.4) assembled within the initial call of `ONE_SOLVER_STEP` at the iteration  $\ell$ .

The quality of the output  $u_\ell$  is then assessed by the module `ESTIMATE` and if necessary, see section 4.3,  $u_\ell$  is passed again as an input to the successive call of module `ONE_SOLVER_STEP`.

### 4.2 The module `ESTIMATE`

The module `ESTIMATE` crucially relies on the theoretical developments of Section 3. It takes as input the current approximation  $u_\ell$  to the exact finite element solution  $u_\ell^{\text{ex}}$ , computes the corresponding flux reconstructions  $\sigma_{\ell,\text{alg}}$ ,  $\sigma_{\ell,\text{dis}}$ ,  $\sigma_{\ell,\text{tot}}$  and the total residual lifting  $\rho_{\ell,\text{tot}}$  defined in Section 3, and finally outputs a collection of local error indicators  $\{\eta_{\text{alg},K}(u_\ell), \eta_{\text{dis},K}(u_\ell), \eta_{\text{osc},K}(u_\ell)\}_{K \in \mathcal{T}_\ell}$  together with the lower bound  $\mu(u_\ell)$  defined in Theorems 3.3 and 3.5, respectively. In what follows, we outline all the necessary details concerning the actual construction of the flux reconstructions  $\sigma_{\ell,\text{alg}}$  and  $\sigma_{\ell,\text{dis}}$ , hence also their sum  $\sigma_{\ell,\text{tot}}$ , and the lifting  $\rho_{\ell,\text{tot}}$ . Once they are all properly constructed, the local error indicators  $\eta_{*,K}(u_\ell)$  of (3.5b) and  $\mu(u_\ell)$  defined in (3.8) are evaluated.

#### 4.2.1 Multilevel construction of algebraic error flux reconstruction $\sigma_{\ell,\text{alg}}$

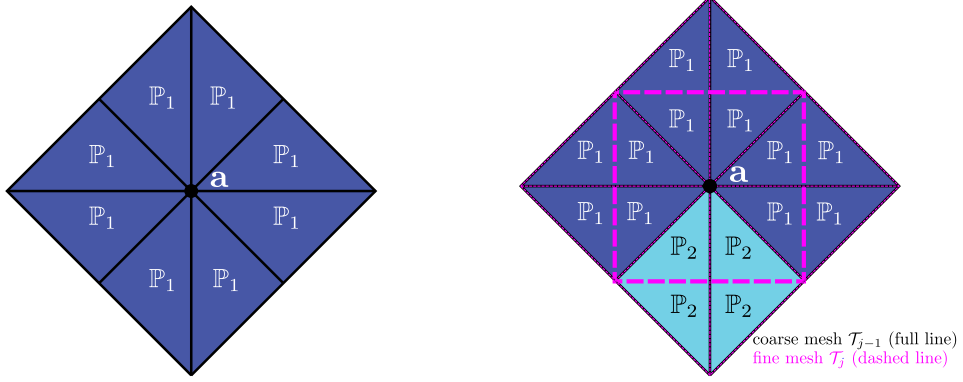
In order to obtain the algebraic error flux reconstruction  $\sigma_{\ell,\text{alg}}$  of Definition 3.2, we use the multilevel approach introduced by Papež *et al.* in [16]. We extend it here to the present conforming  $hp$ -finite element setting. The multilevel approach seems like a natural choice, especially in the present adaptive framework, where for the current space  $V_\ell$ , built up on the pair  $(\mathcal{T}_\ell, \mathbf{p}_\ell)$ , the hierarchy of its nested finite element subspaces  $\{V_j\}_{0 \leq j < \ell}$ , together with the meshes and polynomial degree distributions  $\{(\mathcal{T}_j, \mathbf{p}_j)\}_{0 \leq j < \ell}$ , are readily at hand from the previous iterations of the adaptive loop. We will refer to the mesh levels 0 and  $\ell$  as the coarsest and the finest level, respectively.



Firstly, following [16], for the algebraic residual  $\mathbf{r}_\ell \in \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell)$  given by (2.7) we introduce the *coarsest-level Riesz representer*  $\varphi_{0,\text{alg}} \in V_0$

$$(\nabla\varphi_{0,\text{alg}}, \nabla v_0) = (\mathbf{r}_\ell, v_0) \quad \forall v_0 \in V_0. \quad (4.1)$$

For each  $\mathbf{a} \in \mathcal{V}_j$ ,  $0 \leq j \leq \ell$ , recall the definition of a simplex patch  $\mathcal{T}_j^\mathbf{a}$  with the corresponding subdomain  $\omega_j^\mathbf{a}$ . In addition, let us introduce for each coarse vertex  $\mathbf{a} \in \mathcal{V}_{j-1}$ ,  $1 \leq j \leq \ell$ , a simplex patch  $\mathcal{T}_{j,j-1}^\mathbf{a}$  of all the next finer level simplices  $K \in \mathcal{T}_j$  such that  $K \subset \omega_{j-1}^\mathbf{a}$ , cf. Figure 1, and the local polynomial degree  $p_\mathbf{a}^{\text{alg}} := \max_{K \in \mathcal{T}_{j,j-1}^\mathbf{a}} p_{j,K}$  (any other choice so that  $p_\mathbf{a}^{\text{alg}} \geq \max_{K \in \mathcal{T}_{j,j-1}^\mathbf{a}} p_{j,K}$  can also be considered).



**Figure 1:** Patches of simplices  $\mathcal{T}_{j-1}^\mathbf{a}$  (left) and  $\mathcal{T}_{j,j-1}^\mathbf{a}$  (right) on the subdomain  $\omega_{j-1}^\mathbf{a}$  around a coarse vertex  $\mathbf{a} \in \mathcal{V}_{j-1}$  together with the corresponding polynomial degree distributions. Note that in this case, the local polynomial degree  $p_\mathbf{a}^{\text{alg}} = 2$ .

Let the mesh level  $1 \leq j \leq \ell$  be fixed together with the vertex from the next coarser mesh  $\mathbf{a} \in \mathcal{V}_{j-1}$ . We define the local  $p$ -th order Raviart–Thomas–Nédélec space on the subdomain  $\omega_{j-1}^\mathbf{a}$  with the mesh induced by the next finer mesh  $\mathcal{T}_j$  by

$$\mathbf{RTN}_p(\omega_{j-1}^\mathbf{a}) = \{\mathbf{v}_j \in \mathbf{H}(\text{div}, \omega_{j-1}^\mathbf{a}); \mathbf{v}_j|_K \in \mathbf{RTN}_p(K), \quad \forall K \in \mathcal{T}_{j,j-1}^\mathbf{a}\}, \quad (4.2)$$

where  $\mathbf{RTN}_p(K) = [\mathbb{P}_p(K)]^d + \mathbb{P}_p(K)\mathbf{x}$  is the usual  $p$ -th order Raviart–Thomas–Nédélec space (cf. [26, 27]) on a simplex  $K \in \mathcal{T}_j$ . Furthermore, we consider the pair of local mixed finite element spaces  $(\mathbf{V}_{j,j-1}^\mathbf{a}, Q_{j,j-1}^\mathbf{a})$  which are defined by

$$\begin{aligned} \mathbf{V}_{j,j-1}^\mathbf{a} &:= \{\mathbf{v}_j \in \mathbf{RTN}_{p_\mathbf{a}^{\text{alg}}}(\omega_{j-1}^\mathbf{a}); \mathbf{v}_j \cdot \mathbf{n}_{\omega_{j-1}^\mathbf{a}} = 0 \text{ on } \partial\omega_{j-1}^\mathbf{a}\}, \\ Q_{j,j-1}^\mathbf{a} &:= \{q_j \in \mathbb{P}_{p_\mathbf{a}^{\text{alg}}}(\mathcal{T}_{j,j-1}^\mathbf{a}); (q_j, 1)_{\omega_{j-1}^\mathbf{a}} = 0\}, \end{aligned} \quad \text{if } \mathbf{a} \in \mathcal{V}_{j-1}^{\text{int}}, \quad (4.3a)$$

$$\begin{aligned} \mathbf{V}_{j,j-1}^\mathbf{a} &:= \{\mathbf{v}_j \in \mathbf{RTN}_{p_\mathbf{a}^{\text{alg}}}(\omega_{j-1}^\mathbf{a}); \mathbf{v}_j \cdot \mathbf{n}_{\omega_{j-1}^\mathbf{a}} = 0 \text{ on } \partial\omega_{j-1}^\mathbf{a} \setminus \partial\Omega\}, \\ Q_{j,j-1}^\mathbf{a} &:= \mathbb{P}_{p_\mathbf{a}^{\text{alg}}}(\mathcal{T}_{j,j-1}^\mathbf{a}), \end{aligned} \quad \text{if } \mathbf{a} \in \mathcal{V}_{j-1}^{\text{ext}}, \quad (4.3b)$$

where  $\mathbf{n}_{\omega_{j-1}^\mathbf{a}}$  denotes the unit outward normal to  $\omega_{j-1}^\mathbf{a}$ . For an interior vertex  $\mathbf{a} \in \mathcal{V}_{j-1}^{\text{int}}$ , the degrees of freedom of the spaces  $\mathbf{V}_{j,j-1}^\mathbf{a}$  and  $Q_{j,j-1}^\mathbf{a}$  with the local polynomial degree  $p_\mathbf{a}^{\text{alg}}$  equal to two are illustrated in Figure 2.

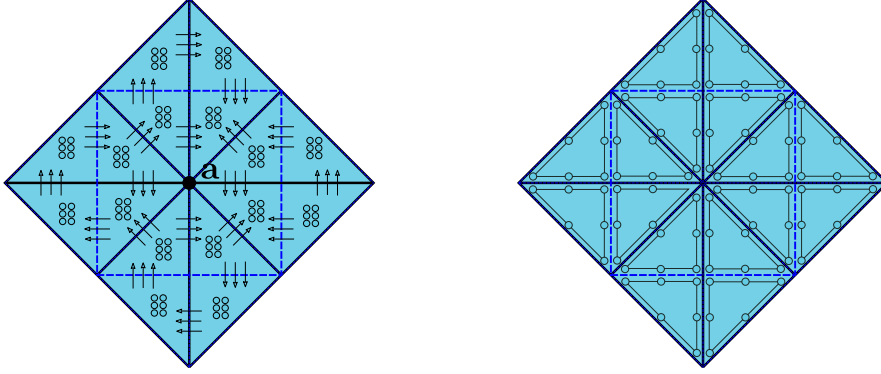
Once the coarsest-level Riesz representer  $\varphi_{0,\text{alg}}$  of (4.1) is computed, the algebraic error flux reconstruction  $\sigma_{\ell,\text{alg}}$  is constructed via solving the local dual mixed finite element problems on simplex patches  $\mathcal{T}_{j,j-1}^\mathbf{a}$  around the coarse vertices  $\mathbf{a} \in \mathcal{V}_{j-1}$ , starting from  $j = 1$  and up to the current finest level  $j = \ell$ :

**Definition 4.1** (Construction of  $\sigma_{\ell,\text{alg}}$ ). *Let  $u_\ell \in V_\ell$  be arbitrary. The algebraic error flux reconstruction  $\sigma_{\ell,\text{alg}}$  is constructed as*

$$\sigma_{\ell,\text{alg}} := \sum_{j=1}^{\ell} \sum_{\mathbf{a} \in \mathcal{V}_{j-1}} \sigma_{j,\text{alg}}^\mathbf{a}, \quad (4.4)$$

where:





**Figure 2:** Degrees of freedom of the local mixed finite element spaces  $\mathbf{V}_{j,j-1}^{\mathbf{a}}$  (left, arrows and bullets) and  $Q_{j,j-1}^{\mathbf{a}}$  (right, bullets) with  $p_{\mathbf{a}}^{\text{alg}} = 2$ ; interior vertex  $\mathbf{a} \in \mathcal{V}_{j-1}^{\text{int}}$ . On the right, leaving out one degree of freedom of the broken space  $\mathbb{P}_{p_{\mathbf{a}}^{\text{alg}}}(\mathcal{T}_{j,j-1}^{\mathbf{a}})$  corresponds to the zero mean value constraint posed on the functions in  $Q_{j,j-1}^{\mathbf{a}}$ . The underlying mesh and the choice of  $p_{\mathbf{a}}^{\text{alg}}$  correspond to the simplex patch  $\mathcal{T}_{j,j-1}^{\mathbf{a}}$  from Figure 1.

- On level  $j = 1$ , for each vertex  $\mathbf{a} \in \mathcal{V}_0$ ,  $(\boldsymbol{\sigma}_{1,\text{alg}}^{\mathbf{a}}, \gamma_1^{\mathbf{a}}) \in \mathbf{V}_{1,0}^{\mathbf{a}} \times Q_{1,0}^{\mathbf{a}}$  solves

$$(\boldsymbol{\sigma}_{1,\text{alg}}^{\mathbf{a}}, \mathbf{v}_1)_{\omega_0^{\mathbf{a}}} - (\gamma_1^{\mathbf{a}}, \nabla \cdot \mathbf{v}_1)_{\omega_0^{\mathbf{a}}} = 0 \quad \forall \mathbf{v}_1 \in \mathbf{V}_{1,0}^{\mathbf{a}}, \quad (4.5a)$$

$$(\nabla \cdot \boldsymbol{\sigma}_{1,\text{alg}}^{\mathbf{a}}, q_1)_{\omega_0^{\mathbf{a}}} = (\boldsymbol{\tau}_\ell \psi_0^{\mathbf{a}} - \nabla \varphi_{0,\text{alg}} \cdot \nabla \psi_0^{\mathbf{a}}, q_1)_{\omega_0^{\mathbf{a}}} \quad \forall q_1 \in Q_{1,0}^{\mathbf{a}}; \quad (4.5b)$$

or, equivalently,

$$\boldsymbol{\sigma}_{1,\text{alg}}^{\mathbf{a}} := \arg \min_{\mathbf{v}_1 \in \mathbf{V}_{1,0}^{\mathbf{a}}, \nabla \cdot \mathbf{v}_1 = \Pi_{Q_{1,0}^{\mathbf{a}}}(\boldsymbol{\tau}_\ell \psi_0^{\mathbf{a}} - \nabla \varphi_{0,\text{alg}} \cdot \nabla \psi_0^{\mathbf{a}})} \|\mathbf{v}_1\|_{\omega_0^{\mathbf{a}}}. \quad (4.6)$$

- On level  $2 \leq j \leq \ell$ , for each vertex  $\mathbf{a} \in \mathcal{V}_{j-1}$ ,  $(\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, \gamma_j^{\mathbf{a}}) \in \mathbf{V}_{j,j-1}^{\mathbf{a}} \times Q_{j,j-1}^{\mathbf{a}}$  solves

$$(\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, \mathbf{v}_j)_{\omega_{j-1}^{\mathbf{a}}} - (\gamma_j^{\mathbf{a}}, \nabla \cdot \mathbf{v}_j)_{\omega_{j-1}^{\mathbf{a}}} = 0 \quad \forall \mathbf{v}_j \in \mathbf{V}_{j,j-1}^{\mathbf{a}}, \quad (4.7a)$$

$$(\nabla \cdot \boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}}, q_j)_{\omega_{j-1}^{\mathbf{a}}} = \left( \boldsymbol{\tau}_\ell \psi_{j-1}^{\mathbf{a}} - \sum_{i=1}^{j-1} \psi_{j-1}^{\mathbf{a}} \nabla \cdot \boldsymbol{\sigma}_{i,\text{alg}}^{\mathbf{a}}, q_j \right)_{\omega_{j-1}^{\mathbf{a}}} \quad \forall q_j \in Q_{j,j-1}^{\mathbf{a}}; \quad (4.7b)$$

where we set  $\boldsymbol{\sigma}_{i,\text{alg}}^{\mathbf{a}} := \sum_{\mathbf{a} \in \mathcal{V}_{i-1}} \boldsymbol{\sigma}_{i,\text{alg}}^{\mathbf{a}}$ , for  $1 \leq i \leq j-1$ . Equivalently, problem (4.7) can be restated as

$$\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}} := \arg \min_{\mathbf{v}_j \in \mathbf{V}_{j,j-1}^{\mathbf{a}}, \nabla \cdot \mathbf{v}_j = \Pi_{Q_{j,j-1}^{\mathbf{a}}}(\boldsymbol{\tau}_\ell \psi_{j-1}^{\mathbf{a}} - \sum_{i=1}^{j-1} \psi_{j-1}^{\mathbf{a}} \nabla \cdot \boldsymbol{\sigma}_{i,\text{alg}}^{\mathbf{a}})} \|\mathbf{v}_j\|_{\omega_{j-1}^{\mathbf{a}}}. \quad (4.8)$$

Here, each vertex contribution  $\boldsymbol{\sigma}_{i,\text{alg}}^{\mathbf{a}}$  is extended by zero outside its initial domain of definition.

Note that the Neumann compatibility condition for problem (4.5) is satisfied for all  $\mathbf{a} \in \mathcal{V}_0^{\text{int}}$  due to the definition of  $\varphi_{0,\text{alg}}$  (take  $v_0 = \psi_0^{\mathbf{a}}$  as a test function in (4.1)). Thus (4.5b) is satisfied for all the test functions from  $\mathbb{P}_{p_{\mathbf{a}}^{\text{alg}}}(\mathcal{T}_{1,0}^{\mathbf{a}})$ , not only those with zero mean value. Due to the discontinuous nature of this broken polynomial space and the fact that for each  $K \in \mathcal{T}_1$ , with the parent element  $\tilde{K} \in \mathcal{T}_0$ ,  $p_{1,K} \leq \min_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} p_{\mathbf{a}}^{\text{alg}}$ , we have

$$\begin{aligned} (\nabla \cdot \boldsymbol{\sigma}_{1,\text{alg}}, q_1)_K &= \sum_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} (\nabla \cdot \boldsymbol{\sigma}_{1,\text{alg}}^{\mathbf{a}}, q_1)_K = \sum_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} (\boldsymbol{\tau}_\ell \psi_0^{\mathbf{a}} - \nabla \varphi_{0,\text{alg}} \cdot \nabla \psi_0^{\mathbf{a}}, q_1)_K \\ &= (\boldsymbol{\tau}_\ell, q_1)_K \quad \forall K \in \mathcal{T}_1 \quad \forall q_1 \in \mathbb{P}_{p_{1,K}}(K), \end{aligned} \quad (4.9)$$

where we have employed the partition of unity via  $\sum_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} \psi_0^{\mathbf{a}}|_K = 1|_K$ . At level  $j = 2$ , for each interior vertex  $\mathbf{a} \in \mathcal{V}_1^{\text{int}}$ , the Neumann compatibility condition for problem (4.7)  $(\boldsymbol{\tau}_\ell - \nabla \cdot \boldsymbol{\sigma}_{1,\text{alg}}, \psi_1^{\mathbf{a}})_{\omega_1^{\mathbf{a}}} = 0$  is a direct consequence of (4.9), since  $\psi_1^{\mathbf{a}}|_K \in \mathbb{P}_{p_{1,K}}(K)$  for each  $K \in \mathcal{T}_1^{\mathbf{a}}$ . Thus, similarly to (4.5b) also (4.7b), so

far only at level  $j = 2$ , is satisfied for all the test functions from  $\mathbb{P}_{p_{\mathbf{a}}^{\text{alg}}}(\mathcal{T}_{2,1}^{\mathbf{a}})$  without any zero-mean value restrictions. Then, similarly to (4.9), we use the discontinuous nature of the broken space  $\mathbb{P}_{p_{\mathbf{a}}^{\text{alg}}}(\mathcal{T}_{2,1}^{\mathbf{a}})$ , our choice of local polynomial degrees  $p_{\mathbf{a}}^{\text{alg}}$  used in definition of the spaces (4.3), for each vertex  $\mathbf{a} \in \mathcal{V}_1$ , such that  $p_{2,K} \leq \min_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} p_{\mathbf{a}}^{\text{alg}}$ , for each  $K \in \mathcal{T}_2$  with the parent element  $\tilde{K} \in \mathcal{T}_1$ , and the partition of unity via  $\sum_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} \psi_1^{\mathbf{a}}|_K = 1$  to obtain

$$\left( \mathbf{r}_\ell - \sum_{i=1}^2 \nabla \cdot \boldsymbol{\sigma}_{i,\text{alg}}, q_2 \right)_K = 0 \quad \forall K \in \mathcal{T}_2 \quad \forall q_2 \in \mathbb{P}_{p_{2,K}}(K). \quad (4.10)$$

Property (4.10) in turn yields the Neumann compatibility condition on the third level. Progressing successively, at remaining levels  $3 \leq j \leq \ell - 1$ , we always have the property

$$\left( \mathbf{r}_\ell - \sum_{i=1}^j \nabla \cdot \boldsymbol{\sigma}_{i,\text{alg}}, q_j \right)_K = 0 \quad \forall K \in \mathcal{T}_j \quad \forall q_j \in \mathbb{P}_{p_{j,K}}(K), \quad (4.11)$$

yielding the Neumann compatibility condition on the next level  $j + 1$ , thus (4.7b) is satisfied for all the test functions from the broken polynomial space  $\mathbb{P}_{p_{\mathbf{a}}^{\text{alg}}}(\mathcal{T}_{j,j-1}^{\mathbf{a}})$ , for each  $\mathbf{a} \in \mathcal{V}_{j-1}$ ,  $2 \leq j \leq \ell$ . Furthermore, as a result of the Neumann compatibility condition on the finest level, on each finest simplex  $K \in \mathcal{T}_\ell$ , the sum of local contributions  $\sum_{\mathbf{a} \in \mathcal{V}_{\ell-1}} \boldsymbol{\sigma}_{\ell,\text{alg}}^{\mathbf{a}}$  admits the following property:

$$\left( \sum_{\mathbf{a} \in \mathcal{V}_{\ell-1}} \nabla \cdot \boldsymbol{\sigma}_{\ell,\text{alg}}^{\mathbf{a}}, q_\ell \right)_K = \sum_{\mathbf{a} \in \mathcal{V}_{\tilde{K}}} (\nabla \cdot \boldsymbol{\sigma}_{\ell,\text{alg}}^{\mathbf{a}}, q_\ell)_K = \left( \mathbf{r}_\ell - \sum_{i=1}^{\ell-1} \nabla \cdot \boldsymbol{\sigma}_{i,\text{alg}}, q_\ell \right)_K \quad \forall q_\ell \in \mathbb{P}_{p_{\ell,K}}(K). \quad (4.12)$$

**Lemma 4.2** (Properties of  $\boldsymbol{\sigma}_{\ell,\text{alg}}$ ). *The algebraic error flux reconstruction  $\boldsymbol{\sigma}_{\ell,\text{alg}}$  constructed in Definition 4.1 satisfies the properties of Definition 3.2.*

*Proof.* For  $1 \leq j \leq \ell$ , each local contribution  $\boldsymbol{\sigma}_{j,\text{alg}}^{\mathbf{a}} \in \mathbf{H}(\text{div}, \omega_{j-1}^{\mathbf{a}})$ ,  $\mathbf{a} \in \mathcal{V}_{j-1}$ , by construction. Imposing the homogeneous Neumann boundary conditions in the definition of the local spaces (4.3) and extending by zero outside of  $\omega_{j-1}^{\mathbf{a}}$  then give the overall  $\mathbf{H}(\text{div}, \Omega)$ -conformity (3.2a) of  $\boldsymbol{\sigma}_{\ell,\text{alg}}$  as it is defined by (4.4). Then, splitting the sum in the definition (4.4) in combination with the property (4.12) yields for each  $K \in \mathcal{T}_\ell$ ,

$$(\nabla \cdot \boldsymbol{\sigma}_{\ell,\text{alg}}, q_\ell)_K = \left( \sum_{\mathbf{a} \in \mathcal{V}_{\ell-1}} \nabla \cdot \boldsymbol{\sigma}_{\ell,\text{alg}}^{\mathbf{a}} + \sum_{i=1}^{\ell-1} \nabla \cdot \boldsymbol{\sigma}_{i,\text{alg}}, q_\ell \right)_K = (\mathbf{r}_\ell, q_\ell)_K \quad \forall q_\ell \in \mathbb{P}_{p_{\ell,K}}(K).$$

Hence, the algebraic error flux reconstruction  $\boldsymbol{\sigma}_{\ell,\text{alg}}$  indeed satisfies (3.2b).  $\square$

**Remark 4.3** (Comparison with previous developments). *The local problems in Definition 4.1 differ from those of [16, Definition 6.3] in two aspects: (i) the local spaces on each simplex patch are assigned a specific polynomial degree along the lines of [15, 22]; (ii) the right-hand sides of the local problems (4.7), namely the divergence constraints in (4.7b), differ from their counterparts in [16, Definition 6.3]. In particular, the first level algebraic error flux reconstruction  $\boldsymbol{\sigma}_{1,\text{alg}}$  is now successively corrected on the finer levels without the need of introducing the  $L^2$ -orthogonal projections onto global coarser spaces, which is not suitable anymore because of the possibly varying polynomial degrees  $p_{\mathbf{a}}^{\text{alg}}$  across the neighboring patches.*

#### 4.2.2 Construction of the discretization flux reconstruction $\boldsymbol{\sigma}_{\ell,\text{dis}}$

Next, we present the details on the actual construction of the  $\mathbf{H}(\text{div}, \Omega)$ -conforming discretization flux reconstruction  $\boldsymbol{\sigma}_{\ell,\text{dis}}$ . Similarly to the above construction of  $\boldsymbol{\sigma}_{\ell,\text{alg}}$ , we construct  $\boldsymbol{\sigma}_{\ell,\text{dis}}$  locally via mixed finite element solves, but this time only on the finest simplex patches  $\mathcal{T}_\ell^{\mathbf{a}}$  around the finest mesh vertices  $\mathbf{a} \in \mathcal{V}_\ell$ . Namely, we follow the approach of [16, Definition 7.1], [18, Sec. 4.4], and [14, Definition 6.9] adapted to the present setting with varying polynomial degree. For each  $\mathbf{a} \in \mathcal{V}_\ell$ , we consider the local polynomial degree  $p_{\mathbf{a}}^{\text{dis}} := \max_{K \in \mathcal{T}_\ell^{\mathbf{a}}} p_{\ell,K}$  (again any other choice so that  $p_{\mathbf{a}}^{\text{dis}} \geq \max_{K \in \mathcal{T}_\ell^{\mathbf{a}}} p_{\ell,K}$  can also be employed). For a fixed finest vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , let

$$\mathbf{RTN}_p(\omega_\ell^{\mathbf{a}}) = \{ \mathbf{v}_\ell \in \mathbf{H}(\text{div}, \omega_\ell^{\mathbf{a}}); \mathbf{v}_\ell|_K \in \mathbf{RTN}_p(K), \quad \forall K \in \mathcal{T}_\ell^{\mathbf{a}} \}. \quad (4.13)$$

Then, we define the local spaces with homogeneous Neumann boundary condition

$$\begin{aligned} \mathbf{V}_\ell^{\mathbf{a}} &:= \{\mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{dis}}}(\omega_\ell^{\mathbf{a}}); \mathbf{v}_\ell \cdot \mathbf{n}_{\omega_\ell^{\mathbf{a}}} = 0 \text{ on } \partial\omega_\ell^{\mathbf{a}}\}, \\ Q_\ell^{\mathbf{a}} &:= \{q_\ell \in \mathbb{P}_{p_{\mathbf{a}}^{\text{dis}}}(\mathcal{T}_\ell^{\mathbf{a}}); (q_\ell, 1)_{\omega_\ell^{\mathbf{a}}} = 0\}, \end{aligned} \quad \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}, \quad (4.14a)$$

$$\begin{aligned} \mathbf{V}_\ell^{\mathbf{a}} &:= \{\mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{dis}}}(\omega_\ell^{\mathbf{a}}); \mathbf{v}_\ell \cdot \mathbf{n}_{\omega_\ell^{\mathbf{a}}} = 0 \text{ on } \partial\omega_\ell^{\mathbf{a}} \setminus \partial\Omega\}, \\ Q_\ell^{\mathbf{a}} &:= \mathbb{P}_{p_{\mathbf{a}}^{\text{dis}}}(\mathcal{T}_\ell^{\mathbf{a}}), \end{aligned} \quad \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}, \quad (4.14b)$$

with  $\mathbf{n}_{\omega_\ell^{\mathbf{a}}}$  denoting the unit outward normal to  $\omega_\ell^{\mathbf{a}}$ .

**Definition 4.4** (Construction of  $\sigma_{\ell, \text{dis}}$ ). *Let  $u_\ell \in V_\ell$  be the approximation used in Definition 4.1. We construct the discretization flux reconstruction  $\sigma_{\ell, \text{dis}}$  as*

$$\sigma_{\ell, \text{dis}} := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \sigma_{\ell, \text{dis}}^{\mathbf{a}}, \quad (4.15)$$

where, for each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ ,  $(\sigma_{\ell, \text{dis}}^{\mathbf{a}}, \gamma_\ell^{\mathbf{a}}) \in \mathbf{V}_\ell^{\mathbf{a}} \times Q_\ell^{\mathbf{a}}$  solves

$$(\sigma_{\ell, \text{dis}}^{\mathbf{a}}, \mathbf{v}_\ell)_{\omega_\ell^{\mathbf{a}}} - (\gamma_\ell^{\mathbf{a}}, \nabla \cdot \mathbf{v}_\ell)_{\omega_\ell^{\mathbf{a}}} = -(\psi_\ell^{\mathbf{a}} \nabla u_\ell, \mathbf{v}_\ell)_{\omega_\ell^{\mathbf{a}}} \quad \forall \mathbf{v}_\ell \in \mathbf{V}_\ell^{\mathbf{a}}, \quad (4.16a)$$

$$(\nabla \cdot \sigma_{\ell, \text{dis}}^{\mathbf{a}}, q_\ell)_{\omega_\ell^{\mathbf{a}}} = (f \psi_\ell^{\mathbf{a}} - \nabla u_\ell \cdot \nabla \psi_\ell^{\mathbf{a}} - \mathbf{r}_\ell \psi_\ell^{\mathbf{a}}, q_\ell)_{\omega_\ell^{\mathbf{a}}} \quad \forall q_\ell \in Q_\ell^{\mathbf{a}}, \quad (4.16b)$$

and where  $\sigma_{\ell, \text{dis}}^{\mathbf{a}}$  is extended by zero outside  $\omega_\ell^{\mathbf{a}}$ .

The Neumann compatibility condition for problem (4.16) is satisfied for all  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$  as a direct consequence of (2.8).

**Lemma 4.5** (Properties of  $\sigma_{\ell, \text{dis}}$ ). *The discretization flux reconstruction  $\sigma_{\ell, \text{dis}}$  from Definition 4.4 belongs to  $\mathbf{H}(\text{div}, \Omega)$  and on each simplex  $K \in \mathcal{T}_\ell$ , it satisfies*

$$(\nabla \cdot \sigma_{\ell, \text{dis}}, q_\ell)_K = (f - \mathbf{r}_\ell, q_\ell)_K \quad \forall K \in \mathcal{T}_\ell, \quad \forall q_\ell \in \mathbb{P}_{p_{\ell, K}}(K). \quad (4.17)$$

*Proof.* Since each local contribution  $\sigma_{\ell, \text{dis}}^{\mathbf{a}}$ ,  $\mathbf{a} \in \mathcal{V}_\ell$ , lies in  $\mathbf{H}(\text{div}, \omega_\ell^{\mathbf{a}})$ , the overall  $\mathbf{H}(\text{div}, \Omega)$ -conformity is a direct consequence of (4.15). Next, as a result of the Neumann compatibility condition, we are allowed to take any function  $q_\ell \in \mathbb{P}_{p_{\mathbf{a}}^{\text{dis}}}(\mathcal{T}_\ell^{\mathbf{a}})$  as a test function in (4.16b), without any zero mean value restriction. On each  $K \in \mathcal{T}_\ell$ , we have  $p_{\ell, K} \leq \min_{\mathbf{a} \in \mathcal{V}_K} p_{\mathbf{a}}^{\text{dis}}$ . Hence, combining (4.15) with (4.16b) and employing the partition of unity  $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_\ell^{\mathbf{a}}|_K = 1|_K$ , we see that

$$\begin{aligned} (\nabla \cdot \sigma_{\ell, \text{dis}}, q_\ell)_K &= \sum_{\mathbf{a} \in \mathcal{V}_K} (\nabla \cdot \sigma_{\ell, \text{dis}}^{\mathbf{a}}, q_\ell)_K = \sum_{\mathbf{a} \in \mathcal{V}_K} (f \psi_\ell^{\mathbf{a}} - \nabla u_\ell \cdot \nabla \psi_\ell^{\mathbf{a}} - \mathbf{r}_\ell \psi_\ell^{\mathbf{a}}, q_\ell)_K \\ &= (f - \mathbf{r}_\ell, q_\ell)_K \quad \forall K \in \mathcal{T}_\ell \quad \forall q_\ell \in \mathbb{P}_{p_{\ell, K}}(K), \end{aligned} \quad (4.18)$$

which concludes the proof.  $\square$

### 4.2.3 Discrete spaces for the total residual lifting $\rho_{\ell, \text{tot}}$

Finally, for each finest vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , we specify our choice for the  $H_*^1(\omega_\ell^{\mathbf{a}})$ -conforming scalar-valued spaces  $V_\ell^{\mathbf{a}}$ , in which we seek the local contributions of the total residual lifting  $\rho_{\ell, \text{tot}}$  of Definition 3.4:

$$V_\ell^{\mathbf{a}} := \{v_\ell \in H^1(\omega_\ell^{\mathbf{a}}); v_\ell \in \mathbb{P}_{p_{\mathbf{a}}^{\text{dis}}}(K), \forall K \in \mathcal{T}_\ell^{\mathbf{a}}, (v_\ell, 1)_{\omega_\ell^{\mathbf{a}}} = 0\} \quad \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}, \quad (4.19a)$$

$$V_\ell^{\mathbf{a}} := \{v_\ell \in H^1(\omega_\ell^{\mathbf{a}}); v_\ell \in \mathbb{P}_{p_{\mathbf{a}}^{\text{dis}}}(K), \forall K \in \mathcal{T}_\ell^{\mathbf{a}}, v_\ell = 0 \text{ on } \partial\omega_\ell^{\mathbf{a}} \setminus \partial\Omega\} \quad \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}. \quad (4.19b)$$

Other choices of  $V_\ell^{\mathbf{a}}$  could be also considered, in particular considering the actual polynomial degree  $p_{\ell, K}$ , on each simplex  $K \in \mathcal{T}_\ell^{\mathbf{a}}$ , instead of the local degree  $p_{\mathbf{a}}^{\text{dis}}$  would be, from a theoretical viewpoint, also sufficient.

### 4.3 Adaptive stopping criteria for the algebraic solver

The output of the `ESTIMATE` module enables us to assess the quality of the current approximation  $u_\ell$  and thus make a reasonable decision if another call of the module `ONE_SOLVER_STEP` is really needed. It is considered to be unnecessary, as discussed e.g. in [13, 24, 8, 14, 18] and the references therein, if the current algebraic error is smaller than the total error by a factor  $0 < \gamma_\ell < 1$  (typically of order 0.1), i.e.

$$\|\nabla(u_\ell^{\text{ex}} - u_\ell)\| \leq \gamma_\ell \|\nabla(u - u_\ell)\|. \quad (4.20)$$

In order to ensure (4.20), we require that our current approximation  $u_\ell$  satisfies the following global (safe) stopping criterion

$$\eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell \mu(u_\ell). \quad (4.21)$$

This typically allows us to avoid possible unnecessary iterations of the algebraic solver within the `ONE_SOLVER_STEP` module in case of the use of the classical stopping criterion for the algebraic solver based on the Euclidean norm of the algebraic residual vector (2.5)

$$\frac{\|\mathbf{R}_\ell\|}{\|\mathbf{F}_\ell\|} \leq \varepsilon, \quad (4.22)$$

with  $\varepsilon$  prescribed at a very small value, thus without exploiting any knowledge of the error estimators.

### 4.4 The module MARK

The module `MARK` takes as input the local error estimators computed within the `ESTIMATE` module, corresponding to the current approximation  $u_\ell$  satisfying the stopping criterion from Section 4.3. It outputs a set of marked vertices  $\tilde{\mathcal{V}}_\ell^\theta \subset \mathcal{V}_\ell$  using a bulk-chasing criterion inspired by the well-known Dörfler's marking criterion [17]. The choice of marking vertices instead of marking simplices directly is motivated by the nature of our  $hp$ -decision criterion in the module `REFINE` (see Section 4.5).

For a fixed threshold parameter  $\theta \in (0, 1]$ , the set of marked vertices  $\tilde{\mathcal{V}}_\ell^\theta$  is selected in such a way that

$$\eta\left(u_\ell, \bigcup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}\right) \geq \theta \eta(u_\ell, \mathcal{T}_\ell), \quad (4.23)$$

where, for a subset  $\mathcal{S} \subset \mathcal{T}_\ell$ , we employ the notation  $\eta(u_\ell, \mathcal{S}) := \{\sum_{K \in \mathcal{S}} \eta_K(u_\ell)^2\}^{1/2}$ . Then, letting

$$\mathcal{M}_\ell^\theta := \bigcup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}} \subset \mathcal{T}_\ell \quad (4.24)$$

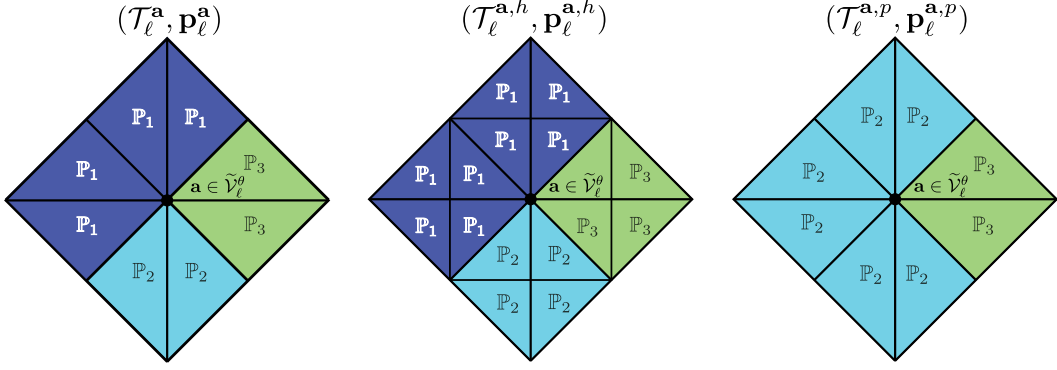
be the collection of all the simplices that belong to a patch associated with a marked vertex, we observe that (4.23) means that  $\eta(u_\ell, \mathcal{M}_\ell^\theta) \geq \theta \eta(u_\ell, \mathcal{T}_\ell)$ . To select a set  $\tilde{\mathcal{V}}_\ell^\theta$  of minimal cardinality, the mesh vertices in  $\mathcal{V}_\ell$  are sorted by comparing the vertex-based error estimators  $\eta(u_\ell, \mathcal{T}_\ell^{\mathbf{a}})$  for all  $\mathbf{a} \in \mathcal{V}_\ell$ , and a greedy algorithm is employed to build the set  $\tilde{\mathcal{V}}_\ell^\theta$ . A possibly slightly larger set  $\tilde{\mathcal{V}}_\ell^\theta$  can be constructed with linear cost in terms of the number of mesh vertices by using the algorithm proposed in [17, Section 5.2].

### 4.5 The module REFINE

The module `REFINE`, as in the previous paper [15], takes as input the set of marked vertices  $\tilde{\mathcal{V}}_\ell^\theta$  and outputs the mesh  $\mathcal{T}_{\ell+1}$  and the polynomial-degree distribution  $\mathbf{p}_{\ell+1}$  to be used at the next iteration of the adaptive loop from Scheme 2. This module proceeds in three steps. First, an  $hp$ -decision is made on all the marked vertices, so that each marked vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$  is flagged either for  $h$ -refinement or for  $p$ -refinement. This means that the set  $\tilde{\mathcal{V}}_\ell^\theta$  is split into two disjoint subsets  $\tilde{\mathcal{V}}_\ell^\theta = \tilde{\mathcal{V}}_\ell^h \cup \tilde{\mathcal{V}}_\ell^p$  with obvious notation. Then, in the second step, the subsets  $\tilde{\mathcal{V}}_\ell^h$  and  $\tilde{\mathcal{V}}_\ell^p$  are used to define subsets  $\mathcal{M}_\ell^h$  and  $\mathcal{M}_\ell^p$  of the set of marked simplices  $\mathcal{M}_\ell^\theta$  (see (4.24)). The subsets  $\mathcal{M}_\ell^h$  and  $\mathcal{M}_\ell^p$  are not necessarily disjoint which means that some simplices can be flagged for  $hp$ -refinement. Finally, the two subsets  $\mathcal{M}_\ell^h$  and  $\mathcal{M}_\ell^p$  are used to construct  $\mathcal{T}_{\ell+1}$  and  $\mathbf{p}_{\ell+1}$ .

### 4.5.1 $hp$ -decision on vertices

Our  $hp$ -decision on marked vertices is made on the basis of two local primal solves on the patch  $\mathcal{T}_\ell^{\mathbf{a}}$  attached to each marked vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$ . The idea is to construct two distinct local patch-based spaces in order to emulate separately the effects of  $h$ - and  $p$ -refinement. We first consider the case where the two local primal solves use Dirichlet conditions. Let us denote the polynomial-degree distribution in the patch  $\mathcal{T}_\ell^{\mathbf{a}}$  by the vector  $\mathbf{p}_\ell^{\mathbf{a}} := (p_{\ell,K})_{K \in \mathcal{T}_\ell^{\mathbf{a}}}$ .



**Figure 3:** An example of patch  $\mathcal{T}_\ell^{\mathbf{a}}$  together with its polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a}}$  (left), its  $h$ -refined version (center), and its  $p$ -refined version (right) from Definitions 4.6 and 4.7, respectively.

**Definition 4.6** ( $h$ -refinement residual – Dirichlet conditions). Let  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$  be a marked vertex with associated patch  $\mathcal{T}_\ell^{\mathbf{a}}$  and polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a}}$ . We set

$$V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_0^1(\omega_\ell^{\mathbf{a}}), \quad (4.25)$$

where  $\mathcal{T}_\ell^{\mathbf{a},h}$  is obtained as a matching simplicial refinement of  $\mathcal{T}_\ell^{\mathbf{a}}$  by dividing each simplex  $K \in \mathcal{T}_\ell^{\mathbf{a}}$  into at least two children simplices, and the polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a},h}$  is obtained from  $\mathbf{p}_\ell^{\mathbf{a}}$  by assigning to each newly-created simplex the same polynomial degree as its parent. Then, we let  $r^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}$  solve

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}.$$

**Definition 4.7** ( $p$ -refinement residual – Dirichlet conditions). Let  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$  be a marked vertex with associated patch  $\mathcal{T}_\ell^{\mathbf{a}}$  and polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a}}$ . We set

$$V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a},p}) \cap H_0^1(\omega_\ell^{\mathbf{a}}), \quad (4.26)$$

where  $\mathcal{T}_\ell^{\mathbf{a},p} := \mathcal{T}_\ell^{\mathbf{a}}$ , and the polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a},p}$  is obtained from  $\mathbf{p}_\ell^{\mathbf{a}}$  by assigning to each simplex  $K \in \mathcal{T}_\ell^{\mathbf{a},p} = \mathcal{T}_\ell^{\mathbf{a}}$  the polynomial degree  $p_{\ell,K} + \delta_K^{\mathbf{a}}$  where

$$\delta_K^{\mathbf{a}} := \begin{cases} 1 & \text{if } p_{\ell,K} = \min_{K' \in \mathcal{T}_\ell^{\mathbf{a}}} p_{\ell,K'}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.27)$$

Then, we let  $r^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}$  solve

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}.$$

The local residual liftings  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  from Definitions 4.6 and 4.7, respectively, are used to define the following two disjoint subsets of the set of marked vertices  $\tilde{\mathcal{V}}_\ell^\theta$ :

$$\tilde{\mathcal{V}}_\ell^h := \{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta \mid \|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}} \geq \|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}\}, \quad (4.28a)$$

$$\tilde{\mathcal{V}}_\ell^p := \{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta \mid \|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}} < \|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}\}, \quad (4.28b)$$

in such a way that

$$\tilde{\mathcal{V}}_\ell^\theta = \tilde{\mathcal{V}}_\ell^h \cup \tilde{\mathcal{V}}_\ell^p, \quad \tilde{\mathcal{V}}_\ell^h \cap \tilde{\mathcal{V}}_\ell^p = \emptyset.$$

The above  $hp$ -decision criterion on vertices means that a marked vertex is flagged for  $h$ -refinement if  $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^\mathbf{a}}$  is larger than  $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^\mathbf{a}}$ ; otherwise, this vertex is flagged for  $p$ -refinement.

For the construction of the residuals  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  in Definitions 4.6 and 4.7, respectively, we considered homogeneous Dirichlet boundary conditions as in [15]. Alternatively, while keeping the local criterion (4.28) unchanged, it is possible to define the  $h$ - and  $p$ -refinement residuals by solving the local problems with homogeneous Neumann boundary conditions. We will also consider these alternative definitions in our numerical experiments.

**Definition 4.8** ( $h$ -refinement residual – Neumann conditions). *Let  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$  be a marked vertex. Let the simplicial submesh  $\mathcal{T}_\ell^{\mathbf{a},h}$  and the corresponding polynomial degree distribution  $\mathbf{p}_\ell^{\mathbf{a},h}$  be as in Definition 4.6. We recall the definition of space  $H_*^1(\omega_\ell^\mathbf{a})$  in (3.7) and we set*

$$V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_*^1(\omega_\ell^\mathbf{a}). \quad (4.29)$$

Then, we let  $r^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}$  solve

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^\mathbf{a}} = (f, w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},h}))_{\omega_\ell^\mathbf{a}} - (\nabla u_\ell, \nabla w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},h}))_{\omega_\ell^\mathbf{a}} \quad \forall v^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}, \quad (4.30)$$

where  $w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},h})$  stands for weighting a function  $v^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}$  by the hat function  $\psi_\ell^\mathbf{a}$  such that

$$w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},h}) \in \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H^1(\omega_\ell^\mathbf{a}) \quad \text{and} \quad w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},h})(\mathbf{x}) = \psi_\ell^\mathbf{a}(\mathbf{x}) \cdot v^{\mathbf{a},h}(\mathbf{x}) \quad (4.31)$$

with the nodes  $\mathbf{x}$  uniquely determining a function in  $\mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H^1(\omega_\ell^\mathbf{a})$ .

**Definition 4.9** ( $p$ -refinement residual – Neumann conditions). *Let  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$  be a marked vertex associated with the simplicial mesh  $\mathcal{T}_\ell^{\mathbf{a},p}$  and corresponding polynomial degree distribution  $\mathbf{p}_\ell^{\mathbf{a},p}$  as in Definition 4.7. We set*

$$V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a},p}) \cap H_*^1(\omega_\ell^\mathbf{a}). \quad (4.32)$$

Then, we let  $r^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}$  solve

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^\mathbf{a}} = (f, w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},p}))_{\omega_\ell^\mathbf{a}} - (\nabla u_\ell, \nabla w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},p}))_{\omega_\ell^\mathbf{a}} \quad \forall v^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}, \quad (4.33)$$

where  $w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},p})$ , similarly to (4.31), stands for weighting a function  $v^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}$  by a hat function  $\psi_\ell^\mathbf{a}$  such that

$$w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},p}) \in \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a},p}) \cap H^1(\omega_\ell^\mathbf{a}) \quad \text{and} \quad w_{\psi_\ell^\mathbf{a}}(v^{\mathbf{a},p})(\mathbf{x}) = \psi_\ell^\mathbf{a}(\mathbf{x}) \cdot v^{\mathbf{a},p}(\mathbf{x})$$

with the nodes  $\mathbf{x}$  uniquely determining a function in  $\mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a},p}) \cap H^1(\omega_\ell^\mathbf{a})$ .

#### 4.5.2 $hp$ -decision on simplices

The second step in the module REFINE is to use the subsets  $\tilde{\mathcal{V}}_\ell^h$  and  $\tilde{\mathcal{V}}_\ell^p$  to decide whether  $h$ -,  $p$ -, or  $hp$ -refinement should be performed on each simplex having at least one flagged vertex. To this purpose, we define the following subsets:

$$\mathcal{M}_\ell^h := \{K \in \mathcal{T}_\ell \mid \mathcal{V}_K \cap \tilde{\mathcal{V}}_\ell^h \neq \emptyset\} \subset \mathcal{M}_\ell^\theta, \quad (4.34a)$$

$$\mathcal{M}_\ell^p := \{K \in \mathcal{T}_\ell \mid \mathcal{V}_K \cap \tilde{\mathcal{V}}_\ell^p \neq \emptyset\} \subset \mathcal{M}_\ell^\theta. \quad (4.34b)$$

In other words, a simplex  $K \in \mathcal{T}_\ell$  is flagged for  $h$ -refinement (resp.,  $p$ -refinement) if it has at least one vertex flagged for  $h$ -refinement (resp.,  $p$ -refinement). Note that the subsets  $\mathcal{M}_\ell^h$  and  $\mathcal{M}_\ell^p$  are not necessarily disjoint since a simplex can have some vertices flagged for  $h$ -refinement and others flagged for  $p$ -refinement; such simplices are then flagged for  $hp$ -refinement. Note also that  $\mathcal{M}_\ell^h \cup \mathcal{M}_\ell^p = \cup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^\mathbf{a} = \mathcal{M}_\ell^\theta$  is indeed the set of marked simplices considered in the module MARK.

### 4.5.3 $hp$ -refinement

In this last and final step, the subsets  $\mathcal{M}_\ell^h$  and  $\mathcal{M}_\ell^p$  are used to produce first the next mesh  $\mathcal{T}_{\ell+1}$  and then the next polynomial-degree distribution  $\mathbf{p}_{\ell+1}$  on the mesh  $\mathcal{T}_{\ell+1}$ .

The next mesh  $\mathcal{T}_{\ell+1}$  is a matching simplicial refinement of  $\mathcal{T}_\ell$  obtained by dividing each flagged simplex  $K \in \mathcal{M}_\ell^h$  into at least two simplices in a way that is consistent with the matching simplicial refinement of  $\mathcal{T}_\ell^{\mathbf{a}}$  considered in Definition 4.6 to build  $\mathcal{T}_\ell^{\mathbf{a},h}$ , i.e., such that  $\mathcal{T}_\ell^{\mathbf{a},h} \subset \mathcal{T}_{\ell+1}$  for all  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^h$ . Note that to preserve the conformity of the mesh, additional refinements beyond the set of flagged simplices  $\mathcal{M}_\ell^h$  may be carried out when building  $\mathcal{T}_{\ell+1}$ . Several algorithms can be considered to refine the mesh. In our numerical experiments, we used the newest vertex bisection algorithm [28, 29].

After having constructed the next mesh  $\mathcal{T}_{\ell+1}$ , we assign the next polynomial-degree distribution  $\mathbf{p}_{\ell+1}$  as follows. For all  $K \in \mathcal{T}_{\ell+1}$ , let  $\tilde{K}$  denote its parent simplex in  $\mathcal{T}_\ell$ . We then set

$$p_{\ell+1,K} := p_{\ell,\tilde{K}} \quad \text{if } \tilde{K} \notin \mathcal{M}_\ell^p, \quad (4.35)$$

that is, we assign the same polynomial degree to the children of a simplex that is not flagged for  $p$ -refinement, whereas we set

$$p_{\ell+1,K} := \max_{\mathbf{a} \in \mathcal{V}_{\tilde{K}} \cap \tilde{\mathcal{V}}_\ell^p} (p_{\ell,\tilde{K}} + \delta_{\tilde{K}}^{\mathbf{a}}) \quad \text{if } \tilde{K} \in \mathcal{M}_\ell^p, \quad (4.36)$$

that is, we assign to the children of a simplex  $\tilde{K} \in \mathcal{M}_\ell^p$  flagged for  $p$ -refinement the largest of the polynomial degrees considered in Definition 4.7 to build the local residual liftings associated with the vertices of  $\tilde{K}$  flagged for  $p$ -refinement.

## 5 Guaranteed bound on the error reduction

In this section we extend the results of our previous work [15, Section 5], where a computable guaranteed bound on the error reduction factor between two consecutive steps of an adaptive procedure with an exact solver has been derived. We recall that the adaptive strategy of [15] generates a sequence of exact finite element solutions  $\{u_\ell^{\text{ex}}\}_{\ell \geq 0}$ . For a fixed  $\ell \geq 0$ , let us denote by  $C_{\text{red}}^{\text{ex}}$  the bound on the energy error reduction factor between  $u_\ell^{\text{ex}} \in V_\ell$  and  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  derived in [15, Theorem 5.2] such that

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\| \leq C_{\text{red}}^{\text{ex}} \|\nabla(u - u_\ell^{\text{ex}})\| \quad \text{with} \quad 0 \leq C_{\text{red}}^{\text{ex}} \leq 1. \quad (5.1)$$

Using the current notation, and letting  $\omega_\ell := \cup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \omega_\ell^{\mathbf{a}}$ , the definition of  $C_{\text{red}}^{\text{ex}}$  reads

$$C_{\text{red}}^{\text{ex}} := \sqrt{1 - \left( \theta \frac{\eta_{\mathcal{M}_\ell^\theta}^{\text{ex}}}{\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell)} \right)^2}, \quad (5.2)$$

with the total error estimator  $\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell)$  of Theorem 3.3 (the local algebraic error estimator  $\eta_{\text{alg},K}(u_\ell^{\text{ex}}) := 0$  for each  $K \in \mathcal{T}_\ell$ ), and the discrete lower bound  $\eta_{\mathcal{M}_\ell^\theta}^{\text{ex}} \leq \|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell^{\text{ex}})\|_{\omega_\ell}$  defined in [15, Lemma 5.1].

The aim of this section is to derive an equivalent of the bound (5.1) between the two *inexact* solutions  $u_\ell \in V_\ell$  and  $u_{\ell+1} \in V_{\ell+1}$  obtained by the iterative procedure of Scheme 2 in the form

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\text{red}} \|\nabla(u - u_\ell)\|.$$

It turns out essential to first estimate a guaranteed bound on the error reduction between the current inexact solution  $u_\ell \in V_\ell$  and the (unavailable) exact solution on the next level  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$ . For this we start by extending the discrete lower bound of [15, Lemma 5.1] to the present setting:

**Lemma 5.1** (Guaranteed lower bound on the incremental error on marked simplices). *Let the mesh  $\mathcal{T}_{\ell+1}$  and the polynomial-degree distribution  $\mathbf{p}_{\ell+1}$  result from the REFINE module of Section 4.5, and recall that  $V_{\ell+1} = \mathbb{P}_{\mathbf{p}_{\ell+1}}(\mathcal{T}_{\ell+1}) \cap H_0^1(\Omega)$  is the finite element space to be used on iteration  $(\ell+1)$  of the adaptive loop of Scheme 2. For all the marked vertices  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$ , let us set, in extension of (4.25), (4.26),*

$$V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H_0^1(\omega_\ell^{\mathbf{a}}), \quad (5.3)$$



and construct the residual lifting  $r^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$  by solving

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}. \quad (5.4)$$

Then, after extending  $r^{\mathbf{a},hp}$  by zero outside  $\omega_\ell^{\mathbf{a}}$ , for the current inexact approximation  $u_\ell \in V_\ell$  and the exact approximation  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  on the next level, the following holds true:

$$\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|_{\omega_\ell} \geq \underline{\eta}_{\mathcal{M}_\ell^\theta}, \quad \underline{\eta}_{\mathcal{M}_\ell^\theta} := \begin{cases} \frac{\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}}^2}{\|\nabla(\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp})\|_{\omega_\ell}} & \text{if } \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.5)$$

*Proof.* We remark that the definition of the residual liftings  $r^{\mathbf{a},hp}$  (5.4) employs directly the inexact approximation  $u_\ell$  unlike in [15, Lemma 5.1] where  $u_\ell^{\text{ex}}$  (in the present notation) was considered. Nevertheless, the arguments to prove the lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\theta}^{\text{ex}}$  of [15, Lemma 5.1] stay valid and can be used to show (5.5).  $\square$

In case of the use of residuals  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  from Definitions 4.8 and 4.9, respectively, in the local criterion (4.28) within the REFINE module, the above lower bound can be adjusted as well.

**Lemma 5.2** (Guaranteed lower bound on the incremental error on marked simplices – alternative definition). *Let the assumptions of Lemma 5.1 be satisfied. Moreover, let us set for all the marked vertices  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$ , in extension of (4.29) and (4.32),*

$$V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H_*^1(\omega_\ell^{\mathbf{a}}). \quad (5.6)$$

Then, for each marked vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta$ , construct the residual lifting  $r^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$  by solving

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} = (f, w_{\psi_\ell^{\mathbf{a}}}(v^{\mathbf{a},hp}))_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla w_{\psi_\ell^{\mathbf{a}}}(v^{\mathbf{a},hp}))_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}, \quad (5.7)$$

where  $w_{\psi_\ell^{\mathbf{a}}}(v^{\mathbf{a},hp})$  stands for weighting a function  $v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$  by hat function  $\psi_\ell^{\mathbf{a}}$  such that

$$w_{\psi_\ell^{\mathbf{a}}}(v^{\mathbf{a},p}) \in V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \quad \text{and} \quad w_{\psi_\ell^{\mathbf{a}}}(v^{\mathbf{a},p})(\mathbf{x}) = \psi_\ell^{\mathbf{a}}(\mathbf{x}) \cdot v^{\mathbf{a},p}(\mathbf{x})$$

with the nodes  $\mathbf{x}$  uniquely determining a function in  $V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}}$ . After extending each  $r^{\mathbf{a},hp}$  by zero outside  $\omega_\ell^{\mathbf{a}}$ , the following lower bound holds true:

$$\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|_{\omega_\ell} \geq \underline{\eta}_{\mathcal{M}_\ell^\theta}, \quad \underline{\eta}_{\mathcal{M}_\ell^\theta} := \begin{cases} \frac{\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}}^2}{\|\nabla(\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} w_{\psi_\ell^{\mathbf{a}}}(r^{\mathbf{a},hp}))\|_{\omega_\ell}} & \text{if } \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.8)$$

*Proof.* Let us note that  $(u_{\ell+1}^{\text{ex}} - u_\ell)|_{\omega_\ell}$  belongs to the space  $V_{\ell+1}(\omega_\ell)$ , a restriction of the finite element space  $V_{\ell+1}$  to  $\omega_\ell$ . However, note that it does not necessarily belong to the homogeneous Dirichlet subspace  $V_{\ell+1}^0(\omega_\ell)$ . The definition of the energy norm  $\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|_{\omega_\ell}$  and the fact that  $v_{\ell+1} \in V_{\ell+1}^0(\omega_\ell)$  extended by zero outside  $\omega_\ell$  is a member of the space  $V_{\ell+1}$  so that it can be used as a test function in the definition (2.3) of  $u_{\ell+1}^{\text{ex}}$  on the mesh  $\mathcal{T}_{\ell+1}$  yield

$$\begin{aligned} \|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|_{\omega_\ell} &= \sup_{v_{\ell+1} \in V_{\ell+1}(\omega_\ell)} \frac{(\nabla(u_{\ell+1}^{\text{ex}} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}} \\ &\geq \sup_{v_{\ell+1} \in V_{\ell+1}^0(\omega_\ell)} \frac{(\nabla(u_{\ell+1}^{\text{ex}} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}} \\ &= \sup_{v_{\ell+1} \in V_{\ell+1}^0(\omega_\ell)} \frac{(f, v_{\ell+1})_{\omega_\ell} - (\nabla u_\ell, \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}. \end{aligned}$$

Now, choose the test function  $v_{\ell+1} := \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} w_{\psi_\ell^\mathbf{a}}(r^{\mathbf{a},hp})$ ; note that, due to the weighting  $w_{\psi_\ell^\mathbf{a}}(\cdot)$ , such choice of  $v_{\ell+1}$  indeed belongs to  $V_{\ell+1}^0(\omega_\ell)$ . Then, we infer that

$$\begin{aligned} & \left( f, \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} w_{\psi_\ell^\mathbf{a}}(r^{\mathbf{a},hp}) \right)_{\omega_\ell} - \left( \nabla u_\ell, \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} w_{\psi_\ell^\mathbf{a}}(r^{\mathbf{a},hp}) \right)_{\omega_\ell} \\ &= \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \{ (f, w_{\psi_\ell^\mathbf{a}}(r^{\mathbf{a},hp}))_{\omega_\ell} - (\nabla u_\ell, \nabla w_{\psi_\ell^\mathbf{a}}(r^{\mathbf{a},hp}))_{\omega_\ell} \} = \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\theta} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^\mathbf{a}}^2, \end{aligned}$$

where we have employed (5.7) with  $r^{\mathbf{a},hp}$  as a test function. This implies the assertion (5.8).  $\square$

We now proceed with an intermediate result giving a guaranteed bound on the error reduction factor between the current inexact approximation  $u_\ell$  and the (unavailable) next level exact solution  $u_{\ell+1}^{\text{ex}}$ .

**Lemma 5.3** (Auxiliary guaranteed bound on the energy error reduction factor). *Let  $\theta$  be the threshold parameter used within the module MARK of Section 4.4 and let the mesh  $\mathcal{T}_{\ell+1}$  and the polynomial degree distribution  $\mathbf{p}_{\ell+1}$  be given by the REFINE module of Section 4.5. Next, let  $V_{\ell+1} := \mathbb{P}_{\mathbf{p}_{\ell+1}}(\mathcal{T}_{\ell+1}) \cap H_0^1(\Omega)$  be the space to be used on iteration  $(\ell+1)$  of the algorithm prescribed by Scheme 2. Let  $\underline{\eta}_{\mathcal{M}_\ell^\theta}$  be the lower bound defined by (5.5) or (5.8), depending on the choice of the construction of residuals  $r^{\mathbf{a},h}$ ,  $r^{\mathbf{a},p}$  used within the REFINE module. Then, unless  $\eta(u_\ell, \mathcal{T}_\ell) = 0$ , in which case  $u_\ell = u$ , and the adaptive loop terminates, the exact finite element solution  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  satisfies*

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\| \leq C_{\text{red}}^* \|\nabla(u - u_\ell)\| \quad \text{with} \quad 0 \leq C_{\text{red}}^* := \sqrt{1 - \frac{\underline{\eta}_{\mathcal{M}_\ell^\theta}^2}{\eta^2(u_\ell, \mathcal{T}_\ell)}} \leq 1. \quad (5.9)$$

*Proof.* Since the Galerkin orthogonality property between the current approximation  $u_\ell \in V_\ell$  and the exact finite element solution  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  holds true, we have

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 = \|\nabla(u - u_\ell)\|^2 - \|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|^2. \quad (5.10)$$

Afterwards, employing the lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\theta}$  and the total error upper bound  $\eta(u_\ell, \mathcal{T}_\ell)$  from (3.5) in (5.10) yields

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 \leq \|\nabla(u - u_\ell)\|^2 - \underline{\eta}_{\mathcal{M}_\ell^\theta}^2 \frac{\|\nabla(u - u_\ell)\|^2}{\|\nabla(u - u_\ell)\|^2} \leq \|\nabla(u - u_\ell)\|^2 \left( 1 - \frac{\underline{\eta}_{\mathcal{M}_\ell^\theta}^2}{\eta^2(u_\ell, \mathcal{T}_\ell)} \right).$$

The assertion (5.9) then follows by taking the square root.  $\square$

Finally, we are ready to present the result on a computable guaranteed bound on the reduction factor in the inexact setting:

**Theorem 5.4** (Guaranteed bound on the energy error reduction factor between two inexact solutions). *Let the assumptions of Lemma 5.3 be satisfied and let  $C_{\text{red}}^*$  be given by (5.9). Moreover, let  $u_{\ell+1} \in V_{\ell+1}$  be the inexact finite element approximation on iteration  $(\ell+1)$  of the procedure prescribed by Scheme 2, satisfying the global stopping criterion*

$$\eta_{\text{alg}}(u_{\ell+1}, \mathcal{T}_{\ell+1}) \leq \gamma_{\ell+1} \mu(u_{\ell+1}), \quad (5.11)$$

with the parameter

$$0 \leq \gamma_{\ell+1} \leq (1 - C_{\text{red}}^*). \quad (5.12)$$

Then, the resulting error reduction between the inexact solution  $u_\ell \in V_\ell$  from the current iteration  $\ell$  and the next approximation  $u_{\ell+1}$  to be computed on the next iteration verifies

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\text{red}} \|\nabla(u - u_\ell)\| \quad \text{with} \quad 0 \leq C_{\text{red}} := \frac{C_{\text{red}}^*}{(1 - \gamma_{\ell+1})} \leq 1. \quad (5.13)$$

*Proof.* We start by adding and subtracting  $\nabla u_{\ell+1}^{\text{ex}}$  inside the norm on the left hand side of (5.13). The triangle inequality then yields

$$\|\nabla(u - u_{\ell+1})\| \leq \|\nabla(u - u_{\ell+1}^{\text{ex}})\| + \|\nabla(u_{\ell+1}^{\text{ex}} - u_{\ell+1})\|. \quad (5.14)$$

For bounding the first term, we employ the auxiliary bound (5.9). The second term of (5.14), the algebraic error on iteration  $(\ell+1)$ , is first bounded from above by the algebraic error estimate (3.6). Then, the stopping criterion (5.11) with the parameter  $\gamma_{\ell+1}$  in combination with the total energy error lower bound (3.8) give

$$\|\nabla(u_{\ell+1}^{\text{ex}} - u_{\ell+1})\| \leq \eta_{\text{alg}}(u_{\ell+1}, \mathcal{T}_{\ell+1}) \leq \gamma_{\ell+1} \mu(u_{\ell+1}) \leq \gamma_{\ell+1} \|\nabla(u - u_{\ell+1})\|,$$

whence we infer (5.13). The condition (5.12) on parameter  $\gamma_{\ell+1}$  then ensures the upper bound  $C_{\text{red}} \leq 1$ .  $\square$

**Remark 5.5** (Extreme case equivalent to the use of an exact solver). *In Theorem 5.4 we do not exclude the extreme case when the auxiliary upper bound  $C_{\text{red}}^* = 1$ . This in turn leads to the stopping criterion (5.11) with the parameter  $\gamma_{\ell+1} = 0$ , which is equivalent to computing the exact finite element solution  $u_{\ell+1}^{\text{ex}}$ . However, we note that in our numerical experiments, reported in Section 6, we never encountered such a situation when the exact solver would be necessary.*

**Remark 5.6** (Motivation). *We believe that, under some additional assumptions on the refinements, such as the interior node property [30], one could actually show  $C_{\text{red}}^* < 1$ . The convergence of the proposed method would then easily follow. We do not address this topic here, however, it represents a subject of our active research.*

## 6 Numerical experiments

We now illustrate the capabilities and robustness of the proposed adaptive solver using two types of test cases in two space dimensions. We consider two problems with a (relatively) smooth weak solution and one with a singular weak solution.

We focus on the influence of the inexact algebraic solver on the performance of the proposed  $hp$ -refinement strategy described in Section 4.5. While employing the adaptive stopping criterion (4.21) with the parameter  $\gamma_{\ell}$  satisfying (5.12), we assess the quality of the guaranteed bound on the reduction factor  $C_{\text{red}}$  from Theorem 5.4 throughout the adaptive process in terms of the effectivity index defined as

$$I_{\text{red}}^{\text{eff}} := \frac{C_{\text{red}}}{\frac{\|\nabla(u - u_{\ell+1})\|}{\|\nabla(u - u_{\ell})\|}}. \quad (6.1)$$

We also verify the sharpness of the underlying discrete lower bound  $\underline{\eta}_{\mathcal{M}_{\ell}^{\theta}}$  given by (5.5) or (5.8) in terms of the effectivity index defined as

$$I_{\text{LB}}^{\text{eff}} := \frac{\|\nabla(u_{\ell+1}^{\text{ex}} - u_{\ell})\|_{\omega_{\ell}}}{\underline{\eta}_{\mathcal{M}_{\ell}^{\theta}}}. \quad (6.2)$$

Besides that, we are interested in the comparison of the different stopping criteria for the algebraic solver with regard to the number of necessary iterations per step of the adaptive loop, the time spent on algebraic computations and their influence on the overall adaptive process. In all the example problems, we use the  $hp$ -multigrid method with 5 pre-smoothing Gauss-Seidel steps and no post-smoothing as the algebraic solver. We always take into account at most 10 last levels available from the current hierarchy of adaptively refined meshes at our disposal for the  $hp$ -multigrid solver, as well as for the algebraic error flux reconstruction  $\sigma_{\ell, \text{alg}}$  from Definition 4.1. In other words, if the adaptive loop iteration counter  $\ell \geq 10$ , we adjust the range of the sum (4.4) in the following way

$$\sigma_{\ell, \text{alg}} := \sum_{j=\ell-8}^{\ell} \sum_{\mathbf{a} \in \mathcal{V}_{j-1}} \sigma_{j, \text{alg}}^{\mathbf{a}},$$

thus we solve problem (4.5) on level  $j = \ell - 8$  and problem (4.7) on levels  $\ell - 7 \leq j \leq \ell$ . Similarly, the multigrid solver uses the hierarchy  $\{\mathcal{T}_j, \mathbf{p}_j\}_{\ell-9 \leq j \leq \ell}$ , for  $\ell \geq 10$ , instead of the complete available hierarchy

$\{\mathcal{T}_j, \mathbf{p}_j\}_{0 \leq j \leq \ell}$ , which is used in case of the iteration counter  $\ell$  being lower than 10. The (well-established) choice  $\theta = 0.5$  for the marking parameter in (4.23) is considered. We examine the proposed  $hp$ -refinement strategy employing the local residuals  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  defined via solving the local problems with either homogeneous Dirichlet (Definitions 4.6 and 4.7) or homogeneous Neumann boundary conditions (Definitions 4.8 and 4.9). As mentioned above, we employ the newest vertex bisection algorithm [28] to perform  $h$ -refinement and we use the polynomial-degree increment (4.27) to perform  $p$ -refinement.

## 6.1 Smooth solution (sharp Gaussian)

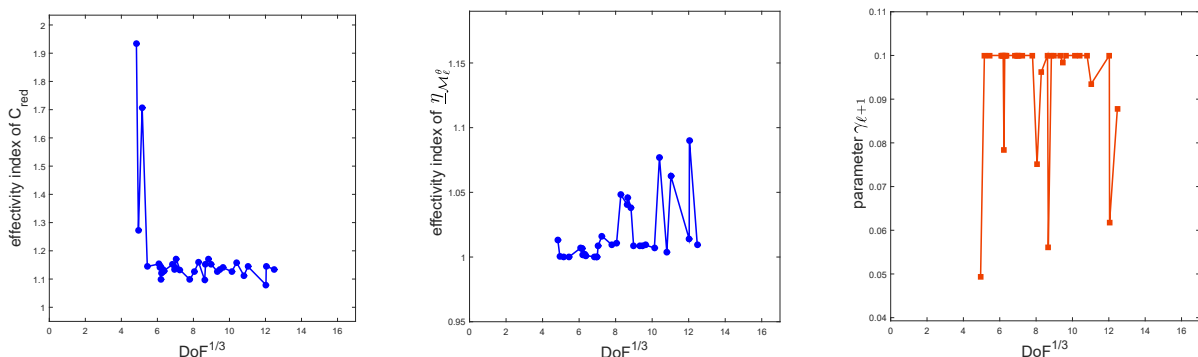
As the first test case, we consider the model problem (1.1) posed on a square domain  $\Omega := (-1, 1) \times (-1, 1)$  with a weak solution containing a rather sharp peak

$$u(x, y) = (x^2 - 1)(y^2 - 1) \exp(-100(x^2 + y^2)).$$

We start the computation with a coarse criss-cross mesh  $\mathcal{T}_0$  with  $\max_{K \in \mathcal{T}_0} h_K = 0.25$  and a uniform polynomial-degree distribution equal to 1 on all triangles. In the following, we present the results obtained using the proposed  $hp$ -refinement strategy employing the local residuals  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  defined by either the local Dirichlet or local Neumann problems.

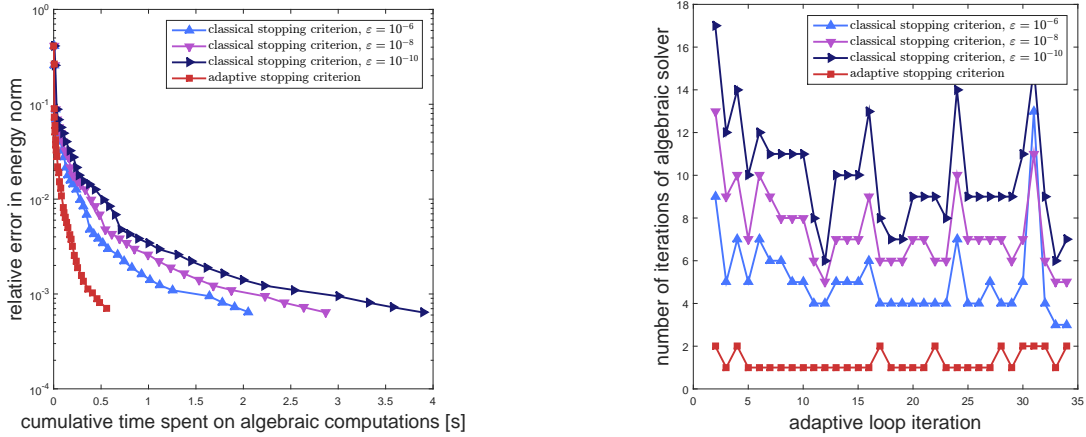
### 6.1.1 Strategy driven by the local Dirichlet problems

Firstly, in Figure 4 we investigate the accuracy of the predicted reduction factor  $C_{\text{red}}$  (left panel) and the lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\theta}$  (center panel) by means of their effectivity indices (6.1) and (6.2) throughout the whole adaptive procedure described in Scheme 2 with the module REFINE driven by solving the local Dirichlet problems from Definitions 4.6 and 4.7. We find the effectivity indices in both cases close to the optimal value of one. The corresponding values of the parameter  $\gamma_{\ell+1}$  used within the employed stopping criterion (5.11) are given in the right panel of Figure 4. A quantitative assessment of the proposed adaptive strategy

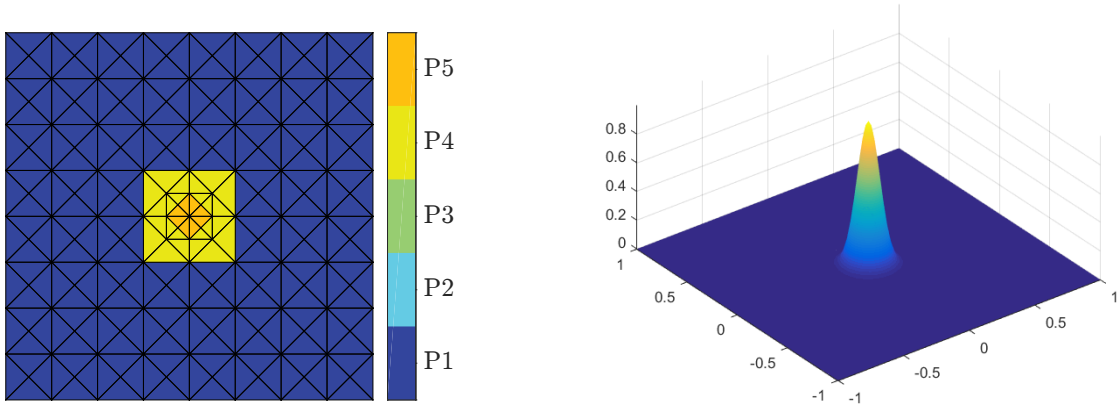


**Figure 4:** [Sharp-Gaussian of Section 6.1] The effectivity index for the error reduction factor estimate  $C_{\text{red}}$  of Theorem 5.4 given by (6.1) (left); the effectivity index of the discrete lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\theta}$  of Lemma 5.1 given by (6.2) (center); corresponding values of the parameter  $\gamma_{\ell+1}$  used in (5.13) (right).

with the inexact solver and various stopping criteria is presented in Figure 5 (left panel) where we plot the relative error  $\|\nabla(u - u_\ell)\| / \|\nabla u\|$  as a function of the cumulative time spent on the algebraic computations in linear-logarithmic scale. We observe that the strategy with inexact solver piloted by the adaptive stopping criterion leads to the steepest error decrease with respect to the computational effort. This is mostly due to cutting off unnecessary algebraic iterations as reported in the right panel of Figure 5. Figure 6 (left panel) displays the mesh and polynomial-degree distribution obtained after 17 steps of the adaptive procedure. On the right panel of Figure 6, we plot the corresponding inexact numerical solution obtained with the adaptive stopping criterion (4.21) for the algebraic solver, i.e. after the second V-cycle of the  $hp$ -multigrid solver. The detailed evolution of the total error lower bound (3.8) and algebraic error upper bound (3.6), the two main ingredients for the stopping criterion (4.21), throughout the iterations of the algebraic solver at the 17th level of refinement is plotted on the left panel of Figure 7. The annotations illustrate when the



**Figure 5:** [Sharp-Gaussian of Section 6.1] The relative energy error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of cumulative time spent on algebraic computations with the stopping criteria (4.21) and (4.22) (left) and the respective numbers of algebraic solver iterations per step of the adaptive procedure of Scheme 2 (right).

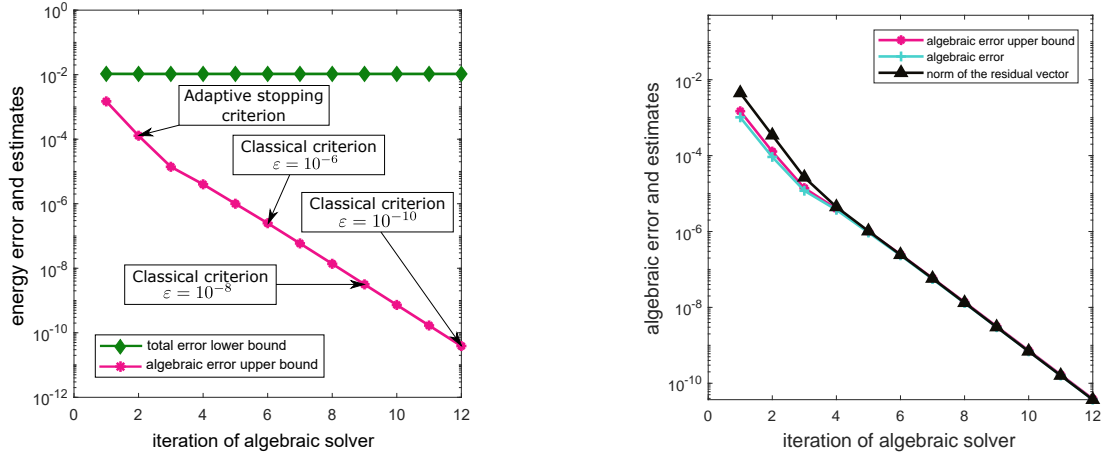


**Figure 6:** [Sharp-Gaussian of Section 6.1] The mesh and polynomial-degree distribution  $(\mathcal{T}_{17}, \mathbf{p}_{17})$  (left) along with the corresponding numerical solution  $u_{17}$  obtained after 17 iterations of the  $hp$ -adaptive procedure (right).

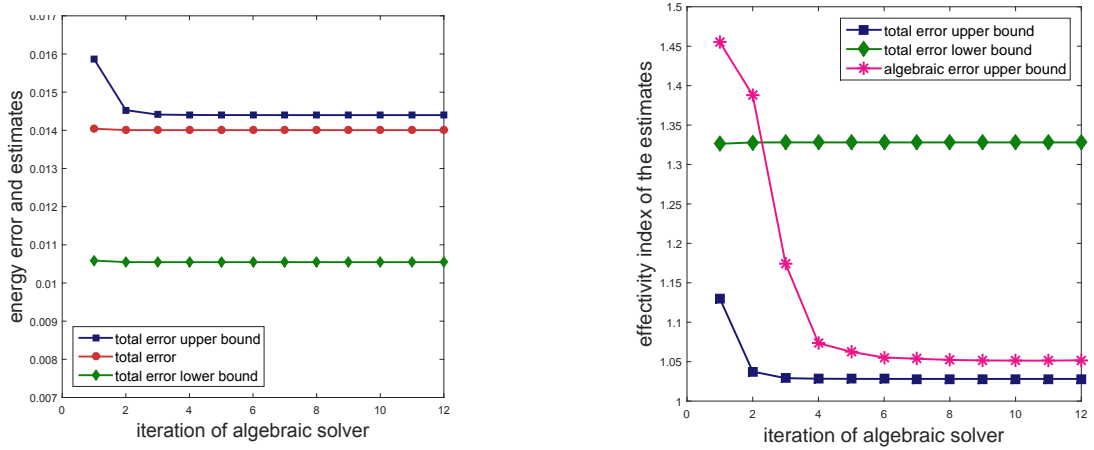
classical stopping criterion (4.22) with various levels for the tolerance  $\varepsilon$  are reached. The corresponding values of the true algebraic error in comparison with the algebraic error upper bound, and the norm of the algebraic residual vector  $\|\mathbf{R}_\ell\|$  are given on the right panel of Figure 7. We observe that the algebraic error upper bound, as well as the norm  $\|\mathbf{R}_\ell\|$ , closely follow the actual value of the algebraic error, with our error estimate giving a slightly tighter bound during the first three multigrid iterations. In Figure 8, left panel, we depict the total energy error along with its upper and lower bounds during the multigrid iterations. The quantitative evaluation of all the estimators computed within the module `ESTIMATE` in terms of their effectivity indices, i.e. the ratio of the estimates over the error for the upper bounds and the reciprocal for the lower bound, is given on the right panel of Figure 8. We note that also these effectivity indices take values close to the optimal value of one. The spatial distributions of the actual total and algebraic errors with the total upper error indicators and algebraic upper error indicators at the moment when the algebraic iterations are stopped on step  $\ell = 17$  of the  $hp$ -adaptive loop, as dictated by the adaptive stopping criterion (4.21), are displayed in Figures 9 and 10. We see that the actual and predicted error distributions match very nicely.

### 6.1.2 Strategy driven by the local Neumann problems

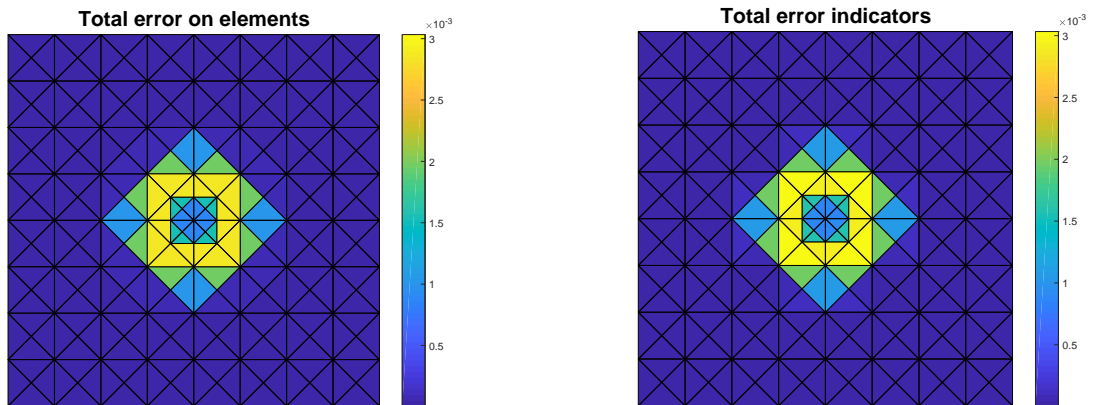
We also present the results obtained while employing in the adaptive procedure the local residuals  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  defined via solving the local Neumann problems as proposed in Definitions 4.8 and 4.9. The use of these residuals leads to slightly different meshes and polynomial-degree distributions during the adaptive process



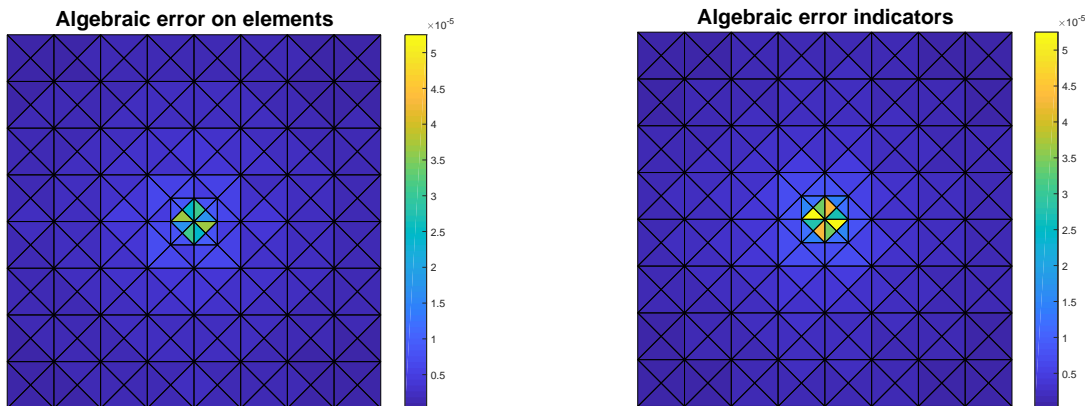
**Figure 7:** [Sharp-Gaussian of Section 6.1] Algebraic error upper bound  $\eta_{\text{alg}}(u_{17}, \mathcal{T}_{17})$  in comparison with total error lower bound  $\mu(u_{17})$  (*left*) and with true algebraic error  $\|\nabla(u_{17}^{\text{ex}} - u_{17})\|$  and norm of algebraic residual vector  $\|R_{17}\|$  (*right*) as a function of algebraic solver iterations.



**Figure 8:** [Sharp-Gaussian of Section 6.1] Total energy error with its upper and lower bound (*left*) and effectivity indices of total error upper bound (3.5), total error lower bound (3.8), and algebraic error upper bound (3.6) (*right*), throughout the iterations of the multigrid solver.



**Figure 9:** [Sharp-Gaussian of Section 6.1] Elementwise distribution of the total energy error  $\|\nabla(u - u_{17})\|$  (*left*) and total upper error indicators  $\eta_K(u_{17})$  (*right*).

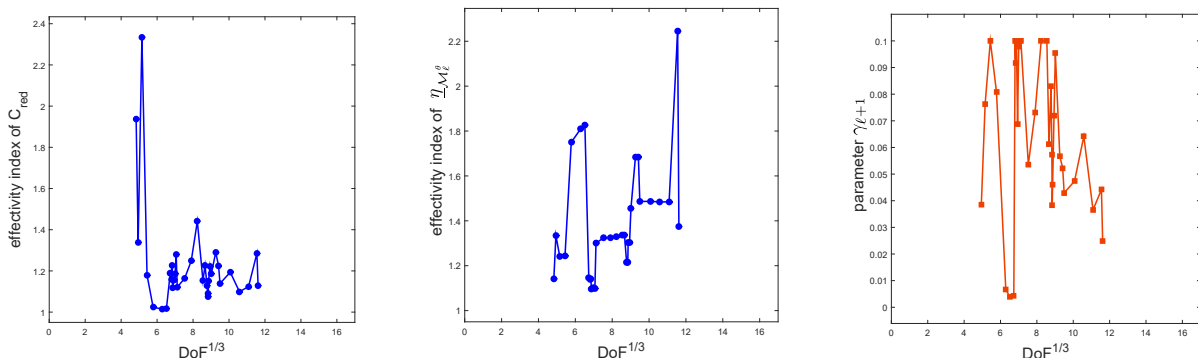


**Figure 10:** [Sharp-Gaussian of Section 6.1] Elementwise distribution of the algebraic energy error  $\|\nabla(u_{17}^{\text{ex}} - u_{17})\|$  (left) and algebraic upper error indicators (right) with the adaptive stopping criterion (4.21) satisfied,  $\gamma_{17} = 0.1$ .

(not presented here). In particular, we plot the effectivity indices of the estimated reduction factor  $C_{\text{red}}$  and of the underlying lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\theta}$  in Figure 11. We find these estimates a little less precise compared to the ones presented in Figure 4, yet the effectivity indices are still quite close to one. The savings when using the adaptive stopping criterion (4.21) compared to the classical stopping criterion (4.22) are then demonstrated in Figure 12 in terms of the time spent on algebraic computations and in terms of the number of necessary iterations of the algebraic solver.

## 6.2 Exponential convergence

In Figure 13 we show that the proposed  $hp$ -refinement strategy (driven by solving either local Dirichlet problems or local Neumann problems) still leads, even in the presence of inexact solver, to meshes and polynomial degree distributions for which the relative error decreases exponentially fast. We plot the relative error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of  $\text{DoF}_\ell^{\frac{1}{3}}$  in logarithmic-linear scale for our strategy with inexact solver, the pure  $h$ -version of the adaptive loop with exact solver as given in Scheme 1, and while using the uniform  $h$ -refinement. For further comparison with different adaptive  $hp$ -refinement strategies (with exact algebraic solver) for this model problem we refer to [15, Section 6.1].

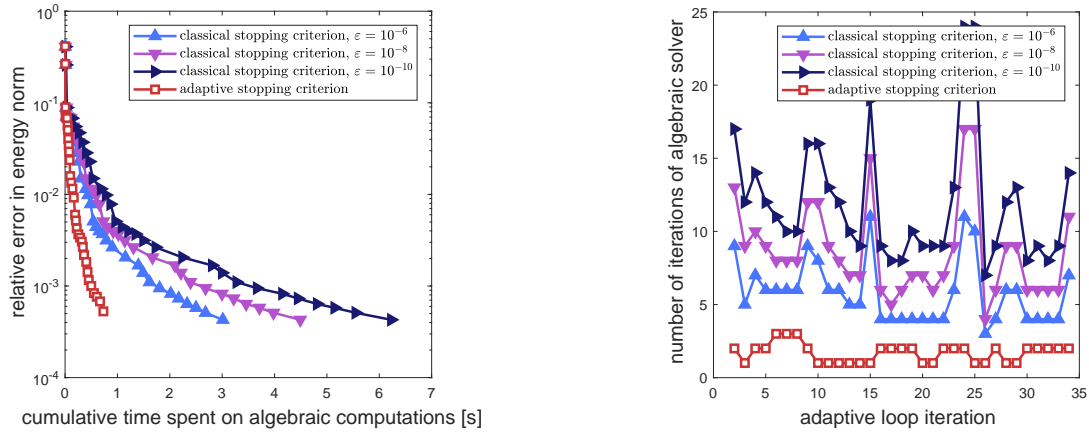


**Figure 11:** [Sharp-Gaussian of Section 6.1, strategy with local Neumann problems] The effectivity index for the error reduction factor estimate  $C_{\text{red}}$  of Theorem 5.4 (left); the effectivity index of the discrete lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\theta}$  of Lemma 5.2 given by (6.2) (center); corresponding values of parameter  $\gamma_{\ell+1}$  used in (5.13) (right).

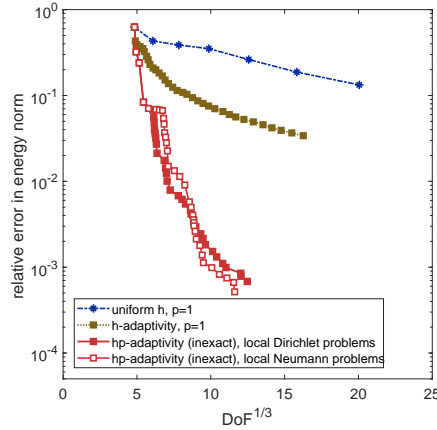
## 6.3 Smooth solution (asymmetric wave front)

Looking at the results of Section 6.1, namely Figures 5 and 12, one could be tempted to employ at each iteration of the adaptive procedure only a single iteration of the algebraic solver with the hope to eventually





**Figure 12:** [Sharp-Gaussian of Section 6.1, strategy with local Neumann problems] The relative energy error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of cumulative time spent on algebraic computations of the stopping criteria (4.21) and (4.22) (left) and the respective numbers of algebraic solver iterations per step of the adaptive procedure of Scheme 2 (right).



**Figure 13:** [Sharp-Gaussian of Section 6.1] Relative energy error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of  $\text{DoF}^{\frac{1}{3}}$ , obtained with our  $hp$ -refinement strategy (driven by solving local Dirichlet problems and also local Neumann problems) with inexact algebraic solver, purely  $h$ -adaptive version with exact solver and using uniform  $h$ -refinement.

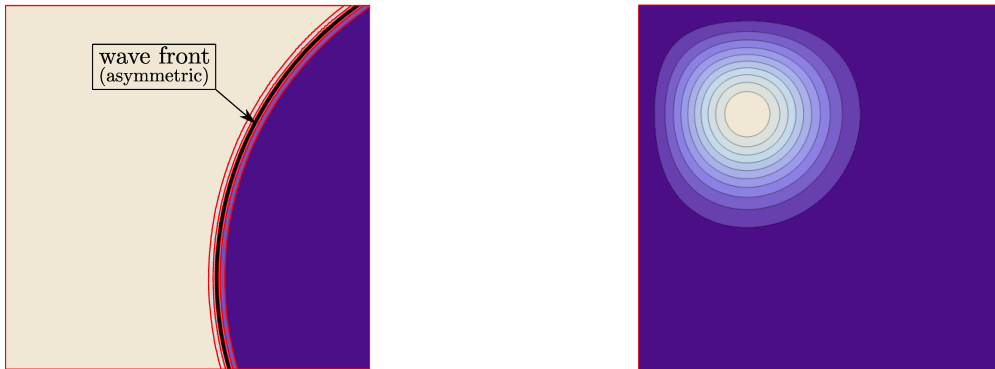
converge to the correct solution, while saving a substantial amount of computational effort. This kind of heuristic approach may actually be beneficial in cases when we launch the adaptive process with a good enough initial guess. However, as we demonstrate here in a fabricated setting, in case of an inaccurate initial guess, it is our adaptive strategy that represents a safe choice, while outperforming both the heuristic approach and the adaptive strategies with algebraic solver piloted by the classical stopping criteria.

To illustrate our point, we consider as the second test case a problem posed on the square domain  $\Omega := (0, 1) \times (0, 1)$  with the exact solution (in polar coordinates)

$$u(r) = \arctan(\alpha(r - r_0)), \quad r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$$

containing a wave front asymmetric within the domain. The parameter  $\alpha := 100$  prescribes the steepness of the circular wave front with radius  $r_0 := 0.92$  centred at the point  $(x_c, y_c) := (1.5, 0.25)$ , see Figure 14 (left panel) (for other variants of the wave front problem, we refer to Mitchell and McClain [31, Sections 5.16–5.19]). For this test case and also the test case of Section 6.4, the total error upper bound  $\eta(u_\ell, \mathcal{T}_\ell)$  employed within the adaptive procedure takes into account the error from the approximation of the inhomogeneous Dirichlet boundary condition prescribed by the exact solution on  $\partial\Omega$ ; to this purpose, we proceed as described in [22, Theorem 3.3] and the references therein.

We start the computation with a criss-cross grid  $\mathcal{T}_0$  with  $\max_{K \in \mathcal{T}_0} h_K = 0.125$ . In contrast to the other test cases and the description of the module `ONE_SOLVER_STEP` in Section 4.1, this time we solve the algebraic system inexactly even at the initial level  $\ell = 0$  using a geometric V-cycle multigrid solver with hierarchy of 4 additional meshes obtained by uniform coarsening of the mesh  $\mathcal{T}_0$ . As the initial guess for the algebraic solver, we consider a vector corresponding to a function which does not approximate the wave front well and contains a peak in the region where the exact solution is more or less flat. We display the contour plot of the initial guess function in the right panel of Figure 14. In the left panel of Figure 15, we plot the obtained mesh and polynomial degree distribution  $(\mathcal{T}_{20}, \mathbf{p}_{20})$  after 20 iterations of the adaptive procedure driven by solving local Neumann problems (Definitions 4.8 and 4.9) and when employing the so-called heuristic approach, i.e. performing only a single iteration of the algebraic solver at each level of refinement. Note the extra refinements present in the region of the peak of the initial guess function: these are not at all present when employing the adaptive stopping criterion (4.21) for the algebraic solver, see the right panel of Figure 15. Figure 16 shows the spatial distribution of the actual total error and the total upper error indicators corresponding to  $(\mathcal{T}_{20}, \mathbf{p}_{20})$  from Figure 15 (right panel) at the moment when the algebraic solver is stopped using (4.21) with  $\gamma_{20} = 0.1$ . Moreover, Figure 17 presents the comparison of our adaptive strategy with different stopping criteria for the algebraic solver in terms of the number of necessary algebraic iterations per iteration of the adaptive loop, and in terms of the amount of time spent on the algebraic computations in order to reach a relative estimated error lower than 0.01. We observe that while using the heuristic approach, compared to the use of the adaptive stopping criterion, nine additional iterations of the adaptive loop were necessary (due to incorrect refinements at the beginning of the adaptive process). Even though only one single iteration is performed per each adaptive loop iteration, we altogether spend approximately 4 times more time on algebraic computations than in the case of using adaptive stopping criterion. The heuristic approach surprisingly turns out to be comparable with the use of the classical criterion with  $\varepsilon = 10^{-6}$  in this overall cost assessment. Then, Figure 18 presents the effectivity indices for the reduction factor  $C_{\text{red}}$  and the lower bound  $\eta_{\mathcal{M}_\ell^q}$  which are quite promising even for this test case. The value of parameter  $\gamma_\ell$  was throughout the whole adaptive procedure equal to 0.1. Lastly, Figure 19 shows the error decay with respect to  $\text{DoF}_\ell^{\frac{1}{3}}$  when using the proposed strategy with inexact algebraic solver and, for comparison, also while using a pure  $h$ -adaptive version of the loop given by Scheme 1, and using simply uniform  $h$ -refinement. We observe that also for this model problem, the proposed strategy leads to an exponential convergence rate.

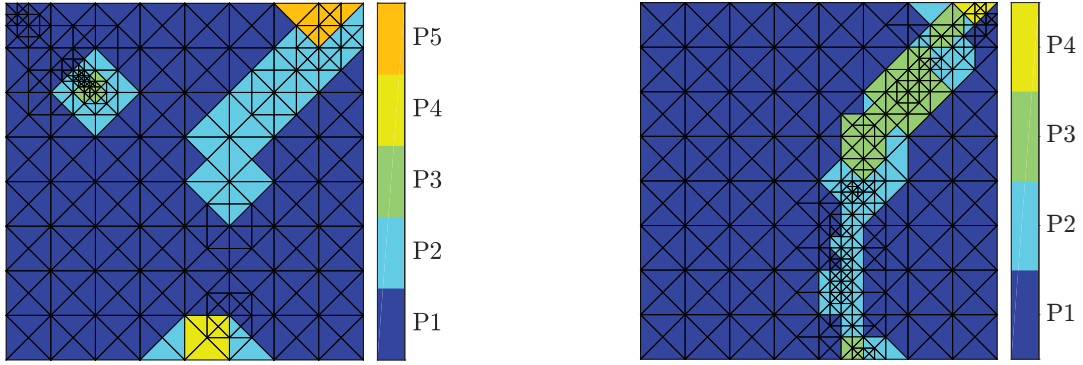


**Figure 14:** [Asymmetric wave front of Section 6.3] Contour plots of the exact solution (*left*) and the function inducing the initial guess for the algebraic solver (*right*).

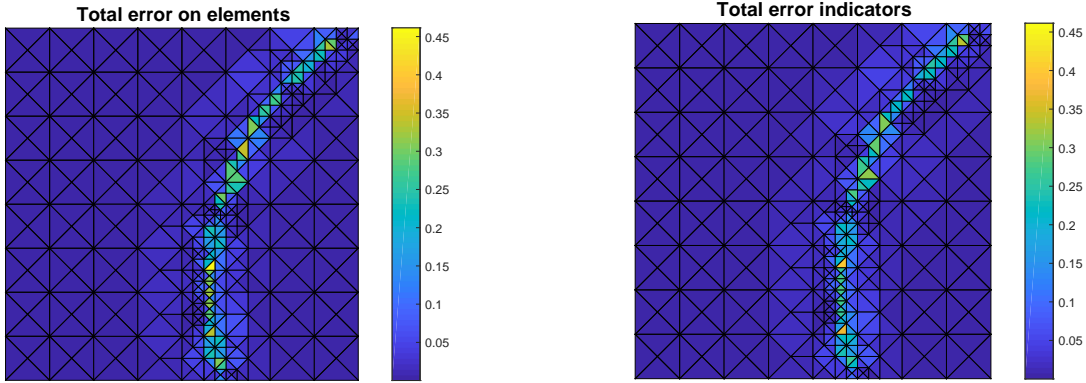
## 6.4 Singular solution (L-shape domain)

As a model problem with singular exact solution, we consider the classic re-entrant corner problem, cf. [31, 22, 15], posed on the L-shape domain  $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$  with  $f = 0$  and the weak solution (in polar coordinates)

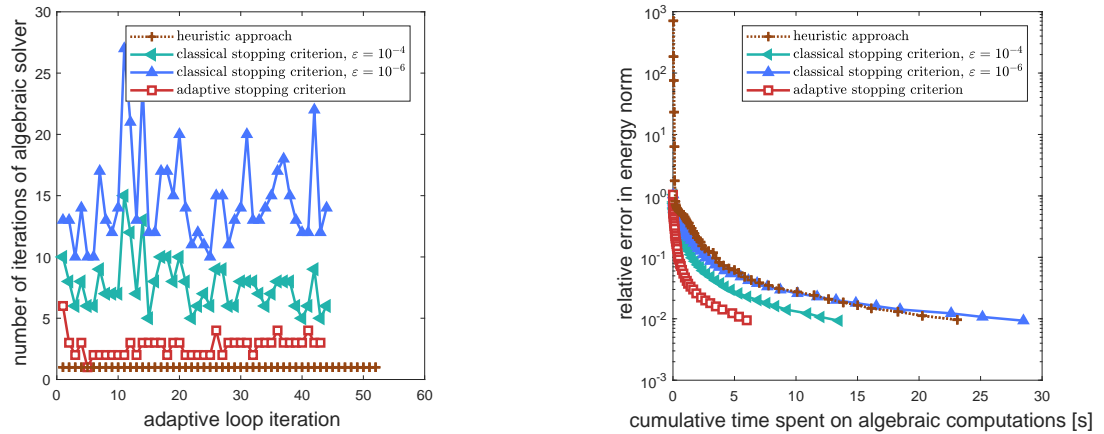
$$u(r, \varphi) = r^{\frac{2}{3}} \sin\left(\frac{2\varphi}{3}\right).$$



**Figure 15:** [Asymmetric wave front of Section 6.3] The mesh and polynomial degree distribution ( $\mathcal{T}_{20}, \mathbf{p}_{20}$ ) obtained while using the so-called heuristic approach (one multigrid iteration on each  $hp$ -adaptive step) (*left*) and while employing the adaptive stopping criterion (4.21) (*right*).

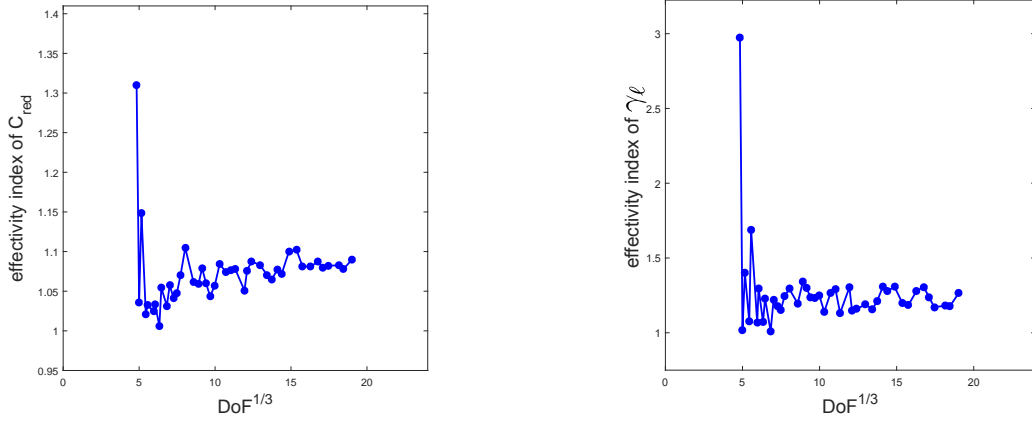


**Figure 16:** [Asymmetric wave front of Section 6.3] Elementwise distribution of the total energy error  $\|\nabla(u - u_{20})\|$  (*left*) and total upper error indicators  $\eta_K(u_{20})$  (*right*). The effectivity index of the estimate is 1.1106.

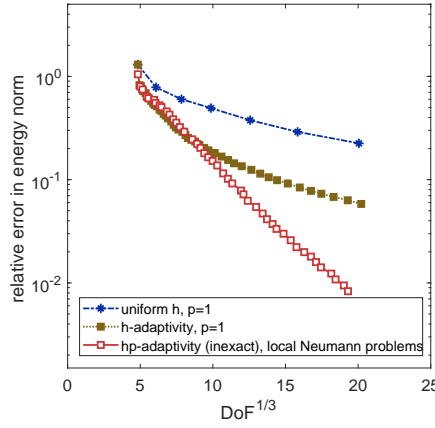


**Figure 17:** [Asymmetric wave front of Section 6.3] The relative energy error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of cumulative time spent on algebraic computations for various stopping criteria of Section 4.3 (*left*) and the respective numbers of algebraic solver iterations per step of the adaptive procedure of Scheme 2 (*right*).

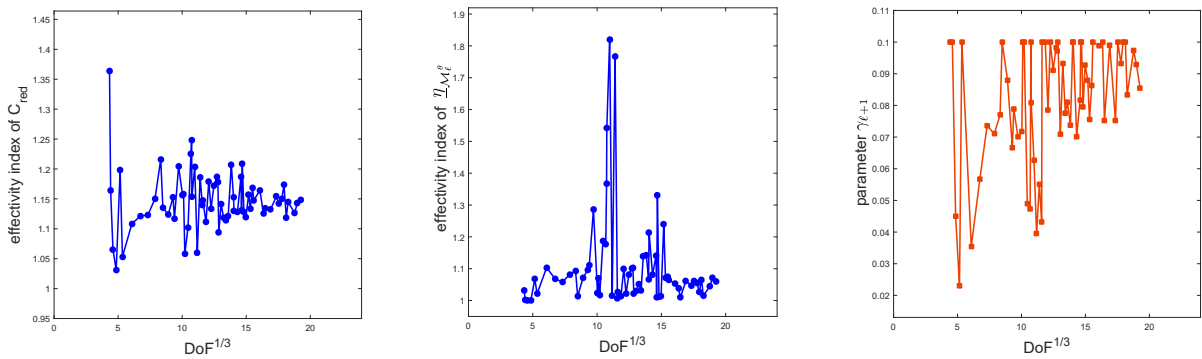
We start the computation on a coarse criss-cross grid  $\mathcal{T}_0$  with  $\max_{K \in \mathcal{T}_0} h_K = 0.25$  and all the polynomial degrees set uniformly to 1. We present here the results obtained with our strategy driven by solving the local Dirichlet problems only. We note that the results obtained with the strategy employing the local residuals  $r^{\mathbf{a},h}$  and  $r^{\mathbf{a},p}$  from Definitions 4.8 and 4.9 are very similar.



**Figure 18:** [Asymmetric wave front of Section 6.3] The effectivity indices for the error reduction factor estimate  $C_{\text{red}}$  of Theorem 5.4 given by (6.1) (left) and the discrete lower bound  $\underline{\eta}_{\mathcal{M}_{\ell}^{\theta}}$  of Lemma 5.2 given by (6.2) (right).

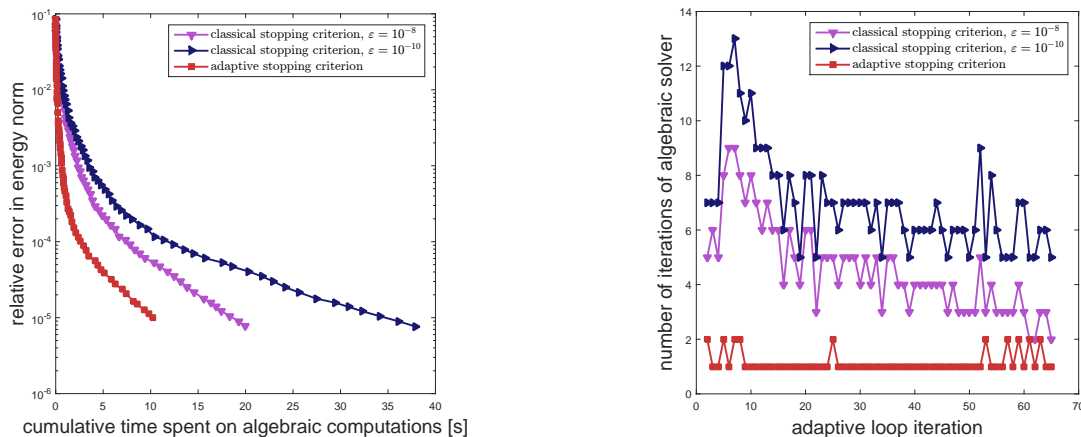


**Figure 19:** [Asymmetric wave front of Section 6.3] Relative energy error  $\|\nabla(u - u_{\ell})\|/\|\nabla u\|$  as a function of  $\text{DoF}^{\frac{1}{3}}$ , obtained with our  $hp$ -refinement strategy (driven by solving local Neumann problems) with inexact algebraic solver, purely  $h$ -adaptive version with exact solver and using uniform  $h$ -refinement.



**Figure 20:** [L-shape domain of Section 6.4] The effectivity index for the error reduction factor estimate  $C_{\text{red}}$  of Theorem 5.4 given by (6.1) (left); the effectivity index of the discrete lower bound  $\underline{\eta}_{\mathcal{M}_{\ell}^{\theta}}$  of Lemma 5.1 given by (6.2) (center); corresponding values of the parameter  $\gamma_{\ell+1}$  used in (5.13) (right).

First, in Figure 20, we assess the quality of the estimated reduction factor  $C_{\text{red}}$  and the lower bound  $\underline{\eta}_{\mathcal{M}_{\ell}^{\theta}}$ . We observe that the effectivity indices remain close to the optimal value of one also for this test case. In the right panel of Figure 20, we plot the corresponding values of the parameter  $\gamma_{\ell+1}$  used within the stopping

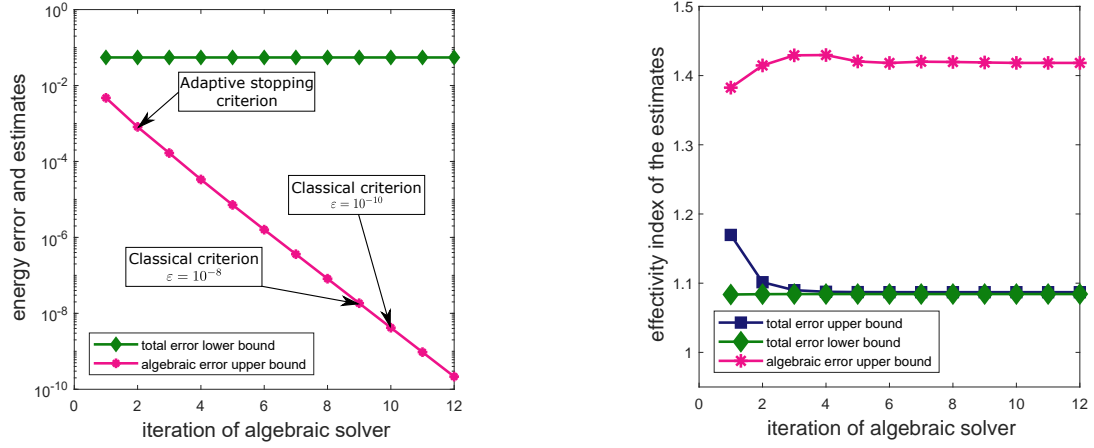


**Figure 21:** [L-shape domain of Section 6.4] The relative energy error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of cumulative time spent on algebraic computations for various stopping criteria of Section 4.3 (left) and the respective numbers of algebraic solver iterations per step of the adaptive procedure of Scheme 2 (right).

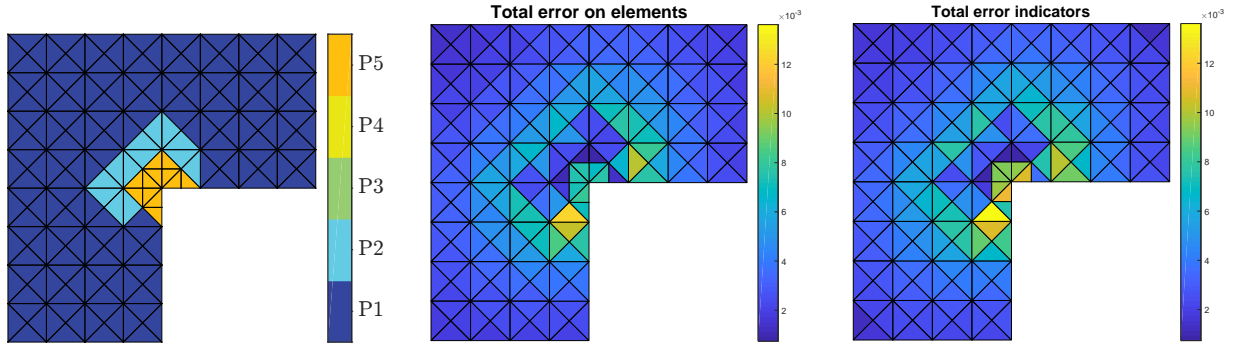
criterion (5.11). Next, Figure 21 demonstrates how the use of the stopping criterion (5.11) allows one to cut off the unnecessary iterations of the multigrid solver and save a substantial portion of the computational time spent on algebraic computations. Using the multigrid solver controlled by (5.11), to reach the relative error lower than  $10^{-5}$ , one saves about 50%, or even 75%, of the computational time dedicated to the algebraic solver in case of the use of the classical stopping criterion (4.22) with  $\epsilon = 10^{-8}$  or  $\epsilon = 10^{-10}$ , respectively. In the left panel of Figure 22, we investigate the evolution of the total error lower bound (3.8) and the algebraic error upper bound (3.6) throughout the iterations of the multigrid solver at the 7th level of refinement. The annotations therein indicate at which iteration of the algebraic solver our adaptive and the classical stopping criteria are satisfied. The quality of all the error bounds computed within the ESTIMATE module, at the same level of refinement, can be appreciated in the right panel of Figure 22. The corresponding mesh and polynomial degree distribution ( $\mathcal{T}_7, \mathbf{p}_7$ ) is displayed in the left panel of Figure 23. In Figure 23 (central and right panels) and Figure 24, we show the spatial distribution of the actual total and algebraic errors along with the total upper error indicators and algebraic upper error indicators after the 2nd iteration of multigrid solver on the 7th step of the  $hp$ -adaptive loop, i.e. at the moment when we stopped the multigrid solver as dictated by the adaptive stopping criterion (4.21). To conclude, we display in Figure 25 the overall decay of the relative error as a function of  $\text{DoF}_\ell^{\frac{1}{3}}$  in logarithmic-linear scale, to illustrate that also for this problem with singular exact solution, the present  $hp$ -adaptive strategy with inexact algebraic solver leads to an asymptotic exponential convergence rate. We display also the results obtained with uniform  $h$ -refinement, with pure  $h$ -version of the adaptive loop from Scheme 1, and its  $hp$ -version based on a priori knowledge of the weak solution leading to the best possible convergence rate. For further comparison with some other  $hp$ -refinement strategies (with exact algebraic solver), we refer to [15, Section 6.2], [31, Section 5.4].

## 7 Conclusions

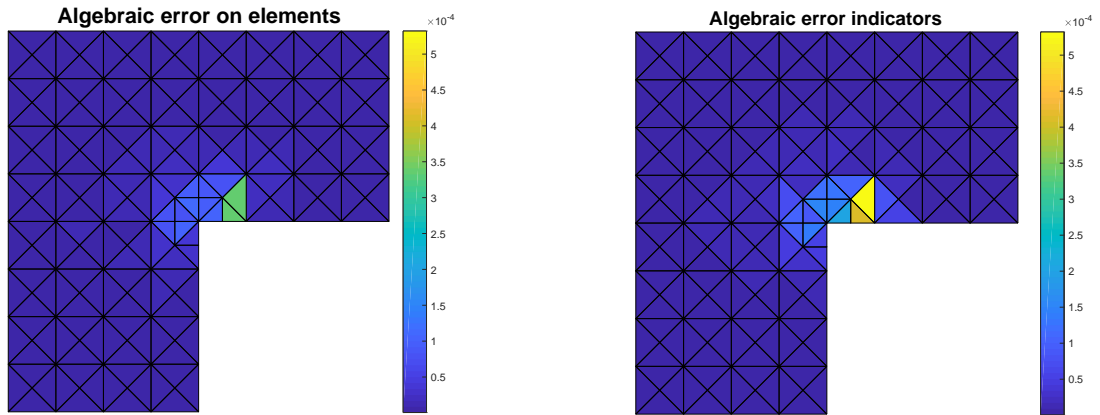
In this work, we extended our adaptive  $hp$ -refinement strategy for solving elliptic problems by taking into account an inexact algebraic solver within the adaptive loop. We constructed flux reconstructions and a total residual lifting by solving small local problems on patches of elements, yielding guaranteed a posteriori error bounds on algebraic and total errors. Then we proposed an adaptive stopping criterion for the iterative algebraic solver ensuring the desired balance between the algebraic and the total error. The total error indicators are employed to mark mesh vertices; the actual  $hp$ -refinement decision is driven by solving additional local problems on the patches of elements associated with the marked vertices. Once the next mesh and polynomial degree distribution have been determined, solving one additional local problem per marked vertex leads to a fully computable guaranteed bound on the error reduction factor between two successive inexact approximations. We considered here two options for the local problems on patches



**Figure 22:** [L-shape domain of Section 6.4] Algebraic error upper bound  $\eta_{\text{alg}}(u_7, \mathcal{T}_7)$  in comparison with total error lower bound  $\mu(u_7)$  (left); effectivity indices of total error upper bound (3.5), total error lower bound (3.8), and algebraic error upper bound (3.6) (right), throughout the iterations of the multigrid solver.

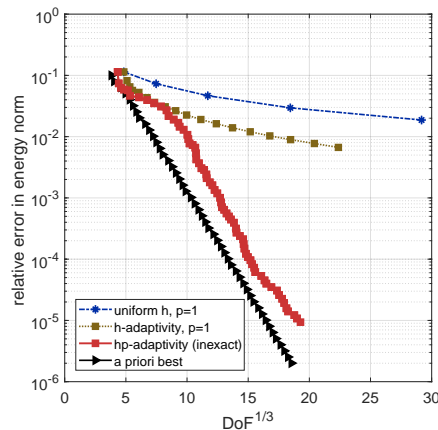


**Figure 23:** [L-shape domain of Section 6.4] The mesh and polynomial degree distribution  $(\mathcal{T}_7, \mathbf{p}_7)$ ; the corresponding elementwise distribution of the total energy error  $\|\nabla(u - u_7)\|$  (left) and the total error indicators  $\eta_{\mathcal{K}}(u_7)$  (right).



**Figure 24:** [L-shape domain of Section 6.4] Elementwise distribution of the algebraic energy error  $\|\nabla(u_7^{\text{ex}} - u_7)\|$  (left) and the algebraic error indicators  $\eta_{\text{alg}, \mathcal{K}}(u_7)$  (right) obtained with the adaptive stopping criterion (4.21) satisfied,  $\gamma_\ell = 0.04$ .

around marked vertices, with homogeneous Dirichlet and homogeneous Neumann boundary conditions. The numerical experiments demonstrated the accuracy of the estimated quantities while highlighting the applicability of the presented strategy. For all the test cases the obtained meshes and polynomial degree



**Figure 25:** [L-shape domain of Section 6.4] Relative energy error  $\|\nabla(u - u_\ell)\|/\|\nabla u\|$  as a function of  $\text{DoF}_\ell^{\frac{1}{3}}$ , obtained with our  $hp$ -refinement strategy (driven by solving local Dirichlet problems) with inexact algebraic solver, using uniform  $h$ -refinement, purely  $h$ -adaptive version of adaptive loop from Scheme 1, and its  $hp$ -version exploiting the a priori knowledge of the weak solution.

distributions lead to asymptotic exponential convergence rates. The further analysis of the reduction factor  $C_{\text{red}}$ , as indicated in Remark 5.6, is on the agenda.

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