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# A Two Phase Investment Game for Competitive Opinion Dynamics in Social Networks

Swapnil Dhamal<sup>1,2,3,\*</sup>, Walid Ben-Ameur<sup>2</sup>, Tijani Chahed<sup>2</sup>, Eitan Altman<sup>1,3</sup>

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## Abstract

We propose a setting for two-phase opinion dynamics in social networks, where a node's final opinion in the first phase acts as its initial biased opinion in the second phase. In this setting, we study the problem of two camps aiming to maximize adoption of their respective opinions, by strategically investing on nodes in the two phases. A node's initial opinion in the second phase naturally plays a key role in determining the final opinion of that node, and hence also of other nodes in the network due to its influence on them. However, more importantly, this bias also determines the effectiveness of a camp's investment on that node in the second phase. In order to formalize this two-phase investment setting, we propose an extension of Friedkin-Johnsen model, and hence formulate the utility functions of the camps. We arrive at a decision parameter which can be interpreted as two-phase Katz centrality. There is a natural tradeoff while splitting the available budget between the two phases. A lower investment in the first phase results in worse initial biases in the network for the second phase. On the other hand, a higher investment in the first phase spares a lower available budget for the second phase, resulting in an inability to fully harness the influenced biases. We first analyze the non-competitive case where only one camp invests,

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\*This is to indicate the corresponding author.

*Email address:* [swapnil.dhamal@gmail.com](mailto:swapnil.dhamal@gmail.com) (Swapnil Dhamal)

<sup>1</sup>Institut National de Recherche en Informatique et en Automatique, Sophia Antipolis-Méditerranée, Valbonne 06902, France

<sup>2</sup>Télécom SudParis, CNRS, Evry 91011, France

<sup>3</sup>Laboratoire Informatique d'Avignon, Avignon 84140, France

for which we present a polynomial time algorithm for determining an optimal way to split the camp’s budget between the two phases. We then analyze the case of competing camps, where we show the existence of Nash equilibrium and that it can be computed in polynomial time under reasonable assumptions. We conclude our study with simulations on real-world network datasets, in order to quantify the effects of the initial biases and the weightage attributed by nodes to their initial biases, as well as that of a camp deviating from its equilibrium strategy. Our main conclusion is that, if nodes attribute high weightage to their initial biases, it is advantageous to have a high investment in the first phase, so as to effectively influence the biases to be harnessed in the second phase.

*Keywords:* Social networks, opinion dynamics, two phases, zero-sum games, Nash equilibrium, Katz centrality

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## 1. Introduction

Studying opinion dynamics in a society is important to understand and influence elections, viral marketing, propagation of ideas and behaviors, etc. Social networks play a prime role in determining the opinions of constituent nodes, since nodes usually update their opinions based on the opinions of their connections [2, 3]. This fact is exploited by camps, intending to influence the opinions of these nodes in their favor. In this paper, we consider two camps who aim to maximize the adoption of their respective opinions in a social network. We consider a strict competition in the space of real-valued opinions, where one camp aims to drive the overall opinion of the network towards being positive while the other camp aims to drive it towards negative; we refer to them as good and bad camps respectively. We consider a well-accepted quantification of the overall opinion of a network: the average or equivalently, the sum of opinion values of the nodes in the social network [4, 5, 6]. Hence, the good and bad camps simultaneously aim to maximize and minimize this sum, respectively.

The average or sum of opinion values is well suited to several applications. For instance, in a fund collection scenario, the magnitude of the opinion value of

a node can be interpreted as the amount of funds it is willing to contribute, and its sign would imply the camp towards which it is willing to contribute. Here, the objective of the good camp would be to drive the sum of opinion values of the nodes to be as high as possible, so as to gather maximum funds for its concerned cause. On the other hand, the objective of the bad camp would be to drive this sum to be as low as possible, that is, to convince the population to contribute for an opposing cause. In such scenarios, the opinion value of a node would be a real-valued number, and the overall opinion of the network is well indicated by the sum or average of the opinion values of its constituent nodes. While fund collection is a particular scenario where a node's opinion is explicitly expressed in the form of its contribution, the underlying principle, in general, applies to scenarios that involve nodes having certain belief or information.

In the literature on opinion dynamics in social networks, there have been efforts to develop models which could determine how the individuals update their opinions based on the opinions of their connections [3]. With such an underlying model of opinion dynamics, we consider that each camp aims to maximize the adoption of its opinion in the social network, while accounting for the presence of the competing camp. A camp could hence act on achieving its objective by strategically investing on selected influential individuals in a social network who could adopt its opinion. This investment could be in the form of money, free products or discounts, attention, convincing discussions, etc. Thus given a budget constraint, the strategy of a camp comprises how much to invest on each node, in the presence of a competing camp who also invests strategically. This results in a game, and we are essentially interested in determining the equilibrium strategies of the two camps, from which neither camp would want to unilaterally deviate. Hence, our focus in this paper will be on determining the Nash equilibrium of this game.

### *1.1. Motivation*

In the popular model by Friedkin and Johnsen [7, 8] (which we will describe later), every node holds an initial bias in opinion. It could have formed owing

to the node’s fundamental views, experiences, information from news and other sources, opinion dynamics in the past, etc. This initial bias plays an important role in determining a node’s final opinion, and consequently the opinions of its neighbors and hence that of its neighbors’ neighbors and so on. If nodes give significant weightage to their biases, the camps would want to manipulate these biases. This could be achieved by campaigning in two phases, wherein the opinion at the conclusion of the first phase would act as the initial biased opinion for the second phase. Such campaigning is often used during elections and marketing, in order to gradually drive the nodes’ opinions.

In real-world scenarios, the initial bias of a node often impacts a camp’s effectiveness on that node. For instance, if the initial bias of a node is positive, the investment made by the good camp is likely to be more effective on it than that made by the bad camp. The reasoning is on similar lines as that of models in which a node pays more attention to opinions that do not differ too much from its own opinion (such as the bounded confidence model [9]). Since a camp’s effectiveness depends on the nodes’ biases, its investment in the first phase not only manipulates the biases for getting a head start in the second phase, but also the effectiveness of its investment in the second phase.

Furthermore, with the possibility of campaigning in two phases, a camp could not only decide which nodes to invest on, but also how to split its available budget between the two phases.

## 1.2. Related Work

*Opinion dynamics in social networks.* The topic of opinion dynamics has received significant attention in the social networks community. Xia, Wang, and Xuan [10] give a multidisciplinary review of the field of opinion dynamics as a combination of the social processes and the analytical and computational tools. A line of work deals with opinion diffusion in social networks under popular models such as the independent cascade and linear threshold [2, 11, 12]. Lorenz [13] surveys several modeling frameworks concerning continuous opinion dynamics. Acemoglu and Ozdaglar [3] review several other models of opinion dynamics,

some noteworthy ones being DeGroot [14], Voter [15], Friedkin-Johnsen [7, 8], bounded confidence [9], etc. In Friedkin-Johnsen model, each node updates its opinion using a weighted combination of its initial bias and its neighbors' opinions. In this paper, we generalize this model to multiple phases, while also incorporating the camps' investments.

*Identifying influential nodes.* Problems related to determining influential nodes for maximizing opinion adoption in social networks have been extensively studied in the literature [2, 12]. For instance, Yildiz, Ozdaglar, and Acemoglu [16] study the problem of optimal placement of stubborn nodes (whose opinion values stay unchanged) in the discrete binary opinions setting. Gionis, Terzi, and Tsaparas [4] study the problem of identifying such nodes whose positive opinions would maximize the overall positive opinion in the network. Kempe, Kleinberg, and Tardos [11] propose approximation algorithms for identifying influential nodes under the independent cascade and linear threshold models, which has since been followed by a plethora of increasingly efficient techniques [12]. The competitive setting has resulted in a number of game theoretic studies [17, 18, 19]. Bharathi, Kempe, and Salek [20] were among the first to study opinion adoption in social networks from a game theoretic viewpoint. Goyal, Heidari, and Kearns [21] present a model for the diffusion of two competing opinions in a social network, in which nodes first choose whether to adopt either of the opinions or none of them, and then choose which opinion to adopt. Anagnostopoulos, Ferraioli, and Leonardi [22] study this model in detail for some of the well-known dynamics. Ghaderi and Srikant [23] study how the equilibrium of the game depends on the network structure, nodes' initial opinions, the location of stubborn nodes and the extent of their stubbornness.

Specific to analytically tractable models such as DeGroot, there have been studies in the competitive setting to identify influential nodes and the amounts to be invested on them [5, 24, 25]. Dhamal et al. [6] study a broader framework with respect to one such model (Friedkin-Johnsen model), while considering a number of practically motivated settings such as those accounting for dimin-

ishing marginal returns on investment, adversarial behavior of the competitor, uncertainty regarding system parameters, and bound on the combined investment by the camps on each node. Our work extends these studies to two phases, by identifying influential nodes in the two phases and how much they should be invested on in each phase.

*Multi-phase opinion diffusion.* There have been a few studies on adaptive selection of influential nodes for opinion diffusion in multiple phases. Singer [26] presents a survey of such adaptive methodologies. Golovin and Krause [27] introduce adaptive submodularity, which facilitates adaptive greedy algorithm to provide a performance guarantee. Seeman and Singer [28] were among the first to dedicatedly study the framework of adaptive node selection. Rubinstein, Seeman, and Singer [29] present adaptive algorithms for selecting nodes with heterogeneous costs. Horel and Singer [30] develop scalable methods for models in which the influence of a set can be expressed as the sum of influence of its members. Correa et al. [31] show that the adaptivity benefit is bounded if every pair of nodes randomly meet at the same rate. Badanidiyuru et al. [32] propose an algorithm based on locally-adaptive policies. Dhamal, Prabuchandran, and Narahari [33] empirically study the problem of optimally splitting the available budget between two phases under the popular independent cascade model, which has been extended to more than two phases in [34]. Tong et al. [35] study adaptive node selection in a dynamic independent cascade model. Yuan and Tang [36] present a framework where nodes can be selected before termination of an ongoing diffusion. Mondal, Dhamal, and Narahari [37] study a setting where the first phase is regular diffusion, while the second phase is boosted using referral incentives.

While the reasoning behind using multiple phases in these studies is adaptation of node selection strategy based on previous observations, we aim to use multiple phases for manipulating the initial biases of nodes. This requires a very different conceptual and analytical treatment from the ones in the literature.

To the best of our knowledge, there has not been an analytical study on a

rich model such as Friedkin-Johnsen, for opinion dynamics in two phases (not even for single camp). The most relevant to this study is our earlier work [38] where, however, a camp’s influence on a node is assumed to be independent of the node’s bias. In this paper, we consider a more realistic setting by relaxing this assumption. An interesting outcome of relaxing this assumption is that, while the camps’ optimal strategies turn out to be mutually independent in [38], these strategies get coupled in our setting. In other words, the setting in [38] results in a *competition*, while the one in this paper results in a *game*.

### 1.3. Our Contributions

Following are the specific contributions of this paper:

- We formulate the two-phase objective function under Friedkin-Johnsen model, where a node’s final opinion in the first phase acts as its initial bias for the second phase, and the effectiveness of a camp’s investment on the node depends on this initial bias. (Section 2)
- For the non-competitive case, we develop a polynomial time algorithm for determining an optimal way to split a camp’s budget between the two phases and the nodes to be invested on in the two phases. (Section 3)
- For the competitive case involving both the camps, we show the existence of Nash equilibrium, and that it can be computed in polynomial time under reasonable assumptions. (Section 4)
- Using simulations, we illustrate our analytically derived results on real-world network datasets, and quantify the effects of the initial biases and the weightage attributed by nodes to their initial biases, as well as that of a camp deviating from its equilibrium strategy. (Section 5)

## 2. Our Model

Given a social network, let  $N$  be the set of nodes and  $E$  be the set of weighted directed edges. Let  $n = |N|$ . Our model can be viewed as a multiphase extension of [6], and more broadly, an extension of Friedkin-Johnsen model [7, 8].



### 2.1. Friedkin-Johnsen Model

In Friedkin-Johnsen model, prior to the process of opinion dynamics, each node holds a certain bias in its opinion. We denote this opinion bias of a node  $i$  by  $v_i^0$ , and the weightage that the node attributes to it by  $w_{ii}^0$ .

The network effect is captured by how much a node is influenced by each of its friends or connections, that is, how much weightage is attributed by a node to the opinion of each of its connections. Let  $v_j$  be the opinion held by node  $j$ , and  $w_{ij}$  be the weightage attributed by node  $i$  to the opinion of node  $j$ . The influence on node  $i$  owing to node  $j$  is given by  $w_{ij}v_j$ , thus the net influence on  $i$  owing to all of its connections is  $\sum_{j \in N} w_{ij}v_j$  (where  $w_{ij} \neq 0$  only if  $j$  is a connection of  $i$ ). Note that the edge weights could be negative as well (as justified in [39, 40]). A negative edge weight  $w_{ij}$  can be interpreted as distrust that node  $i$  holds on node  $j$ , that is,  $i$  would be driven towards adopting an opinion that is opposite to that held or suggested by  $j$ .

Since in Friedkin-Johnsen model, each node updates its opinion using a weighted convex combination of its bias and its neighbors' opinions, the update rule is given by

$$\forall i \in N : v_i \leftarrow w_{ii}^0 v_i^0 + \sum_{j \in N} w_{ij} v_j$$

where  $|w_{ii}^0| + \sum_{j \in N} |w_{ij}| \leq 1, \forall i \in N$ . Since the model follows an opinion update rule, convergence is often a desirable property. A standard assumption for guaranteeing convergence is  $\sum_{j \in N} |w_{ij}| < 1$ . We will later see how we use this condition in our analysis.

### 2.2. Our Extended Model

As our opinion dynamics runs in two phases, most parameters have two values, one for each phase. For such a parameter, we denote its value corresponding to phase  $p$  using superscript  $(p)$ , where  $p = 1$  for the first phase and  $p = 2$  for the second phase.

*Initial Bias.* Consistent with Friedkin-Johnsen model, every node  $i$  holds an initial bias in opinion  $v_i^0 \in \mathbb{R}$  prior to the opinion dynamics process, and attributes a weightage of  $w_{ii}^0$  to it. In the two-phase setting,  $v_i^0$  acts as the initial bias for the first phase. We denote the opinion value of node  $i$  at the conclusion of the first phase by  $v_i^{(1)}$ . This acts as its initial bias for the second phase. So by convention, we have  $v_i^{(0)} = v_i^0$ .

*Network Effect.* Consistent with Friedkin-Johnsen model, we consider that the influence on node  $i$  owing to node  $j$  in phase  $p$ , is  $w_{ij}v_j^{(p)}$ , where  $w_{ij}$  is the weightage attributed by node  $i$  to the opinion of its connection  $j$ . So the net influence on node  $i$  owing to all of its connections is  $\sum_{j \in N} w_{ij}v_j^{(p)}$ .

*Weightage to Campaigning.* We denote the weightage that node  $i$  attributes to the good and bad campaigning in phase  $p$  by  $w_{ig}^{(p)}$  and  $w_{ib}^{(p)}$ , respectively. Since we consider that the initial bias of a node impacts the effectiveness of camps' investments,  $v_i^{(p-1)} > 0$  would likely result in  $w_{ig}^{(p)} > w_{ib}^{(p)}$ . Note that  $w_{ii}^0$  would also play a role since it quantifies the weightage given by node  $i$  to its initial bias. We hence propose a model on this line, wherein  $w_{ig}^{(p)}$  is a monotone non-decreasing function of  $w_{ii}^0 v_i^{(p-1)}$ , and  $w_{ib}^{(p)}$  is a monotone non-increasing function of  $w_{ii}^0 v_i^{(p-1)}$ . Let node  $i$  attribute a total of  $\theta_i$  to the influence weights of the camps, that is,  $w_{ig}^{(p)} + w_{ib}^{(p)} = \theta_i$ . We propose the following natural model:

$$w_{ig}^{(p)} = \theta_i \left( \frac{1 + w_{ii}^0 v_i^{(p-1)}}{2} \right), \quad w_{ib}^{(p)} = \theta_i \left( \frac{1 - w_{ii}^0 v_i^{(p-1)}}{2} \right) \quad (1)$$

*Camp Investments.* The good and bad camps attempt to directly influence the nodes so that their opinions are driven towards being positive and negative, respectively. We denote the investments made by the good and bad camps on node  $i$  in phase  $p$  by  $x_i^{(p)}$  and  $y_i^{(p)}$  respectively ( $x_i^{(p)}, y_i^{(p)} \geq 0, \forall i \in N$  for  $p = 1, 2$ ). Since the influence of good camp (positive opinion) on node  $i$  in phase  $p$  would be an increasing function of both  $x_i^{(p)}$  and  $w_{ig}^{(p)}$ , we assume the influence to be  $+w_{ig}^{(p)} x_i^{(p)}$  (maintaining the linearity of Friedkin-Johnsen model). Similarly,  $-w_{ib}^{(p)} y_i^{(p)}$  is the influence of bad camp (negative opinion)

on node  $i$  in phase  $p$ . Let  $k_g$  and  $k_b$  be the respective budgets of the good and bad camps. Hence the camps should invest in the two phases such that  $\sum_{i \in N} (x_i^{(1)} + x_i^{(2)}) \leq k_g$  and  $\sum_{i \in N} (y_i^{(1)} + y_i^{(2)}) \leq k_b$ .

*Matrix Forms.* Let  $\mathbf{W}$  be the matrix consisting of weights  $w_{ij}$  for each pair of nodes  $(i, j)$ . Let  $\mathbf{v}^0, \mathbf{w}^0, \Theta, \mathbf{w}_g^{(\mathbf{p})}, \mathbf{w}_b^{(\mathbf{p})}, \mathbf{x}^{(\mathbf{p})}, \mathbf{y}^{(\mathbf{p})}, \mathbf{v}^{(\mathbf{p})}$  be the vectors consisting of elements  $v_i^0, w_{ii}^0, \theta_i, w_{ig}^{(p)}, w_{ib}^{(p)}, x_i^{(p)}, y_i^{(p)}, v_i^{(p)}$ , respectively. Let operation  $\circ$  denote Hadamard vector product, that is,  $(\mathbf{a} \circ \mathbf{b})_i = a_i b_i$ .

*Opinion Update Rule.* Recall that the update rule in Friedkin-Johnsen model is given by

$$\forall i \in N : v_i \leftarrow w_{ii}^0 v_i^0 + \sum_{j \in N} w_{ij} v_j$$

Extending to multiple phases, the update rule in the  $p^{\text{th}}$  phase is

$$\forall i \in N : v_i^{(p)} \leftarrow w_{ii}^0 v_i^{(p-1)} + \sum_{j \in N} w_{ij} v_j^{(p)}$$

Accounting for camps' investments, we get

$$\begin{aligned} \forall i \in N : v_i^{(p)} &\leftarrow w_{ii}^0 v_i^{(p-1)} + \sum_{j \in N} w_{ij} v_j^{(p)} + w_{ig}^{(p)} x_i^{(p)} - w_{ib}^{(p)} y_i^{(p)} \\ \iff \mathbf{v}^{(\mathbf{p})} &\leftarrow \mathbf{w}^0 \circ \mathbf{v}^{(\mathbf{p}-1)} + \mathbf{W} \mathbf{v}^{(\mathbf{p})} + \mathbf{w}_g^{(\mathbf{p})} \circ \mathbf{x}^{(\mathbf{p})} - \mathbf{w}_b^{(\mathbf{p})} \circ \mathbf{y}^{(\mathbf{p})} \end{aligned} \quad (2)$$

In phase  $p$ , the vectors  $\mathbf{x}^{(\mathbf{p})}, \mathbf{y}^{(\mathbf{p})}, \mathbf{v}^{(\mathbf{p}-1)}$  stay unchanged; the weights  $\mathbf{w}_g^{(\mathbf{p})}, \mathbf{w}_b^{(\mathbf{p})}$  (which depend on  $\mathbf{v}^{(\mathbf{p}-1)}$ ) and  $\mathbf{w}^0$  also stay unchanged; while  $\mathbf{v}^{(\mathbf{p})}$  gets updated. Hence, writing the update rule as recursion (with iterating integer  $\tau \geq 0$ ), we get

$$\mathbf{v}_{\langle \tau \rangle}^{(\mathbf{p})} = \mathbf{W} \mathbf{v}_{\langle \tau-1 \rangle}^{(\mathbf{p})} + \mathbf{w}^0 \circ \mathbf{v}^{(\mathbf{p}-1)} + \mathbf{w}_g^{(\mathbf{p})} \circ \mathbf{x}^{(\mathbf{p})} - \mathbf{w}_b^{(\mathbf{p})} \circ \mathbf{y}^{(\mathbf{p})}$$

Solving the recursion simplifies it to

$$\mathbf{v}_{\langle \tau \rangle}^{(\mathbf{p})} = \mathbf{W}^\tau \mathbf{v}_{\langle 0 \rangle}^{(\mathbf{p})} + \left( \sum_{\eta=0}^{\tau-1} \mathbf{W}^\eta \right) (\mathbf{w}^0 \circ \mathbf{v}^{(\mathbf{p}-1)} + \mathbf{w}_g^{(\mathbf{p})} \circ \mathbf{x}^{(\mathbf{p})} - \mathbf{w}_b^{(\mathbf{p})} \circ \mathbf{y}^{(\mathbf{p})})$$

Now, the initial bias for phase  $p$  :  $\mathbf{v}_{(0)}^{(p)} = \mathbf{v}^{(p-1)}$ . Also, since  $\sum_{j \in N} |w_{ij}| < 1$ , we have that  $\mathbf{W}$  is a strictly substochastic matrix (sum of each row strictly less than 1). Hence its spectral radius is less than 1. So when  $\tau \rightarrow \infty$ , we have  $\lim_{\tau \rightarrow \infty} \mathbf{W}^\tau = \mathbf{0}$  and  $\lim_{\tau \rightarrow \infty} \sum_{\eta=0}^{\tau-1} \mathbf{W}^\eta = (\mathbf{I} - \mathbf{W})^{-1}$  [5]. Hence,

$$\lim_{\tau \rightarrow \infty} \mathbf{v}_{(\tau)}^{(p)} = (\mathbf{I} - \mathbf{W})^{-1} (\mathbf{w}^0 \circ \mathbf{v}^{(p-1)} + \mathbf{w}_g^{(p)} \circ \mathbf{x}^{(p)} - \mathbf{w}_b^{(p)} \circ \mathbf{y}^{(p)})$$

which is a constant vector. So the dynamics in phase  $p$  converges to the steady state

$$\mathbf{v}^{(p)} = (\mathbf{I} - \mathbf{W})^{-1} (\mathbf{w}^0 \circ \mathbf{v}^{(p-1)} + \mathbf{w}_g^{(p)} \circ \mathbf{x}^{(p)} - \mathbf{w}_b^{(p)} \circ \mathbf{y}^{(p)}) \quad (3)$$

### 2.3. Formulation of Two-Phase Objective Function

We now derive the objective function  $\sum_{i \in N} v_i^{(2)}$ , the sum of opinion values of all the nodes at the end of the second phase. Premultiplying Equation (3) by  $\mathbf{1}^T$  gives

$$\mathbf{1}^T \mathbf{v}^{(p)} = \mathbf{1}^T (\mathbf{I} - \mathbf{W})^{-1} (\mathbf{w}^0 \circ \mathbf{v}^{(p-1)} + \mathbf{w}_g \circ \mathbf{x}^{(p)} - \mathbf{w}_b \circ \mathbf{y}^{(p)})$$

Let  $\Delta = (\mathbf{I} - \mathbf{W})^{-1}$  and  $\mathbf{r}^T = \mathbf{1}^T (\mathbf{I} - \mathbf{W})^{-1}$ , that is,  $r_i = \sum_{j \in N} \Delta_{ji}$ . Since  $\Delta = \sum_{\eta=0}^{\infty} \mathbf{W}^\eta$ , we have that  $\Delta_{ji}$  is the influence that  $j$  receives from  $i$  through walks of all possible lengths. So  $r_i = \sum_{j \in N} \Delta_{ji}$  can be viewed as overall influencing power of  $i$ . Substituting these in the above equation, we get

$$\sum_{i \in N} v_i^{(p)} = \sum_{i \in N} r_i (w_{ii}^0 v_i^{(p-1)} + w_{ig} x_i^{(p)} - w_{ib} y_i^{(p)})$$

When  $p = 1$ , this is the sum of opinions at the end of the first phase:

$$\sum_{i \in N} v_i^{(1)} = \sum_{i \in N} r_i (w_{ii}^0 v_i^0 + w_{ig}^{(1)} x_i^{(1)} - w_{ib}^{(1)} y_i^{(1)}) \quad (4)$$

Similarly, when  $p = 2$ , the sum of opinion values at the end of the second phase is (by also using Equations (1)):

$$\begin{aligned}
\sum_{j \in N} v_j^{(2)} &= \sum_{j \in N} r_j (w_{jj}^0 v_j^{(1)} + w_{jj}^{(2)} x_j^{(2)} - w_{jb}^{(2)} y_j^{(2)}) \\
&= \sum_{j \in N} r_j \left( w_{jj}^0 v_j^{(1)} + \frac{\theta_j}{2} (1 + w_{jj}^0 v_j^{(1)}) x_j^{(2)} - \frac{\theta_j}{2} (1 - w_{jj}^0 v_j^{(1)}) y_j^{(2)} \right) \\
&= \sum_{j \in N} r_j w_{jj}^0 v_j^{(1)} \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) + \sum_{j \in N} r_j \frac{\theta_j}{2} (x_j^{(2)} - y_j^{(2)}) \quad (5)
\end{aligned}$$

The first term  $\sum_{j \in N} r_j w_{jj}^0 v_j^{(1)} \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right)$  can be obtained by pre-multiplying (4) by  $(\mathbf{r} \circ \mathbf{w}^0 \circ (\mathbf{1} + \frac{\Theta}{2} \circ \mathbf{x}^{(2)} + \frac{\Theta}{2} \circ \mathbf{y}^{(2)}))^T$  and using (1). Hence,

$$\begin{aligned}
&\sum_{j \in N} r_j w_{jj}^0 v_j^{(1)} \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) \\
&= \left( \mathbf{r} \circ \mathbf{w}^0 \circ (\mathbf{1} + \frac{\Theta}{2} \circ \mathbf{x}^{(2)} + \frac{\Theta}{2} \circ \mathbf{y}^{(2)}) \right)^T (\mathbf{I} - \mathbf{W})^{-1} (\mathbf{w}^0 \circ \mathbf{v}^{(0)} + \mathbf{w}_{\mathbf{g}}^{(1)} \circ \mathbf{x}^{(1)} - \mathbf{w}_{\mathbf{b}}^{(1)} \circ \mathbf{y}^{(1)}) \\
&= \sum_{i \in N} \left( \sum_{j \in N} r_j w_{jj}^0 \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) \Delta_{ji} \right) (w_{ii}^0 v_i^0 + w_{ig}^{(1)} x_i^{(1)} - w_{ib}^{(1)} y_i^{(1)}) \\
&= \sum_{i \in N} \left( \sum_{j \in N} r_j w_{jj}^0 \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) \Delta_{ji} \right) \left( w_{ii}^0 v_i^0 + \frac{\theta_i}{2} (1 + w_{ii}^0 v_i^0) x_i^{(1)} - \frac{\theta_i}{2} (1 - w_{ii}^0 v_i^0) y_i^{(1)} \right) \\
&= \sum_{i \in N} \sum_{j \in N} \left( w_{ii}^0 v_i^0 \left( 1 + \frac{\theta_i}{2} x_i^{(1)} + \frac{\theta_i}{2} y_i^{(1)} \right) + \frac{\theta_i}{2} (x_i^{(1)} - y_i^{(1)}) \right) \left( r_j w_{jj}^0 \Delta_{ji} \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) \right)
\end{aligned}$$

Substituting this in (5), we get we get that  $\sum_{i \in N} v_i^{(2)}$  equals

$$\begin{aligned}
&\sum_{i \in N} \sum_{j \in N} \left( w_{ii}^0 v_i^0 \left( 1 + \frac{\theta_i}{2} x_i^{(1)} + \frac{\theta_i}{2} y_i^{(1)} \right) + \frac{\theta_i}{2} (x_i^{(1)} - y_i^{(1)}) \right) \left( r_j w_{jj}^0 \Delta_{ji} \left( 1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) \right) \\
&\quad + \sum_{j \in N} r_j \frac{\theta_j}{2} (x_j^{(2)} - y_j^{(2)}) \quad (6)
\end{aligned}$$

For notational simplicity, let  $b_{ji} = r_j w_{jj}^0 \Delta_{ji}$ ,  $c_i = w_{ii}^0 v_i^0$ . We saw that  $r_i = \sum_{j \in N} \Delta_{ji}$  indicates the influencing power of node  $i$ . Now,  $b_{ji} = \Delta_{ji} w_{jj}^0 r_j$  quantifies the overall influence  $\Delta_{ji}$  of  $i$  on  $j$ , which would give weightage  $w_{jj}^0$  to its initial bias in the next phase, and have an influencing power of  $r_j$  in the next phase. Hence  $b_{ji}$  can be interpreted as the influence of node  $i$  on the

network through node  $j$ , looking one phase ahead. So  $\sum_{j \in N} b_{ji}$  can be viewed as the overall influencing power of node  $i$ , looking one phase ahead. We denote  $s_i = \sum_{j \in N} b_{ji}$  and identify its role in our analysis.

*Two-phase Katz centrality.* Note that  $r_i = ((\mathbf{I} - \mathbf{W}^T)^{-1} \mathbf{1})_i$  can be interpreted as a form of *Katz centrality* [41] of node  $i$ . The original Katz centrality is defined for an adjacency matrix  $\mathcal{A}$  with all edges having the same weight  $\alpha$ . Katz centrality of node  $i$  is defined as the  $i^{\text{th}}$  element of vector  $\left( (\mathbf{I} - \alpha \mathcal{A}^T)^{-1} - \mathbf{I} \right) \mathbf{1} = (\mathbf{I} - \alpha \mathcal{A}^T)^{-1} \mathbf{1} - \mathbf{1}$ , for  $0 < \alpha < \frac{1}{|\rho|}$  where  $\rho$  is the largest eigenvalue of  $\mathcal{A}$ . In our case where  $\mathbf{r} = (\mathbf{I} - \mathbf{W}^T)^{-1} \mathbf{1}$ ,  $\mathcal{A}$  is replaced by the weighted adjacency matrix  $\mathbf{W}$ , for which  $|\rho| < 1$  (since  $\mathbf{W}$  is strictly substochastic), and we have  $\alpha = 1$ . The subtraction of vector  $\mathbf{1}$  is common for all nodes, so its relative effect can be ignored. Hence,  $r_i$  measures node  $i$ 's relative influence in a social network when there are is no subsequent phase to follow. In the two-phase setting, this applies to the second phase since it is the terminal phase. However, while selecting optimal nodes in the first phase, when there is a subsequent phase to follow, the effectiveness of node  $i$  depends on its influencing power over those nodes ( $j$ ), which would give good weightage ( $w_{jj}^0$ ) to their initial biases in the second phase, as well as have a good influencing power over other nodes ( $r_j$ ) in the second phase. This is precisely captured by  $s_i$ , and so it can be viewed as the two-phase Katz centrality.

Now, using this simplified notation, Equation (6) can be written as

$$\begin{aligned}
\sum_{i \in N} v_i^{(2)} &= \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} \left( \sum_{i \in N} c_i b_{ji} + r_j \right) + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} \left( \sum_{i \in N} c_i b_{ji} - r_j \right) \\
&\quad + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \\
&\quad - \sum_{i \in N} y_i^{(1)} \frac{\theta_i}{2} (1 - c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \tag{7}
\end{aligned}$$

#### 2.4. The Two-phase Investment Game

Following our model, we have that  $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$  is the strategy of the good camp for the two phases, and  $(\mathbf{y}^{(1)}, \mathbf{y}^{(2)})$  is the strategy of the bad camp. Given an investment strategy profile  $((\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}))$ , let  $u_g((\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}))$  be the good camp's utility and  $u_b((\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}))$  be the bad camp's utility. The good camp aims to maximize the value of (7), while the bad camp aims to minimize it. So,

$$\begin{aligned} u_g((\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), (\mathbf{y}^{(1)}, \mathbf{y}^{(2)})) &= \sum_{i \in N} v_i^{(2)} \\ \text{and } u_b((\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), (\mathbf{y}^{(1)}, \mathbf{y}^{(2)})) &= - \sum_{i \in N} v_i^{(2)} \end{aligned} \quad (8)$$

with the following constraints on the investment strategies:

$$\begin{aligned} \sum_{i \in N} (x_i^{(1)} + x_i^{(2)}) &\leq k_g, \quad \sum_{i \in N} (y_i^{(1)} + y_i^{(2)}) \leq k_b \\ \forall i \in N : x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)} &\geq 0 \end{aligned}$$

The game can thus be viewed as a two-player zero-sum game, where the players determine their investment strategies  $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$  and  $(\mathbf{y}^{(1)}, \mathbf{y}^{(2)})$ ; the good camp invests as per  $\mathbf{x}^{(1)}$  in the first phase and as per  $\mathbf{x}^{(2)}$  in the second phase, and the bad camp invests as per  $\mathbf{y}^{(1)}$  in the first phase and as per  $\mathbf{y}^{(2)}$  in the second phase. Our objective essentially is to find the Nash equilibrium strategies of the two camps.

First, we consider a simplified yet interesting (and not-yet-studied-in-literature) case where budget of one of the camps is 0 (say  $k_b = 0$ ); so effectively we have only the good camp.

### 3. The Non-Competitive Case

For the non-competitive case, when there is only one camp (say the good camp, without loss of analytical generality), we have  $y_i^{(1)} = y_i^{(2)} = 0, \forall i \in N$  in Equation (7). So we get

$$\begin{aligned}
\sum_{i \in N} v_i^{(2)} &= \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} \left( \sum_{i \in N} c_i b_{ji} + r_j \right) + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \\
&= \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} \left( \sum_{i \in N} c_i b_{ji} + r_j \right) \\
&\quad + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} \tag{9}
\end{aligned}$$

$$= \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) + (\text{terms independent of } x_i^{(1)}) \tag{10}$$

$$= \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} \left( r_j + \sum_{i \in N} c_i b_{ji} + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) b_{ji} \right) + (\text{terms independent of } x_j^{(2)}) \tag{11}$$

Deducing from Equations (10) and (11) that  $\sum_{i \in N} v_i^{(2)}$  is a bilinear function in  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$ , we prove our next result.

Let the budget  $k_g$  be split such that  $k_g^{(1)}$  and  $k_g^{(2)}$  are the first and second phase investments, respectively. Our method of finding optimal  $k_g^{(1)}$  and  $k_g^{(2)}$  is to first search for them in the search space  $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$  and then compare the thus obtained value of  $\sum_{i \in N} v_i^{(2)}$  with that corresponding to  $k_g^{(1)} = k_g^{(2)} = 0$ , so as to get the optimal value of  $\sum_{i \in N} v_i^{(2)}$ .

**Proposition 1.** *In the search space  $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$ , it is optimal for the good camp to exhaust its entire budget ( $k_g^{(1)} + k_g^{(2)} = k_g$ ), and to invest on at most one node in each phase.*

*Proof.* Given any  $\mathbf{x}^{(2)}$ , Expression (10) can be maximized w.r.t.  $\mathbf{x}^{(1)}$  by allocating  $k_g - \sum_{j \in N} x_j^{(2)}$  to a single node  $i$  that maximizes  $\frac{\theta_i}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} \right)$ , if this value is positive. In case of multiple such nodes, one node can be chosen at random. If this value is non-positive for all nodes, it is optimal to have  $\mathbf{x}^{(1)} = \mathbf{0}$ . When  $\mathbf{x}^{(1)} = \mathbf{0}$ , Expression (11) now implies that it is optimal to allocate the entire budget  $k_g$  in second phase to a single node  $j$  that maximizes  $\frac{\theta_j}{2} \left( r_j + \sum_{i \in N} c_i b_{ji} \right)$ , if this value is positive. If this value is non-positive for all nodes, it is optimal to have  $\mathbf{x}^{(2)} = \mathbf{0}$ . This is the case where starting with an  $\mathbf{x}^{(2)}$ , we conclude that it is optimal to either invest  $k_g - \sum_{j \in N} x_j^{(2)}$  on a single



node in first phase, or invest the entire budget  $k_g$  on a single node in second phase, or invest in neither phase.

Similarly using (11), starting with an  $\mathbf{x}^{(1)}$ , we can conclude that it is optimal to either invest  $k_g - \sum_{j \in N} x_j^{(1)}$  on a single node in the second phase, or invest the entire  $k_g$  on a single node in the first phase, or invest in neither phase.

So starting from any  $\mathbf{x}^{(1)}$  or  $\mathbf{x}^{(2)}$ , we can iteratively improve (need not be strictly) on the value of (9) by investing on at most one node in a given phase. Furthermore, it is suboptimal to have  $k_g^{(1)} + k_g^{(2)} < k_g$  unless  $k_g^{(1)} = k_g^{(2)} = 0$ .

Note that, in case of multiple nodes maximizing the respective coefficients (for instance,  $\frac{\theta_i}{2}(1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} \right)$  while determining optimal  $\mathbf{x}^{(1)}$ ) we mentioned that one such node can be chosen at random. So, there may be optimal strategies in which the camp could invest on multiple nodes in a phase. However, since investing on one node per phase suffices to achieve the optimum, it is an optimal strategy (not the only optimal strategy) to invest on at most one node in each phase.  $\square$

Following Proposition 1, there exist optimal vectors  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  that maximize (9), such that  $x_\alpha^{(1)} = k_g^{(1)}, x_\beta^{(2)} = k_g^{(2)}, x_{i \neq \alpha}^{(1)} = x_{j \neq \beta}^{(2)} = 0$ . Now the next step is to find nodes  $\alpha$  and  $\beta$  that maximize (9). By incorporating  $\alpha$  and  $\beta$  in (9), we have  $\sum_{i \in N} v_i^{(2)}$  equals

$$\begin{aligned} \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + k_g^{(1)} \frac{\theta_\alpha}{2} (1 + c_\alpha) s_\alpha + k_g^{(2)} \frac{\theta_\beta}{2} \left( \sum_{i \in N} c_i b_{\beta i} + r_\beta \right) \\ + k_g^{(1)} k_g^{(2)} \frac{\theta_\alpha \theta_\beta}{4} (1 + c_\alpha) b_{\beta \alpha} \end{aligned} \quad (12)$$

Now, for a given pair  $(\alpha, \beta)$ , we will find the optimal values of  $k_g^{(1)}$  and  $k_g^{(2)}$  from (12). From Proposition 1, we first consider  $k_g^{(2)} = k_g - k_g^{(1)}$ . So the expression to be maximized is

$$\begin{aligned} \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + k_g^{(1)} \frac{\theta_\alpha}{2} (1 + c_\alpha) s_\alpha + (k_g - k_g^{(1)}) \frac{\theta_\beta}{2} \left( \sum_{i \in N} c_i b_{\beta i} + r_\beta \right) \\ + k_g^{(1)} (k_g - k_g^{(1)}) \frac{\theta_\alpha \theta_\beta}{4} (1 + c_\alpha) b_{\beta \alpha} \end{aligned}$$

Equating its first derivative w.r.t.  $k_g^{(1)}$  to zero, we get the candidate  $k_g^{(1)}$  for pair  $(\alpha, \beta)$  to be

$$\frac{k_g}{2} + \frac{s_\alpha}{\theta_\beta b_{\beta\alpha}} - \frac{\sum_{i \in N} b_{\beta i} c_i + r_\beta}{\theta_\alpha b_{\beta\alpha} (1 + c_\alpha)}$$

A valid value of  $k_g^{(1)}$  can be obtained only if the denominators in above expression are non-zero. However, a zero denominator would mean that Expression (12) is linear, resulting in the optimal  $k_g^{(1)}$  being either 0 or  $k_g$ . Also, if the second derivative with respect to  $k_g^{(1)}$  is positive, that is,  $-\theta_\alpha \theta_\beta (1 + c_\alpha) b_{\beta\alpha} > 0$ , optimal  $k_g^{(1)}$  is either 0 or  $k_g$ . If the second derivative with respect to  $k_g^{(1)}$  is negative:  $-\theta_\alpha \theta_\beta (1 + c_\alpha) b_{\beta\alpha} = -\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha} (w_{\alpha\alpha}^0 v_\alpha^0 + 1) < 0$ , and since  $k_g^{(1)}$  is bounded in  $[0, k_g]$ , optimal  $k_g^{(1)}$  for pair  $(\alpha, \beta)$  is (since  $c_i = w_{ii}^0 v_i^0$ ,  $b_{ji} = r_j w_{jj}^0 \Delta_{ji}$ ,  $s_i = \sum_{j \in N} b_{ji}$ ):

$$\min \left\{ \max \left\{ \frac{k_g}{2} + \frac{s_\alpha}{\theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha}} - \frac{1 + w_{\beta\beta}^0 \sum_{i \in N} \Delta_{\beta i} w_{ii}^0 v_i^0}{\theta_\alpha w_{\beta\beta}^0 \Delta_{\beta\alpha} (1 + w_{\alpha\alpha}^0 v_\alpha^0)}, 0 \right\}, k_g \right\} \quad (13)$$

and the corresponding optimal value of  $k_g^{(2)}$  for pair  $(\alpha, \beta)$  is

$$\min \left\{ \max \left\{ \frac{k_g}{2} - \frac{s_\alpha}{\theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha}} + \frac{1 + w_{\beta\beta}^0 \sum_{i \in N} \Delta_{\beta i} w_{ii}^0 v_i^0}{\theta_\alpha w_{\beta\beta}^0 \Delta_{\beta\alpha} (1 + w_{\alpha\alpha}^0 v_\alpha^0)}, 0 \right\}, k_g \right\} \quad (14)$$

When we assumed  $k_g^{(1)}$  and  $k_g^{(2)}$  to be fixed, we had to iterate through all  $(\alpha, \beta)$  pairs to determine the one that gives the optimal value of Expression (12). Now, whenever we look at an  $(\alpha, \beta)$  pair, we can determine the corresponding optimal values of  $k_g^{(1)}$  and  $k_g^{(2)}$  using (13) and (14), and hence determine the value of Expression (12) by plugging in the optimal  $k_g^{(1)}$  and  $k_g^{(2)}$  and that  $(\alpha, \beta)$  pair. The optimal pair  $(\alpha, \beta)$  can thus be obtained as the pair that maximizes (12).

Above analysis holds when  $k_g^{(1)} + k_g^{(2)} = k_g$ . From Proposition 1, we need to consider one more possibility that  $k_g^{(1)} = k_g^{(2)} = 0$ , which gives a constant value  $\sum_{i \in N} \sum_{j \in N} c_i b_{ji}$  for Expression (12). Let  $(0, 0)$  correspond to this additional

possibility. It is hence optimal to invest  $k_g^{(1)}$  (obtained using (13)) on node  $\alpha$  in the first phase and  $k_g^{(2)}$  (obtained using (14)) on node  $\beta$  in the second phase, subject to it giving a value greater than  $\sum_{i \in N} \sum_{j \in N} c_i b_{ji}$  to Expression (12).

Since we iterate through  $(n^2 + 1)$  possibilities (namely,  $(\alpha, \beta) \in N \times N \cup \{(0, 0)\}$ ), the above procedure gives a polynomial time algorithm for determining the optimal budget split and the optimal investments on nodes in two phases.

**Remark 1.** *For non-negative values of parameters, (13) indicates that for a given  $(\alpha, \beta)$  pair, the good camp would want to invest more in the first phase for a higher  $s_\alpha$ . This is intuitive from our understanding of  $s_\alpha$  being viewed as the influencing power of node  $\alpha$  looking one phase ahead. Similarly, (14) indicates that it would want to invest more in second phase for a higher  $r_\beta$ , since  $r_\beta$  can be viewed as the influencing power of node  $\beta$  in the immediate phase. Also, (13) and (14) indicate that a higher  $\theta_\alpha$  drives the camp to invest in first phase and a higher  $\theta_\beta$  drives it to invest in second phase. Since  $w_{ig}$  is an increasing function of  $\theta_i$ , this implicitly means that a node with a higher  $w_{ig}$  drives the good camp to invest in the phase in which that node is selected. Further, we illustrate the role of  $w_{ii}^0$  using simulations in Section 5.*

#### 4. The Case of Competing Camps

We now analyze the general scenario involving the two competing camps. We first prove the following result.

**Proposition 2.** *In the search space  $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$ , it is optimal for the good camp to have  $k_g^{(1)} + k_g^{(2)} = k_g$ , and to invest on at most one node in each phase. In the search space  $k_b^{(1)} + k_b^{(2)} \in (0, k_b]$ , it is optimal for the bad camp to have  $k_b^{(1)} + k_b^{(2)} = k_b$ , and to invest on at most one node in each phase.*

*Proof.* We show that  $\sum_{i \in N} v_i^{(2)}$  is a multilinear function, since it can be written as a linear function in  $\mathbf{x}^{(1)}, \mathbf{y}^{(1)}, \mathbf{x}^{(2)}, \mathbf{y}^{(2)}$  individually, as follows:

$$\begin{aligned}
\sum_{i \in N} v_i^{(2)} &= \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} \left( \sum_{i \in N} c_i b_{ji} + r_j \right) + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} \left( \sum_{i \in N} c_i b_{ji} - r_j \right) \\
&\quad + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \\
&\quad - \sum_{i \in N} y_i^{(1)} \frac{\theta_i}{2} (1 - c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \\
&= \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \\
&\quad + (\text{terms independent of } x_i^{(1)}) \\
&= - \sum_{i \in N} y_i^{(1)} \frac{\theta_i}{2} (1 - c_i) \left( s_i + \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji} \right) \\
&\quad + (\text{terms independent of } y_i^{(1)}) \\
&= \sum_{j \in N} x_j^{(2)} \frac{\theta_j}{2} \left( r_j + \sum_{i \in N} c_i b_{ji} + \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) b_{ji} - \sum_{i \in N} y_i^{(1)} \frac{\theta_i}{2} (1 - c_i) b_{ji} \right) \\
&\quad + (\text{terms independent of } x_j^{(2)}) \\
&= - \sum_{j \in N} y_j^{(2)} \frac{\theta_j}{2} \left( r_j - \sum_{i \in N} c_i b_{ji} - \sum_{i \in N} x_i^{(1)} \frac{\theta_i}{2} (1 + c_i) b_{ji} + \sum_{i \in N} y_i^{(1)} \frac{\theta_i}{2} (1 - c_i) b_{ji} \right) \\
&\quad + (\text{terms independent of } y_j^{(2)})
\end{aligned}$$

The rest of the proof follows on similar lines as Proposition 1. (Note the negative signs assigned to the coefficients  $y_i^{(1)}$  and  $y_j^{(2)}$  since the bad camp computes the values of these parameters so as to minimize  $\sum_{i \in N} v_i^{(2)}$ ).  $\square$

From Proposition 2, there exist optimal vectors  $\mathbf{x}^{(1)}$ ,  $\mathbf{x}^{(2)}$  for good camp and optimal vectors  $\mathbf{y}^{(1)}$ ,  $\mathbf{y}^{(2)}$  for bad camp, such that  $x_\alpha^{(1)} = k_g^{(1)}$ ,  $x_\beta^{(2)} = k_g^{(2)}$ ,  $y_\gamma^{(1)} = k_b^{(1)}$ ,  $y_\delta^{(2)} = k_b^{(2)}$ , and  $x_{i \neq \alpha}^{(1)} = x_{j \neq \beta}^{(2)} = y_{i \neq \gamma}^{(1)} = y_{j \neq \delta}^{(2)} = 0$ . Assuming such profile of nodes  $((\alpha, \beta), (\gamma, \delta))$ , we first find  $((x_\alpha^{(1)}, x_\beta^{(2)}), (y_\gamma^{(1)}, y_\delta^{(2)}))$ , or equivalently, the optimal  $((k_g^{(1)}, k_g^{(2)}), (k_b^{(1)}, k_b^{(2)}))$  corresponding to such a profile. By incorporating  $((\alpha, \beta), (\gamma, \delta))$ , Expression (7) for  $\sum_{i \in N} v_i^{(2)}$  simplifies to

$$\begin{aligned}
&\sum_{i \in N} \sum_{j \in N} c_i b_{ji} + k_g^{(2)} \frac{\theta_\beta}{2} \left( \sum_{i \in N} c_i b_{\beta i} + r_\beta \right) + k_b^{(2)} \frac{\theta_\delta}{2} \left( \sum_{i \in N} c_i b_{\delta i} - r_\delta \right) \\
&\quad + k_g^{(1)} \frac{\theta_\alpha}{2} (1 + c_\alpha) \left( s_\alpha + k_g^{(2)} \frac{\theta_\beta}{2} b_{\beta \alpha} + k_b^{(2)} \frac{\theta_\delta}{2} b_{\delta \alpha} \right) \\
&\quad - k_b^{(1)} \frac{\theta_\gamma}{2} (1 - c_\gamma) \left( s_\gamma + k_g^{(2)} \frac{\theta_\beta}{2} b_{\beta \gamma} + k_b^{(2)} \frac{\theta_\delta}{2} b_{\delta \gamma} \right) \tag{15}
\end{aligned}$$

First, we consider the case when  $k_g^{(1)} + k_g^{(2)} = k_g$  and  $k_b^{(1)} + k_b^{(2)} = k_b$ . Now, for a given profile of nodes  $((\alpha, \beta), (\gamma, \delta))$ , we will find the optimal values of  $k_g^{(1)}, k_g^{(2)}, k_b^{(1)}, k_b^{(2)}$ . In this case, we have  $k_g^{(2)} = k_g - k_g^{(1)}$  and  $k_b^{(2)} = k_b - k_b^{(1)}$ .

Substituting this in (15) and equating  $\frac{\partial \sum_{i \in N} v_i^{(2)}}{\partial k_g^{(1)}} = 0$ , we get

$$-\frac{\theta_\beta}{2} \left( \sum_{i \in N} c_i b_{\beta i} + r_\beta \right) + \frac{\theta_\alpha}{2} (1 + c_\alpha) \left( s_\alpha + (k_g - 2k_g^{(1)}) \frac{\theta_\beta}{2} b_{\beta\alpha} + (k_b - k_b^{(1)}) \frac{\theta_\delta}{2} b_{\delta\alpha} \right) + k_b^{(1)} \frac{\theta_\gamma}{2} (1 - c_\gamma) \frac{\theta_\beta}{2} b_{\beta\alpha} = 0$$

Let  $A = \theta_\gamma \theta_\delta (1 - c_\gamma) b_{\delta\gamma}$ ,  $D = \theta_\alpha \theta_\beta (1 + c_\alpha) b_{\beta\alpha}$ , and

$B = \frac{1}{2} (\theta_\alpha \theta_\delta (1 + c_\alpha) b_{\delta\alpha} - \theta_\gamma \theta_\beta (1 - c_\gamma) b_{\beta\gamma})$ . So, the above can be written as

$$Dk_g^{(1)} + Bk_b^{(1)} = -\theta_\beta \left( \sum_{i \in N} c_i b_{\beta i} + r_\beta \right) + \theta_\alpha (1 + c_\alpha) \left( s_\alpha + k_g \frac{\theta_\beta}{2} b_{\beta\alpha} + k_b \frac{\theta_\delta}{2} b_{\delta\alpha} \right) \quad (16)$$

Similarly, equating  $\frac{\partial \sum_{i \in N} v_i^{(2)}}{\partial k_b^{(1)}} = 0$ , we get

$$Bk_g^{(1)} - Ak_b^{(1)} = -\theta_\delta \left( \sum_{i \in N} c_i b_{\delta i} - r_\delta \right) - \theta_\gamma (1 - c_\gamma) \left( s_\gamma + k_g \frac{\theta_\beta}{2} b_{\beta\gamma} + k_b \frac{\theta_\delta}{2} b_{\delta\gamma} \right) \quad (17)$$

Solving Equations (16) and (17) simultaneously, we obtain the candidate  $k_g^{(1)}$  for profile  $((\alpha, \beta), (\gamma, \delta))$  to be

$$\begin{aligned} \frac{1}{B^2 + DA} & \left[ s_\alpha \theta_\alpha (1 + c_\alpha) A - r_\beta \theta_\beta A - s_\gamma \theta_\gamma (1 - c_\gamma) B + r_\delta \theta_\delta B \right. \\ & + k_g \left( \frac{\theta_\alpha \theta_\beta}{2} (1 + c_\alpha) b_{\beta\alpha} A - \frac{\theta_\gamma \theta_\beta}{2} (1 - c_\gamma) b_{\beta\gamma} B \right) \\ & + k_b \left( \frac{\theta_\alpha \theta_\delta}{2} (1 + c_\alpha) b_{\delta\alpha} A - \frac{\theta_\gamma \theta_\delta}{2} (1 - c_\gamma) b_{\delta\gamma} B \right) \\ & \left. - \theta_\beta A \sum_{i \in N} c_i b_{\beta i} - \theta_\delta B \sum_{i \in N} c_i b_{\delta i} \right] \end{aligned}$$

We can similarly obtain the candidate  $k_b^{(1)}$  for profile  $((\alpha, \beta), (\gamma, \delta))$  to be

$$\begin{aligned} \frac{1}{B^2 + DA} & \left[ s_\alpha \theta_\alpha (1 + c_\alpha) B - r_\beta \theta_\beta B + s_\gamma \theta_\gamma (1 - c_\gamma) D - r_\delta \theta_\delta D \right. \\ & + k_g \left( \frac{\theta_\alpha \theta_\beta}{2} (1 + c_\alpha) b_{\beta\alpha} B + \frac{\theta_\gamma \theta_\beta}{2} (1 - c_\gamma) b_{\beta\gamma} D \right) \\ & + k_b \left( \frac{\theta_\alpha \theta_\delta}{2} (1 + c_\alpha) b_{\delta\alpha} B + \frac{\theta_\gamma \theta_\delta}{2} (1 - c_\gamma) b_{\delta\gamma} D \right) \\ & \left. - \theta_\beta B \sum_{i \in N} c_i b_{\beta i} + \theta_\delta D \sum_{i \in N} c_i b_{\delta i} \right] \end{aligned}$$

If second derivative w.r.t.  $k_g^{(1)}$ , that is,  $-\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha} (1 + w_{\alpha\alpha}^0 v_\alpha^0) < 0$  and that w.r.t.  $k_b^{(1)}$ , that is,  $\theta_\gamma \theta_\delta r_\delta w_{\delta\delta}^0 \Delta_{\delta\gamma} (1 - w_{\gamma\gamma}^0 v_\gamma^0) > 0$ , and the obtained solution is such that  $k_g^{(1)} \in [0, k_g]$  and  $k_b^{(1)} \in [0, k_b]$ , then neither the good camp can change  $k_g^{(1)}$  to increase  $\sum_{i \in N} v_i^{(2)}$ , nor the bad camp can change  $k_b^{(1)}$  to decrease  $\sum_{i \in N} v_i^{(2)}$ . So we can effectively write  $u_g((\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}))$  as  $u_g((\alpha, \beta), (\gamma, \delta))$ , where  $u_g((\alpha, \beta), (\gamma, \delta))$  is the value of  $\sum_{i \in N} v_i^{(2)}$ , which corresponds to the strategy profile where good camp invests on nodes  $(\alpha, \beta)$  with optimal budget split  $(k_g^{(1)}, k_g^{(2)})$ , and bad camp invests on nodes  $(\gamma, \delta)$  with optimal budget split  $(k_b^{(1)}, k_b^{(2)})$ .

For general case where the solution  $k_g^{(1)}, k_b^{(1)}$  obtained above may not satisfy  $k_g^{(1)} \in [0, k_g]$  and  $k_b^{(1)} \in [0, k_b]$ , we make a practically reasonable assumption so as to determine  $u_g((\alpha, \beta), (\gamma, \delta))$ . It is easy to show that if  $w_{ij} \geq 0, \forall (i, j)$ , then  $\Delta_{ij} \geq 0, \forall (i, j)$  and  $r_i \geq 1, \forall i \in N$ . So if we assume  $w_{ij} \geq 0, \forall (i, j)$  and  $w_{ii}^0 \geq 0, \theta_i \geq 0, v_i^0 \in [-1, 1], \forall i \in N$ , we would have that  $-\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha} (1 + w_{\alpha\alpha}^0 v_\alpha^0) \leq 0$  and  $\theta_\gamma \theta_\delta r_\delta w_{\delta\delta}^0 \Delta_{\delta\gamma} (1 - w_{\gamma\gamma}^0 v_\gamma^0) \geq 0$ . That is, we would have  $\sum_{i \in N} v_i^{(2)}$  to be a convex-concave function, which is concave w.r.t.  $k_g^{(1)}$  and convex w.r.t.  $k_b^{(1)}$ . So in the domain  $([0, k_g], [0, k_b])$ , we can find a  $(k_g^{(1)}, k_b^{(1)})$  such that, neither the good camp can change  $k_g^{(1)}$  to increase  $\sum_{i \in N} v_i^{(2)}$ , nor the bad camp can change  $k_b^{(1)}$  to decrease  $\sum_{i \in N} v_i^{(2)}$  [42, 43]. So we can assign this value  $\sum_{i \in N} v_i^{(2)}$  to  $u_g((\alpha, \beta), (\gamma, \delta))$ .

Thus using above technique, we obtain  $u_g((\alpha, \beta), (\gamma, \delta))$  for all profiles of nodes  $((\alpha, \beta), (\gamma, \delta))$  when  $k_g^{(1)} + k_g^{(2)} = k_g$  and  $k_b^{(1)} + k_b^{(2)} = k_b$ . From Proposition 2, the only other cases to consider are  $k_b^{(1)} = k_b^{(2)} = 0$  and  $k_g^{(1)} = k_g^{(2)} = 0$ . Let the profile  $((\alpha, \beta), (0, 0))$  correspond to  $k_b^{(1)} = k_b^{(2)} = 0$ . Note that when  $k_b^{(1)} = k_b^{(2)} = 0$ , it reduces to non-competitive case with only good camp (Section 3); the value of  $\sum_{i \in N} v_i^{(2)}$  for an  $(\alpha, \beta)$  pair can hence be assigned to  $u_g((\alpha, \beta), (0, 0))$ . Thus we can obtain  $u_g((\alpha, \beta), (\gamma, \delta))$  for all profiles of nodes  $((\alpha, \beta), (0, 0))$ . Similarly, we can obtain  $u_g((\alpha, \beta), (\gamma, \delta))$  for all profiles of nodes  $((0, 0), (\gamma, \delta))$ . And from Equation (15),  $u_g((0, 0), (0, 0)) = \sum_{i \in N} \sum_{j \in N} c_i b_{ji}$ .

So we have that the good camp has  $(n^2 + 1)$  possible pure strategies to choose

from, namely,  $(\alpha, \beta) \in N \times N \cup \{(0, 0)\}$ . Similarly, the bad camp has  $(n^2 + 1)$  possible pure strategies to choose from, namely,  $(\gamma, \delta) \in N \times N \cup \{(0, 0)\}$ . We thus have a two-player zero-sum game, for which the utilities of the players can be computed for each strategy profile  $((\alpha, \beta), (\gamma, \delta))$  as explained above. Though we cannot ensure the existence of a pure strategy Nash equilibrium, the finiteness of the number of strategies ensures the existence of a mixed strategy Nash equilibrium. Further, owing to it being a two-player zero-sum game, the Nash equilibrium can be found efficiently by solving a linear program [44].

Summarizing, under practically reasonable assumptions ( $w_{ij} \geq 0, \forall(i, j)$  and  $w_{ii}^0 \geq 0, \theta_i \geq 0, v_i^0 \in [-1, 1], \forall i \in N$ ), we transformed the problem into a two-player zero-sum game with each player having  $(n^2 + 1)$  pure strategies, and showed how the players' utilities can be computed for each strategy profile. We thus deduced the existence of Nash equilibrium and that it can be found efficiently using linear programming.

## 5. Simulations and Results

For determining implications of our analytical results on real-world network datasets, we conducted simulations on NetHEPT dataset (an academic collaboration network obtained from co-authorships in the ‘‘High Energy Physics - Theory’’ papers published on the e-print arXiv from 1991 to 2003) consisting of 15,233 nodes and 31,376 edges. It is widely used for experimental justifications in the literature on opinion adoption [11, 45, 46]. For the purpose of graphical illustration as well as running the computationally intensive algorithm for determining Nash equilibrium for the case of competing camps, we use the popular Zachary’s Karate club dataset consisting of 34 nodes and 78 edges [47]. In all of our simulations, we assume the value of  $w_{ii}^0$  to be the same for all nodes, in order to systematically study the effect of this value.

### 5.1. Simulation Results: The Non-Competitive (Single Camp) Case

For the single camp case, we consider  $v_i^0 = 0, \forall i \in N$  (unless specified otherwise) to start with a neutral network. This would help us to reliably study the effects of critical parameters  $w_{ii}^0, r_i$ , and hence also  $s_j$ .

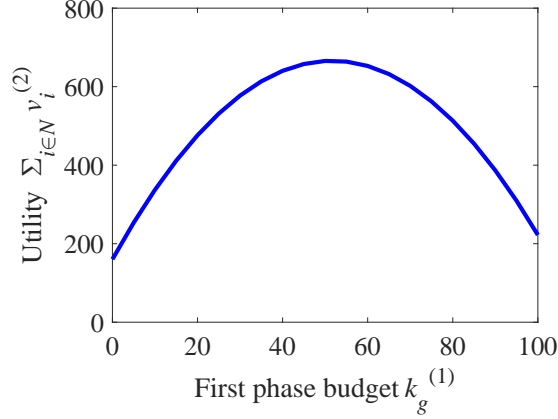


Figure 1: The effect of different budget splits on the good camp’s utility (NetHEPT) with  $k_g = 100$  for  $w_{ii}^0 = 0.5, \forall i \in N$  ( $v_i^0 = 0, \forall i \in N$ )

Figure 1 shows how the good camp’s utility  $\sum_{i \in N} v_i^{(2)}$  changes as it increases the budget allotment for the first phase (while decreasing the budget allotment for the second phase) for NetHEPT dataset with  $k_g = 100$  for  $w_{ii}^0 = 0.5, \forall i \in N$ . A natural tradeoff is observed in the plot since investing heavily in either phase results in a low utility; the optimal budget allocation for the first phase is thus an intermediate value. The tradeoff arises since a lower investment in the first phase results in worse initial biases for the second phase in the network, thus resulting a poorer opinion values to start with in the second phase, and also lower effectiveness of its investment in the second phase. On the other hand, a higher investment in the first phase spares a lower available budget for second phase, resulting in the camp being unable to fully harness the influenced biases.

### 5.1.1. The effect of $w_{ii}^0$

Figure 2 presents the optimal budget that should be allotted for the first phase as a function of  $w_{ii}^0$ . In our simulations, the optimal values obtained are such that  $k_g^{(2)} = k_g - k_g^{(1)}$ . For low values of  $w_{ii}^0$ , the optimal strategy is to invest almost entirely in the second phase. This is because the effect of the first phase diminishes in the second phase when  $w_{ii}^0$  is low. Remark 1 states that a high  $s_j$  value (influencing power of  $j$  looking one phase ahead) would attract high investment in the first phase. The value  $s_j = \sum_{i \in N} r_i w_{ii}^0 \Delta_{ij}$  would be



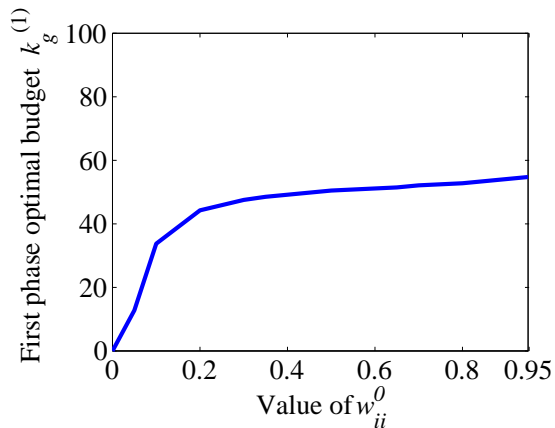


Figure 2: The effect of  $w_{ii}^0$  (NetHEPT) with  $k_g = 100$  ( $v_i^0 = 0, \forall i \in N$ )

significant only if  $j$  influences nodes  $i$  with significant values of  $w_{ii}^0$ . With low  $w_{ii}^0$ , we are less likely to have node  $j$  with high value of  $s_j$  since it requires it to be influential towards significant number of nodes  $i$  with significant values of  $w_{ii}^0$ . Hence, allotting a significant budget for the first phase is advantageous only if we have nodes with significant value of  $w_{ii}^0$ .

### 5.1.2. The effect of $v_i^0$

To supplement the above observation, we observed the effects of  $v_i^0 \neq 0$  (initially biased network). These plots did not differ noticeably from the one in Figure 2, even for higher magnitudes of  $v_i^0$ . There are subtle differences, however. If  $v_i^0$ 's are positive, the good camp invests less in the first phase. This is because  $v_i^0$ 's already give a head start for a healthy value at the end of the first phase (which is bias for the second phase), and so the budget could rather be invested in the second phase. On the other hand, if  $v_i^0$ 's are negative, it invests more in the first phase so as to nullify the initial disadvantage.

### 5.1.3. The effect of camp being myopic

We generally consider that the camp is *farsighted*, that is, it computes its strategy considering that there would be a second phase, that is, it considers the objective function  $\sum_{i \in N} v_i^{(2)}$ . A camp can be called *myopic* if it computes its strategy greedily by considering the near-sighted objective function  $\sum_{i \in N} v_i^{(1)}$ .

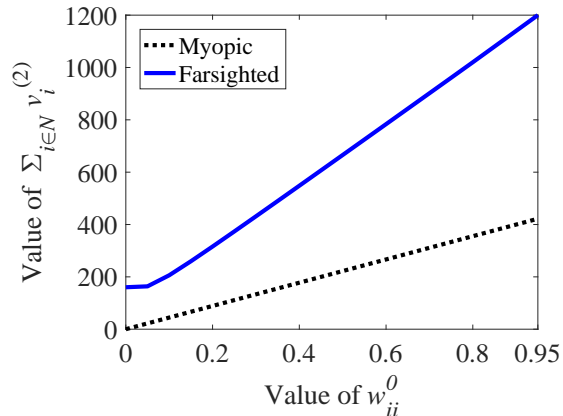


Figure 3: The effect of using myopic investment strategy for different values of  $w_{ii}^0$  (NetHEPT) with  $k_g = 100$  ( $v_i^0 = 0, \forall i \in N$ )

In other words, when the camp is myopic, it perceives its utility as  $\sum_i v_i^{(1)}$  and devises its strategy to invest greedily in the first phase, even though its actual utility is  $\sum_i v_i^{(2)}$ .

Figure 3 shows the effect of the good camp using myopic strategy (investing the entire budget in the first phase) over the wide range of  $w_{ii}^0$  values for NetHEPT dataset with  $k_g = 100$ . Note that when  $w_{ii}^0 = 0$ , the myopic strategy results in zero utility as it invests its entire budget in the first phase (which plays no role when  $w_{ii}^0 = 0$ ), while the farsighted strategy suggests the camp to invest its entire budget in the second phase (thus effectively leading to a single phase opinion dynamics). The plot also suggests that the loss incurred by playing myopic strategy could be considerably high for high values of  $w_{ii}^0$ . This emphasizes that, though high  $w_{ii}^0$ 's are suitable for high investments in the first phase so as to influence the biases for the second phase (as seen in Figure 2), it is important to spare a certain fraction of the budget so as to be invested in the second phase in order to harness the influenced biases.

#### 5.1.4. Phasewise progression of opinion values

In order to illustrate the phasewise progression of opinion values of nodes, we use the small-sized Zachary's Karate club dataset for visualization. Figure 4 shows the computed values of  $s_i$  and  $r_i$  for the nodes in the dataset. The size and

color saturation of a node  $i$  represent the value of the corresponding parameter (bigger size and higher saturation implies higher value). For this dataset, the investment was made on a single node common to both the phases. This node visibly stands out in Figures 5 and 6 with its size and color saturation; we refer to this as the prime node in our discussion.

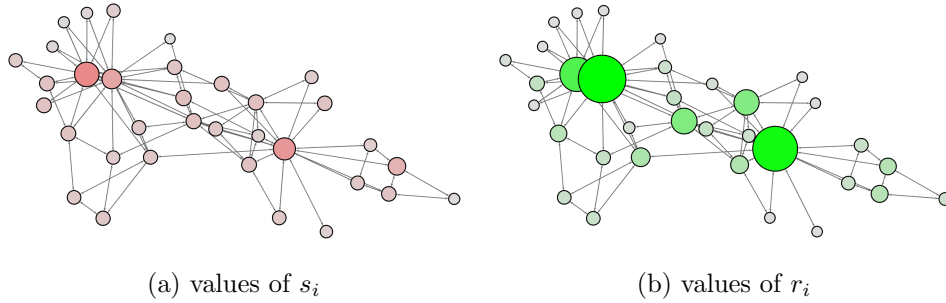


Figure 4: Nodes' relative values of  $s_i$  and  $r_i$

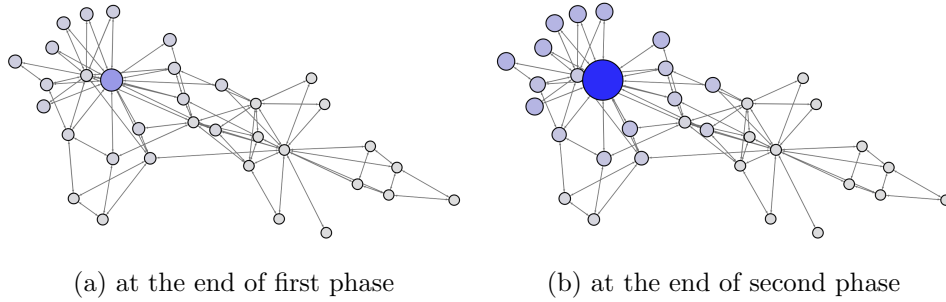


Figure 5: Illustration of phasewise progression of opinion values when  $w_{ii}^0 = 0.5$  (Karate) with  $k_g = 5$  ( $v_i^0 = 0, \forall i \in N$ )

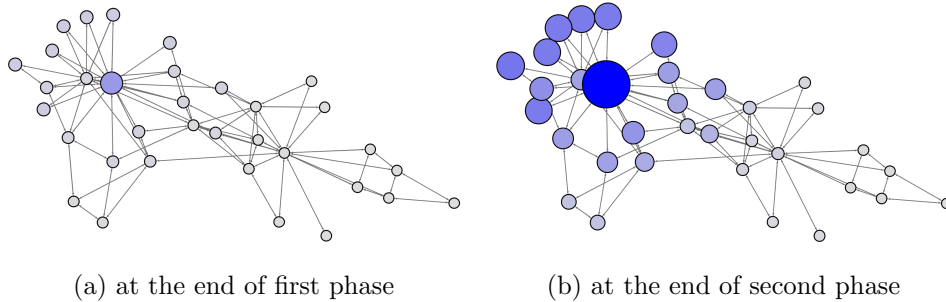


Figure 6: Illustration of phasewise progression of opinion values when  $w_{ii}^0 = 0.9$  (Karate) with  $k_g = 5$  ( $v_i^0 = 0, \forall i \in N$ )

In Figures 5 and 6, the size and color saturation of a node  $i$  represent its opinion value (bigger size and higher saturation implies higher opinion value). It can be seen that a higher value of  $w_{ii}^0$  results in a significant change in opinion values in the second phase. This is owing to the fact that the good camp's investment is more effective in the second phase when nodes attribute higher weightage to their initial biases in the second phase, or equivalently, their opinions at the end of the first phase (Equation (1)) (assuming positive opinion values which is the case here).

## 5.2. Simulation Results: The Case of Competing Camps

Though we presented a polynomial time algorithm for determining Nash equilibrium when there are two camps, which is of theoretical interest, it is computationally expensive to run it on larger networks. Hence, for the purpose of studying the case of competing camps, we consider the Zachary's Karate club dataset (34 nodes, 78 edges) [47].

Note that for  $v_i^0 = 0$ , the two-phase investment game turns out to be symmetric, since both camps would have the same effectiveness in the first phase, that is,  $w_{ig}^{(1)} = w_{ib}^{(1)}, \forall i \in N$ . The other parameters such as  $w_{ij}$  and  $w_{ii}^0$ , and hence  $s_i$  and  $r_i$ , are common to both the camps. So, the Nash equilibrium strategy played by the camps is symmetric. Owing to the game being zero-sum, this results in both camps receiving zero utility, that is,  $\sum_{i \in N} v_i^{(2)} = 0$ .

### 5.2.1. The effect of $w_{ii}^0$

Since  $v_i = 0$  results in trivial observations, we study the effect of  $w_{ii}^0$ , while first considering  $v_i = +0.1$ . In general, we observed that high values of  $w_{ii}^0$  resulted in a pure strategy Nash equilibrium in which the camps invested their entire budget on the prime node mentioned in Section 5.1.4. For low and intermediate values of  $w_{ii}^0$ , we computed mixed strategy Nash equilibrium, describing the probability with which the good camp would invest on the pair  $(\alpha, \beta)$  (node  $\alpha$  in the first phase and node  $\beta$  in the second phase) and the probability with which the bad camp would invest on the pair  $(\gamma, \delta)$ , with the corresponding

saddle point budget splits as derived in Section 4. Hence, in order to study the budget allotted to the first phase for a given value of  $w_{ii}^0$ , we compute the expectation (weighted average where the weights correspond to the probabilities) of the first phase investments corresponding to the aforementioned pairs of nodes.

Formally, in mixed strategy Nash equilibrium, let the good camp play its strategy  $(\alpha, \beta)$  with probability  $\mathbb{P}_g(\alpha, \beta)$  and the bad camp play  $(\gamma, \delta)$  with probability  $\mathbb{P}_b(\gamma, \delta)$ . Let the corresponding saddle point suggest the good camp to invest  $k_g^{(1)}((\alpha, \beta), (\gamma, \delta))$  on node  $\alpha$ , and the bad camp to invest  $k_b^{(1)}((\alpha, \beta), (\gamma, \delta))$  on node  $\gamma$ , in the first phase. So the expected (weighted average) first phase investment by the good camp is  $\sum_{((\alpha, \beta), (\gamma, \delta))} \mathbb{P}_g(\alpha, \beta) \cdot \mathbb{P}_b(\gamma, \delta) \cdot k_g^{(1)}((\alpha, \beta), (\gamma, \delta))$  and that by the bad camp is  $\sum_{((\alpha, \beta), (\gamma, \delta))} \mathbb{P}_g(\alpha, \beta) \cdot \mathbb{P}_b(\gamma, \delta) \cdot k_b^{(1)}((\alpha, \beta), (\gamma, \delta))$ .

For various values of  $w_{ii}^0$  in general, the camps invested significantly on the prime node (mentioned in Section 5.1.4) with high probability. Investments were also made with considerable probability on the node with the highest value of  $s_i$  in the first phase and the second highest value of  $r_i$  in the second phase, in Figure 4. Certain other nodes with good enough values of  $s_i$  and  $r_i$  were invested on with low non-zero probabilities.

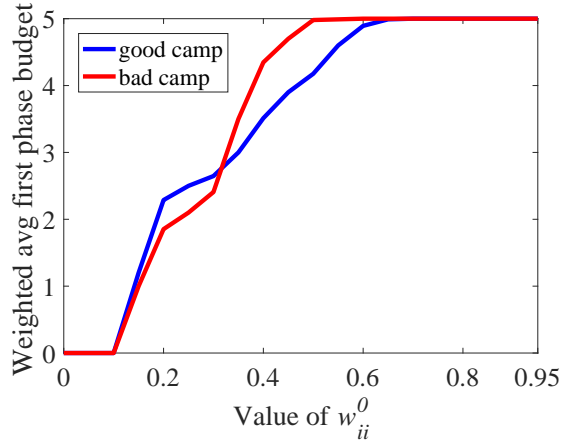


Figure 7: The effect of  $w_{ii}^0$  (Karate) with  $k_g = k_b = 5$  ( $v_i^0 = +0.1, \forall i \in N$ )

Figure 7 presents the weighted average (expected) investments in the first phase by the two camps in the mixed strategy Nash equilibrium, corresponding

to different values of  $w_{ii}^0$ . Similar to the single camp case, the trend is similar in that, the budget allotted to the first phase increases with  $w_{ii}^0$ . A high  $w_{ii}^0$  encourages the camps to influence the initial biases for the second phase, which not only plays a key role in determining the final opinion owing to the high weightage attributed to the biases in the second phase (which are same as the opinions at the end of the first phase), but also enhances the effectiveness of the camps' investments in the second phase.

### 5.2.2. The effect of $v_i^0$

Recall from Section 4 that  $u_g((\alpha, \beta), (\gamma, \delta))$  is the good camp's utility when it plays the pure strategy  $(\alpha, \beta)$  and the bad camp plays  $(\gamma, \delta)$  with the corresponding saddle point budget splits. Following our above discussion on the probabilities in mixed strategy Nash equilibrium, the value of  $\sum_{i \in N} v_i^{(2)}$ , quantifying the expected utility of the good camp in mixed strategy Nash equilibrium, is  $\sum_{((\alpha, \beta), (\gamma, \delta))} \mathbb{P}_g(\alpha, \beta) \cdot \mathbb{P}_b(\gamma, \delta) \cdot u_g((\alpha, \beta), (\gamma, \delta))$ .

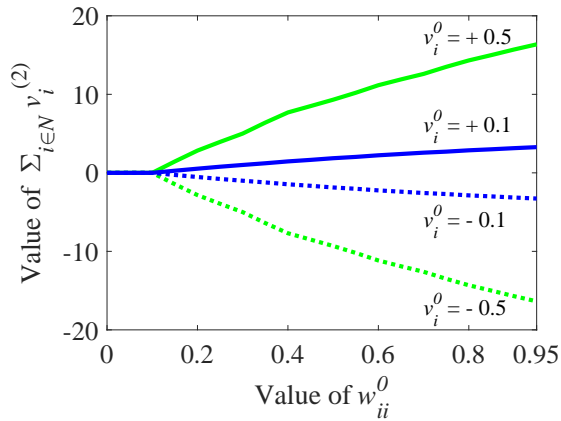


Figure 8: The effect of  $w_{ii}^0$  for different values of  $v_i^0$  (Karate) with  $k_g = k_b = 5$

Figures 8 and 9 present the effect of  $v_i^0$  in conjunction with  $w_{ii}^0$ , on the value of  $\sum_{i \in N} v_i^{(2)}$ . We observed that a higher value of initial bias in the first phase ( $v_i^0$ ) results in a better utility for the good camp (and hence worse utility for the bad camp). This is not only because of a head start offered to the good camp, but also because of its investment being more effective. This is further

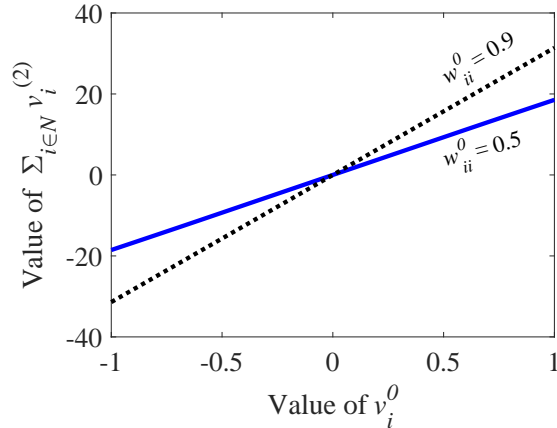


Figure 9: The effect of  $v_i^0$  for different values of  $w_{ii}^0$  (Karate) with  $k_g = k_b = 5$

magnified for higher  $w_{ii}^0$ , that is, if nodes attribute higher weightage to their biases (similar to Section 5.1.4 where a higher  $w_{ii}^0$  results in a more significant change in opinion values).

It is also noteworthy to see that, if we change the sign of  $v_i^0$  without changing its magnitude, the value of  $\sum_{i \in N} v_i^{(2)}$  also changes its sign while maintaining its magnitude. This is owing to the fact that, changing the sign of  $v_i^0$  interchanges the two camps' roles, that is, the values of  $w_{ig}^{(1)}$  and  $w_{ib}^{(1)}$  (the effectiveness of their investments in the first phase) get interchanged (while other parameters such as  $w_{ij}$  and  $w_{ii}^0$  are common to both camps).

### 5.2.3. The effect of a camp deviating from Nash equilibrium strategy

We study the loss incurred by a camp if it deviates from its Nash equilibrium strategy, to: (a) a myopic strategy (investing by perceiving its utility function as  $\sum_{i \in N} v_i^{(1)}$ ), or (b) the farsighted single camp strategy (investing by perceiving its utility function as  $\sum_{i \in N} v_i^{(2)}$  but ignoring the presence of the competing camp). In our simulations, we consider the good camp deviating from its equilibrium strategy.

Note that if the good camp decides to play myopic strategy, it perceives its utility as per Equation (4), where its optimal strategy is independent of the strategy played by the bad camp. In Figure 10, we observe that the loss incurred

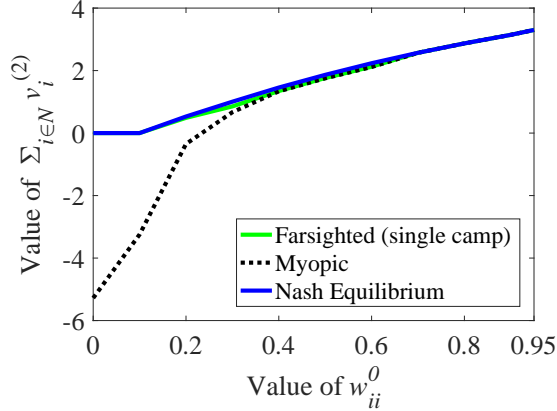


Figure 10: The effect of the good camp deviating from its Nash equilibrium strategy (Karate) with  $k_g = k_b = 5$  ( $v_i^0 = +0.1, \forall i \in N$ )

by playing myopic strategy is significant for low values of  $w_{ii}^0$ , in which range, it is actually optimal to invest most of the budget in the second phase. There is no loss incurred for high values of  $w_{ii}^0$ , since in this range, its equilibrium strategy is to invest the entire budget in the first phase (Figure 7), which also coincides with the myopic strategy (the nodes selected for investing on in both strategies are the same for high  $w_{ii}^0$ ).

The loss incurred by playing the single camp farsighted strategy is observed to be relatively insignificant (the values of  $\sum_{i \in N} v_i^{(2)}$  corresponding to it almost coincide with the equilibrium values in Figure 10). However, it is likely to be a property of the small-sized Karate club network, rather than a generalizable observation. It would be interesting to conduct a study on larger networks, for which more efficient algorithms need to be developed for determining Nash equilibrium.

## 6. Conclusion

Using Friedkin-Johnsen model of opinion dynamics, we proposed a framework for two-phase investment on nodes in a social network, where a node's final opinion in the first phase acts as its bias in the second phase, and the effectiveness of a camp's investment on that node in the second phase depends on



this bias. We hence formulated a two-phase investment game, where the camps' utilities involved a decision parameter which can be interpreted as two-phase Katz centrality.

For the case when there is one investing camp, we derived polynomial time algorithm for determining an optimal way to split the budget between the two phases. We observed a natural tradeoff, since a lower investment in the first phase results in worse initial biases for the second phase, while a higher investment in the first phase spares a lower available budget for the second phase. Our simulations quantified the impact of the weightage that nodes attribute to their biases. A high weightage necessitated high investment in the first phase, so as to effectively influence the biases to be harnessed in the second phase. We also showed the loss incurred by a camp when it uses a myopic strategy instead of the derived optimal one, thus highlighting the importance of an optimal budget split. We also illustrated the phasewise progression of opinion values, and hence observed the significant change in opinion values in the second phase when nodes attribute high weightage to their biases.

For the case when there are two competing camps, we showed existence of Nash equilibrium and its polynomial time computability under reasonable assumptions ( $w_{ij} \geq 0, \forall(i, j)$  and  $w_{ii}^0 \geq 0, \theta_i \geq 0, v_i^0 \in [-1, 1], \forall i \in N$ ). Using simulations, we computed the mixed strategy Nash equilibrium investments of the camps, and hence observed the expectation of the first phase investments with respect to the weightage attributed by nodes to their biases. A similar trend was seen as the single camp case, that a high weightage necessitates high investment in the first phase. We also observed that a higher value of initial bias in the first phase results in a better utility for the good camp (and hence worse utility for the bad camp), not only because of a head start but also because of its investment being more effective. This is further magnified if nodes attribute higher weightage to their biases. We concluded by showing the loss incurred by a camp if it deviated from its Nash equilibrium strategy.

### *Future Work*

This work has several interesting directions for future work, of which we mention a few. The two camps setting can be extended to multiple camps where each camp would attempt to drive the opinion of the population towards its own. Also, the two phase study can be generalized to multiple phases to see if any additional insights or benefits can be obtained. The two-phase investment game can be studied under other models of opinion dynamics and investigate if it is possible to arrive at closed form expressions under them. It would be interesting to study the problem with bounds on investment on each node by the two camps (such as  $\forall i, x_i + y_i \leq 1$ ). It is also worth exploring the possibility of more efficient algorithms for determining the Nash equilibrium in the competitive case.

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