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To cite this version:

Hongyang Dong, Qinglei Hu, Maruthi Akella, Frédéric Mazenc. Partial Lyapunov Strictification: Dual Quaternion based Observer for 6-DOF Tracking Control. IEEE Transactions on Control Systems Technology, 2018, 14, pp.1-17. 10.1109/tcst.2018.2864723 . hal-01918284v1

HAL Id: hal-01918284

https://inria.hal.science/hal-01918284v1

Submitted on 14 Nov 2018 (v1), last revised 1 Nov 2019 (v2)

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Partial Lyapunov Strictification: Dual Quaternion Based Observer for 6-DOF Tracking Control

Hongyang Dong, Qinglei Hu, *Member, IEEE,*
Maruthi R. Akella, *Senior Member, IEEE,* and Frédéric Mazenc, *Senior Member, IEEE*

***Abstract*—Based on the dual-quaternion description, a smooth six-degree-of-freedom observer is proposed to estimate the incorporating linear and angular velocity, called dual angular velocity, for a rigid body. To establish the observer, some novel properties of dual vector and dual quaternion are established, additionally, the kinematics of dual transformation matrix is deduced, and the transition relationship between dual quaternion and dual transformation matrix is subsequently analyzed. An important feature of the observer is that all estimated states are ensured to be $\mathcal{C}^\infty$ continuous, and estimation errors are shown to exhibit asymptotic convergence. Furthermore, to achieve tracking control objective, the proposed observer is combined with an independently designed proportional-derivative-like feedback control law (using full-state feedback), and a special Lyapunov "strictification" process is employed to ensure a separation property between the observer and the controller, which further guarantees almost global asymptotic stability of the closed-loop dynamics. Numerical simulation results for a prototypical spacecraft hovering mission application are presented to illustrate the effectiveness and robustness of the proposed method.**

***Index Terms*—Dual quaternion, Observer, Lyapunov strictification, 6-DOF control, Tracking control.**

I. Introduction

FOR various motion control problems of practical importance, such as on-orbit missions of spacecraft (including monitoring, surveillance, refueling, on-orbit assembly), a pursuer/follower is often required to track both the time-varying relative position and the reference attitude trajectories accurately and synchronously with respect to a target/chief (i.e., six-degree-of-freedom (6-DOF) tracking or pose tracking). Beard et al. [1] proposed a coordination architecture for spacecraft formation tracking control problem, in which leader-following strategy and virtual-structure approaches are introduced. Ref. [2] presented a 6-DOF synchronization scheme for spacecraft formations for deep space mission applications. Ref. [3] introduced backstepping and passivity control methods for 6-DOF tracking problem, and Lv et al. [4] addressed input constraint problem under the similar background.

H. Dong is with the Department of Control Science and Engineering, Harbin Institute of Technology, Harbin, 150001, China. E-mail: donghongyang91@gmail.com.

Q. Hu is with the School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, China. E-mail: huql_buaa@buaa.edu.cn.

M. R. Akella is with the Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, Texas 78751, USA. Email: makella@mail.utexas.edu.

F. Mazenc is with the Equipe-Project INRIA DISCO, Laboratory of Signal and Systems, 3 rue Joliot-Curie, 91192 Gif sur Yvette cedex, France. E-mail: Frederic.MAZENC@lss.supelec.fr.

Six-degree-of freedom spacecraft controllers based on dual-number/dual-quaternion description have recently drawn significant attention in the literature. Qualitatively speaking, dual quaternion is the extension of traditional Euler quaternion, but different in the sense that it can be used to describe not only the rotational motion of the rigid body, but also the translational motion synchronously. Wang et al. [5], [6] provided detailed discussion of a dual-quaternion-based modeling process for the integrated rotational and translational kinematics and dynamics of rigid body, and they also proposed several finite-time control laws by using sliding mode methods. Based on the dual quaternion formulation, Filipe [7] presented a robust control method with additional mass and inertia identification mechanisms. Refs. [8] and [9] further introduced several fault-tolerant control schemes for 6-DOF tracking operations of spacecraft in the presence of actuator faults.

From a practical standpoint, due to the slow sampling rates and intrinsic noise characteristics of sensors, and also possible strict constraints on the cost and space (volume) available for sensors installation in case of cube/nano-satellites, reliable linear and angular velocity measurements may not always be feasible/available. For attitude tracking control problems, when angular velocities are unavailable, the passivity property of rigid-body attitude dynamics allows for control objective to still be accomplished using only output (dynamic) feedback [10], [11], [12]. Another widely-studied method to solve this problem is designing observers to estimate angular velocities and establishing proper observer-controller architectures to achieve tracking control goals [13], [14], [15]. To our best knowledge, the first nonlinear angular velocity observer for rigid body under quaternion kinematics description was proposed by Salcudean [16]. Zou derived distributed [17] and finite-time stable [18] observers for the attitude tracking control problem. But because the fact that separation property is usually difficult to establish for nonlinear systems, stability analyses of closed-loop systems with observer-controller architectures always involved nontrivial theoretical complexity. Nicosia et al. [19] presented a nonlinear observer and guaranteed asymptotically bounded stability for closed-loop system. Recently, based on a switching logic, the separation property of attitude tracking system with a nonlinear observer and an independently designed proportional-derivative (PD) controller was synthesized in Ref. [20], and the closed-loop system was rendered to be almost globally stable. Ref. [21] further extended this result and guaranteed $\mathcal{C}^\infty$ continuity for all estimated states by the way of circumventing the need for switching within the angular velocity observer structure.

However, when concerning the observer design for 6-DOF tracking control problems without both linear and angular velocities measurements, relevant studies are relatively limited. Based on vectrix formalism, a 6-DOF output feedback control law was introduced in Ref. [22], and the closed-loop dynamics was rendered to be locally asymptotically stable. Ref. [23] combined the passivity control method with dual-quaternion-based dynamics, and guaranteed almost globally asymptotic stability.

In this paper, significantly building upon former results given in Refs. [20] and [21], a novel 6-DOF observer with dual-quaternion-based description is proposed to estimate the incorporating linear and angular velocity (i.e., dual angular velocity) of a rigid body. Compared to Refs. [20] and [21], in which only 3-DOF attitude tracking problems are considered, the dual-quaternion parameterization and also the intrinsic coupling (which is considered in dual-quaternion-based motion model) between orientational motion and translational motion fundamentally increase the difficulty of observer design in this paper. To establish the observer, some important mathematical properties of dual vector, dual quaternion and dual transformation matrix are presented and proved, the kinematics of dual transformation matrix and the transition relationship between dual quaternion and dual transformation matrix are subsequently deduced. To our best knowledge, these results are formally derived in this paper for the first time, and help provide important building blocks for both the observer design and the other main results of our paper. In addition, the special structure of the new observer presented in this paper ensures $\mathcal{C}^\infty$ continuity of all estimated states, and global asymptotic convergence of estimation errors is guaranteed irrespective of prescribed control inputs.

Furthermore, the proposed observer is shown to satisfy "separation" property when combined with an independently designed proportional-derivative-like (PD-like) controller. The property is achieved by utilizing a partial Lyapunov "strictification" strategy [24], [25], [26], in which a nonstrict Lyapunov function could be transformed into a strict one whose derivative contains additional non-positive terms of system states. Specifically, a strictification-like analysis is carried out for the controller part Lyapunov-like function to guarantee the boundedness of all tracking state errors. Subsequently, this result is employed to conduct a further strictification process to obtain a partially strict Lyapunov-like function for observer. The word "partially" is used to emphasize that, for a dual quaternion, only the vector component of it is contained in the "strictified" Lyapunov-like function's time derivative. Finally, by employing a composite function, consisting of both the new partially strict observer Lyapunov-like function and the controller one, the combined observer-controller scheme is proofed to result in almost global[1] asymptotic convergence of all tracking errors, and the separation property is guaranteed accordingly. To the best knowledge of the authors, this is the first time a dual velocity observer is designed, and also the first time separation property is established for rigid-body dynamics under dual-quaternion description.

The remainder of paper is organized as follows. In Section II, commonly-used definitions and operations of dual number and dual quaternion are reviewed, and some important properties of them are introduced and proofed, the kinematics of dual transformation matrix and the transit relationship between dual quaternion and dual transformation matrix are also derived, and then the dual-quaternion-based relative kinematics and dynamics of rigid body are introduced. Based on these mathematical preliminaries, a novel dual angular velocity observer is proposed in Section III, and the convergence of estimation errors are analyzed. In Section IV, the observer is combined with a PD-like controller, by a special strictification process, the proof of separation property and the stability analysis for the closed-loop system are presented. Subsequently, numerical simulations for a prototypical hovering task of spacecraft are presented in Section V to illustrate the effectiveness of the proposed method. Finally the paper is end with some conclusions in Section VI.

II. Mathematical Preliminaries

A. Dual Number and Dual Vector

The concept of dual number was firstly introduced by Clifford [29], then named and perfected by Study [30]. The definition of a dual number is given as follows.

$$\hat{a} = a_r + \varepsilon a_d \tag{1}$$

where $\hat{a} \in \hat{\mathbb{R}}$ is called a dual number, $a_r,\ a_d \in \mathbb{R}$ are called as the real part and the dual part of $\hat{a}$, respectively. Throughout the paper, using $\hat{\mathbb{R}}$ denote the set of dual numbers, superscript $\hat{\cdot}$ denotes the corresponding quantity is in dual number set, and subscript $(\cdot)_r$ and $(\cdot)_d$ denote the real part and the dual part of dual numbers, respectively. ε is referred to a special dual unit with rules:

$$\varepsilon^2 = 0 \text{ but } \varepsilon \neq 0 \tag{2}$$

Dual vector is a class of dual number which real and dual parts are vectors:

$$\hat{\boldsymbol{a}} = \boldsymbol{a}_r + \varepsilon \boldsymbol{a}_d \tag{3}$$

where $\hat{\boldsymbol{a}} = \boldsymbol{a}_r + \varepsilon \boldsymbol{a}_d \in \hat{\mathbb{R}}^n$, and $\boldsymbol{a}_r,\ \boldsymbol{a}_d \in \mathbb{R}^n$ are the real part and dual part of $\hat{\boldsymbol{a}}$, respectively. Some basic operations of dual number and dual vector are introduced as follows.

$$\lambda \hat{\boldsymbol{a}} = \lambda \boldsymbol{a}_r + \varepsilon \lambda \boldsymbol{a}_d \tag{4}$$

$$\hat{\boldsymbol{a}}^{\mathrm{T}} = \boldsymbol{a}_r^{\mathrm{T}} + \varepsilon \boldsymbol{a}_d^{\mathrm{T}} \tag{5}$$

$$\hat{a}\hat{\boldsymbol{a}} = a_r \boldsymbol{a}_r + \varepsilon (a_r \boldsymbol{a}_d + a_d \boldsymbol{a}_r) \tag{6}$$

$$\hat{\boldsymbol{a}}^s = \boldsymbol{a}_d + \varepsilon \boldsymbol{a}_r \tag{7}$$

$$\|\hat{\boldsymbol{a}}\| = \|\boldsymbol{a}_r\| + \varepsilon \|\boldsymbol{a}_d\| \tag{8}$$

$$\hat{\boldsymbol{a}}_1 \pm \hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{r1} \pm \boldsymbol{a}_{r2} + \varepsilon (\boldsymbol{a}_{d1} \pm \boldsymbol{a}_{d2}) \tag{9}$$

$$\hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{a}}_2 = \hat{\boldsymbol{a}}_1^{\mathrm{T}} \hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{1r} \cdot \boldsymbol{a}_{2r} + \varepsilon (\boldsymbol{a}_{1r} \cdot \boldsymbol{a}_{2d} + \boldsymbol{a}_{1d} \cdot \boldsymbol{a}_{2r}) \tag{10}$$

[1]Due to topological obstructions (noncontractible) of the configuration space of attitude motion $SO(3)$, it is impossible for any continuous state-feedback controller to render global asymptotic stability [27], [28], so the notion "almost global" is adopted here to imply the stability of attitude motion over an open and dense set in $SO(3)$.

$$\begin{aligned}\hat{\boldsymbol{a}}_1 \times \hat{\boldsymbol{a}}_2 &= -\hat{\boldsymbol{a}}_2 \times \hat{\boldsymbol{a}}_1 \\ &= \boldsymbol{a}_{1r} \times \boldsymbol{a}_{2r} + \varepsilon(\boldsymbol{a}_{1r} \times \boldsymbol{a}_{2d} + \boldsymbol{a}_{1d} \times \boldsymbol{a}_{2r})\end{aligned} \tag{11}$$

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{a}} = \boldsymbol{A}_r\boldsymbol{a}_r + \varepsilon(\boldsymbol{A}_r\boldsymbol{a}_d + \boldsymbol{A}_d\boldsymbol{a}_r) \tag{12}$$

$$\hat{\boldsymbol{a}}_1 \circ \hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{1r}^{\mathrm{T}}\boldsymbol{a}_{2r} + \boldsymbol{a}_{1d}^{\mathrm{T}}\boldsymbol{a}_{2d} \tag{13}$$

where $\lambda \in \mathbb{R}$, $\hat{\boldsymbol{A}} = \boldsymbol{A}_r + \varepsilon\boldsymbol{A}_d \in \hat{\mathbb{R}}^{m\times n}$ is called a dual matrix, and $\hat{\boldsymbol{a}}_1 = \boldsymbol{a}_{1r} + \varepsilon\boldsymbol{a}_{1d} \in \hat{\mathbb{R}}^n$, $\hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{2r} + \varepsilon\boldsymbol{a}_{2d} \in \hat{\mathbb{R}}^n$.

For any $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{c}} \in \hat{\mathbb{R}}^3$, the following important properties are presented.

Lemma 1: $\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) + \hat{\boldsymbol{b}} \times (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{a}}) + \hat{\boldsymbol{c}} \times (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \boldsymbol{0}$.

Proof: See Appendix A.

Lemma 2: $\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) = \hat{\boldsymbol{b}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{c}}) - \hat{\boldsymbol{c}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}})$.

Proof: See Appendix B.

Lemma 3: $(\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) \cdot \hat{\boldsymbol{a}} = \boldsymbol{0}$, $\hat{\boldsymbol{a}} \cdot (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \boldsymbol{0}$.

Proof: See Appendix C.

Lemma 4: $\hat{\boldsymbol{a}}^s \circ (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \boldsymbol{0}$.

Proof: See Appendix D.

Lemma 5 [23]: $\hat{\boldsymbol{a}} \circ (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) = \hat{\boldsymbol{b}}^s \circ (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{a}}^s) = \hat{\boldsymbol{c}}^s \circ (\hat{\boldsymbol{a}}^s \times \hat{\boldsymbol{b}})$.

Another important property of dual vectors is that the dual cross product can also be written as the multiplication of a skew-symmetric matrix and a vector: $\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}} = \hat{\boldsymbol{S}}(\hat{\boldsymbol{a}})\hat{\boldsymbol{b}}$, where $\hat{\boldsymbol{S}}(\hat{\boldsymbol{a}}) = -\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{a}}) = \boldsymbol{S}(\boldsymbol{a}_r) + \varepsilon\boldsymbol{S}(\boldsymbol{a}_d)$ is called the dual skew-symmetric matrix of dual vector $\hat{\boldsymbol{a}}$ with the form

$$\begin{aligned}\hat{\boldsymbol{S}}(\hat{\boldsymbol{a}}) = &\begin{bmatrix} 0 & -\hat{a}_3 & \hat{a}_2 \\ \hat{a}_3 & 0 & -\hat{a}_1 \\ -\hat{a}_2 & \hat{a}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_{3r} & a_{2r} \\ a_{3r} & 0 & -a_{1r} \\ -a_{2r} & a_{1r} & 0 \end{bmatrix} \\ &+ \varepsilon \begin{bmatrix} 0 & -a_{3d} & a_{2d} \\ a_{3d} & 0 & -a_{1d} \\ -a_{2d} & a_{1d} & 0 \end{bmatrix}\end{aligned} \tag{14}$$

and here $\hat{a}_i = a_{ir} + \varepsilon a_{id}, i = 1, 2, 3$ are components of $\hat{\boldsymbol{a}}$, with $\hat{\boldsymbol{a}} = [\hat{a}_1, \hat{a}_2, \hat{a}_3]^{\mathrm{T}}$.

To further explain the physical significance of dual vector, definition of the well-known Plücker line is given as follows (as illustrated in Fig. 1).

Definition 1 [31] (Plücker line): Any single line $\mathcal{L}$ can be uniquely determined by, 1) the unit direction vector of $\mathcal{L}$: $\boldsymbol{l}$; 2) the moment of $\mathcal{L}$ with respect to the origin of an arbitrary 3-dimensional reference frame: $\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{l}$.

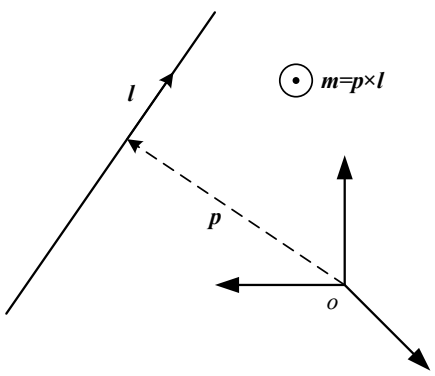


Fig. 1: Definition of Plücker line

Notice that by this definition, not only the direction, but also the position of $\mathcal{L}$ are determined, so actually Plücker line is a 6-DOF description of any line $\mathcal{L}$.

For a 3-dimensional dual vector $\hat{\boldsymbol{a}} \in \hat{\mathbb{R}}^3$, $\hat{\boldsymbol{a}}$ can be used to describe a Plücker line $\mathcal{L}$ if one chooses $\boldsymbol{a}_r = \boldsymbol{l}$ and $\boldsymbol{a}_d = \boldsymbol{m}$, and then $\hat{\boldsymbol{a}}$ is called a "twist". The concept of twist is crucial for dual-quaternion based motion description.

Twists are very suitable to describe the axes of reference frames. Consider a inertial reference frame $\mathcal{N}$ and a body-fixed reference frame $\mathcal{B}$ of a rigid body, then one can use dual vector sets:

$$\{\hat{\boldsymbol{n}}\} = \left\{ \begin{array}{c} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{array} \right\}, \{\hat{\boldsymbol{b}}\} = \left\{ \begin{array}{c} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{array} \right\}$$

to denote the axes of $\mathcal{N}$ and $\mathcal{B}$ respectively (as shown in Fig. 2), where $\hat{\boldsymbol{b}}_i = \boldsymbol{b}_i + \varepsilon\boldsymbol{p} \times \boldsymbol{b}_i$, and $\hat{\boldsymbol{n}}_i = \boldsymbol{n}_i + \varepsilon\boldsymbol{0}$, $i = 1, 2, 3$. By this description, not only the relative orientation, but also the relative position of frame $\mathcal{B}$ with respect to $\mathcal{N}$ can be uniquely determined, and any dual vector $\hat{\boldsymbol{v}} \in \hat{\mathbb{R}}^3$ can be expressed as

$$\hat{\boldsymbol{v}}^B = \hat{v}_1^B\hat{\boldsymbol{b}}_1 + \hat{v}_2^B\hat{\boldsymbol{b}}_2 + \hat{v}_3^B\hat{\boldsymbol{b}}_3 \text{ or } \hat{\boldsymbol{v}}^N = \hat{v}_1^N\hat{\boldsymbol{n}}_1 + \hat{v}_2^N\hat{\boldsymbol{n}}_2 + \hat{v}_3^N\hat{\boldsymbol{n}}_3$$

in frame $\mathcal{B}$ or frame $\mathcal{N}$, respectively, where $\hat{v}_i^B, \hat{v}_i^N \in \hat{\mathbb{R}}$, $i = 1, 2, 3$. Throughout the paper, using $\boldsymbol{a}^X$ or $\hat{\boldsymbol{a}}^X$ denote the corresponding vector or dual vector are expressed in frame $\mathcal{X}$.

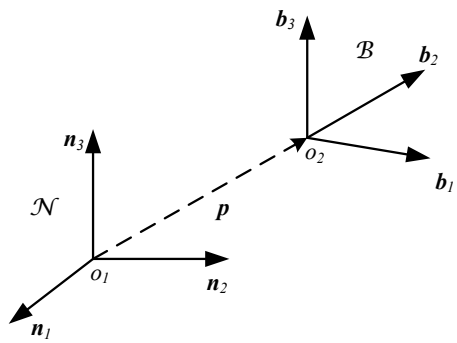


Fig. 2: Reference frames illustration

B. Dual Quaternion and Dual Transformation Matrix

Quaternion is the most commonly-used method to describe the relative attitude motion between two reference frames. The definition of unit quaternion is $\boldsymbol{q} = [\eta, \boldsymbol{\xi}^{\mathrm{T}}]^{\mathrm{T}}$, where $\eta \in \mathbb{R}$ and $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]^{\mathrm{T}} \in \mathbb{R}^3$ are called the scalar part and the vector part of $\boldsymbol{q}$, respectively, and satisfying $\eta^2 + \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = 1$. The multiplication of two quaternions $\boldsymbol{q}_1 = [\eta_1, \boldsymbol{\xi}_1^{\mathrm{T}}]^{\mathrm{T}}$ and $\boldsymbol{q}_2 = [\eta_2, \boldsymbol{\xi}_2^{\mathrm{T}}]^{\mathrm{T}}$ is defined as:

$$\boldsymbol{q}_1 \otimes \boldsymbol{q}_2 = [\eta_1\eta_2 - \boldsymbol{\xi}_1^{\mathrm{T}}\boldsymbol{\xi}_2, (\eta_1\boldsymbol{\xi}_2 + \eta_2\boldsymbol{\xi}_1 + \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2)^{\mathrm{T}}]^{\mathrm{T}} \tag{15}$$

For a rigid body, the relative attitude of its body-fixed frame $\mathcal{B}$ with respect to a inertia frame $\mathcal{N}$ can be decribed by a unit quaternion $\boldsymbol{q}_b$, and the transformation of any real vector $\boldsymbol{a} \in \mathbb{R}^3$ from frame $\mathcal{N}$ to frame $\mathcal{B}$ can be expressed as

$$\boldsymbol{a}^B = \boldsymbol{q}_b^* \otimes \boldsymbol{a}^N \otimes \boldsymbol{q}_b \tag{16}$$

where $\boldsymbol{q}_b^* = [\eta_b, -\boldsymbol{\xi}_b^{\mathrm{T}}]^{\mathrm{T}}$ is the conjugate quaternion of $\boldsymbol{q}_b$. Throughout the paper, when a 3-dimensional vector is multiplied with a quaternion, regard the vector as an equivalent

quaternion with a vanishing and zero scalar part (for example: $\boldsymbol{a}^B \Leftrightarrow [0, (\boldsymbol{a}^B)^{\mathrm{T}}]^{\mathrm{T}}$), to guarantee the dimensions are matched.

The definition of dual quaternion is based on both quaternion and dual vector, as an example, the dual quaternion of $\mathcal{B}$ with respect to $\mathcal{N}$ is [6], [32]

$$\hat{\boldsymbol{q}}_b = \boldsymbol{q}_b + \varepsilon \frac{1}{2} \boldsymbol{q}_b \otimes \boldsymbol{r}_b^B \tag{17}$$

where $\boldsymbol{r}_b^B$ is the position vector from the origin of frame $\mathcal{N}$ to the origin of frame $\mathcal{B}$. $\hat{\boldsymbol{q}}_b$ can also be written as the combination of a dual scalar and a dual vector: $\hat{\boldsymbol{q}}_b = [\hat{\eta}_b, \hat{\boldsymbol{\xi}}_b^{\mathrm{T}}]^{\mathrm{T}}$, where $\hat{\eta}_b \in \hat{\mathbb{R}}$ and $\hat{\boldsymbol{\xi}}_b \in \hat{\mathbb{R}}^3$ are called the scalar part and the vector part of $\hat{\boldsymbol{q}}_b$, respectively. Throughout the paper, use $\mathbb{H}$ and $\hat{\mathbb{H}}$ denote the sets of quaternion and dual quaternion, respectively, and use $\hat{\mathbb{H}}_s$ and $\hat{\mathbb{H}}_v$ denote the sets of scalar part and vector part of dual quaternion, respectively.

Dual quaternion follows the operations of dual vector, and some other special operations of it are given as follows.

$$\mathrm{vec}(\hat{\boldsymbol{q}}) = \hat{\boldsymbol{\xi}} \tag{18}$$

$$\hat{\boldsymbol{q}}^* = [\hat{\eta}, -\hat{\boldsymbol{\xi}}^{\mathrm{T}}]^{\mathrm{T}} \tag{19}$$

$$\hat{\boldsymbol{q}}_1 \otimes \hat{\boldsymbol{q}}_2 = [\hat{\eta}_1\hat{\eta}_2 - \hat{\boldsymbol{\xi}}_1^{\mathrm{T}}\hat{\boldsymbol{\xi}}_2, (\hat{\eta}_1\hat{\boldsymbol{\xi}}_2 + \hat{\eta}_2\hat{\boldsymbol{\xi}}_1 + \hat{\boldsymbol{\xi}}_1 \times \hat{\boldsymbol{\xi}}_2)^{\mathrm{T}}]^{\mathrm{T}} \tag{20}$$

Furthermore, for any dual quaternion $\hat{\boldsymbol{q}}_1, \hat{\boldsymbol{q}}_2, \hat{\boldsymbol{q}}_3 \in \hat{\mathbb{H}}$, the following property can be hold,

Lemma 6 [23]: $\hat{\boldsymbol{q}}_1 \circ (\hat{\boldsymbol{q}}_2 \otimes \hat{\boldsymbol{q}}_3) = \hat{\boldsymbol{q}}_3^s \circ (\hat{\boldsymbol{q}}_2^* \otimes (\hat{\boldsymbol{q}}_1^s))$.

The kinematics of dual quaternion is proofed to be [6], [32]

$$\dot{\hat{\boldsymbol{q}}}_b = \frac{1}{2}\hat{\boldsymbol{q}}_b \otimes \hat{\boldsymbol{\omega}}_b^B = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_b)\hat{\boldsymbol{\omega}}_b^B \tag{21}$$

where

$$\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_b) = \begin{bmatrix} -\hat{\boldsymbol{\xi}}_b^{\mathrm{T}} \\ \hat{\eta}_b \mathbf{I}_{3\times3} + S(\hat{\boldsymbol{\xi}}_b) \end{bmatrix} \tag{22}$$

and $\hat{\boldsymbol{\omega}}_b^B = \boldsymbol{\omega}_b^B + \varepsilon(\boldsymbol{v}_b^B + \boldsymbol{\omega}_b^B \times \boldsymbol{r}_b^B)$ is called the dual angular velocity of frame $\mathcal{B}$ with respect to frame $\mathcal{N}$, and here $\boldsymbol{\omega}_b^B$ and $\boldsymbol{v}_b^B$ are the angular velocity and the linear velocity of the rigid body, respectively.

Differential with quaternion, dual quaternion can be used to describe both the rotational motion and the translational motion integrally of frame $\mathcal{B}$ with respect to frame $\mathcal{N}$. And any dual vector $\hat{\boldsymbol{a}}$ in 3-dimensional space satisfies [6], [32]:

$$\hat{\boldsymbol{a}}^B = \hat{\boldsymbol{q}}_b^* \otimes \hat{\boldsymbol{a}}^N \otimes \hat{\boldsymbol{q}}_b \tag{23}$$

By Eqs. (10), (11) and (20), and utilize Lemma 3, one can get,

$$\begin{aligned}
\hat{\boldsymbol{a}}^B &= [\hat{\eta}_b, -\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}]^{\mathrm{T}} \otimes [\hat{0}, (\hat{\boldsymbol{a}}^N)^{\mathrm{T}}]^{\mathrm{T}} \otimes [\hat{\eta}_b, \hat{\boldsymbol{\xi}}_b^{\mathrm{T}}]^{\mathrm{T}} \\
&= [\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{a}}^N, (\hat{\eta}_b\hat{\boldsymbol{a}}^N - \hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{a}}^N)^{\mathrm{T}}]^{\mathrm{T}} \otimes [\hat{\eta}_b, \hat{\boldsymbol{\xi}}_b^{\mathrm{T}}]^{\mathrm{T}} \\
&= [\hat{0}, ((\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{a}}^N)\hat{\boldsymbol{\xi}}_b + \hat{\eta}_b(\hat{\eta}_b\hat{\boldsymbol{a}}^N - \hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{a}}^N) + ... \\
&\quad (\hat{\eta}_b\hat{\boldsymbol{a}}^N - \hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{a}}^N) \times \hat{\boldsymbol{\xi}}_b)^{\mathrm{T}}]^{\mathrm{T}} \\
&= [\hat{0}, ((\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{a}}^N)\hat{\boldsymbol{\xi}}_b - \hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{a}}^N) + \hat{\eta}_b\hat{\eta}_b\hat{\boldsymbol{a}}^N - ... \\
&\quad 2\hat{\eta}_b(\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{a}}^N) + 2\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{a}}^N))^{\mathrm{T}}]^{\mathrm{T}}
\end{aligned} \tag{24}$$

Use Lemma 2 and rewrite $\hat{\boldsymbol{a}}^B$ to the standard 3-dimensional form, one can obtain:

$$\hat{\boldsymbol{a}}^B = ((\hat{\eta}_b\hat{\eta}_b + \hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\xi}}_b)\mathbf{I}_3 - 2\hat{\eta}_b\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b) + 2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b))\hat{\boldsymbol{a}}^N \tag{25}$$

Since $\eta_b\eta_b + \boldsymbol{\xi}_b^{\mathrm{T}}\boldsymbol{\xi}_b = 1$, and by (17), one can proof

$$\hat{\eta}_b\hat{\eta}_b + \hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\xi}}_b = \hat{\boldsymbol{q}}_b^{\mathrm{T}}\hat{\boldsymbol{q}}_b = 1 \tag{26}$$

Substituting (26) into (25) yields

$$\hat{\boldsymbol{a}}^B = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b)\hat{\boldsymbol{a}}^N \tag{27}$$

Here:

$$\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b) = \mathbf{I}_3 - 2\hat{\eta}_b\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b) + 2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b) \tag{28}$$

is called the dual transformation matrix from frame $\mathcal{N}$ to frame $\mathcal{B}$. It can be used to describe the 6-DOF transformation of any dual vector from frame $\mathcal{N}$ to frame $\mathcal{B}$.

From (28), an important property of dual transformation matrix can be obtained,

$$\begin{aligned}
\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b^*) &= \mathbf{I}_3 + 2\hat{\eta}_b\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b) + 2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b) \\
&= \mathbf{I}_3^{\mathrm{T}} - 2\hat{\eta}_b\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_b) + 2\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_b) \\
&= \hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_b)
\end{aligned} \tag{29}$$

So the converse transformation, from $\mathcal{B}$ to $\mathcal{N}$, can be described by $\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_b)$, and furthermore, it can be verified that

$$\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_b)\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b) = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b)\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_b) = \mathbf{I}_3 \tag{30}$$

To analyze the kinematics of dual transformation matrix, consider a fixed dual vector $\hat{\boldsymbol{n}}_i$ in frame $\mathcal{N}$, due to $\mathcal{N}$ is chosen as the inertial frame, so $\mathrm{d}\hat{\boldsymbol{n}}_i/\mathrm{d}t = 0$. For ease of notation, use $\hat{\boldsymbol{C}}$ denote $\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b)$, and then we have,

$$\begin{aligned}
\frac{\mathrm{d}(\hat{\boldsymbol{C}}\hat{\boldsymbol{n}}_i)}{\mathrm{d}t} &= \frac{\mathrm{d}\hat{\boldsymbol{C}}}{\mathrm{d}t}\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{C}}\frac{\mathrm{d}\hat{\boldsymbol{n}}_i}{\mathrm{d}t} = \frac{\mathrm{d}\hat{\boldsymbol{C}}}{\mathrm{d}t}\hat{\boldsymbol{n}}_i \\
&= \frac{\mathrm{d}(\hat{\boldsymbol{n}}_i - 2\hat{\eta}_b\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i + 2\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))}{\mathrm{d}t}
\end{aligned} \tag{31}$$

Substitute (21) and (22) into (31), and consider Lemma 1, one can get

$$\begin{aligned}
\frac{\mathrm{d}\hat{\boldsymbol{C}}}{\mathrm{d}t}\hat{\boldsymbol{n}}_i =& -2\dot{\hat{\eta}}_b\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i - 2\hat{\eta}_b\dot{\hat{\boldsymbol{\xi}}}_b \times \hat{\boldsymbol{n}}_i + 2\dot{\hat{\boldsymbol{\xi}}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) \\
&+ 2\hat{\boldsymbol{\xi}}_b \times (\dot{\hat{\boldsymbol{\xi}}}_b \times \hat{\boldsymbol{n}}_i) \\
=&(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i - \hat{\eta}_b(\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B + \hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times \hat{\boldsymbol{n}}_i \\
&+ (\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B + \hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) \\
&+ \hat{\boldsymbol{\xi}}_b \times ((\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B + \hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times \hat{\boldsymbol{n}}_i) \\
=&(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i - \hat{\eta}_b^2\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i \\
&- \hat{\eta}_b(\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times \hat{\boldsymbol{n}}_i + \hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) \\
&+ (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) + \hat{\eta}_b\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i) \\
&+ \hat{\boldsymbol{\xi}}_b \times ((\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times \hat{\boldsymbol{n}}_i) \\
=&(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i - \hat{\eta}_b^2\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i \\
&+ 2\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) + (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) \\
&+ \hat{\boldsymbol{\xi}}_b \times ((\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times \hat{\boldsymbol{n}}_i)
\end{aligned} \tag{32}$$

Using Lemma 1 again, one has:

$$\begin{aligned}
&\hat{\boldsymbol{\xi}}_b \times ((\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times \hat{\boldsymbol{n}}_i) \\
&= (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) - \hat{\boldsymbol{n}}_i \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B))
\end{aligned} \tag{33}$$

Similarly

$$\begin{aligned}&(\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)\\ &= \hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)) + \hat{\boldsymbol{\omega}}_b^B \times ((\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) \times \hat{\boldsymbol{\xi}}_b) \quad (34)\\ &= \hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)) - \hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\end{aligned}$$

Substituting (33) and (34) into (32) yields

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{C}}}{\mathrm{d}t}\hat{\boldsymbol{n}}_i =&(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i - \hat{\eta}_b^2\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i\\ &+ 2\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) + 2(\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B) \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)\\ &- \hat{\boldsymbol{n}}_i \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B))\\ =&(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i - \hat{\eta}_b^2\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i\\ &+ 2\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) - 2\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\\ &+ 2\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\\ &- \hat{\boldsymbol{n}}_i \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B))\end{aligned} \quad (35)$$

then by Lemma 2 and Lemma 3, one has,

$$\begin{aligned}&\hat{\boldsymbol{n}}_i \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{\omega}}_b^B))\\ &= \hat{\boldsymbol{n}}_i \times ((\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b - (\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{\omega}}_b^B) \quad (36)\\ &= -(\hat{\boldsymbol{\xi}}_b^T\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i + (\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i\end{aligned}$$

$$\begin{aligned}&\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\\ &= (\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\hat{\boldsymbol{\omega}}_b^B - (\hat{\boldsymbol{\xi}}_b^T\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i) \quad (37)\\ &= -(\hat{\boldsymbol{\xi}}_b^T\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i\end{aligned}$$

Substituting Eqs. (36) and (37) into Eq. (35) yields,

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{C}}}{\mathrm{d}t}\hat{\boldsymbol{n}}_i =& -(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\xi}}_b + \hat{\eta}_b^2)\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i + 2\hat{\eta}_i\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)\\ &- 2\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)) + 2(\hat{\boldsymbol{\xi}}_b^{\mathrm{T}}\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i\\ &+ 2\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\\ =& -\hat{\boldsymbol{\omega}}_b^B \times \hat{\boldsymbol{n}}_i + 2\hat{\eta}_b\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i)\\ &- 2\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{\xi}}_b \times (\hat{\boldsymbol{\xi}}_b \times \hat{\boldsymbol{n}}_i))\\ =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\mathbf{I}_3 - 2\hat{\eta}_b\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b) + 2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_b))\hat{\boldsymbol{n}}_i\\ =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{C}}\hat{\boldsymbol{n}}_i\end{aligned} \quad (38)$$

So that,

$$(\frac{\mathrm{d}\hat{\boldsymbol{C}}}{\mathrm{d}t} + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{C}})\hat{\boldsymbol{n}}_i = \mathbf{0} \quad (39)$$

Due to the arbitrariness of $\hat{\boldsymbol{n}}_i$, finally we can get,

$$\frac{\mathrm{d}\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b)}{\mathrm{d}t} = -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b) \quad (40)$$

Remark 1: Notice that the definitions and properties given above can be extended to any dual transformation matrix between two arbitrary reference frames, and $\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b)$ is just a special example for deduction. Furthermore, it's readily to demonstrate that one can also use the kinematics of dual transformation matrix given in (40) to deduce the kinematics of dual quaternion shown in (21), the fundamental reason is that dual quaternion and dual transformation matrix are two equivalent methods to describe the 6-DOF transformation between two reference frames, so they can be converted to each other.

C. Dual Quaternion Based Relative Kinematics and Dynamics

By the theorem of moment of momentum, the dynamics of a rigid body under dual-quaternion-based description can be written as [6]:

$$\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_b^B = -\hat{\boldsymbol{\omega}}_b^B \times (\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{u}}^B \quad (41)$$

where $\hat{\boldsymbol{u}}^B = \boldsymbol{f}^B + \varepsilon\boldsymbol{\tau}^B$ is called the total dual input applied to the rigid body, and here $\boldsymbol{f}^B \in \mathbb{R}^3$ and $\boldsymbol{\tau}^B \in \mathbb{R}^3$ are the total force and the total torque applied to the rigid body, respectively. Thus,

$$\hat{\boldsymbol{M}}_b = m_b\mathbf{I}_3\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \varepsilon\boldsymbol{J}_b \quad (42)$$

is called the dual mass matrix, with $m_b \in \mathbb{R}$ is the mass of the rigid body, and $\boldsymbol{J}_b \in \mathbb{R}^{3\times3}$ is inertia of the rigid body in body fixed frame. An important property of dual mass is that it doesn't satisfy the law of association, formalized as

$$\hat{\boldsymbol{M}}(\hat{\boldsymbol{A}}\hat{\boldsymbol{B}}) \neq (\hat{\boldsymbol{M}}\hat{\boldsymbol{A}})\hat{\boldsymbol{B}}, \hat{\boldsymbol{A}}(\hat{\boldsymbol{M}}\hat{\boldsymbol{B}}) \neq (\hat{\boldsymbol{A}}\hat{\boldsymbol{M}})\hat{\boldsymbol{B}} \quad (43)$$

where $\hat{\boldsymbol{M}}$ is a dual mass, and $\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}$ are dual matrices or dual vectors. But the following equation can still be hold:

$$\hat{\boldsymbol{M}}(\hat{\boldsymbol{M}}^{-1}\hat{\boldsymbol{A}}) = \hat{\boldsymbol{M}}^{-1}(\hat{\boldsymbol{M}}\hat{\boldsymbol{A}}) = \hat{\boldsymbol{A}} \quad (44)$$

where $\hat{\boldsymbol{M}}^{-1} = \boldsymbol{J}^{-1}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{1}{m}\mathbf{I}_3\varepsilon$ is called the inverse of $\hat{\boldsymbol{M}}$.

Consider a reference frame $\mathcal{T}$ (which is a desired frame to track with), then the relative position and attitude of frame $\mathcal{T}$ with respect to frame $\mathcal{N}$ can be represented by a dual quaternion $\hat{\boldsymbol{q}}_t$, with the kinematics:

$$\dot{\hat{\boldsymbol{q}}}_t = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_t)\hat{\boldsymbol{\omega}}_t^T \quad (45)$$

where $\hat{\boldsymbol{\omega}}_t^T \in \hat{\mathbb{R}}^3$ is the dual angular velocity of frame $\mathcal{T}$ with respect to frame $\mathcal{N}$, in this paper, we assume $\hat{\boldsymbol{\omega}}_t^T$ is bounded and differentiable.

Then, the relative dual quaternion (or called the error dual quaternion) between frame $\mathcal{B}$ and frame $\mathcal{T}$ is defined as:

$$\hat{\boldsymbol{q}}_{bt} = [\hat{\eta}_{bt}, \hat{\boldsymbol{\xi}}_{bt}^{\mathrm{T}}]^{\mathrm{T}} = \hat{\boldsymbol{q}}_t^* \otimes \hat{\boldsymbol{q}}_b = \boldsymbol{q}_{bt} + \varepsilon\frac{1}{2}\boldsymbol{q}_{bt} \otimes \boldsymbol{r}_{bt}^B \quad (46)$$

where $\boldsymbol{q}_{bt}$ is the quaternion of $\mathcal{B}$ with respect to $\mathcal{T}$, and $\boldsymbol{r}_{bt}^B \in \mathbb{R}^3$ is the relative position vector from the origin of $\mathcal{T}$ to the origin of $\mathcal{B}$, and expressed in $\mathcal{B}$. Furthermore, the relative dual angular velocity can be described as:

$$\begin{aligned}\hat{\boldsymbol{\omega}}_{bt}^B &= \hat{\boldsymbol{\omega}}_b^B - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\hat{\boldsymbol{\omega}}_t^T\\ &= \hat{\boldsymbol{\omega}}_b^B - \hat{\boldsymbol{\omega}}_t^B = \boldsymbol{\omega}_{bt}^B + \varepsilon(\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B)\end{aligned} \quad (47)$$

where $\boldsymbol{\omega}_{bt}^B$ and $\boldsymbol{v}_{bt}^B$ are the relative angular velocity and the relative linear velocity of $\mathcal{B}$ with respect to $\mathcal{R}$, respectively. And $\hat{\boldsymbol{\omega}}_t^B = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\hat{\boldsymbol{\omega}}_t^T$ is defined for ease of notation.

Then by substituting (21), (45) and (41) into the time differentials of (46) and (47), the 6-DOF relative kinematics and dynamics of frame $\mathcal{B}$ with respect to frame $\mathcal{T}$ can be presented as [5], [6],

$$\dot{\hat{\boldsymbol{q}}}_{bt} = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_{bt})\hat{\boldsymbol{\omega}}_{bt}^B \quad (48)$$

$$\begin{aligned}\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_{bt}^B =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B)\\ &+ \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_t^T) + \hat{\boldsymbol{u}}^B\end{aligned} \quad (49)$$

D. Proportional-Derivative-Like Controller

Similar with the traditional attitude tracking control problem, one can also use a PD-like controller to achieve the 6-DOF tracking goal for system in (48-49), as summarized in Proposition 1.

Proposition 1 [23]: Consider the system given in (48) and (49), design the total dual input applied to the rigid body as

$$\begin{aligned}\hat{\boldsymbol{u}}^B =& -k_p \mathrm{vec}(\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) - k_d(\hat{\boldsymbol{\omega}}_{bt}^B)^s \\ &+ \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_t^T) + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_t^B)\end{aligned} \tag{50}$$

with $k_p, k_d > 0$, and $\hat{\boldsymbol{q}}_I = [1,0,0,0]^{\mathrm{T}} + \varepsilon[0,0,0,0]^{\mathrm{T}}$. It can be guaranteed for any $\hat{\boldsymbol{q}}_{bt} \in \hat{\mathbb{H}}$, when $t \to \infty$, $\hat{\boldsymbol{\xi}}_{bt}(t) \to 0$, and $\hat{\boldsymbol{\omega}}_{bt}^B(t) \to 0$.

III. Dual Angular Velocity Observer Development

In this section, a smooth observer is designed to estimate the dual angular velocity, and the convergence of estimation errors is analyzed by Lyapunov-based method.

Consider an observer-related reference frame $\mathcal{O}$, in which the estimates for $\hat{\boldsymbol{\omega}}_b^B$ are generated. Let $\hat{\boldsymbol{q}}_o = [\hat{\eta}_o, \hat{\boldsymbol{\xi}}_o^{\mathrm{T}}]^{\mathrm{T}} \in \hat{\mathbb{H}}$ and $\hat{\boldsymbol{\omega}}_o^O \in \hat{\mathbb{R}}^3$ denote the estimated values of $\hat{\boldsymbol{q}}_b$ and $\hat{\boldsymbol{\omega}}_b^B$, respectively. Furthermore, define $\hat{\boldsymbol{q}}_e = [\hat{\eta}_e, \hat{\boldsymbol{\xi}}_e^{\mathrm{T}}]^{\mathrm{T}}$ and $\hat{\boldsymbol{\omega}}_e^B$ as the estimation error states:

$$\hat{\boldsymbol{q}}_e = \hat{\boldsymbol{q}}_o^* \otimes \hat{\boldsymbol{q}}_b = \boldsymbol{q}_e + \varepsilon\frac{1}{2}\boldsymbol{q}_e \otimes \boldsymbol{r}_e^B \tag{51}$$

$$\hat{\boldsymbol{\omega}}_e^B = \hat{\boldsymbol{\omega}}_b^B - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{\omega}}_o^O = \hat{\boldsymbol{\omega}}_b^B - \hat{\boldsymbol{\omega}}_o^B \tag{52}$$

where $\boldsymbol{q}_e = \boldsymbol{q}_o^* \otimes \boldsymbol{q}_b$ and $\boldsymbol{r}_e^B = \boldsymbol{r}_b^B - \boldsymbol{r}_o^B$ are the estimation errors of relative quaternion and position, respectively.

Then, one of the main results of this paper, an smooth observer which guarantees the asymptotic convergence of estimation errors, is summarized in the following theorem.

Theorem 1: Consider the following smooth observer construction,

$$\dot{\hat{\boldsymbol{q}}}_o = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_o)(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s) \tag{53}$$

$$\begin{aligned}\dot{\hat{\boldsymbol{\omega}}}_o^O =& \hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{M}}_b^{-1}\cdot \\ &(\gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) - \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) - \ldots \\ &\lambda\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}((\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\hat{\boldsymbol{\omega}}_o^B) + \hat{\boldsymbol{u}}^B)\end{aligned} \tag{54}$$

where $\lambda, \gamma > 0$ and $\hat{\boldsymbol{M}}_b^{-1} = \boldsymbol{J}_b^{-1}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{1}{m_b}\mathbf{I}_3\varepsilon$ is the inverse of $\hat{\boldsymbol{M}}_b$. Then for any $\hat{\boldsymbol{q}}_o(0) \in \hat{\mathbb{H}}$, and $\hat{\boldsymbol{\omega}}_o^O(0) \in \hat{\mathbb{R}}^3$, it can be guaranteed $\hat{\boldsymbol{\xi}}_e \to \mathbf{0}(t)$ and $\hat{\boldsymbol{\omega}}_e^B(t) \to \mathbf{0}$, when $t \to \infty$.

Proof: According to the definition of dual transformation matrix and the property given in (29), one has:

$$\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e) = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_b)\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_o) \tag{55}$$

Consider the time derivative of (55), and take into account (40) and (53), one can get:

$$\begin{aligned}\frac{\mathrm{d}(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e))}{\mathrm{d}t} =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e) \\ &+ \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\end{aligned} \tag{56}$$

For any dual vector $\hat{\boldsymbol{a}}^O \in \hat{\mathbb{R}}_3$, it satisfies:

$$\begin{aligned}&\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\hat{\boldsymbol{a}}^O \\ &= \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)((\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s) \times \hat{\boldsymbol{a}}^O) \\ &= (\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)) \\ &\quad \times (\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{a}}^O) \\ &= \hat{\boldsymbol{S}}(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)) \\ &\quad \cdot \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{a}}^O\end{aligned} \tag{57}$$

Due to the arbitrariness of $\hat{\boldsymbol{a}}^O$, one has:

$$\begin{aligned}&\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s) \\ &= \hat{\boldsymbol{S}}(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)(\hat{\boldsymbol{\omega}}_o^O + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)) \\ &\quad \cdot \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\end{aligned} \tag{58}$$

Substitute (58) into (56) and consider (52), yields:

$$\frac{\mathrm{d}(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e))}{\mathrm{d}t} = -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e) \tag{59}$$

As discussed in Remark 1, (59) can be converted to a equivalent dual-quaternion kinematics:

$$\begin{aligned}\dot{\hat{\boldsymbol{q}}}_e =& \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_e)(\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s) \\ =& \frac{1}{2}\hat{\boldsymbol{q}}_e \otimes (\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\end{aligned} \tag{60}$$

Based on these analyses, consider the following Lyapunov-like function candidate:

$$V_o = \gamma(\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I) \circ (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I) + \frac{1}{2}(\hat{\boldsymbol{\omega}}_e^B)^s \circ (\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B) \tag{61}$$

Since $V_o(\hat{\boldsymbol{q}}_e = \hat{\boldsymbol{q}}_I, \hat{\boldsymbol{\omega}}_e^B = \mathbf{0}) = 0$, and $V_o > 0$ for any $\hat{\boldsymbol{q}}_e \neq \hat{\boldsymbol{q}}_I$ and $\hat{\boldsymbol{\omega}}_e^B \neq 0$, so V_o is a valid Lyapunov-like candidate. The time differential of V is,

$$\dot{V}_o = 2\gamma(\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I) \circ \dot{\hat{\boldsymbol{q}}}_e + (\hat{\boldsymbol{\omega}}_e^B)^s \circ (\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_e^B) \tag{62}$$

By (52), (54) and (59), we have

$$\begin{aligned}&\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_e^B = \\ &\quad \hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_b^B - \hat{\boldsymbol{M}}_b(-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\cdot \\ &\quad \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{\omega}}_o^O + \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\dot{\hat{\boldsymbol{\omega}}}_o^O) \\ &= -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{u}}^B \\ &\quad + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{\omega}}_o^O) \\ &\quad - (\gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^*(\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) - \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) \\ &\quad - \lambda\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}((\mathrm{vec}(\hat{\boldsymbol{q}}_e^*(\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)\hat{\boldsymbol{\omega}}_o^B) + \hat{\boldsymbol{u}}^B) \\ &= -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) \\ &\quad + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) - \gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^*(\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)\end{aligned} \tag{63}$$

Substituting (60) and (63) into (62), and considering Lemma 6, results in,

$$\begin{aligned}
&\dot{V}_o = \\
&\quad \gamma(\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I) \circ (\hat{\boldsymbol{q}}_e \otimes (\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)) \\
&\quad + (\hat{\boldsymbol{\omega}}_e^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) \\
&\quad + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) - \gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)) \\
&= \gamma(\hat{\boldsymbol{\omega}}_e^B - \lambda(\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s)^s \circ (\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \\
&\quad + (\hat{\boldsymbol{\omega}}_e^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) + ... \\
&\quad \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) - \gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)) \\
&= -\gamma\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \circ \mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \\
&\quad + (\hat{\boldsymbol{\omega}}_e^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) + ... \\
&\quad \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B))
\end{aligned} \tag{64}$$

Rewrite the following term as,

$$\begin{aligned}
\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) =& \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) \\
&+ \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B)
\end{aligned} \tag{65}$$

Consider Lemma 4, and substitute (65) into (64) yields

$$\begin{aligned}
\dot{V}_o =& -\gamma\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \circ \mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \\
&+ (\hat{\boldsymbol{\omega}}_e^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B))
\end{aligned} \tag{66}$$

Since

$$\begin{aligned}
&-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) \\
&= -(\boldsymbol{\omega}_{or}^B + \varepsilon\boldsymbol{\omega}_{od}^B) \times (m_b\boldsymbol{\omega}_{ed}^B + \varepsilon\boldsymbol{J}_b\boldsymbol{\omega}_{er}^B) \\
&\quad + \hat{\boldsymbol{M}}_b(\boldsymbol{\omega}_{er}^B \times \boldsymbol{\omega}_{or}^B + \varepsilon(\boldsymbol{\omega}_{er}^B \times \boldsymbol{\omega}_{od}^B + \boldsymbol{\omega}_{ed}^B \times \boldsymbol{\omega}_{or}^B)) \\
&= -m_b\boldsymbol{\omega}_{or}^B \times \boldsymbol{\omega}_{ed}^B - \varepsilon(m_b\boldsymbol{\omega}_{od}^B \times \boldsymbol{\omega}_{ed}^B + \boldsymbol{\omega}_{or}^B \times (\boldsymbol{J}_b\boldsymbol{\omega}_{er}^B)) \\
&\quad + \varepsilon\boldsymbol{J}_b(\boldsymbol{\omega}_{er}^B \times \boldsymbol{\omega}_{or}^B) + m_b(\boldsymbol{\omega}_{er}^B \times \boldsymbol{\omega}_{od}^B + \boldsymbol{\omega}_{ed}^B \times \boldsymbol{\omega}_{or}^B)
\end{aligned} \tag{67}$$

where $\boldsymbol{\omega}_{or}^B$ and $\boldsymbol{\omega}_{od}^B$ are the real part and dual part of $\hat{\boldsymbol{\omega}}_o^B$, respectively, while $\boldsymbol{\omega}_{er}^B$ and $\boldsymbol{\omega}_{ed}^B$ are the real part and dual part of $\hat{\boldsymbol{\omega}}_e^B$, respectively. So

$$\begin{aligned}
&(\hat{\boldsymbol{\omega}}_e^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B)) \\
&= m_b\boldsymbol{\omega}_{ed}^B \cdot (\boldsymbol{\omega}_{er}^B \times \boldsymbol{\omega}_{od}^B) - m_b\boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{\omega}_{od}^B \times \boldsymbol{\omega}_{ed}^B) \\
&\quad - \boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{\omega}_{or}^B \times (\boldsymbol{J}_b\boldsymbol{\omega}_{er}^B)) - \boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{J}_b(\boldsymbol{\omega}_{er}^B \times \boldsymbol{\omega}_{or}^B)) \\
&= -\boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)\boldsymbol{J}_b\boldsymbol{\omega}_{er}^B) - \boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)\boldsymbol{\omega}_{er}^B)
\end{aligned} \tag{68}$$

Notice $\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)\boldsymbol{J}_b + \boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)$ is a skew-symmetric matrix, so that:

$$\begin{aligned}
&-\boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)\boldsymbol{J}_b\boldsymbol{\omega}_{er}^B) - \boldsymbol{\omega}_{er}^B \cdot (\boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)\boldsymbol{\omega}_{er}^B) = \\
&\quad -\boldsymbol{\omega}_{er}^B \cdot ((\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)\boldsymbol{J}_b + \boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{or}^B)) \times \boldsymbol{\omega}_{er}^B) = 0
\end{aligned} \tag{69}$$

and (66) can be rewritten as,

$$\begin{aligned}
\dot{V}_o =& -\gamma\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \circ \mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) \\
=& -\gamma\lambda\mathrm{vec}(\frac{1}{2}\boldsymbol{r}_e^B + \varepsilon(1 - \boldsymbol{q}_e^* + \frac{1}{4}\|\boldsymbol{r}_e\|^2)) \\
&\circ \mathrm{vec}(\frac{1}{2}\boldsymbol{r}_e^B + \varepsilon(1 - \boldsymbol{q}_e^* + \frac{1}{4}\|\boldsymbol{r}_e\|^2)) \\
=& -\gamma\lambda(\frac{1}{2}\boldsymbol{r}_e^B + \varepsilon\boldsymbol{\xi}_e) \circ (\frac{1}{2}\boldsymbol{r}_e^B + \varepsilon\boldsymbol{\xi}_e) \\
=& -\gamma\lambda(\frac{1}{4}\|\boldsymbol{r}_e^B\|^2 + \|\boldsymbol{\xi}_e\|^2)
\end{aligned} \tag{70}$$

Because $V_o(t) \geq 0$ and $\dot{V}_o(t) \leq 0$, so $V_o \in \mathcal{L}_\infty$, and by the definition of V_o, we have $\boldsymbol{\xi}_e, \boldsymbol{r}_e \in \mathcal{L}_\infty$. And due to $\int_0^\infty \dot{V}_o(\sigma)\mathrm{d}\sigma$ exists, by (70), $\boldsymbol{\xi}_e, \boldsymbol{r}_e \in \mathcal{L}_2$. Furthermore, $V_o \in \mathcal{L}_\infty$ also guarantees that $\hat{\boldsymbol{\omega}}_e^B \in \mathcal{L}_\infty$, and consider (60), one has $\dot{\boldsymbol{\xi}}_e, \dot{\boldsymbol{r}}_e \in \mathcal{L}_\infty$. Actually to the corollary of Barbalat's lemma, one can guarantee $\lim_{t\to\infty} \boldsymbol{\xi}_e(t) = 0$ and $\lim_{t\to\infty} \boldsymbol{r}_e(t) = 0$, and it is readily to demonstrate $\lim_{t\to\infty} \hat{\boldsymbol{\xi}}_e(t) = 0$ accordingly.

Consider (60), the vector part of $\dot{\hat{\boldsymbol{q}}}_e$ can be written as:

$$\dot{\hat{\boldsymbol{\xi}}}_e = -\frac{\lambda}{2}\hat{\boldsymbol{\xi}}_e + \hat{\boldsymbol{\delta}} \tag{71}$$

where

$$\begin{aligned}
\hat{\boldsymbol{\delta}} =& -\frac{\lambda}{2}(\hat{\eta}_e\mathbf{I}_{3\times3} + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e))\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s \\
&- \frac{\lambda}{2}\hat{\boldsymbol{\xi}}_e + \frac{1}{2}(\hat{\eta}_e\mathbf{I}_{3\times3} + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e))\hat{\boldsymbol{\omega}}_e^B
\end{aligned}$$

Eq. (70) shows that $\hat{\boldsymbol{\delta}}$ is a bounded signal consisting of the bounded states $\hat{\boldsymbol{\xi}}_e$ and $\hat{\boldsymbol{\omega}}_e^B$. Thus (71) ia actually an asymptotically stable linear filter with bounded input. Since $\lim_{t\to\infty} \hat{\boldsymbol{\xi}}_e(t) = 0$, so $\lim_{t\to\infty} \hat{\boldsymbol{\delta}}(t) = 0$, which results in $\lim_{t\to\infty} \hat{\eta}_e\hat{\boldsymbol{\omega}}_e^B(t) = 0$, and cause $\lim_{t\to\infty} \hat{\eta}_e(t) = \pm1$, so finally we can guarantee $\lim_{t\to\infty} \hat{\boldsymbol{\omega}}_e^B(t) = 0$, and the proof is completed.

IV. Lyapunov-Like Function Strictification

In this section, based on the estimation results of observer, an independently designed PD-like controller is introduced to achieve 6-DOF relative tracking objective, and then the Lyapunov strictification strategy is employed to proof the separation property of the observer-controller construction, and to guarantee the almost global asymptotic convergence of tracking errors.

A. Necessity Analysis

The definition of strict Lyapunov function is given firstly.

Definition 2 [33]: For a nonautonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) \tag{72}$$

A Lyapunov function $V(t, \boldsymbol{x})$ is a strict Lyapunov function for system (72), if there exists a positive definite function $\alpha(\cdot)$ such that

$$\frac{\partial V(t,\boldsymbol{x})}{\partial t} + \frac{\partial V(t,\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(t,\boldsymbol{x}) \leq -\alpha(\|\boldsymbol{x}\|) \tag{73}$$

From this definition, it can be known that if one can find a strict Lyapunov function for system (72), then the asymptotic convergence result of $\boldsymbol{x}$ can be directly guaranteed.

As mentioned in Sec. II. D, when the true dual angular velocity $\hat{\boldsymbol{\omega}}_b^B$ is available, one can use a PD-like controller given in Proposition 1 to stabilize the closed-loop system. When $\hat{\boldsymbol{\omega}}_b^B$ can't be measured, a direct idea is using the estimation value $\hat{\boldsymbol{\omega}}_o^B$ to replace $\hat{\boldsymbol{\omega}}_b^B$, and the new controller is,

$$\begin{aligned}
\hat{\boldsymbol{u}}^B =& -k_p\mathrm{vec}(\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) - k_d(\hat{\boldsymbol{\omega}}_{ot}^B)^s \\
&+ \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_t^T) + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_t^B)
\end{aligned} \tag{74}$$

where $k_p, k_d > 0$, $\hat{\boldsymbol{\omega}}_{ot}^B = \hat{\boldsymbol{\omega}}_o^B - \hat{\boldsymbol{\omega}}_t^B$. To analysis the stability of the closed-loop system, consider the following Lyapunov-like function candidate,

$$V_c = k_p(\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I) \circ (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I) + \frac{1}{2}(\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) \quad (75)$$

Consider Lemma 6, and substituting (49) into the time differential of V_c yields,

$$\begin{aligned}\dot{V}_c =& k_p(\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) \\ &+ (\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_b^B) + ... \\ &\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_t^T) + \hat{\boldsymbol{u}}^B)\end{aligned} \quad (76)$$

Then by Lemma 4, and substituting control input (74) into (76), we can get,

$$\begin{aligned}\dot{V}_c =& (\hat{\boldsymbol{\omega}}_{ot}^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_b^B) + ... \\ &\hat{\boldsymbol{M}}_b \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B - k_d(\hat{\boldsymbol{\omega}}_{ot}^B)^s + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_t^B)) \\ =& - k_d(\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (\hat{\boldsymbol{\omega}}_{ot}^B)^s \\ &- (\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B))\end{aligned} \quad (77)$$

Similar with the analysis given in (67) to (69), it can be proofed that

$$(\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B)) = 0 \quad (78)$$

Combine this result with (77), and notice $\hat{\boldsymbol{\omega}}_{bt}^B = \hat{\boldsymbol{\omega}}_{ot}^B + \hat{\boldsymbol{\omega}}_e^B$, we have,

$$\dot{V}_c = -k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_{bt}^B + k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_e^B \quad (79)$$

It can be seen that the cross term $k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_e^B$ is a big obstacle for stability analysis.

Furthermore, consider a composite Lyapunov-like function candidate $V_{oc} = \nu V_o + V_c$, where $\nu \in \mathbb{R}$ is a positive constant. By (70) and (79), the time differential of V_{oc} is,

$$\dot{V}_{oc} = -\nu\gamma\lambda \hat{\boldsymbol{\varsigma}}_e \circ \hat{\boldsymbol{\varsigma}}_e - k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_{bt}^B + k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_e^B \quad (80)$$

where $\hat{\boldsymbol{\varsigma}}_e = (\mathrm{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s))^s = \boldsymbol{\xi}_e + \frac{1}{2}\varepsilon \boldsymbol{r}_e$ is defined for ease of notation.

It can be seen from (80) that the cross term $k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_e^B$ still can't be canceled or dominated because there is no negative term in $\hat{\boldsymbol{\omega}}_e^B$, which presents itself as a serious obstacle for further stability analysis.

The strictification strategy will be carried out to solve this problem in next subsection. The objective of strictification is judiciously modifying the Lyapunov-like function candidate V_o by introducing a negative term in $\hat{\boldsymbol{\omega}}_e^B$ in the right part of (80), so one can dominate the cross term $k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_e^B$. But to achieve this goal, we need to proof the boundedness of $\hat{\boldsymbol{\omega}}_b^B$ firstly. What's more, if we can also introduce a negative term of $\hat{\boldsymbol{q}}_{bt}$ into the time differential of V_c, then (80) could directly show the result that both the estimation errors and the tracking errors will converge to zero asymptotically, and consequently the separation property can be established.

B. Boundedness Analysis for the Closed-Loop System

To further analyze the convergence of system states under the controller given in (74), introduce a cross term N_c,

$$N_c = (\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) \quad (81)$$

where

$$\hat{\boldsymbol{\varsigma}}_{bt} = (\mathrm{vec}(\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s))^s = \boldsymbol{\xi}_{bt} + \frac{1}{2}\varepsilon \boldsymbol{r}_{bt}^B \quad (82)$$

The differential of N_c with respect to time is,

$$\begin{aligned}\dot{N}_c =& (\frac{1}{2}\dot{\boldsymbol{r}}_{bt}^B + \varepsilon\dot{\boldsymbol{\xi}}_{bt}) \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) + (\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (\hat{\boldsymbol{M}}_b \dot{\hat{\boldsymbol{\omega}}}_{bt}^B) \\ =& (\frac{1}{2}\boldsymbol{v}_{bt}^B + \varepsilon(\eta_{bt}\boldsymbol{\omega}_{bt}^B + \boldsymbol{S}(\boldsymbol{\xi}_{bt})\boldsymbol{\omega}_{bt}^B) \circ \\ &(m_b(\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) + \varepsilon \boldsymbol{J}_b \boldsymbol{\omega}_{bt}^B) \\ &+ (\hat{\boldsymbol{\varsigma}}_{bt})^s(-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_b^B) + ... \\ &\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_t^T) + \hat{\boldsymbol{u}}^B)\end{aligned} \quad (83)$$

Substituting $\hat{\boldsymbol{u}}_B$ given in (74) into (83) yields,

$$\begin{aligned}\dot{N}_c =& \frac{m_b}{2}(\boldsymbol{v}_{bt}^B)^{\mathrm{T}}(\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) + (\eta_{bt}\boldsymbol{\omega}_{bt}^B)^{\mathrm{T}}(\boldsymbol{J}_b \boldsymbol{\omega}_{bt}^B) \\ &+ (\boldsymbol{S}(\boldsymbol{\xi}_{bt})\boldsymbol{\omega}_{bt}^B)^{\mathrm{T}}(\boldsymbol{J}_b \boldsymbol{\omega}_{bt}^B) \\ &+ (\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B) \\ &- k_p \mathrm{vec}(\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) - k_d(\hat{\boldsymbol{\omega}}_{ot}^B)^s) \\ =& - k_p(\hat{\boldsymbol{\varsigma}}_{bt}) \circ (\hat{\boldsymbol{\varsigma}}_{bt}) + (\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) + ... \\ &\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B) - k_d(\hat{\boldsymbol{\omega}}_{ot}^B)^s) \\ &+ \frac{m_b}{2}(\boldsymbol{v}_{bt}^B)^{\mathrm{T}}(\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) \\ &+ (\eta_{bt}\boldsymbol{\omega}_{bt}^B)^{\mathrm{T}}(\boldsymbol{J}_b \boldsymbol{\omega}_{bt}^B) + (\boldsymbol{S}(\boldsymbol{\xi}_{bt})\boldsymbol{\omega}_{bt}^B)^{\mathrm{T}}(\boldsymbol{J}_b \boldsymbol{\omega}_{bt}^B)\end{aligned} \quad (84)$$

Define $\hat{c}_1 = m_b \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \varepsilon J_{b\max}$ and $\hat{c}_2 = \frac{1}{J_{b\min}}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \varepsilon\frac{1}{m_b}$, where $J_{b\max}$ and $J_{b\min}$ are the maximum and minimum eigenvalues of $\boldsymbol{J}_b$, respectively. And since $\hat{\boldsymbol{\omega}}_t^B$ is bounded, there also exists a positive dual constant $\hat{c}_3 = c_{3r} + \varepsilon c_{3d}$, satisfies $\|\hat{\boldsymbol{\omega}}_t^B\| \le \hat{c}_3$. Then by applying the Cauchy-Schwarz inequality, one can get,

$$\begin{aligned}&(\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_t^B)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B)) \le \\ &\frac{1}{2}m_b c_{3r}\|\boldsymbol{r}_{bt}^B\|\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + J_{b\max}c_{3r}\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{\omega}_{bt}^B\| \\ &+ m_b c_{3d}\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|\end{aligned} \quad (85)$$

$$\begin{aligned}&(\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^B)\hat{\boldsymbol{\omega}}_t^B)) \le \frac{1}{2}m_b c_{3d}\|\boldsymbol{r}_{bt}^B\|\|\boldsymbol{\omega}_{bt}^B\| \\ &+ \frac{1}{2}m_b c_{3r}\|\boldsymbol{r}_{bt}^B\|\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + J_{b\max}c_{3r}\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{\omega}_{bt}^B\|\end{aligned} \quad (86)$$

Notice $\|\eta_{bt}\|, \|\boldsymbol{\xi}_{bt}\| \le 1$, and $\hat{\boldsymbol{\omega}}_{bt}^B - \hat{\boldsymbol{\omega}}_e^B = \hat{\boldsymbol{\omega}}_{ot}^B$, one can further get,

$$\begin{aligned}\dot{N}_c \le& - k_p(\hat{\boldsymbol{\varsigma}}_{bt}) \circ (\hat{\boldsymbol{\varsigma}}_{bt}) - k_d(\hat{\boldsymbol{\varsigma}}_{bt}) \circ (\hat{\boldsymbol{\omega}}_e^B) \\ &+ l_1\|\boldsymbol{r}_{bt}^B\|\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + l_2\|\boldsymbol{r}_{bt}^B\|\|\boldsymbol{\omega}_{bt}^B\| \\ &+ l_3\|\boldsymbol{\xi}_{br}\|\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + l_4\|\boldsymbol{\xi}_{br}\|\|\boldsymbol{\omega}_{bt}^B\| \\ &+ p_1(\boldsymbol{v}_{bt}^B)^{\mathrm{T}}(\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) + p_2\|\boldsymbol{\omega}_{bt}^B\|^2\end{aligned} \quad (87)$$

where $l_1 = m_b c_{3r} + k_d/2$, $l_2 = m_b c_{3d}/2$, $l_3 = m_b c_{3d}$, $l_4 = 2J_{b\max}c_{3r} + k_d$, $p_1 = m_b/2$, $p_2 = 2J_{b\max}$.

Proposition 2: Consider the tracking dynamics in (48) and (49), and the control input given in (74) (in which the estimated value $\hat{\boldsymbol{\omega}}_o^B$ is employed to replace the unavailable dual angular velocity $\hat{\boldsymbol{\omega}}_b^B$), then $\hat{\boldsymbol{q}}_{bt}$, $\hat{\boldsymbol{\omega}}_{bt}^B$, $\hat{\boldsymbol{\omega}}_b^B$ and $\hat{\boldsymbol{\omega}}_o^B$ are guaranteed to be uniformly bounded.

Proof: Consider the following augmented Lyapunov-like function candidate,

$$\begin{aligned} V_{c1} =& \rho V_c + N_c = \rho k_p(\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I) \circ (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I) \\ &+ \frac{\rho}{2}(\hat{\boldsymbol{\omega}}_{bt}^B)^s \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) + (\hat{\boldsymbol{\varsigma}}_{bt})^s \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^B) \end{aligned} \tag{88}$$

where $\rho \in \mathbb{R}$ is a constant. By the definition of $\hat{\boldsymbol{q}}_{bt}$, $\hat{\boldsymbol{\varsigma}}_{bt}$, and $\hat{\boldsymbol{\omega}}_{bt}^B$, one has,

$$\begin{aligned} V_{c1} =& \rho k_p((\eta_{bt} - 1)^2 + \boldsymbol{\xi}_{bt}^{\mathrm{T}} \boldsymbol{\xi}_{bt} + \frac{1}{4}(\boldsymbol{\xi}_{bt}^{\mathrm{T}} \boldsymbol{r}_{bt}^B)^2 \\ &+ \frac{1}{4}(\eta_{bt} \boldsymbol{r}_{bt}^B + \boldsymbol{\xi}_{bt} \times \boldsymbol{r}_{bt}^B)^{\mathrm{T}} (\eta_{bt} \boldsymbol{r}_{bt}^B + \boldsymbol{\xi}_{bt} \times \boldsymbol{r}_{bt}^B)) \\ &+ \frac{\rho m_b}{2} \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 + \frac{\rho J_{b\min}}{2} \|\boldsymbol{\omega}_{bt}^B\|^2 \\ &+ \frac{m_b}{2} (\boldsymbol{r}_{bt}^B)^{\mathrm{T}} (\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) + (\boldsymbol{\xi}_{bt}^B)^{\mathrm{T}} (\boldsymbol{J}_b \boldsymbol{\omega}_{bt}^B) \end{aligned} \tag{89}$$

Then by Binet-Cauchy identity of cross product along with Cauchy-Schwarz inequality, one can get,

$$\begin{aligned} V_{c1} \geq& \rho k_p((\eta_{bt} - 1)^2 + \|\boldsymbol{\xi}_{bt}\|^2 + \frac{1}{4}\|\boldsymbol{r}_{bt}\|^2) \\ &+ \frac{\rho m_b}{2} \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 + \frac{\rho J_{b\min}}{2} \|\boldsymbol{\omega}_{bt}^B\|^2 \\ &- \frac{m_b}{4} \|\boldsymbol{r}_{bt}^B\|^2 - \frac{m_b}{4} \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \\ &- \frac{J_{b\max}}{2} \|\boldsymbol{\xi}_{bt}\|^2 - \frac{J_{b\max}}{2} \|\boldsymbol{\omega}_{bt}^B\|^2 \\ \geq& \rho k_p(\eta_{bt} - 1)^2 + (\rho k_p - \frac{J_{b\max}}{2}) \|\boldsymbol{\xi}_{bt}\|^2 \\ &+ (\frac{\rho k_p}{4} - \frac{m_b}{4}) \|\boldsymbol{r}_{bt}^B\|^2 + (\frac{\rho J_{b\min}}{2} - \frac{J_{b\max}}{2}) \|\boldsymbol{\omega}_{bt}^B\|^2 \\ &+ (\frac{\rho m_b}{2} - \frac{m_b}{4}) \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \end{aligned} \tag{90}$$

Choosing

$$\rho \geq \max\{\frac{J_{b\max}}{k_p}, \frac{2m_b}{k_p}, \frac{2J_{b\max}}{J_{b\min}}, 1\}$$

then one has,

$$\begin{aligned} V_{c1} \geq& \rho k_p(\eta_{bt} - 1)^2 + \frac{J_{b\max}}{2} \|\boldsymbol{\xi}_{bt}\|^2 + \frac{m_b}{4} \|\boldsymbol{r}_{bt}^B\|^2 \\ &+ \frac{J_{b\max}}{2} \|\boldsymbol{\omega}_{bt}^B\|^2 + \frac{m_b}{4} \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \end{aligned} \tag{91}$$

Eq. (91) guarantees $V_{c1} \geq 0$, and $V_{c1} = 0$ only when $\hat{\boldsymbol{q}}_{bt} = \hat{\boldsymbol{q}}_I$ and $\hat{\boldsymbol{\omega}}_{bt}^B = 0$, so V_{c1} is a valid Lyapunov function.

Substituting (79) and (87) into the time differential of V_{c1}, and utilizing arithmetic-geometric average inequality, results in,

$$\begin{aligned} \dot{V}_{c1} \leq& - k_p \hat{\boldsymbol{\varsigma}}_{bt} \circ \hat{\boldsymbol{\varsigma}}_{bt} - \rho k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_{bt}^B + \rho k_d \hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_e^B \\ &- k_d \hat{\boldsymbol{\varsigma}}_{bt} \circ \hat{\boldsymbol{\omega}}_e^B + l_1 \|\boldsymbol{r}_{bt}^B\| \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| \\ &+ l_2 \|\boldsymbol{r}_{bt}^B\| \|\boldsymbol{\omega}_{bt}^B\| + l_3 \|\boldsymbol{\xi}_{br}\| \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| \\ &+ l_4 \|\boldsymbol{\xi}_{br}\| \|\boldsymbol{\omega}_{bt}^B\| + p_1 (\boldsymbol{v}_{bt}^B)^{\mathrm{T}} (\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) \\ &+ p_2 \|\boldsymbol{\omega}_{bt}^B\|^2 \\ \leq& - \kappa_1 (\|\boldsymbol{\xi}_{bt}\|^2 + \frac{1}{4}\|\boldsymbol{r}_{bt}^B\|^2) \\ &- \frac{\rho k_d}{2} (\|\boldsymbol{\omega}_{bt}^B\|^2 + \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2) + \kappa_2 \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B \\ &+ l_1 \|\boldsymbol{r}_{bt}^B\| \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + l_2 \|\boldsymbol{r}_{bt}^B\| \|\boldsymbol{\omega}_{bt}^B\| \\ &+ l_3 \|\boldsymbol{\xi}_{br}\| \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + l_4 \|\boldsymbol{\xi}_{br}\| \|\boldsymbol{\omega}_{bt}^B\| \\ &+ p_1 (\boldsymbol{v}_{bt}^B)^{\mathrm{T}} (\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) + p_2 \|\boldsymbol{\omega}_{bt}^B\|^2 \end{aligned} \tag{92}$$

where $\kappa_1 = k_p - k_d \varpi/2$ and $\kappa_2 = k_d/(2\varpi) + \rho k_d/2$, and here ϖ is a positive constant. To guarantee $\kappa_1, \kappa_2 > 0$, choosing $\varpi = k_p/k_d$.

Furthermore, by completing squares, we have

$$\begin{aligned} \dot{V}_{c1} \leq& - \frac{\kappa_1}{2} \|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{8} \|\boldsymbol{r}_{bt}^B\|^2 \\ &- (\frac{\rho k_d}{2} - \frac{4l_2^2}{\kappa_1} - \frac{l_4^2}{\kappa_1} - p_2) \|\boldsymbol{\omega}_{bt}^B\|^2 \\ &- (\frac{\rho k_d}{2} - \frac{4l_1^2}{\kappa_1} - \frac{l_3^2}{\kappa_1} - 2p_1 p_3) \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \\ &+ \kappa_2 \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B - (\frac{4l_1^2}{\kappa_1} \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 - ... \\ &l_1 \|\boldsymbol{r}_{bt}^B\| \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + \frac{\kappa_1}{16} \|\boldsymbol{r}_{bt}^B\|^2) \\ &- (\frac{4l_2^2}{\kappa_1} \|\boldsymbol{\omega}_{bt}^B\|^2 - l_2 \|\boldsymbol{r}_{bt}^B\| \|\boldsymbol{\omega}_{bt}^B\| + \frac{\kappa_1}{16} \|\boldsymbol{r}_{bt}^B\|) \\ &- (\frac{l_3^2}{\kappa_1} \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 - ... \\ &l_3 \|\boldsymbol{\xi}_{br}\| \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\| + \frac{\kappa_1}{4} \|\boldsymbol{\xi}_{bt}\|^2) \\ &- (\frac{l_4^2}{\kappa_1} \|\boldsymbol{\omega}_{bt}^B\|^2 - l_4 \|\boldsymbol{\xi}_{br}\| \|\boldsymbol{\omega}_{bt}^B\| + \frac{\kappa_1}{4} \|\boldsymbol{\xi}_{bt}\|^2) \\ &- (2p_1 p_3 \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 - ... \\ &p_1 (\boldsymbol{v}_{bt}^B)^{\mathrm{T}} (\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B) + \frac{p_1}{4p_3} \|\boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2) \\ &+ \frac{p_1}{4p_3} \|\boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \\ \leq& - \frac{\kappa_1}{2} \|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{8} \|\boldsymbol{r}_{bt}^B\|^2 \\ &- (\frac{\rho k_d}{2} - \frac{4l_2^2}{\kappa_1} - \frac{l_4^2}{\kappa_1} - p_2) \|\boldsymbol{\omega}_{bt}^B\|^2 \\ &- (\frac{\rho k_d}{2} - \frac{4l_1^2}{\kappa_1} - \frac{l_3^2}{\kappa_1} - 2p_1 p_3) \|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \\ &+ \kappa_2 \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B + \frac{p_1}{4p_3} \|\boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \end{aligned} \tag{93}$$

where $p_3 \in \mathbb{R}$ is a positive constant. For further analysis, consider the dual part (attitude control part) of (74), we have,

$$\boldsymbol{u}_d^B = -k_p \boldsymbol{\xi}_{bt} - k_d \boldsymbol{\omega}_{ot}^B + \boldsymbol{J}_b \boldsymbol{C}(\boldsymbol{q}_{bt}) \dot{\boldsymbol{\omega}}_t^T + \boldsymbol{S}(\boldsymbol{\omega}_t^B)(\boldsymbol{J}_b \boldsymbol{\omega}_t^B) \tag{94}$$

So actually the dual control part (94) is an observer-based PD-like controller for the attitude of the rigid body.

In Theorem 1, it is proofed that $\forall t \geq 0, \dot{V}_o \leq 0$, so V_o is bounded, consequently $\hat{\boldsymbol{q}}_e^B$, $\hat{\boldsymbol{\omega}}_e^B$ and $\boldsymbol{\omega}_e^B$ are uniformly bounded. Then by straightforward Lyapunov-based analysis, one can proof that $\boldsymbol{\omega}_{bt}^B$ is also bounded (see [21] for details). So there should exists a positive constant l_5, satisfying $\|\boldsymbol{\omega}_{bt}^B\|^2 < l_5$, and we can further obtain,

$$\begin{aligned}\dot{V}_{c1} \leq & -\frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - (\frac{\kappa_1}{8} - \frac{p_1 l_5}{4p_3})\|\boldsymbol{r}_{bt}^B\|^2 \\ & - (\frac{\rho k_d}{2} - \frac{4l_2^2}{\kappa_1} - \frac{l_4^2}{\kappa_1} - p_2)\|\boldsymbol{\omega}_{bt}^B\|^2 \\ & - (\frac{\rho k_d}{2} - \frac{4l_1^2}{\kappa_1} - \frac{l_3^2}{\kappa_1} - 2p_1p_3)\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \\ & + \kappa_2 \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B\end{aligned} \tag{95}$$

If we further modify the value of ρ to

$$\begin{aligned}\rho = & \max\{\frac{J_{\max}}{k_p}, \frac{2m_b}{k_p}, \frac{2J_{b\max}}{J_{b\min}}, 1, \frac{16l_1^2}{\kappa_1 k_d} + \frac{4l_3^2}{\kappa_1 k_d} + \frac{8p_1p_3}{k_d}, ... \\ & \frac{16l_2^2}{\kappa_1 k_d} + \frac{4l_4^2}{\kappa_1 k_d} + \frac{4p_2}{k_d}\}\end{aligned}$$

and choose $p_3 = 4p_1 l_5/\kappa_1$, we have,

$$\begin{aligned}\dot{V}_{c1} \leq & -\frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{16}\|\boldsymbol{r}_{bt}^B\|^2 - \frac{\rho k_d}{4}\|\boldsymbol{\omega}_{bt}^B\|^2 \\ & - \frac{\rho k_d}{4}\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 + \kappa_2 \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B\end{aligned} \tag{96}$$

By Cauchy-Schwarz inequality, an upper bound of V_{c1} can be obtained from (89),

$$\begin{aligned}V_{c1} \leq & \rho k_p + \frac{J_{b\max}}{2} + \rho k_p\|\boldsymbol{\xi}_{bt}\|^2 + \frac{5\rho k_p + m_b}{4}\|\boldsymbol{r}_{bt}^B\|^2 \\ & + \frac{(2\rho+1)m_b}{4}\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 \\ & + \frac{\rho J_{b\min} + J_{b\max}}{2}\|\boldsymbol{\omega}_{bt}^B\|^2\end{aligned} \tag{97}$$

Since it has been proofed in Theorem 1 that $\hat{\boldsymbol{\omega}}_e^B$ is bounded, so there should exist positive constant p_4, satisfying $\hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B \leq p_4$. Then by (96) and (97), we can get,

$$\dot{V}_{c1} \leq -\chi V_{c1} + \sigma \tag{98}$$

where

$$\begin{aligned}\chi = & \min\{\frac{\kappa_1}{2\rho k_p}, \frac{\kappa_1}{4(5\rho k_p + m_b)}, \frac{\rho k_d}{2(\rho J_{b\min} + J_{b\max})}, ... \\ & \frac{\rho k_d}{(2\rho+1)m_b}\}, \ \sigma = \chi(\rho k_p + \frac{J_{b\max}}{2}) + \kappa_2 p_4\end{aligned}$$

So that V_{c1} is uniformly bounded, which guarantees $\hat{\boldsymbol{q}}_{bt}$ and $\hat{\boldsymbol{\omega}}_{bt}^B$ are uniformly bounded. What's more, since $\hat{\boldsymbol{\omega}}_t^B$ and $\hat{\boldsymbol{\omega}}_e^B$ are bounded, so $\hat{\boldsymbol{\omega}}_b^B$ and $\hat{\boldsymbol{\omega}}_o^B$ are also bounded. The proof is completed.

Remark 2: Though we can't obtain a strict controller Lyapunov-like function due to the exist of term $\kappa_2 \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B$, but the strictification-like process given in Proposition 2 helps us guarantee the uniformly bounded results of $\hat{\boldsymbol{\omega}}_b^B$ and $\hat{\boldsymbol{\omega}}_o^B$, and these results are crucial for the subsequent analysis to get a partially strict Lyapunov-like function for the observer, and ultimately toward the establishment of the separation property.

C. Toward a strict observer Lyapunov-Like Function

To conduct the strictification process, consider a new cross term,

$$N_o = -2(\hat{\eta}_e \hat{\boldsymbol{\xi}}_e) \circ \boldsymbol{\omega}_e^B \tag{99}$$

Notice $\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\varsigma}}_e = \hat{\boldsymbol{0}}$ and $\hat{\eta}_e\hat{\eta}_e + \hat{\boldsymbol{\xi}}_e^{\mathrm{T}}\hat{\boldsymbol{\xi}}_e = 1$, the time differential of N_o can be written as:

$$\begin{aligned}\dot{N}_o = & ((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}(\hat{\boldsymbol{\omega}}_e^B - \lambda\hat{\boldsymbol{\varsigma}}_e))\hat{\boldsymbol{\xi}}_e) \circ \hat{\boldsymbol{\omega}}_e^B \\ & - (\hat{\eta}_e(\hat{\eta}_e \mathbf{I}_{3\times3} + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e))(\hat{\boldsymbol{\omega}}_e^B - \lambda\hat{\boldsymbol{\varsigma}}_e)) \circ \hat{\boldsymbol{\omega}}_e^B \\ & - 2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e) \circ \dot{\hat{\boldsymbol{\omega}}}_e^B \\ = & ((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}(\hat{\boldsymbol{\omega}}_e^B - \lambda\hat{\boldsymbol{\varsigma}}_e))\hat{\boldsymbol{\xi}}_e) \circ \hat{\boldsymbol{\omega}}_e^B - (\hat{\eta}_e\hat{\eta}_e\hat{\boldsymbol{\omega}}_e^B) \circ \hat{\boldsymbol{\omega}}_e^B \\ & + (\lambda\hat{\eta}_e\hat{\eta}_e\hat{\boldsymbol{\varsigma}}_e) \circ \hat{\boldsymbol{\omega}}_e^B - (\hat{\eta}_e\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^B) \circ \hat{\boldsymbol{\omega}}_e^B \\ & - 2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e) \circ \dot{\hat{\boldsymbol{\omega}}}_e^B \\ = & - \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B + ((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^B) \circ \hat{\boldsymbol{\omega}}_e^B \\ & + ((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}(\hat{\boldsymbol{\omega}}_e^B - \lambda\hat{\boldsymbol{\varsigma}}_e))\hat{\boldsymbol{\xi}}_e) \circ \hat{\boldsymbol{\omega}}_e^B + (\lambda\hat{\eta}_e\hat{\eta}_e\hat{\boldsymbol{\varsigma}}_e) \circ \hat{\boldsymbol{\omega}}_e^B \\ & - (\hat{\eta}_e\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^B) \circ \hat{\boldsymbol{\omega}}_e^B - 2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e) \circ \dot{\hat{\boldsymbol{\omega}}}_e^B\end{aligned} \tag{100}$$

Since $\|\boldsymbol{\xi}_e\| \leq 1$, so $\|\hat{\boldsymbol{\xi}}_e\| \leq \|\hat{\boldsymbol{\varsigma}}_e\|$. And cause $\hat{\boldsymbol{\omega}}_e^B$, $\hat{\boldsymbol{\xi}}_e$, $\hat{\eta}_e$, $\hat{\boldsymbol{\varsigma}}_e$, $\hat{\boldsymbol{\omega}}_b^B$ and $\hat{\boldsymbol{\omega}}_o^B$ are bounded, so there exists positive constant dual numbers $\hat{b}_i, i = 1, 2, ..., 6$, satisfy $\|\hat{\boldsymbol{\xi}}_e\| \leq \hat{b}_1$, $\|\hat{\boldsymbol{\omega}}_e^B\| \leq \hat{b}_2$, , $\|\hat{\boldsymbol{\varsigma}}_e\| \leq \hat{b}_3$, $\|\hat{\eta}_e\| \leq \hat{b}_4$, $\|\hat{\boldsymbol{\omega}}_b^B\| < \hat{b}_5$ and $\|\hat{\boldsymbol{\omega}}_o^B\| < \hat{b}_6$. Then the following inequalities can be established:

$$\begin{aligned}& ((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}(\hat{\boldsymbol{\omega}}_e^B - \lambda\hat{\boldsymbol{\varsigma}}_e))\hat{\boldsymbol{\xi}}_e) \circ \hat{\boldsymbol{\omega}}_e^B \leq \\ & \quad (\hat{b}_1\hat{b}_2\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\| + (\lambda\hat{b}_1^2\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\|\end{aligned} \tag{101}$$

$$((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^B) \circ \hat{\boldsymbol{\omega}}_e^B \leq (\hat{b}_1\hat{b}_2\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\| \tag{102}$$

$$(\lambda\hat{\eta}_e\hat{\eta}_e\hat{\boldsymbol{\varsigma}}_e) \circ \hat{\boldsymbol{\omega}}_e^B \leq (\lambda\hat{b}_4^2\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\| \tag{103}$$

$$-(\hat{\eta}_e\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^B) \circ \hat{\boldsymbol{\omega}}_e^B \leq (\hat{b}_2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\| \tag{104}$$

$$-2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e) \circ \dot{\hat{\boldsymbol{\omega}}}_e^B \leq (2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\dot{\hat{\boldsymbol{\omega}}}_e^B\| \tag{105}$$

According to (63), and recall the definition of $\hat{c}_1$ and $\hat{c}_c$, the bound of $\dot{\hat{\boldsymbol{\omega}}}_e^B$ can be obtained,

$$\begin{aligned}\|\dot{\hat{\boldsymbol{\omega}}}_e^B\| = & \|\hat{\boldsymbol{M}}_b^{-1}[-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_b^B) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) + ... \\ & \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_o^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) - \gamma\hat{\boldsymbol{\varsigma}}_e^s]\| \\ = & \|\hat{\boldsymbol{M}}_b^{-1}[-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_b^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B) - \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_o^B) + ... \\ & \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^B)\hat{\boldsymbol{\omega}}_o^B) - \gamma\hat{\boldsymbol{\varsigma}}_e^s]\| \\ \leq & \hat{c}_2(\hat{b}_5\hat{c}_1\|\hat{\boldsymbol{\omega}}_e^B\|) + \hat{c}_2(\|\hat{\boldsymbol{\omega}}_e^B\|(\hat{c}_1\hat{b}_6)) \\ & + \hat{b}_6\|\hat{\boldsymbol{\omega}}_e^B\| + \gamma\hat{c}_2\|\hat{\boldsymbol{\varsigma}}_e\|^s\end{aligned} \tag{106}$$

So by (101)-(106), one can get:

$$\begin{aligned}\dot{N}_o \leq & - \hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B + (\hat{b}_7\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\| \\ & + (2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|) \circ (\hat{c}_2(\hat{b}_5\hat{c}_1\|\hat{\boldsymbol{\omega}}_e^B\|) + \hat{c}_2(\|\hat{\boldsymbol{\omega}}_e^B\|(\hat{c}_1\hat{b}_6)) + ... \\ & \hat{b}_6\|\hat{\boldsymbol{\omega}}_e^B\| + \gamma\hat{c}_2\|\hat{\boldsymbol{\varsigma}}_e\|^s)\end{aligned} \tag{107}$$

where $\hat{b}_7 = 2\hat{b}_1\hat{b}_2 + \lambda\hat{b}_1^2 + \lambda\hat{b}_4^2 + \hat{b}_2\hat{b}_4$. Using $\boldsymbol{\omega}_{er}^B$ and $\boldsymbol{\omega}_{ed}^B$ denote the real part and dual part of $\hat{\boldsymbol{\omega}}_e^B$, respectively. Then we can further get,

$$\begin{aligned}& (\hat{b}_7\|\hat{\boldsymbol{\varsigma}}_e\|) \circ \|\hat{\boldsymbol{\omega}}_e^B\| = \\ & \quad b_{7r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\| + \frac{1}{2}b_{7r}\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{ed}^B\| + b_{7d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\|\end{aligned} \tag{108}$$

$$(2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|)\circ(\hat{c}_2(\hat{b}_5\hat{c}_1\|\hat{\boldsymbol{\omega}}_e^B\|)= \frac{2m_b}{J_{b\min}}b_{4r}b_{5d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\|+\frac{2J_{b\max}}{J_{b\min}}b_{4r}b_{5r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\| + 2b_{4d}b_{5r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\|+b_{4r}b_{5r}\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{ed}^B\| \tag{109}$$

$$(2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|)\circ\hat{c}_2(\|\hat{\boldsymbol{\omega}}_e^B\|(\hat{c}_1\hat{b}_6))= \frac{2m_b}{J_{b\min}}b_{4r}b_{6d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\|+\frac{2J_{b\max}}{J_{b\min}}b_{4r}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\| + 2b_{4d}b_{6d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\|+b_{4r}b_{6d}\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{er}^B\| \tag{110}$$

$$(2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|)\circ\hat{b}_6\|\hat{\boldsymbol{\omega}}_e^B\|= 2b_{4r}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\|+2b_{4d}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\| + 2b_{4d}b_{6d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\|+b_{4r}b_{6r}\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{ed}^B\| + b_{4r}b_{6d}\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{er}^B\| \tag{111}$$

$$(2\hat{b}_4\|\hat{\boldsymbol{\varsigma}}_e\|)\circ\gamma\hat{c}_2\|\hat{\boldsymbol{\varsigma}}_e\|^s= \frac{2\gamma}{J_{b\min}}b_{4r}\|\boldsymbol{\xi}_e\|^2+\frac{\gamma}{m_b}b_{4d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{r}_e^B\|+\frac{\gamma}{2m_b}b_{4r}\|\boldsymbol{r}_e^B\|^2 \tag{112}$$

So (107) can be rewritten as:

$$\begin{aligned}\dot{N}_o\leq&-\hat{\boldsymbol{\omega}}_e^B\circ\hat{\boldsymbol{\omega}}_e^B+\alpha_1\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\|+\alpha_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\| \\ &+\alpha_3\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{er}^B\|+\alpha_4\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{ed}^B\|+\beta_1\|\boldsymbol{\xi}_e\|^2 \\ &+\beta_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{r}_e^B\|+\beta_3\|\boldsymbol{r}_e^B\|^2\end{aligned} \tag{113}$$

where $\alpha_1=b_{7r}+\frac{2J_{b\max}}{J_{b\min}}b_{4r}(b_{5r}+b_{6r})+4b_{4d}b_{6d}+2b_{4r}b_{6r}$, $\alpha_2=b_{7d}+\frac{2m_b}{J_{b\min}}b_{4r}(b_{5d}+b_{6d})+2b_{4d}b_{5r}+2b_{4d}b_{6r}$, $\alpha_3=2b_{4r}b_{6d}$, $\alpha_4=\frac{1}{2}b_{7r}+b_{4r}(b_{5r}+b_{6r})$, $\beta_1=\frac{2\gamma}{J_{b\min}}b_{4r}$, $\beta_2=\frac{\gamma}{m_b}b_{4d}$ and $\beta_3=\frac{\gamma}{2m_b}b_{4r}$. The importance of cross term N_o is shown in the following proposition.

Proposition 3: For the observer construction given in (53) and (54), the following augmented function V_{o1},

$$V_{o1}=\mu V_o+N_o \tag{114}$$

is a partially strict Lyapunov-like function for the estimation error dynamics. Where $\mu>0$ is a positive constant designed as,

$$\begin{aligned}\mu=&\max\{\frac{2}{\gamma},\frac{6}{J_{b\min}},\frac{4}{m_b},\frac{4(2\alpha_3^2+2\alpha_4^2+\beta_2+2\beta_3)}{\gamma\lambda},\ldots \\ &\frac{2\alpha_1^2+2\alpha_2^2+2\beta_1+\beta_2}{\gamma\lambda}\}\end{aligned} \tag{115}$$

Proof: Use $\boldsymbol{\xi}_{er}$ and $\boldsymbol{\xi}_{ed}$ denote the real part and dual part of $\hat{\boldsymbol{\xi}}_e$, respectively. Notice that $\|\eta_{er}\|=\|\eta_e\|\leq 1$ and $\|\boldsymbol{\xi}_{er}\|=\|\boldsymbol{\xi}_e\|\leq 1$, we have,

$$\begin{aligned}V_{o1}=&\mu\gamma(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)\circ(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)+\mu\frac{1}{2}(\hat{\boldsymbol{\omega}}_e^B)^s\circ(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^B) \\ &-2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e)\circ\hat{\boldsymbol{\omega}}_e^B \\ \geq&\mu\gamma((\eta_{er}-1)^2+\eta_{ed}^2+\|\boldsymbol{\xi}_{er}\|^2+\|\boldsymbol{\xi}_{ed}\|^2) \\ &+\frac{\mu}{2}(J_{b\min}\|\boldsymbol{\omega}_{er}^B\|^2+m_b\|\boldsymbol{\omega}_{ed}^B\|^2)-2\|\boldsymbol{\xi}_{er}\|\|\boldsymbol{\omega}_{er}^B\| \\ &-2\|\boldsymbol{\xi}_{ed}\|\|\boldsymbol{\omega}_{er}^B\|-2|\eta_{ed}|\|\boldsymbol{\omega}_{ed}^B\| \\ \geq&\mu\gamma(\eta_{er}-1)^2+(\mu\gamma-1)(\eta_{ed}^2+\|\boldsymbol{\xi}_{er}\|^2+\|\boldsymbol{\xi}_{ed}\|^2) \\ &+(\frac{J_{b\min}}{2}\mu-2)\|\boldsymbol{\omega}_{er}^B\|^2+(\frac{m_b}{2}\mu-1)\|\boldsymbol{\omega}_{ed}^B\|^2 \\ \geq&\mu\gamma(\eta_{er}-1)^2+\eta_{ed}^2+\|\boldsymbol{\xi}_{er}\|^2+\|\boldsymbol{\xi}_{ed}\|^2 \\ &+\|\boldsymbol{\omega}_{er}^B\|^2+\|\boldsymbol{\omega}_{ed}^B\|^2\end{aligned} \tag{116}$$

So V_{o1} is a valid Lyapunov-like function candidate. According to (70) and (113), the differential of V_{o1} with respect to time satisfies,

$$\begin{aligned}\dot{V}_{o1}=&\mu\dot{V}_o+\dot{N}_o \\ \leq&-\mu\gamma\lambda(\frac{1}{4}\|\boldsymbol{r}_e^B\|^2+\|\boldsymbol{\xi}_e\|^2)-\|\boldsymbol{\omega}_{er}^B\|^2-\|\boldsymbol{\omega}_{ed}^B\|^2 \\ &+\alpha_1\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\|+\alpha_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\|+\alpha_3\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{er}^B\| \\ &+\alpha_4\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{ed}^B\|+\beta_1\|\boldsymbol{\xi}_e\|^2 \\ &+\beta_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{r}_e^B\|+\beta_3\|\boldsymbol{r}_e^B\|^2\end{aligned} \tag{117}$$

Utilize Cauchy-Schwarz inequality and complete squares, one has,

$$\begin{aligned}\dot{V}_{o1}\leq&-\mu\gamma\lambda(\frac{1}{8}\|\boldsymbol{r}_e^B\|^2+\frac{1}{2}\|\boldsymbol{\xi}_e\|^2)-\frac{1}{2}(\|\boldsymbol{\omega}_{er}^B\|^2+\|\boldsymbol{\omega}_{ed}^B\|^2) \\ &-(\frac{\mu\gamma\lambda}{8}-\alpha_3^2-\alpha_4^2-\frac{1}{2}\beta_2-\beta_3)\|\boldsymbol{r}_e^B\|^2 \\ &-(\frac{\mu\gamma\lambda}{2}-\alpha_1^2-\alpha_2^2-\beta_1-\frac{1}{2}\beta_2)\|\boldsymbol{\xi}_e\|^2 \\ &-(\alpha_1^2\|\boldsymbol{\xi}_e\|^2-\alpha_1\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^B\|+\frac{1}{4}\|\boldsymbol{\omega}_{er}^B\|) \\ &-(\alpha_2^2\|\boldsymbol{\xi}_e\|^2-\alpha_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^B\|+\frac{1}{4}\|\boldsymbol{\omega}_{ed}^B\|) \\ &-(\alpha_3^2\|\boldsymbol{r}_e^B\|^2-\alpha_3\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{er}^B\|+\frac{1}{4}\|\boldsymbol{\omega}_{er}^B\|) \\ &-(\alpha_4^2\|\boldsymbol{r}_e^B\|^2-\alpha_4\|\boldsymbol{r}_e^B\|\|\boldsymbol{\omega}_{ed}^B\|+\frac{1}{4}\|\boldsymbol{\omega}_{ed}^B\|)\end{aligned} \tag{118}$$

Consider (115), one can further get,

$$\begin{aligned}\dot{V}_{o1}\leq&-\frac{\mu\gamma\lambda}{8}(\|\boldsymbol{r}_e^B\|^2+4\|\boldsymbol{\xi}_e\|^2)-\frac{1}{2}(\|\boldsymbol{\omega}_{er}^B\|^2+\|\boldsymbol{\omega}_{ed}^B\|^2) \\ \leq&-\frac{\mu\gamma\lambda}{2}\hat{\boldsymbol{\varsigma}}_e\circ\hat{\boldsymbol{\varsigma}}_e-\frac{1}{2}\hat{\boldsymbol{\omega}}_e^B\circ\hat{\boldsymbol{\omega}}_e^B\end{aligned} \tag{119}$$

So V_{o1} is now a partially strict Lyapunov-like function, the asymptotic convergence of $\hat{\boldsymbol{\xi}}_e$ and $\hat{\boldsymbol{\omega}}_e$ can be directly guaranteed by (119), through the use of the corollary of Barbalat's lamma. The proof is completed.

V. Observer-based Controller Design and Separation Property Proof

After the strictification process, another important contribution of this paper is organized in Theorem 2.

Theorem 2: Consider the dual-quaternion based relative tracking dynamics in (48) and (49), with the observer proposed in (53) and (54), and the total control input designed in (74), then the tracking and estimation errors satisfy $\lim_{t\to\infty}[\hat{\boldsymbol{\xi}}_e(t), \hat{\boldsymbol{\omega}}_e^B(t), \hat{\boldsymbol{\xi}}_{bt}(t), \hat{\boldsymbol{\omega}}_{bt}^B(t)] = 0$.

Proof: Define the following Lyapunov-like function candidate, which is the combination of an observer part and a controller part,

$$V_{oc1} = \underbrace{\nu V_{o1}}_{\text{observer part}} + \underbrace{V_{c1}}_{\text{controller part}} \tag{120}$$

where $\nu = 4\kappa_2$ is a positive constant.

According to (96) and (119), the time derivative of V_{oc1} satisfies

$$\begin{aligned}\dot{V}_{oc1} \leq & -\frac{\nu\mu\gamma\lambda}{2}\hat{\boldsymbol{\xi}}_e \circ \hat{\boldsymbol{\xi}}_e - \frac{\nu}{2}\hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B \\ & - \frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{16}\|\boldsymbol{r}_{bt}^B\|^2 - \frac{\rho k_d}{4}\|\boldsymbol{\omega}_{bt}^B\|^2 \\ & - \frac{\rho k_d}{4}\|\boldsymbol{v}_{bt}^B + \boldsymbol{\omega}_{bt}^B \times \boldsymbol{r}_{bt}^B\|^2 + \kappa_2\hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B \\ \leq & -\frac{\nu\mu\gamma\lambda}{2}\hat{\boldsymbol{\varsigma}}_e \circ \hat{\boldsymbol{\varsigma}}_e - \frac{\nu}{4}\hat{\boldsymbol{\omega}}_e^B \circ \hat{\boldsymbol{\omega}}_e^B \\ & - \frac{\kappa_1}{4}\hat{\boldsymbol{\varsigma}}_{bt} \circ \hat{\boldsymbol{\varsigma}}_{bt} - \frac{\rho k_d}{4}\hat{\boldsymbol{\omega}}_{bt}^B \circ \hat{\boldsymbol{\omega}}_{bt}^B\end{aligned} \tag{121}$$

Because $V_{oc1} \geq 0$ and $\dot{V}_{oc1} \leq 0$, so $V_{oc1} \in \mathcal{L}_\infty$, and by the definition of V_{oc1}, we have $\hat{\boldsymbol{\varsigma}}_e, \hat{\boldsymbol{\omega}}_e^B, \hat{\boldsymbol{\varsigma}}_{bt}, \hat{\boldsymbol{\omega}}_{bt}^B \in \mathcal{L}_\infty$. Additionally, due to $\int_0^\infty \dot{V}_{oc1}(\sigma)\mathrm{d}\sigma$ exists, we can get $\hat{\boldsymbol{\varsigma}}_e, \hat{\boldsymbol{\omega}}_e^B, \hat{\boldsymbol{\varsigma}}_{bt}, \hat{\boldsymbol{\omega}}_{bt}^B \in \mathcal{L}_2$. Furthermore, recall (48), (49), (60) and (63), we have $\dot{\hat{\boldsymbol{\varsigma}}}_e, \dot{\hat{\boldsymbol{\omega}}}_e^B, \dot{\hat{\boldsymbol{\varsigma}}}_{bt}, \dot{\hat{\boldsymbol{\omega}}}_{bt}^B \in \mathcal{L}_\infty$. By the corollary of Barbalat's Lemma, these results guarantee

$$\lim_{t\to\infty}[\hat{\boldsymbol{\varsigma}}_e(t), \hat{\boldsymbol{\omega}}_e^B(t), \hat{\boldsymbol{\varsigma}}_{bt}(t), \hat{\boldsymbol{\omega}}_{bt}^B(t)] = 0 \tag{122}$$

And it is readily to demonstrate that $\hat{\boldsymbol{\varsigma}}_e(t) = 0$ and $\hat{\boldsymbol{\varsigma}}_{bt}(t) = 0$ actually guarantee $\hat{\boldsymbol{\xi}}_e(t) = 0$ and $\hat{\boldsymbol{\xi}}_{bt}(t) = 0$, respectively. So finally we get

$$\lim_{t\to\infty}[\hat{\boldsymbol{\xi}}_e(t), \hat{\boldsymbol{\omega}}_e^B(t), \hat{\boldsymbol{\xi}}_{bt}(t), \hat{\boldsymbol{\omega}}_{bt}^B(t)] = 0 \tag{123}$$

The proof is completed.

Remark 3: It's noteworthy that, except the controller and observer parameters k_p, k_d, λ and γ, all the other introduced parameters in this section (like ν, ρ, χ, l_i, p_i, κ_i, and so on) are employed purely for analysis. Which means these parameters put no restriction on the controller or the observer, and won't influence the performance of the closed-loop system.

Remark 4: The logic procedure in this section is crucial and rigorously ordered. Firstly, Theorem 1 guarantees the boundedness of the estimation error $\hat{\boldsymbol{\omega}}_e^B$. And then, as shown in Proposition 2, this result is combined with a strictification-like process, and enables $\hat{\boldsymbol{\omega}}_b^B$ and $\hat{\boldsymbol{\omega}}_o^B$ to be uniformly bounded. Subsequently, in Proposition 3, the boundedness of $\hat{\boldsymbol{\omega}}_b^B$ and $\hat{\boldsymbol{\omega}}_o^B$ are utilized to help us get a partially strict observer Lyapunov-like function, and finally the separation property is proofed in Theorem 2.

VI. Numerical Simulation

For observation and surveillance concerns, a pursuer spacecraft sometimes is required to maintain hovering with respect to an observation target (always an another spacecraft), which means the pursuer should maintain fixed relative distance and attitude with respect to the target. In this section, a prototypical spacecraft hovering task mission application is employed and simulated to illustrate the effectiveness of the proposed method.

Assume the observation target is running on a low-Earth orbit, and regard the Local-Vertical-Local-Horizontal (LVLH) reference frame of it as the inertia frame $\mathcal{N} = \{X_N, Y_N, Z_N\}$, wherein X_N is along the radius vector of the target, Z_N is along the angular momentum vector, and Y_N completes the right handed system. The initial attitude of the target is same with frame $\mathcal{N}$, and the angular velocity of it is $[0, 0, \pi/180]^{\mathrm{T}}$rad/s. In this simulation, the desired relative distance between two spacecrafts is 500m, and the pursuer should keep the same attitude with the target. Additionally, there is a vision sensor installed on the x-axis of the pursuer's body-fixed frame (i. e., frame $\mathcal{B}$), and to observe the target, the x-axis is required to always point to the target. So the pursuer should achieve a 6-DOF relative motionless with respect to the target (or rather, the pursuer should track a reference frame $\mathcal{T}$, which has the same attitude with the observation target, and proceeds a circular motion around it.).

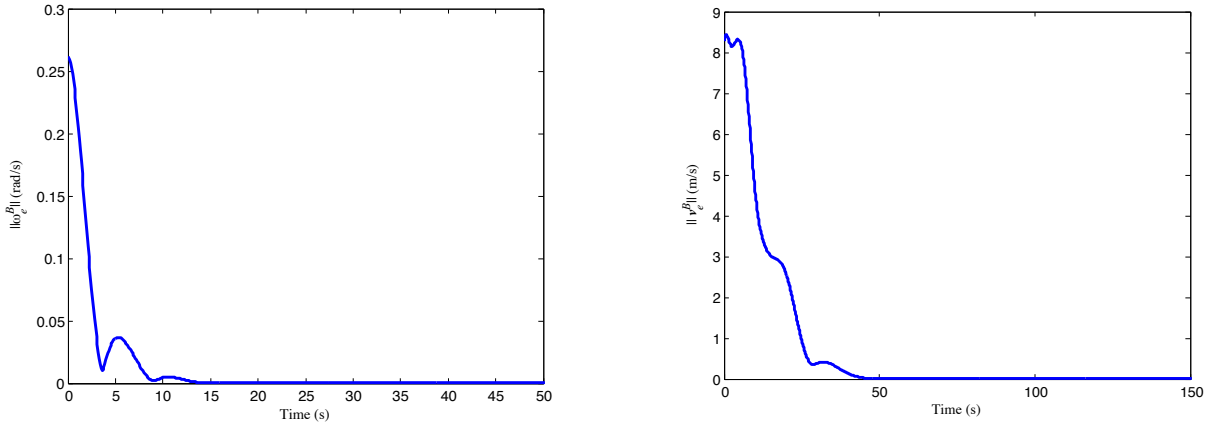
The mass and inertia of the pursuer are chosen as [34]: $m_b = 15$kg and

$$\boldsymbol{J}_b = \begin{bmatrix} 3.0514 & 0 & 0 \\ 0 & 2.6628 & 0 \\ 0 & 0 & 2.1879 \end{bmatrix} \mathrm{kg}\cdot\mathrm{m}^2 \tag{124}$$

The initial position and attitude of the pursuer is $\boldsymbol{r}_b^N(0) = [-400, -20, 20]^{\mathrm{T}}$m and $\boldsymbol{q}_b(0) = [0.8718, 0.4359, -0.2, 0.1]^{\mathrm{T}}$, respectively. And the initial velocity and angular velocity are set as $\boldsymbol{v}_b^N(0) = [0, 0, 0]^{\mathrm{T}}$m/s and $\boldsymbol{\omega}_b^N(0) = [\pi/12, 0, 0]^{\mathrm{T}}$rad/s, respectively. The pursuer can't measure both the linear and angular velocities of itself with respect to $\mathcal{N}$. The control parameters are chosen as $k_p = 2$, $k_d = 8$, $\lambda = 1.5$ and $\gamma = 3$. For comparison, the simulation results of the full-state PD-like feedback (denote as "PD") and the passivity-based velocity-free method in Ref. [23] (denote as "VF", and the linear time-invariant (LTI) system used in this controller is chosen similar with Ref. [23]) are also illustrated.

A. Normal Case

This subsection demonstrates the simulation results when there is no measurement noise of dual quaternions. Figure 3 shows the estimation errors of dual angular velocity (real part and dual part are in Fig. 3(a) and Fig. 3(b), respectively), it can be seen that the designed observer has a good performance, all estimation errors can converge to zero within 50s, with smooth trajectories and high precisions. The tracking errors of relative quaternion and relative position are shown in Fig. 4 and Fig. 5, respectively, and Fig. 6 further demonstrates the tracking errors of linear and angular velocities. These results show that all the three control methods can achieve the control objectives, and the performance of full-state feedback case is



(a) Angular velocity

(b) Linear velocity

Fig. 3: Estimation errors of angular and linear velocity

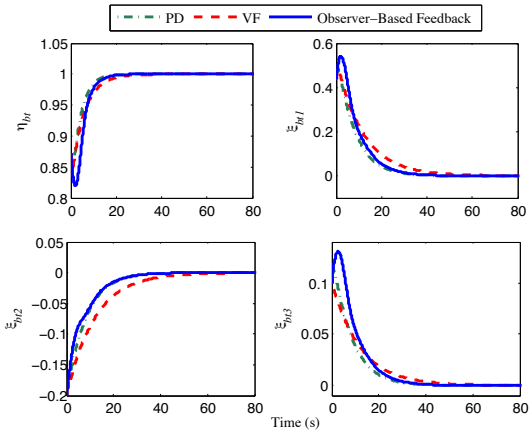
just slightly better than the other two methods. Comparing the two velocity-free methods, it is shown that the observer-based method proposed in this paper results in faster convergence processes and higher precisions with respect to the method given in Ref. [23], but due to the indispensable procedure to eliminate estimation errors, our method also leads to slight fluctuations, especially in the initial part of simulation. The reason why "VF" method needs a longer convergence time maybe become this method is based on the output of a special linear time-invariant system which can be regarded as a low-pass filter, this structure results in a smoother control process but also a extended convergence time. To further demonstrate the performance of the proposed method in this paper, 3D illustrations of tracking process is provided in Fig. 7, wherein the observation object and the pursuer are represented by spheres, and the mutually perpendicular lines pointing from the pursuer are employed to denote the instantaneous axis directions of frame $\mathcal{B}$. And instantaneous positions of the pursuer (denoted by light blue spheres) at $t = 0s$, 100s and 200s are given in Fig. 7. It can be seen that the proposed method has an excellent performance, the hovering task is totally achieved after 50s, then the pursuer can maintain 6-DOF relative motionless with respect to the target. Overall, the method proposed in this paper successfully completes the estimation and tracking task with reasonable convergence time and high precision.

B. Dual Quaternion Measurements with Noises

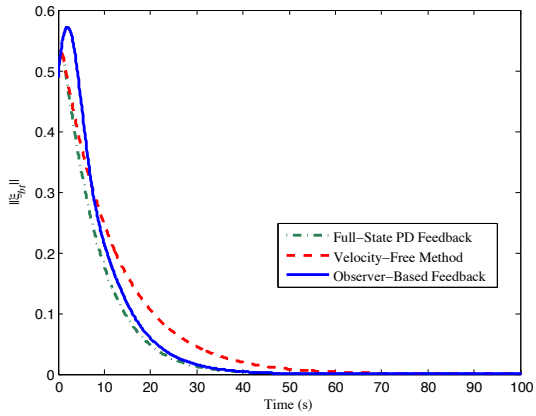
In this subsection, to further illustrate the robustness of the proposed observer-controller construction, measurement noises of dual quaternions are considered in simulation. To without loss of generality, zero-mean gaussian noises are employed, and the variance values of quaternion measurement noises and position measurement noises are chosen as 0.001 and 0.01, respectively. To clearly show the performance differences between different methods, in this subsection, all the results are shown in norms and are plotted on semi-logarithmic scales. Figures 8 and 9 illustrate the tracking errors of the closed-loop system, it can be seen that the overall performance of all the three control methods suffer from the presence of measurement noises, and tracking errors can only converge to a small residual set around the origin. For the noise tolerance concerns, the proposed method has a almost same steady-state performance with the full-state feedback case, and is substantially superior to the velocity-free method given in Ref. [23]. The estimation errors of observer are shown in Fig. 10, though the values of steady-state errors are influenced by noisy measurements, but the proposed observer can still render the estimation errors converge to a small bounded value with a rapid convergence process. Additionally, extensive simulations have also been performed under measurement noises with various control parameters and initial conditions, which confirm that the observer-controller construction proposed in this paper is not adversely affected any worse than the full-state feedback control law in the presence of measurement noises.

VII. Conclusion

This paper addresses the dual angular velocity observer design problem for the 6-DOF tracking control of rigid body in the absence of both linear and angular velocity measurements. To establish the observer construction, some important properties of dual vector and dual quaternion are studied, the kinematics of dual transformation matrix is derived, and the transition relationship between dual quaternion and dual transformation matrix is subsequently analyzed. Based on these mathematical foundations, a smooth observer is proposed to estimate the dual angular velocity of rigid body, and the estimation errors are guaranteed to be global asymptotic convergence irrespective of control inputs. Subsequently, the observer is combined with a separately designed PD-like controller. A partial Lyapunov strictification process is employed to transform nonstrict Lyapunov functions into strict ones whose derivatives contain additional non-positive terms, then the separation property is proofed and the almost global asymptotic convergence of the closed-loop system is guaranteed accordingly. Numerical simulation for a 6-DOF spacecraft hovering mission is carried out, which shows that the proposed observer-controller construction has an excellent

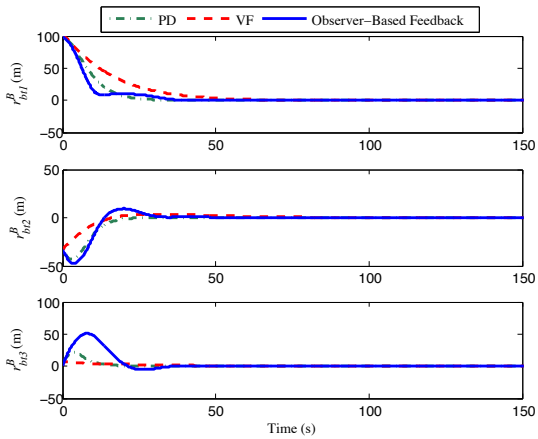


(a) Tracking errors

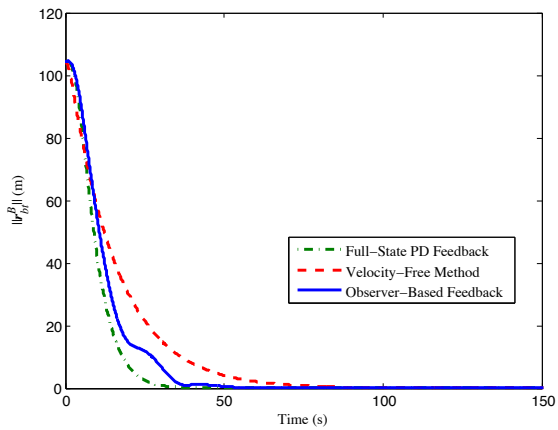


(b) Norm of tracking errors

Fig. 4: Tracking errors of relative quaternion

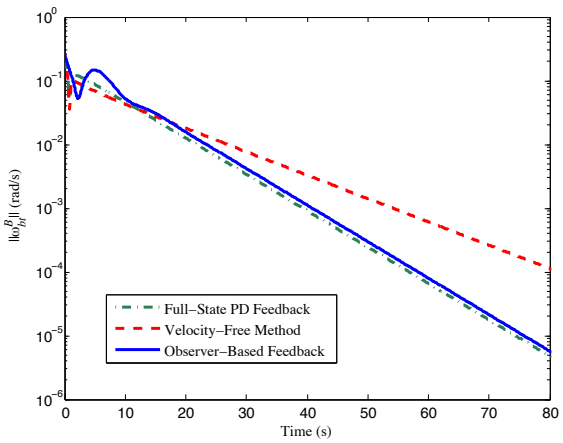


(a) Tracking errors

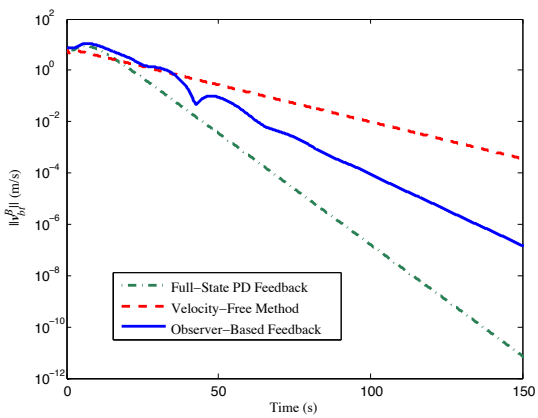


(b) Norm of tracking errors

Fig. 5: Tracking errors of relative position

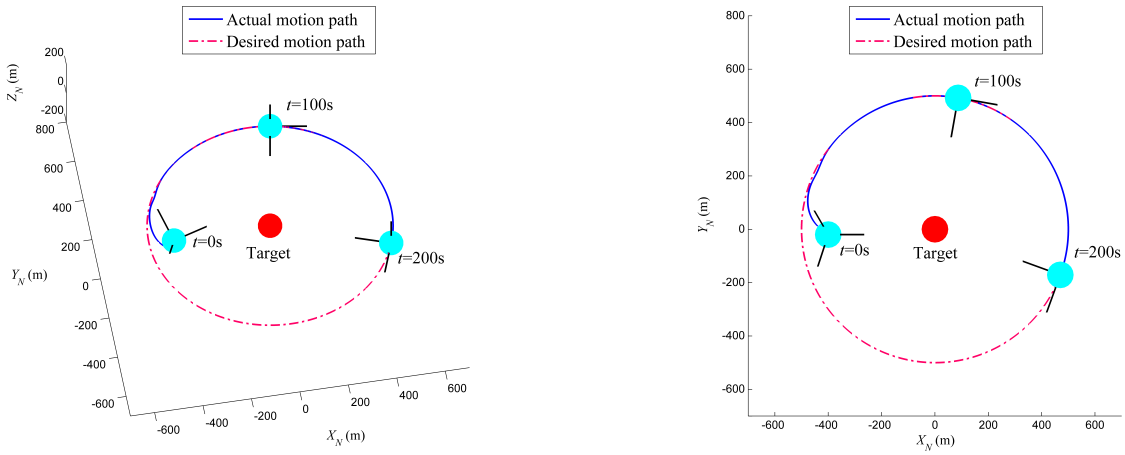


(a) Relative angular velocity



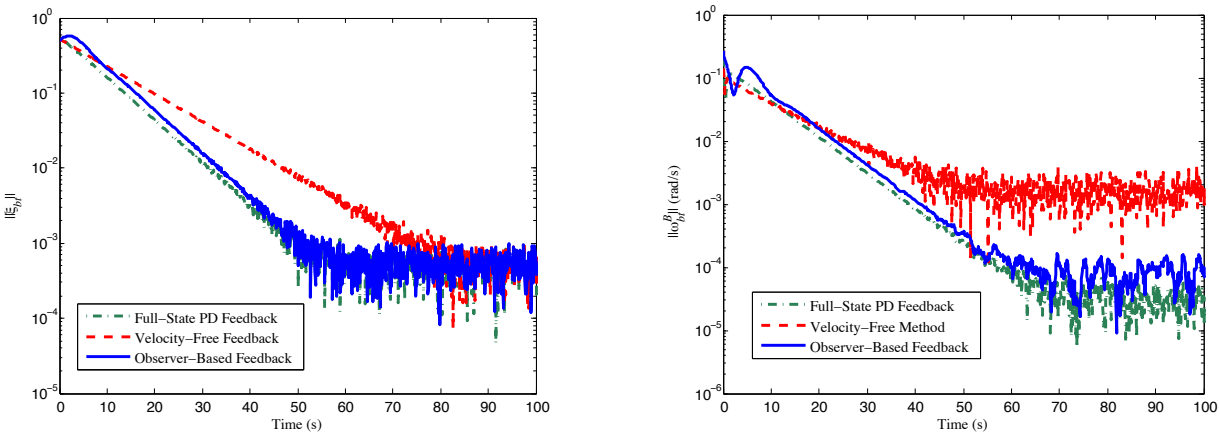
(b) Relative linear velocity

Fig. 6: Tracking errors of relative angular and linear velocity



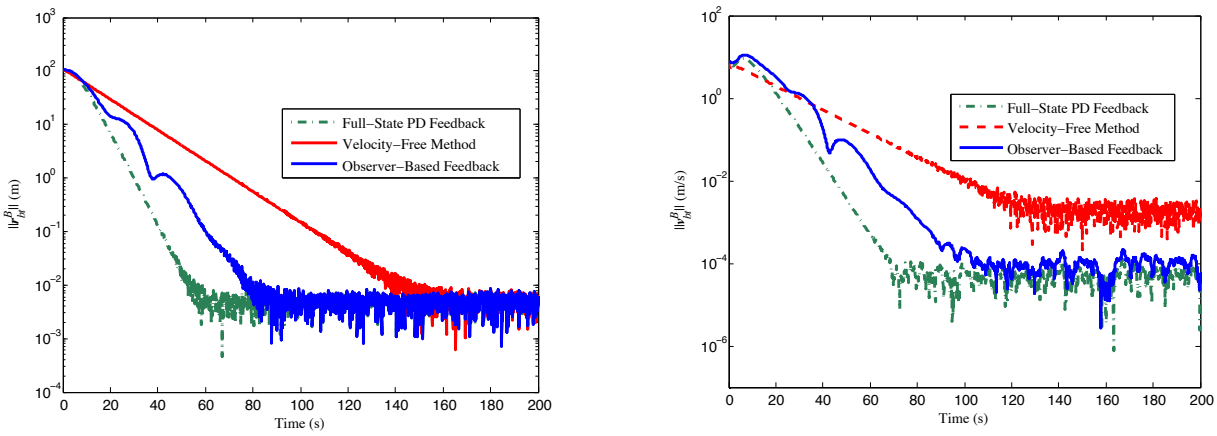
(a) (b)

Fig. 7: 3D illustration



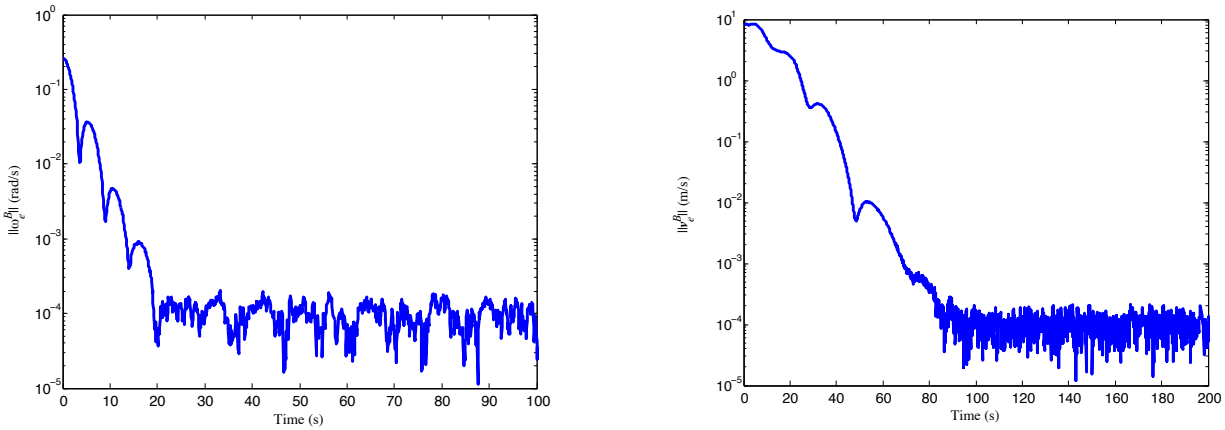
(a) Relative quaternion (b) Relative angular velocity

Fig. 8: Tracking errors of relative quaternion and angular velocity



(a) Relative position (b) Relative velocity

Fig. 9: Tracking errors of relative position and velocity



(a) Angular velocity (b) Linear velocity

Fig. 10: Estimation errors of dual angular velocity

performance even in the presence of noisy measurements of dual quaternion.

Appendix A
Proof of Lemma 1

For $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{c}} \in \hat{\mathbb{R}}^3$, from Eq. (11), one has:

$$\begin{aligned}&\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) = \\ &\quad (\boldsymbol{a}_r + \varepsilon \boldsymbol{a}_d) \times (\boldsymbol{b}_r \times \boldsymbol{c}_r + \varepsilon(\boldsymbol{b}_r \times \boldsymbol{c}_d + \boldsymbol{b}_d \times \boldsymbol{c}_r)) \\ &= \boldsymbol{a}_r \times (\boldsymbol{b}_r \times \boldsymbol{c}_r) + \varepsilon(\boldsymbol{a}_r \times (\boldsymbol{b}_r \times \boldsymbol{c}_d) + ... \\ &\quad \boldsymbol{a}_r \times (\boldsymbol{b}_d \times \boldsymbol{c}_r) + \boldsymbol{a}_d \times (\boldsymbol{b}_r \times \boldsymbol{c}_r))\end{aligned} \tag{125}$$

Similarly:

$$\begin{aligned}\hat{\boldsymbol{b}} \times (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{a}}) =& \boldsymbol{b}_r \times (\boldsymbol{c}_r \times \boldsymbol{a}_r) + \varepsilon(\boldsymbol{b}_r \times (\boldsymbol{c}_r \times \boldsymbol{a}_d) + ... \\ & \boldsymbol{b}_r \times (\boldsymbol{c}_d \times \boldsymbol{a}_r) + \boldsymbol{b}_d \times (\boldsymbol{c}_r \times \boldsymbol{a}_r))\end{aligned} \tag{126}$$

and

$$\begin{aligned}\hat{\boldsymbol{c}} \times (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) =& \boldsymbol{c}_r \times (\boldsymbol{a}_r \times \boldsymbol{b}_r) + \varepsilon(\boldsymbol{c}_r \times (\boldsymbol{a}_r \times \boldsymbol{b}_d) + ... \\ & \boldsymbol{c}_r \times (\boldsymbol{a}_d \times \boldsymbol{b}_r) + \boldsymbol{c}_d \times (\boldsymbol{a}_r \times \boldsymbol{b}_r))\end{aligned} \tag{127}$$

Consider the Jacobi identity of real 3-dimensional vectors:

$$\boldsymbol{a}_* \times (\boldsymbol{b}_\star \times \boldsymbol{c}_\circ) + \boldsymbol{b}_\star \times (\boldsymbol{c}_\circ \times \boldsymbol{a}_*) + \boldsymbol{c}_\circ \times (\boldsymbol{a}_* \times \boldsymbol{b}_\star) = \boldsymbol{0} \tag{128}$$

where $*, \star, \circ$ denote r or d, so by iteratively using this identity, finally one can get:

$$\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) + \hat{\boldsymbol{b}} \times (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{a}}) + \hat{\boldsymbol{c}} \times (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \boldsymbol{0} \tag{129}$$

The lemma is proved.

Appendix B
Proof of Lemma 2

By Eqs. (6) and (10), one has:

$$\begin{aligned}\hat{\boldsymbol{b}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{c}}) =& (\boldsymbol{b}_r + \varepsilon \boldsymbol{b}_d)(\boldsymbol{a}_r \cdot \boldsymbol{c}_r + \varepsilon(\boldsymbol{a}_r \cdot \boldsymbol{c}_d + \boldsymbol{a}_d \cdot \boldsymbol{c}_r)) \\ =& \boldsymbol{b}_r(\boldsymbol{a}_r \cdot \boldsymbol{c}_r) + \varepsilon(\boldsymbol{b}_r(\boldsymbol{a}_r \cdot \boldsymbol{c}_d) + ... \\ & \boldsymbol{b}_d(\boldsymbol{a}_r \cdot \boldsymbol{c}_r) + \boldsymbol{b}_r(\boldsymbol{a}_d \cdot \boldsymbol{c}_r))\end{aligned} \tag{130}$$

and

$$\hat{\boldsymbol{c}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}) = \boldsymbol{c}_r(\boldsymbol{a}_r \cdot \boldsymbol{b}_r) + \varepsilon(\boldsymbol{c}_d(\boldsymbol{a}_r \cdot \boldsymbol{b}_r) + \boldsymbol{c}_r(\boldsymbol{a}_r \cdot \boldsymbol{b}_d) + \boldsymbol{c}_r(\boldsymbol{a}_d \cdot \boldsymbol{b}_r)) \tag{131}$$

The real 3-dimensional vectors follows the property:

$$\boldsymbol{a}_* \times (\boldsymbol{b}_\star \times \boldsymbol{c}_\circ) = \boldsymbol{b}_\star(\boldsymbol{a}_* \cdot \boldsymbol{c}_\circ) - \boldsymbol{c}_\circ(\boldsymbol{a}_* \cdot \boldsymbol{b}_\star) \tag{132}$$

So according to Eqs. (125), (130) and (131), and by iteratively using the above property, one can obtain:

$$\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) = \hat{\boldsymbol{b}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{c}}) - \hat{\boldsymbol{c}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}) \tag{133}$$

The lemma is proved.

Appendix C
Proof of Lemma 3

By Eqs. (10) and (11), and consider the properties of cross multiplication of real vector, we have:

$$\begin{aligned}(\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) \cdot \hat{\boldsymbol{a}} =& (\boldsymbol{a}_r \times \boldsymbol{b}_r + \varepsilon(\boldsymbol{a}_r \times \boldsymbol{b}_d + \boldsymbol{a}_d \times \boldsymbol{b}_r)) \cdot (\boldsymbol{a}_r + \varepsilon \boldsymbol{a}_d) \\ =& (\boldsymbol{a}_r \times \boldsymbol{b}_r) \cdot \boldsymbol{a}_r + \varepsilon((\boldsymbol{a}_r \times \boldsymbol{b}_r) \cdot \boldsymbol{a}_d + ... \\ & (\boldsymbol{a}_r \times \boldsymbol{b}_d) \cdot \boldsymbol{a}_r + (\boldsymbol{a}_d \times \boldsymbol{b}_r) \cdot \boldsymbol{a}_r)) \\ =& \boldsymbol{0} + \varepsilon((\boldsymbol{a}_r \times \boldsymbol{b}_r) \cdot \boldsymbol{a}_d + (\boldsymbol{a}_d \times \boldsymbol{b}_r) \cdot \boldsymbol{a}_r)) \\ =& \boldsymbol{0}\end{aligned} \tag{134}$$

Furthermore:

$$\hat{\boldsymbol{a}} \cdot (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = -(\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) \cdot \hat{\boldsymbol{a}} = \boldsymbol{0} \tag{135}$$

The lemma is proved.

Appendix D
Proof of Lemma 4

By (7), (11) and (13),

$$\begin{aligned}\hat{\boldsymbol{a}}^s \circ (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) =& (\boldsymbol{a}_d + \varepsilon \boldsymbol{a}_r) \circ (\boldsymbol{a}_r \times \boldsymbol{b}_r + ... \\ & \varepsilon(\boldsymbol{a}_r \times \boldsymbol{b}_d + \boldsymbol{a}_d \times \boldsymbol{b}_r)) \\ =& \boldsymbol{a}_d^{\mathrm{T}}(\boldsymbol{a}_r \times \boldsymbol{b}_r) + \boldsymbol{a}_r^{\mathrm{T}}(\boldsymbol{a}_d \times \boldsymbol{b}_r) \\ =& \boldsymbol{0}\end{aligned} \tag{136}$$

The lemma is proved.

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