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► **To cite this version:**

Qinghua Zhang, Liangquan Zhang. State Estimation for Stochastic Time Varying Systems with Disturbance Rejection. SYSID 2018 - 18th IFAC Symposium on System Identification, Jul 2018, Stockholm, Sweden. pp.55-59, 10.1016/j.ifacol.2018.09.090 . hal-01909572

**HAL Id: hal-01909572**

**<https://inria.hal.science/hal-01909572>**

Submitted on 31 Oct 2018

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# State Estimation for Stochastic Time Varying Systems with Disturbance Rejection<sup>\*</sup>

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**Abstract:** State estimation in the presence of unknown disturbances is useful for the design of robust systems in different engineering fields. Most results available on this topic are restricted to linear time invariant (LTI) systems, whereas linear time varying (LTV) systems have been studied to a lesser extent. Existing results on LTV systems are mainly based on the minimization of the state estimation error covariance, ignoring the important issue of the stability of the state estimation error dynamics, which has been a main focus of the studies in the LTI case. The purpose of this paper is to propose a numerically efficient algorithm for state estimation with disturbance rejection, in the general framework of LTV stochastic systems, including linear parameter varying (LPV) systems, with easily checkable conditions guaranteeing the stability of the algorithm. The design method is conceptually simple: disturbance is first rejected from the state equation by appropriate output injection, then the Kalman filter is applied to the resulting state-space model after the output injection.

*Keywords:* disturbance rejection, state estimation, LTV/LPV system, Kalman filter.

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## 1. INTRODUCTION

State estimation is a common task in different engineering fields where dynamic systems are described in state-space form, for instance, in automatic control for feedback control, in fault diagnosis for detecting abnormal state trajectories, and in chemical or bioprocess engineering as soft sensors for production monitoring. Among classical state estimation methods, the Kalman filter (Kalman, 1963) and the Luenberger observer (Luenberger, 1971) are well known algorithms.

More recently, some studies are focused on state estimation in the presence of (unknown) *disturbances*, also known as *unknown inputs*, in order to develop algorithms robust to disturbances or unknown inputs. The earliest works on this topic were mainly about linear time invariant (LTI) *deterministic* systems, for example, (Yang and Wilde, 1988; Darouach et al., 1994; Chen and Patton, 1999). These works aimed at designing observers under some matrix constraints, ensuring *stable* state estimation error equations, which are not affected by disturbances. To explore the degrees of freedom left by such matrix constraints, techniques of linear matrix inequalities (LMI) are used in some works to minimize criterions based on  $H_\infty$  and  $L_2$  norms (Gao et al., 2016; Bezzaoucha et al., 2017). Some other studies have considered LTI *stochastic* systems, by designing state estimators based on the minimum variance criterion (Darouach et al., 1995; Keller et al., 1998).

Compared to LTI systems, linear time varying (LTV) systems have been studied to a lesser extent. Existing results on LTV systems have been mainly focused on the design of algorithms minimizing the state estimation error covariance (Kitanidis, 1987; Darouach and Zasadzinski, 1997; Hsieh, 2000; Gillijns and De Moor, 2007), ignoring the important issue of the stability of the state estimation error equations. In (Darouach and Zasadzinski, 1997), the stability is studied for LTI systems only, though the algorithm proposed in the paper is formulated for general LTV systems. This lack of stability result is due to the difficulty for analyzing LTV systems. As a matter of fact, transfer functions, as a widely used tool for LTI system analysis, are not available for general LTV systems. Nevertheless, it is important to study LTV systems, including linear parameter varying (LPV) systems, since a large class of nonlinear systems can be addressed through LTV/LPV reformulation and approximation (Tóth, 2010).

Because an LTV model corresponds to an LTI model *at every time instant*, a naive solution for LTV systems would be to apply any method developed for LTI systems, by renewing the designed algorithm at every time instant. However, the fact that at every time instant an LTV system is stable in the LTI sense does not ensure the stability of the whole LTV system. This fact is well known in the classical system theory. It is thus a non trivial task to generalize results developed for LTI systems to LTV systems.

*The purpose* of this paper is to propose a numerically efficient algorithm for state estimation with disturbance rejection, in the general framework of LTV stochastic systems, *with easily checkable conditions guaranteeing the stability*

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<sup>\*</sup> L. Zhang acknowledges the financial support partly by the National Nature Science Foundation of China (11701040) and Innovation Foundation of BUPT for Youth (500417024).

of the algorithm. Notably, this algorithm is applicable to LPV systems, which are viewed as a particular class of LPV systems.

The method for designing this algorithm is conceptually simple: disturbances are rejected from the state equation by output injection, then the classical Kalman filter is applied to the new state-space model obtained after the output injection. To the best of our knowledge, this is the first result for LTV systems with disturbance rejection ensuring the stability of the state estimation error equation.

This paper is organized as follows. In Section 2 the considered problem is formulated. In Section 3 the output injection for disturbance rejection is presented. In Section 4 the Kalman filter is applied to the system after output injection. Numerical examples are presented in Section 5. Finally, conclusions are drawn in Section 6.

## 2. PROBLEM FORMULATION

Consider LTV stochastic systems in the form of

$$x_{k+1} = A_k x_k + B_k u_k + E_k d_k + w_k \quad (1a)$$

$$y_k = C_k x_k + v_k \quad (1b)$$

where  $x_k \in \mathbb{R}^n$  is the state,  $y_k \in \mathbb{R}^m$  the output,  $u_k \in \mathbb{R}^p$  the (known) input,  $d_k \in \mathbb{R}^q$  the *disturbance* (or unknown input),  $w_k \in \mathbb{R}^n$  the state noise,  $v_k \in \mathbb{R}^m$  the output noise, and  $A_k, B_k, C_k, E_k$  are known matrices of appropriate sizes. The purpose of this paper is to design an algorithm for the estimation of the state  $x_k$  from the known input  $u_k$ , the output  $y_k$  and the known matrices  $A_k, B_k, C_k, E_k$ . While the noises  $w_k, v_k$  are random variables, the disturbance  $d_k$  is completely arbitrary, notably, no probability distribution, nor upper bound of  $d_k$ , is assumed.

The fact that  $E_k$  is known, but  $d_k$  is completely unknown, means that, at each instant  $k$ , a completely arbitrary disturbance can only affect a known subspace of the state-space, and this subspace may vary with  $k$ . The arbitrary character of  $d_k$  means that there is a void subspace in the state equation. Typically,  $E_k$  is filled with 0's and 1's. For example, at the instant  $k$ , if  $E_k = [1, 0, \dots, 0]^T$ , then the first component of the state equation (1a) is void. It may appear impossible to estimate the state  $x_k$  when part of the state equation is missing. Of course, compared to the classical case where the term  $E_k d_k$  did not exist, here some extra assumptions are required so that state estimation remains feasible, like the last two assumptions listed below.

*Basic assumptions.*

- (i)  $A_k, B_k, C_k, E_k$  are bounded matrix sequences.
- (ii) The initial state  $x_0 \in \mathbb{R}^n$  is a random vector following the Gaussian distribution  $\mathcal{N}(\bar{x}_0, P_0)$ , with a mean vector  $\bar{x}_0$  and a positive definite covariance matrix  $P_0$ .
- (iii)  $w_k$  and  $v_k$  are zero mean white Gaussian noises independent of each other and of  $x_0$ , with bounded covariance matrices  $E(w_k w_k^T) = Q_k$  and  $E(v_k v_k^T) = R_k$ .

These assumptions are usually made in the classical LTV system Kalman filter theory, apart from the involved matrix  $E_k$  that did not exist in the classical case.

*Disturbance subspace assumptions.*

- (iv)  $q \leq m$  (no more disturbances than output sensors).
- (v) There exists a positive constant  $\gamma$  such that, for all  $k = 0, 1, 2, \dots$ , the smallest singular value of the matrix product  $C_{k+1} E_k$  is not smaller than  $\gamma$ .

These assumptions are necessary for numerically reliable rejection of disturbances. Notice that Assumptions (iv) and (v) imply that  $C_{k+1} E_k$  has a full column rank.

In the classical Kalman filter theory, the stability of the Kalman filter mainly relies on observability and controllability assumptions. Similar assumptions will be formulated in the next section, after the disturbance rejection by means of output injection.

## 3. OUTPUT INJECTION FOR DISTURBANCE REJECTION

The purpose this section is to reject the disturbance term  $E_k d_k$  from the state equation by means of output injection, so that the Kalman filter can be applied without being affected by the disturbance.

In (Kitanidis, 1987), linear filters in the form of

$$\hat{x}_{k+1|k+1} = A_k \hat{x}_{k|k} + L_{k+1}(y_{k+1} - C_{k+1} A_k \hat{x}_{k|k}) \quad (2)$$

were considered, neglecting the *known* input term  $B_k u_k$ , which is not essential in linear filter design problems. The filter gain  $L_{k+1}$  was determined by minimizing the state estimation error covariance, under the constraint

$$L_{k+1} C_{k+1} E_k - E_k = 0. \quad (3)$$

See the equation (8) in (Kitanidis, 1987), formulated with different notations. This constraint ensures that the state estimation error is not affected by the disturbance term  $E_k d_k$ . The solution to this constrained optimization problem leads to a filter gain that is considerably more sophisticated than the gain of the Kalman filter, and there is no obvious way to analyze the stability of the resulting filter.

The main idea of the present paper is to reject the disturbance term before the filter design. In the first step, the disturbance rejection is achieved with an output injection satisfying a condition similar to constraint (3). Then the filter design is made as if there was no disturbance term in the considered problem, simplifying the stability analysis of the resulting filter.

Let  $G_k \in \mathbb{R}^{n \times m}$  be a bounded matrix sequence to be specified later. The following equation holds due to (1b):

$$0 = G_{k+1}(y_{k+1} - C_{k+1} x_{k+1} - v_{k+1}). \quad (4)$$

Add each side of this equation to the corresponding side of (1a), then

$$x_{k+1} = A_k x_k + B_k u_k + E_k d_k + w_k + G_{k+1}(y_{k+1} - C_{k+1} x_{k+1} - v_{k+1}). \quad (5)$$

For the purpose of disturbance rejection in the following steps, it is important that this output injection is made with the output equation at instant  $k+1$ , instead of  $k$ .

At the right hand side of equation (5), substitute  $x_{k+1}$  with (1a), then, after some rearrangement (with  $I_n$  denoting the  $n \times n$  identity matrix),

$$x_{k+1} = (I_n - G_{k+1} C_{k+1}) A_k x_k + (I_n - G_{k+1} C_{k+1}) B_k u_k + (I_n - G_{k+1} C_{k+1}) E_k d_k + G_{k+1} y_{k+1} + (I_n - G_{k+1} C_{k+1}) w_k - G_{k+1} v_{k+1}. \quad (6)$$

If the matrix sequence  $G_k$  is chosen such that, at every time instant  $k$ ,

$$(I_n - G_{k+1}C_{k+1})E_k = 0, \quad (7)$$

then equation (6) becomes

$$x_{k+1} = \bar{A}_k x_k + \bar{B}_k u_k + G_{k+1} y_{k+1} + \bar{w}_k, \quad (8)$$

with

$$\bar{A}_k \triangleq (I_n - G_{k+1}C_{k+1})A_k \quad (9)$$

$$\bar{B}_k \triangleq (I_n - G_{k+1}C_{k+1})B_k \quad (10)$$

$$\bar{w}_k \triangleq (I_n - G_{k+1}C_{k+1})w_k - G_{k+1}v_{k+1}. \quad (11)$$

Now the disturbance term  $E_k d_k$  has disappeared from equation (8).

*Remark 1.* Condition (7) that should be satisfied by  $G_k$  is the same as (3), which was imposed to the Kitanidis filter gain  $L_k$ . However, here more degrees of freedom are left to the choice of  $G_k$ , which is not determined by the minimum covariance criterion, unlike the Kitanidis filter gain  $L_k$ .  $\square$

The covariance matrix of the new state noise  $\bar{w}_k$  is

$$\bar{Q}_k \triangleq (I_n - G_{k+1}C_{k+1})Q_k(I_n - G_{k+1}C_{k+1})^T + G_{k+1}R_{k+1}G_{k+1}^T. \quad (12)$$

Rewrite equation (7) as

$$G_{k+1}C_{k+1}E_k = E_k, \quad (13)$$

which should be solved for the unknown matrix  $G_{k+1}$ .

Due to Assumption (iv), the matrix product  $C_{k+1}E_k$  has no more columns than rows, thus in most situations, solutions of  $G_{k+1}$  exist. A sufficient and necessary condition for the existence of solutions is that each row of  $E_k$  is a linear combination of the rows of the matrix product  $C_{k+1}E_k$ . This condition is ensured by assumption (v).

Given an  $m \times n$  matrix  $M$ , another  $n \times m$  matrix  $M^g$  is a generalized inverse of  $M$  if  $MM^gM = M$ . With this notation, the solution

$$G_{k+1} = E_k(C_{k+1}E_k)^g \quad (14)$$

with any generalized inverse  $(C_{k+1}E_k)^g$  satisfies equation (13).

The particular solution

$$G_{k+1} = E_k[(C_{k+1}E_k)^T(C_{k+1}E_k)]^{-1}(C_{k+1}E_k)^T \quad (15)$$

will be adopted in the remaining part of this paper. The boundedness of this solution  $G_{k+1}$  is then ensured by Assumptions (i) and (v). Consequently, with this particular sequence  $G_k$ , the matrices  $\bar{A}_k$  and  $\bar{Q}_k$  defined respectively in (9) and (12) are also bounded.

Now it is time to formulate the last assumptions ensuring the stability of the state estimator proposed in this paper.

#### 4. KALMAN FILTER, STABILITY AND OPTIMALITY

Consider the state-space model composed of the state equation (8) and the output equation (1b), which are grouped below for ease of later references:

$$x_{k+1} = \bar{A}_k x_k + \bar{B}_k u_k + G_{k+1} y_{k+1} + \bar{w}_k, \quad (16a)$$

$$y_k = C_k x_k + v_k. \quad (16b)$$

The state  $x_k$ , the input  $u_k$  and the output  $y_k$  of this new model are the same as in the original system model (1). In other words, these two models represent the same system. In this sense, the two models are equivalent.

By treating  $G_{k+1}y_{k+1}$  as a known input term, the classical Kalman filter can be applied to the state-space model (16), yielding a state estimator for the system originally described by (1). It is indeed an application of the *classical* Kalman filter as originally published in (Kalman, 1963), as no disturbance term appears in the state-space model (16). This is an indirect way for addressing the initial state estimation problem formulated with (1).

For the sake of simplicity, let us neglect the correlation between  $\bar{w}_k$  and  $v_k$ . Then the classical Kalman filter applied to (16) is as follows. After the initialization

$$\hat{x}_{0|0} = \bar{x}_0 \quad (17a)$$

$$P_{0|0} = P_0, \quad (17b)$$

the state estimate  $\hat{x}_{k|k}$  and its covariance matrix  $P_{k|k}$  are then recursively computed, for  $k = 0, 1, 2, \dots$ ,

$$P_{k+1|k} = \bar{A}_k P_{k|k} \bar{A}_k^T + \bar{Q}_k \quad (18a)$$

$$\Sigma_{k+1} = C_{k+1} P_{k+1|k} C_{k+1}^T + R_{k+1} \quad (18b)$$

$$K_{k+1} = P_{k+1|k} C_{k+1}^T \Sigma_{k+1}^{-1} \quad (18c)$$

$$P_{k+1|k+1} = (I_n - K_{k+1} C_{k+1}) P_{k+1|k} \quad (18d)$$

$$\hat{x}_{k+1|k} = \bar{A}_k \hat{x}_{k|k} + \bar{B}_k u_k + G_{k+1} y_{k+1} \quad (18e)$$

$$\tilde{y}_{k+1} = y_{k+1} - C_{k+1} \hat{x}_{k+1|k} \quad (18f)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \tilde{y}_{k+1}. \quad (18g)$$

In order to ensure the stability of this Kalman filter, two more assumptions will be formulated below, in addition to the assumptions already made in Section 2. These assumptions are slightly different from those assumed in the classical Kalman filter theory, which is not suitable for the particular case considered here, as explained later in the two remarks following the statements of Properties 1 and 2.

*Observability and controllability assumptions.*

(vi) The matrix sequence pair  $[\bar{A}_k, C_k]$  is uniformly completely observable.

(vii) The matrix sequence pair  $[\bar{A}_k(I_n - K_k C_k), \bar{Q}_k^{\frac{1}{2}}]$  is uniformly completely controllable, where  $\bar{Q}_k^{\frac{1}{2}}$  is the matrix square root of  $\bar{Q}_k$ .

The uniform complete observability and controllability are defined with the aid of Gramian matrices. See (Kalman, 1963; Jazwinski, 1970).

Notice that these observability and controllability assumptions involve the matrix  $\bar{A}_k$  depending on  $G_k$ , which in turn depends on  $E_k$ . Like in the classical Kalman filter theory, if the involved matrices are known in advance (typically constant or periodical), then these assumptions can be checked in advance, otherwise they have to be checked in real time (e.g. for LPV systems).

*Property 1.* (Boundedness). Under Assumptions (i)-(vii), the recursively computed matrices  $P_{k|k}$  and  $P_{k+1|k}$  are bounded, so are the innovation covariance  $\Sigma_k$  and the Kalman gain  $K_k$ .  $\square$

This property ensures the boundedness of all the variables involved in the recursive computations of the Kalman filter except the states estimates. As Gaussian noises are not bounded, the state estimation errors are (in principle) not

bounded, but the error covariance matrices are bounded. Obviously, boundedness is crucial for any recursive algorithm in real time applications.

*Property 2.* (Stability). Under Assumptions (i)-(vii), the error dynamics of the state estimate  $\hat{x}_{k|k}$  of the Kalman filter (18) is exponentially stable.  $\square$

*Remark 2.* Recall that the boundedness of the sequence  $\bar{G}_k$  is ensured by Assumptions (i) and (v). When applying the Kalman filter to (16),  $G_{k+1}y_{k+1}$  is treated as a known *bounded* input term. Then it may appear that Properties 1 and 2 can be ensured by the classical Kalman filter theory (Kalman, 1963; Jazwinski, 1970). In fact, one of the important conditions required by the classical Kalman filter theory is not satisfied in the present case: when applied to system (16), the classical theory would assume that the matrix  $\bar{A}_k$  is invertible for all time instant  $k$ , whereas this matrix defined in (9) is always *singular*, because of equality (7). Fortunately, a new proof of the Kalman filter stability, ensuring Properties 1 and 2, has been worked out, without requiring invertible  $\bar{A}_k$ . As this complete remake of the Kalman filter stability proof is much longer than the present paper, it will be reported elsewhere.  $\square$

*Remark 3.* Despite the fact that the correlation between  $\bar{w}_k$  and  $v_k$  has been neglected by the Kalman filter applied to system (16), Properties 1 and 2 hold for the following reasons.

The computations of  $P_{k+1|k}$ ,  $P_{k+1|k+1}$ ,  $\Sigma_{k+1}$  and  $K_{k+1}$  are fully determined by  $\bar{A}_k$ ,  $C_{k+1}$ ,  $\bar{Q}_k$  and  $R_{k+1}$  through (18a)-(18d), hence Property 1 is a consequence of the assumptions made on  $\bar{A}_k$ ,  $C_k$ ,  $\bar{Q}_k$ ,  $R_k$ , no matter  $\bar{w}_k$  and  $v_k$  are correlated or not.

Let the state estimation error be denoted by

$$\tilde{x}_{k|k} \triangleq x_k - \hat{x}_{k|k}, \quad (19)$$

then the error dynamics equation is

$$\tilde{x}_{k+1|k+1} = (I_n - K_{k+1}C_{k+1})\bar{A}_k\tilde{x}_{k|k} \quad (20)$$

$$+ (I_n - K_{k+1}C_{k+1})\bar{w}_k - K_{k+1}v_{k+1}. \quad (21)$$

Like in (Kalman, 1963; Jazwinski, 1970), the stability of the error dynamics concerns only the deterministic part of this error equation, fully characterized by the matrix  $(I_n - K_{k+1}C_{k+1})\bar{A}_k$ . As the computation of  $K_{k+1}$  is fully determined by  $\bar{A}_k$ ,  $C_{k+1}$ ,  $\bar{Q}_k$  and  $R_{k+1}$  through (18a)-(18d), the stability of the error dynamics is fully ensured by the assumptions made on  $\bar{A}_k$ ,  $C_k$ ,  $\bar{Q}_k$ ,  $R_k$ . Therefore, Property 2 is not affected by the correlation between  $\bar{w}_k$  and  $v_k$ .  $\square$

Another well known property of the Kalman filter is its optimality. As the correlation between the noises  $\bar{w}_k$  and  $v_k$  was neglected when formulating the Kalman filter (18), the optimality does not exactly hold, but the result should not be far from being optimal. The correlation between the state noise and the output noise is often ignored in Kalman filter applications. In the present case, it is not trivial to take into account the correlation caused by output injection. Further investigations are necessary in order to improve the optimality of the proposed algorithm.

## 5. NUMERICAL EXAMPLES

In order to illustrate the state estimator proposed in this paper, the results of its application to two examples are presented below.

### 5.1 LTI example

Let us borrow the example presented in Section 3.2.2, page 76, of (Chen and Patton, 1999). The example was originally in continuous time, with

$$A_c = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_c = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

To apply the discrete time algorithm, this continuous time system is discretized with the sample time  $T = 1$ , then

$$A = e^{A_c T} \approx \begin{bmatrix} 0.1262 & 0.5335 & 0 \\ -0.5335 & 0.6597 & 0 \\ 0.2417 & -0.5335 & 0.3679 \end{bmatrix} \quad (22)$$

$$C = C_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (23)$$

$$E = A_c^{-1}(A - I_n)E_c \approx [-0.5335 \quad 0.3403 \quad -0.0986]^T.$$

The noise covariance matrices are  $Q = 0.1I_3$ ,  $R = 0.05I_2$ .

The disturbance is simulated with a deterministic part and a random part as

$$d_k = 5 \sin(0.1k) + \omega_k \quad (24)$$

where  $\omega_k$  is an independent and identically distributed random sequence following the Gaussian distribution  $N(0, 1)$ .

For the system model after output injection,

$$G = E[(CE)^T(CE)]^{-1}(CE)^T \approx \begin{bmatrix} 0.9670 & 0.1787 \\ -0.6168 & -0.1140 \\ 0.1787 & 0.0330 \end{bmatrix}$$

$$\bar{A} = (I_n - GC)A \approx \begin{bmatrix} -0.0390 & 0.1130 & -0.0658 \\ -0.4281 & 0.9279 & 0.0419 \\ 0.2111 & -0.6112 & 0.3557 \end{bmatrix}$$

$$\begin{aligned} \bar{Q} &= (I_n - GC)Q(I_n - GC)^T + GRG^T \\ &\approx \begin{bmatrix} 0.0517 & -0.0308 & -0.0089 \\ -0.0308 & 0.1590 & -0.0057 \\ -0.0089 & -0.0057 & 0.0983 \end{bmatrix}. \end{aligned}$$

The initial state  $\bar{x}_0 = 0$  and the initial state estimate covariance  $P_0 = I_3$  (the  $3 \times 3$  identity matrix).

The simulated state and the estimated state are displayed in Figure 1. The state vector is correctly estimated, despite the presence of the disturbance.

### 5.2 LTV example

The previous example is now modified so that the system becomes time varying. A sinusoid term is added to the central entry of the matrix  $A$ , and a triangular wave is added to the second component of the vector  $E$ . More precisely,

$$A_k = \begin{bmatrix} 0.1262 & 0.5335 & 0 \\ -0.5335 & 0.6597 + \sin(k) & 0 \\ 0.2417 & -0.5335 & 0.3679 \end{bmatrix} \quad (25)$$

$$E_k = \begin{bmatrix} -0.5335 \\ 0.3403 + \Delta(k) \\ -0.0986 \end{bmatrix}^T \quad (26)$$

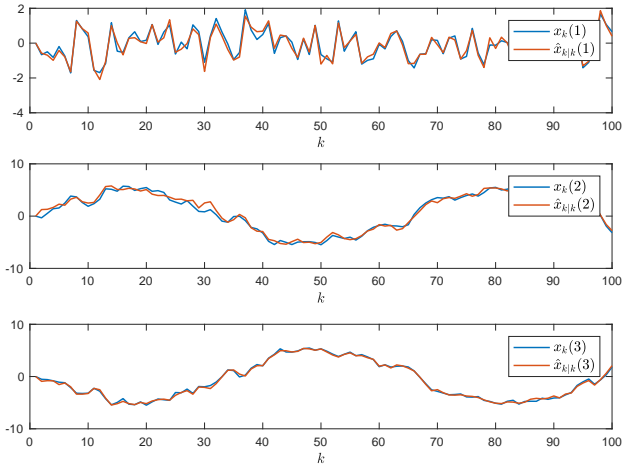


Fig. 1. LTI (time invariant) system example. The 3 components of the simulated state vector and their estimates are plotted.

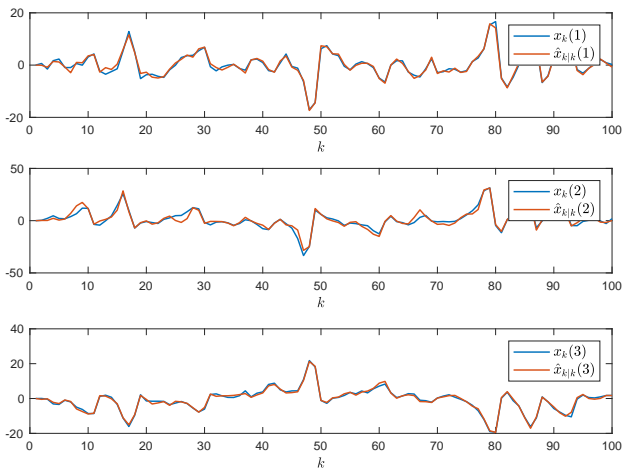


Fig. 2. LTV (time varying) system example. The 3 components of the simulated state vector and their estimates are plotted.

with the triangular wave

$$\Delta(k) = 4 \left\lfloor \frac{x}{10} - \left\lfloor \frac{x}{10} + \frac{1}{2} \right\rfloor \right\rfloor - 1,$$

where  $\lfloor \cdot \rfloor$  denotes the floor function, *i.e.* for any  $x \in \mathbb{R}$ ,  $\lfloor x \rfloor$  is the largest integer less than or equal to  $x$ . The two noise covariance matrices are multiplied by 10, because the signal magnitudes become larger due to the added time varying terms, resulting in  $Q = I_3$ ,  $R = 0.5I_2$ . The other data are the same as in the previous example.

The simulated state and the estimated state are displayed in Figure 2. Again the state vector is correctly estimated, despite the presence of the disturbance.

## 6. CONCLUSION

The stability of state estimators is of primary importance, as demonstrated by classical studies on LTI system state observers. For LTV systems, the stability of state estimators is less often discussed, including in the case of the Kalman filter, despite the fact that its stability has

been established in the original work by Kalman. For LTV systems with disturbance rejection, to our knowledge, no result ensuring the stability of state estimators has been reported, hence we believe that the result presented in this paper represents an important progress in studies on this topic.

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