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In this document, we consider a *median*-based calculus to represent monotone Boolean functions efficiently. We study an equational specification of median forms and extend it from the domain of monotone Boolean functions to the domain of polynomial functions over distributive lattices. This specification is *sound* and *complete*. We illustrate its usefulness when simplifying median formulas algebraically. Furthermore, we propose a definition of *median normal forms* (MNF), that are thought of as minimal median formulas with respect to a structural ordering of expressions. We investigate related complexity issues and show that the problem of deciding whether a formula is in MNF, that is the problem of minimizing the median form of a monotone Boolean function, is in Σ_2^{P}. Moreover, we show that it still holds for arbitrary Boolean functions, not necessarily monotone.

Key words: Boolean function; Monotone Boolean functions; Lattice polynomials; Normal form; Median; Median algebra; Majority; Structural representation; Efficient representation; Complexity; Polynomial hierarchy;

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1 INTRODUCTION

Motivation   Representing Boolean functions using different connectives may yield representations that are of drastically different sizes. In the case of Boolean functions, it has been shown that a median-based representation, that is, a representation based on the ternary median operator, yields asymptotically smaller formulas – in terms of number of connectives – than the classical DNF, CNF, and polynomial normal forms [9]. Moreover, algorithmic procedures to obtain such median representations were given in [10]. However, these procedures may not produce median formulas of the lowest possible complexity (size). This fact asks for procedures to simplify median formulas, in analogy with well known resolution procedures for DNF and CNF expressions.

Since median expressions can be translated into the language of bounded distributive lattices and conversely, lattice polynomials can also be represented by median expressions. Thus, in addition to applications in logic and circuit design, these simplifications become useful when efficiently representing noteworthy lattice functions and other aggregation functions ([15]).

Outline and contribution   Motivated by these observations and following the work of [9], in this paper we investigate the use of the ternary median connective to efficiently represent monotone Boolean functions.

In Section 2 we recall basic background on lattice functions and median algebras that will be used throughout this paper. In Section 3 we recall a finite equational specification (System 1) that allows the simplification of median formulas while preserving logical equivalence. As an immediate consequence of the soundness and completeness of this equational specification (Theorem 1), we may rewrite any formula into any other that is logically equivalent to it. Furthermore, median formulas can be simplified algebraically according to this equational specification. We then propose a structural and lexicographic ordering of formulas in Section 4, and introduce *median normal forms* (MNFs) as being median formulas that are minimal with respect to this ordering. In Section 5 we consider two decision problems related to the task of finding an MNF of a given formula, and we show that both are at most *moderately intractable*, by which we mean that they are likely to be intractable, but not beyond Σ_2^{P}. We also explore a different approach to the potential intractability issue by providing a *term rewriting system* based on the equational specification (System 1). Even though the resulting system is not complete, it runs in polynomial time, thus highlighting a trade-off in complexity: either

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we sacrifice completeness but preserve tractability, or we keep completeness at the cost of high complexity. In Section 6, we discuss the previous results with regard to arbitrary Boolean functions, not necessarily monotone. Naturally, allowing negations in median normal forms does not increase the complexity. However, we show that even in the case of monotone functions, allowing negated variables at the leaves may provide representations strictly smaller than the ones obtained with only variables and constants.

A preliminary version of the current paper appeared in the Proceedings of the 47th International Symposium on Multi-Valued Logic, [2].

2   LATTICE POLYNOMIALS AND MEDIAN FORMULAS

Our first motivation for the systematic study of median formulas is the fact that they can represent lattice polynomials.

2.1   Notation and Preliminaries

In this subsection we recall definitions and notations on lattice and lattice functions, while adopting the terminology of [11]. A *lattice* is an algebraic structure $\langle L, \wedge, \vee \rangle$ where L is a nonempty set, called *universe*, and where $\wedge$ and $\vee$ are two binary operations that satisfy the laws of commutativity, associativity, absorption, and idempotence; a lattice is said to be *distributive* if the two laws distribute over one another. With no danger of ambiguity, we will denote a lattice $\langle L, \wedge, \vee \rangle$ by its universe L.

In what follows, L will always denote an arbitrary bounded distributive lattice with least and greatest elements $\bot$ and $\top$, respectively. For $a, b \in L$, $a \leq b$ means that $a \wedge b = a$ or, equivalently, $a \vee b = b$. For any integer $n \geq 1$, we set $[n] = \{1, \ldots, n\}$. For an arbitrary nonempty set A and a lattice L, the set L^A of all functions from A to L also constitutes a lattice under the operations

$$(f \wedge g)(x) = f(x) \wedge g(x) \quad \text{and} \quad (f \vee g)(x) = f(x) \vee g(x)$$

for every $f, g \in L^A$. In particular, any lattice L induces a lattice structure on the Cartesian product L^n, $n \geq 1$, by defining $\wedge$ and $\vee$ componentwise, i.e.,

$$(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n) = (a_1 \wedge b_1, \ldots, a_n \wedge b_n),$$
$$(a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) = (a_1 \vee b_1, \ldots, a_n \vee b_n).$$

We denote the elements of L by lower case letters $a, b, c, \ldots$ and the elements of L^n, $n > 1$, by bold face letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

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We now recall the notion of lattice polynomial function. The class of *lattice polynomial functions* (or simply, *polynomial functions*) from L^n to L is defined inductively as the set of functions represented by expressions constructed in the language of lattices: the projections $\mathbf{x} \mapsto x_i$, the constant functions $\mathbf{x} \mapsto c$, $c \in L$, and if f and g are polynomial functions, then so are $f \wedge g$ and $f \vee g$.

For instance, the ternary *median* m, i.e., the function given by:

$$m(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1)$$
$$= (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_1)$$

is an example of such a polynomial function.

2.2 Equational specification and term rewriting system

We make use of the notations of [14] and [5] to introduce equational specifications. An *equational specification*, (sometimes called *equational system* within this document) is a pair (Σ, E) of an *alphabet* or *signature* Σ and a set of equations E. The alphabet Σ consists of a countably infinite set of *variables* $x_1, x_2, \ldots$ and a nonempty set of *function symbols* or *operator symbols*. In the current setting, this set contains m and constants.

The set of *terms* (or *expressions*) over Σ is denoted by $\mathrm{Ter}(\Sigma)$ and it is defined inductively as follows:

(1) every variable and constant in Σ is in $\mathrm{Ter}(\Sigma)$, and

(2) if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is in $\mathrm{Ter}(\Sigma)$.

An *equation* is then an expression of the form $s = t$ where $s, t \in \mathrm{Ter}(\Sigma)$. The set of all equations is denoted by E.

The set of *median terms* (also referred to as *median expressions* or *median formulas* in [9]) $\mathbf{M}$ will denote the set of all formulas that are constructed using variables, constants and the median m. Even though [9] focuses on median normal expressions for Boolean functions, this notion adapts to lattice polynomial functions, as the syntax of the language used to represent the monotone Boolean functions and lattice polynomial functions is the same.

Given a formula ϕ in $\mathbf{M}$, its *depth* is denoted by $d(\phi)$ and defined recursively as follows:

(i) for every variable or constant a, $d(a) = 0$,

(ii) for every formula $\phi = \mathrm{m}(a, b, c) \in \mathbf{M}$,

$$d(\phi) = \max\{d(a), d(b), d(c)\} + 1.$$

The *size* $|\phi|$ of a median term ϕ is the number of medians in it.

Example 1. *If* $\phi = \mathrm{m}(\mathrm{m}(x, x, y), \mathrm{m}(x, y, z), v)$ *then* $d(\phi) = 2$ *and* $|\phi| = 3$.

Two median terms ϕ and ψ are said to be *equivalent*, denoted by $\phi \equiv \psi$, if they represent the same function.

Example 2. *For instance, the median terms:*

$$\phi_1 = \mathrm{m}(\mathrm{m}(\mathrm{m}(x, u, v), \mathrm{m}(x, y, v), \mathrm{m}(y, u, v)), x, v)$$
$$\phi_2 = \mathrm{m}(\mathrm{m}(u, y, v), x, v)$$

are equivalent. However, the median terms:

$$\phi_3 = \mathrm{m}(\mathrm{m}(x, y, z), u, v) \quad \text{and} \quad \phi_4 = \mathrm{m}(x, \mathrm{m}(y, z, u), v)$$

are not equivalent. The latter expresses the fact that m *is not* associative *in the sense of [3, 12, 21].*

A *substitution* is a map σ from $\mathrm{Ter}(\Sigma)$ to $\mathrm{Ter}(\Sigma)$ that satisfies

$$\sigma(F(t_1, \ldots, t_n)) = F(\sigma(t_1), \ldots, \sigma(t_n))$$

for every n-ary function symbol (here $n \geq 0$).

Substitution together with the other rules recalled in Table 1, give rise to the so-called *derivable equations*, i.e., equations obtained by applying a finite combination of these rules. If an equation $s = t$ is derivable from equations in E, than we write $(\Sigma, E) \vdash s = t$ or $s \vdash_E t$ for short.

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$$(\Sigma, E) \vdash t = t \quad \text{if } t \in \mathrm{Ter}(\Sigma)$$

$$(\Sigma, E) \vdash s = t \quad \text{if } s = t \in E$$

$$\frac{(\Sigma, E) \vdash s = t}{(\Sigma, E) \vdash t = s}$$

$$\frac{(\Sigma, E) \vdash t_1 = t_2, (\Sigma, E) \vdash t_2 = t_3}{(\Sigma, E) \vdash t_1 = t_3}$$

$$\frac{(\Sigma, E) \vdash s = t}{(\Sigma, E) \vdash \sigma(s) = \sigma(t)} \quad \text{for every substitution } \sigma$$

$$\frac{(\Sigma, E) \vdash s_1 = t_1, \ldots, (\Sigma, E) \vdash s_n = t_n}{(\Sigma, E) \vdash F(s_1, \ldots, s_n) = F(t_1, \ldots, t_n)} \quad \text{for every } n\text{-ary } F \in \Sigma$$

Table 1
Equational inference system.

We now recall definition and notation on term rewriting systems. A *Term Rewriting System (TRS)* is an equational specification with all its equations oriented. A pair (l, r) of terms in $\mathrm{Ter}(\Sigma)$, written as $l \longrightarrow r$, is a *reduction rule* if r is not a variable and all the variables in l are already contained in r.

Each term rewriting system yields a rewrite relation defined to be the closure by substitution and context of its reduction rules.

3 AXIOMATIZATION OF LATTICE POLYNOMIALS

In this section, we present an equational system for median calculus which is both sound and complete, and which we then use to manipulate median expressions.

First, recall that lattice polynomial functions $f \colon L^n \to L$ are exactly the solutions of the *median decomposition system* [19]:

$$f(\mathbf{x}) = \mathrm{m}(f(\mathbf{x}_k^{\perp}), x_k, f(\mathbf{x}_k^{\top})) \tag{1}$$

for all $\mathbf{x} = (x_1, \ldots, x_n)$, $k \in [n]$ and $c \in L$, and where

$$\mathbf{x}_k^c := (x_1, \ldots, x_{k-1}, c, x_{k+1}, \ldots, x_n).$$

A direct consequence of this result is that every polynomial function can be represented by a median term. Indeed, recursive applications of (1) on each $x_1, \ldots, x_n$ of an n-ary polynomial function f produces a median formula representing f, see, e.g., [10].

However, this procedure may not produce optimal formulas (size wise), and this fact motivates the current study.

Example 3. *Consider the 5-ary majority operator*

$$\mathrm{m}_5 : \boldsymbol{x} \mapsto \mathrm{m}_5(x_1, x_2, x_3, x_4, x_5)$$

over Boolean variables. Using the median decomposition algorithm mentioned above we can construct a median formula representation of this function using its values on every point of $\{\bot, \top\}^5$. This produces a representation of size $1 + 2 + 4 + 8 + 16 = 31$ that is not optimal. Indeed, there exists a much smaller representation of size 4, as shown in [6, 20]:

$$\mathrm{m}_5(x_1, x_2, x_3, x_4, x_5) = \mathrm{m}(\mathrm{m}(\mathrm{m}(x_2, x_3, x_4), x_4, x_5), \mathrm{m}(x_2, x_3, x_5), x_1).$$

Let us now recall an axiomatization of the algebraic structure $\langle L, \mathrm{m}, \bot, \top \rangle$, that is the set L with the ternary median function m and the 0-ary functions (constants) $\bot$ and $\top$, by the following equational system. It was given in, e.g., [7].

System 1.

$$(M1)\, \mathrm{m}(x, y, z) = \mathrm{m}(x, z, y) = \mathrm{m}(z, x, y),$$
$$(M2)\, \mathrm{m}(x, x, y) = x,$$
$$(M3)\, \mathrm{m}(\mathrm{m}(x, u, v), \mathrm{m}(y, u, v), z) = \mathrm{m}(\mathrm{m}(x, y, z), u, v),$$
$$(M4)\, \mathrm{m}(\bot, \top, x) = x,$$

for all x, y, z, u, v in L.

Equations $(M1), (M2), (M3)$ are known as *symmetry*, *majority*, *distributivity* laws, respectively.

Remark 1. *An axiomatization of the Boolean algebra $\langle \{0, 1\}, \wedge, \vee, \bar{\cdot}, 0, 1 \rangle$ was used in [4] to simplify Boolean circuits. System 1 is an adaptation of these results to the case of lattice polynomials.*

In order to manipulate median formulas, we need to know whether the axiomatization given by System 1 is both *sound* and *complete*. Soundness

means that every equation $s = t$ that can be derived from System 1 is valid, i.e., that the formulas s and t are equivalent. Completeness means that every equation $s = t$ that is valid can be derived from the axioms of the system. For our purposes, having a sound and complete system is interesting in order to rewrite median terms, and hopefully simplify them. Soundness indeed ensures that whatever simplification we do by applying an equation to a median term will preserve logical equivalence, while completeness ensures that we can infer a median term from an equivalent one by using System 1.

Theorem 1 ([7]). *The algebra $\langle L, \mathrm{m}, \bot, \top \rangle$ together with the axioms of System 1 is sound and complete.*

Sketch of Proof. Soundness is provable algebraically by deriving the axioms of the system using the properties of the lattice L. For instance, the symmetry $(M1)$ stems from the symmetry of the meet and join of L. Completeness is an immediate consequence of the Birkhoff's Completeness Theorem for equational logic, see, e.g., [8, 14]. □

4   MEDIAN NORMAL FORMS

In this section, we propose a structural description of median formulas and introduce the notion of *median normal forms (MNF)* that, as we shall see, correspond to median formulas that are "minimal" with respect to the lexicographical ordering of their structural description.

Orders   A binary relation $\preceq$ on a set S is a *quasi-order*, or *preorder*, if it is reflexive and transitive. A quasi-order is said to be a *partial order* if it is antisymmetric. If $\preceq$ is a partial order, the structure $\langle S, \preceq \rangle$ is called a *partially ordered set* (or *poset*). A quasi-ordered set is *well-founded* if it satisfies the descending chain condition, i.e., there exists no infinite decreasing sequence $\cdots < t_2 < t_1$ of elements of $\mathcal{T}$. If for every pair (a, b), either $a \leq b$ or $b \leq a$, then $(\mathcal{T}, \leq)$ is said to be *totally ordered*. A well-founded and totally ordered set is said to be a *well-ordered set*.

Let n be a positive integer, and T_n the set of all ordered n-tuples over S. Let $T = \bigcup_{n \leq 1} T_n$. The *lexicographical extension* on T denoted by $\preceq_{lex}$, is defined by: $(x_1, \ldots, x_m) \preceq_{lex} (y_1, \ldots, y_n)$ if

- $m \leq n$ and for all $k \in [m], \quad x_k = y_k$[*], or

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[*] I.e., $(x_1, \ldots, x_m)$ is a *prefix* of $(y_1, \ldots, y_n)$.

- there is $k \in [\min(m,n)]$ such that for all $j \in [k-1]$, we have $x_j = y_j$ and $x_k \prec y_k$ (i.e., $x_k \preceq y_k, x_k \neq y_k$)

Example 4. *The* lexicographic order *defined on a finite set of words is well-founded. On the other hand, for $S = \{a,b\}$ with $a \preceq b$, the lexicographical extension on the infinite product S^* is not a well-order:*

$$\ldots \preceq aaab \preceq aab \preceq ab \preceq b.$$

Median normal forms   We now define a way to partially describe the structure of median terms that induces a well-order on the set of median formulas. We then show that for every polynomial function, there exists a set of minimal representations with respect to this order, which we call *median normal forms (MNF)*. As it is the case for DNF and CNF representations, this representation is unique modulo some properties like commutativity or associativity. However, the general structure of these minimal representations still eludes us.

Definition 1. *Let ϕ be a median term of depth d. Let $n_0, \ldots, n_d$ be nonnegative integers, such that for all $i \in \{0\} \cup [d]$, n_i is the number of medians at depth $\leq i$. The* structural representation *of ϕ is the tuple*

$$S_\phi = (n_d, \ldots, n_0).$$

Let $\leq_S$ be the ordering of median formulas defined by:

$$\phi_1 \leq_S \phi_2 \quad if \quad S_{\phi_1} \leq_{lex} S_{\phi_2}.$$

Remark 2. *Note that S_ϕ is a decreasing sequence and that $n_d = |\phi|$. Also, the order $\leq_S$ prioritizes the size of the formula over its depth. For instance, consider the following equivalent formulas*

$$\phi_1 = \mathrm{m}(x_1, x_2, \mathrm{m}(x_3, x_4, \mathrm{m}(x_5, x_6, x_7)))$$
$$\phi_2 = \mathrm{m}(\mathrm{m}(x_1, x_2, x_3), \mathrm{m}(x_1, x_2, x_4), \mathrm{m}(x_5, x_6, x_7)).$$

Clearly, $|\phi_1| = 3 < 4 = |\phi_2|$ while $d(\phi_1) = 3 > 2 = d(\phi_1)$. Looking at their structural representation, we have

$$S_{\phi_1} = (3, 2, 1) \quad whereas \quad S_{\phi_1} = (4, 3),$$

and hence $\phi_1 \leq_S \phi_2$.

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We now give a definition of a median normal form as a minimal median representation.

Definition 2. *We say that a median term ϕ is a* median normal form (MNF) *if for every median term $\phi' \equiv \phi$, we have*

$$\phi \leq_S \phi'.$$

Example 5. *The formula $\phi = \mathrm{m}(\mathrm{m}(x, x, y), y, z)$ is not a median normal form since $\phi' = \mathrm{m}(x, y, z)$ is an equivalent formula, and $S_{\phi'} = (1) \leq_{lex} (2, 1) = S_\phi$.*

Remark 3. *As it has been defined, the structural order cannot account for permutations of variables. For instance, the formulas $\mathrm{m}(x, y, z)$ and $\mathrm{m}(x, z, y)$ have the same structural tuple (1), but they are also equivalent, and both are median normal forms. Thus, a formula does not have a single median normal form, but rather a* set *of median normal forms: the formula ϕ from Example 5 has for set of normal forms $\{\mathrm{m}(x, y, z), \mathrm{m}(x, z, y), \mathrm{m}(y, z, x)\}$.*

5 CONSIDERATIONS ON COMPLEXITY

In this section we address the question of finding median normal forms. To this end, we formalize two decision problems that express the tasks of finding median formulas of smaller structural representation. We show that both problems are at most moderately intractable, and we propose a term rewriting system as a tool for approximating solutions to them.

5.1 Structurally smaller formulas

Although, the definition of the median normal form is expressed simply, a procedure to convert an input formula into an equivalent formula in median normal form is likely intractable. Indeed, we will show that the mere task of checking if a given formula is in MNF seems to be expensive. We formalize this decision problem in Definition 3.

Definition 3. *Consider the decision problem* MONOTONE SMALLMED*:*
Input: a median term ϕ and a decreasing sequence S
Output: succeeds if there exists a formula ψ equivalent to ϕ whose structural representation is strictly smaller than S. Fails if none exists.

Before proving that MONOTONE SMALLMED is intractable, we give a Σ_2^P upper-bound on its hardness. Recall that the Σ_2^P complexity class is on the second level of the polynomial hierarchy, between NP and PSPACE [22].

Theorem 2. MONOTONE SMALLMED *is in the class* Σ_2^{P}.

Proof. A convenient characterization of Σ_2^{P} is that it contains decision problems such that the accepting instances can be expressed as a set of words $\{x : \exists c_1 \forall c_2 F(x, c_1, c_2)\}$, where c_1 and c_2 are certificates whose lengths are polynomial in $|x|$ and F is computable in polynomial time. Consider Algorithm 1 which solves MONOTONE SMALLMED. The size of the first certificate ψ, is indeed polynomial in the size of the input, $|\phi|$, because its structural representation is bounded by that of ϕ. The size of the second certificate σ is also polynomial in the input. Therefore, Algorithm 1 ensures MONOTONE SMALLMED is in Σ_2^{P}. $\qquad \square$

Algorithm 1 Finding a smaller equivalent median form.

Input: A formula ϕ, a decreasing sequence S.

 1: Existentially guess a formula ψ such that $S(\psi) < S$
 2:     Universally guess an assignment $\varsigma^{\dagger}$
 3:        Ensure that $\varsigma(\phi) = \varsigma(\psi)$
 4:        If so, **return** SUCCESS
 5: If none exist, then FAIL

Note that Theorem 2 simply provides an asymptotic upper bound on the complexity of MONOTONE SMALLMED, but a corresponding lower-bound still eludes us.

Definition 3 assumed that the desired formula size was given as part of the input. If instead we assume a constant target size, then it is possible to obtain a better complexity bound.

Definition 4. *For any fixed decreasing sequence S, we define the decision problem* MONOTONE SMALLMED$_S$ *as:*

Input: a median term ϕ

Output: succeeds if there is a formula ψ equivalent to ϕ whose structural representation is smaller than S. Fails if none exists.

Remark 4. *Definition 4 is independent from any interpretation of the median formulas mentioned, in the sense that the objects manipulated are syntactic (i.e. median formulas and their structure). In fact, all results can be applied to, e.g., median representations of both lattice polynomials and monotone*

Boolean functions, since in both cases any function can be represented by a median formula (see, e.g., the definition of the set of median terms **M** *above). In Remark 5 below (Section 6) we discuss equivalent decision problems related to, this time, the representation of Boolean functions. In that case, we consider a language where we allow negations.*

Theorem 3. *For any decreasing sequence S, MONOTONE SMALLMED$_S$ is in the class co-NP.*

Proof. Let s be the first element of the sequence S, n be the number of variables occurring in ϕ, and V_ϕ be the set of variables occurring in ϕ. Any formula ψ of structural representation smaller than S, has no more than s medians. Hence, such a formula ψ cannot involve more than $N = 2s + 1$ variables. If ϕ is equivalent to a formula of structural representation smaller than S, then at most N variables among the ones that occur in ϕ are relevant.

For every subset of variables $V \subset V_\phi$ of size at most N, there is a constant number of formulas smaller than S with variables drawn from V. For each such formula ψ, universally guess a variable assignment σ for the variables occurring in ϕ and check if ϕ and ψ agree on σ. If so, ψ is equivalent to ϕ and we can succeed. If no formula triggers a success, we can end the algorithm and fail.

The number of ways to specify the set V is bounded by n^N. So the total number of universal guesses is bounded by $O(n^N)$. Since N is constant, we conclude that we can determine if ϕ admits an equivalent formula of size smaller than S with a polynomial number of universal guesses. Therefore, MONOTONE SMALLMED$_S$ is in co-NP. $\qquad\square$

Algorithm 1 does not find an MNF for the input formula directly, but it can be used as a subroutine to an algorithm that does. Let ϕ be an input formula and let S_ϕ be its structural representation, then a kind of binary search allows to identify the smallest structural representation such that Algorithm 1 succeeds. The output of the final call to Algorithm 1 is then an MNF of ϕ.

A binary search performs a number of comparison calls that is logarithmic in the size of the ordered domain. In our case, the domain is the set of sequences lexicographically smaller than S_ϕ, and its cardinality is at most exponential in the size of ϕ. Therefore, the binary search performs a number of calls to Algorithm 1 that is polynomial in the input formula.

5.2 A term rewriting approach

A naive implementation of Algorithm 1 amounts to an exhaustive search for equivalent formulas among structurally smaller ones. A possible alternative

would be to search for a structurally smaller formula among equivalent ones.

Recall from Theorem 1 that System 1 is sound and complete. In other words, for any two formulas ϕ and ψ, the formulas are equivalent if and only if there exists a rewriting of ϕ into ψ by means of a sequence of equations from System 1. This idea is implemented in Algorithm 2.

Algorithm 2 Bringing a formula closer to a median normal form.

Input: A formula ϕ, a decreasing sequence S.
Output: A formula ψ with structural representation $S_\psi < S$, diverges if
    none exists.
  1: Set $\psi \leftarrow \phi$
  2: While $\psi \geq_S \phi$ do
  3:    Existentially guess a rewriting from System 1: $\psi = \psi'$
  4:       Update $\psi \leftarrow \psi'$
  5: **return** ψ

This non-deterministic algorithm uses very little space, namely its space complexity is linear in the size of the input, and solves MONOTONE SMALLMED. In fact, MONOTONE SMALLMED is in the class NPSPACE, as this algorithm can be implemented in a Turing Machine using little space. However, it might not always stop and return a result.

From Savitch's theorem (see, e.g., [22]) we know that NPSPACE = PSPACE, but Σ_2^P is contained in PSPACE so Theorem 2 is stronger.

We do not know whether Algorithm 2 is guaranteed to terminate in polynomial time, in the best case, when a structurally smaller formula exists. Indeed, equational reasoning to transform a formula into some smaller one may conceivably require an exponential number of rewriting steps. On the contrary, were Algorithm 2 to always terminate in polynomial time, it would constitute a proof of MONOTONE SMALLMED being in NP.

A solution to the problem of finding a derivation without unbounded searches is to orient the equations of the system using the order on the size of the formula: from bigger to smaller. As a result, applying any rule (except commutativity) to a formula will simplify it with regard to $\leq_S$: every derivation will be a simplification.

Let us then consider the following "term rewriting system" extracted from System 1 by orienting its equations according to the decreasing structural ordering.

System 2.

$$(R1)\, \mathrm{m}(x, y, z) = \mathrm{m}(x, z, y) = \mathrm{m}(z, x, y),$$
$$(R2)\, \mathrm{m}(x, x, y) \longrightarrow x,$$
$$(R3)\, \mathrm{m}(\mathrm{m}(x, u, v), \mathrm{m}(y, u, v), z) \longrightarrow \mathrm{m}(\mathrm{m}(x, y, z), u, v),$$
$$(R4)\, \mathrm{m}(\bot, \top, x) \longrightarrow x,$$

for all variables x, y, z, u, v in L.

The rewrite rule $(R1)$, which is in fact $(M1)$, is kept the same, without orientation. Such systems are sometimes called *rewriting systems modulo commutativity* ([5]). A similar situation takes place when dealing with systems that involve commutative binary operations like $\vee$ or $\wedge$, and in such cases commutativity and associativity may be kept as equational rules. As explained in [23], in the case of median terms, commutativity $(M1)$ can be oriented by defining a total order on terms. Some effects of this orientation are the sorting of the terms by the application of the now-oriented commutativity rules, as well as the occasional blockage of derivation proofs (in these cases we thus lose completeness of the system).

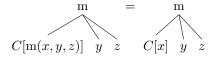By orientating the rules, we can more easily test a derivation between formulas.

Lemma 1. *Let ϕ and ψ be median formulas. If it exists, the derivation between ϕ and ψ is polynomial in the size of ϕ.*

Proof. Every rewrite rule from System 2 save from the permutation $(R1)$ removes a median m from any formula it is applied to. If such a derivation exists, necessarily ϕ has more medians than ψ. The derivation between ϕ and ψ thus contains n steps, with n being the difference between the number of medians in ϕ and the number of medians in ψ. $\qquad\square$

However, we may not be able to rewrite a formula into another equivalent one using System 2 (e.g., it is not possible to rewrite x into $\mathrm{m}(\bot, \top, x)$), much less into a normal form.

Example 6. *Consider $\phi = \mathrm{m}(\mathrm{m}(\mathrm{m}(x, y, z), u, v), y, z)$, which has for canonical form $\phi' = \mathrm{m}(\mathrm{m}(x, u, v), y, z)$. No rule from System 2 can be applied to formula ϕ. Thus, it is not possible to simplify ϕ into ϕ'. Let us now prove that $\phi' = \mathrm{m}(\mathrm{m}(x, u, v), y, z)$ is the canonical form of ϕ' (up to permutations). The proof of equivalence between the two formulas ϕ and ϕ' is a particular case of a certain generalized rule we give hereafter. Recall that* contexts *are*

"terms" containing one occurrence of a special symbol $\diamond$, denoting an empty space. A context is generally denoted by $C[\,.\,]$. If $t \in \mathrm{Ter}(\Sigma)$ and t is substituted in $\diamond$, the result is $C[t] \in \mathrm{Ter}(\Sigma)$; t is said to be a subterm *of $C[t]$. The generalized rule is*

$$
\begin{array}{ccc}
\quad\mathrm{m} & = & \mathrm{m} \\
\diagup\,\big|\,\diagdown & & \diagup\,\big|\,\diagdown \\
C[\mathrm{m}(x,y,z)] \quad y \quad z & & C[x] \quad y \quad z
\end{array}
$$

with C a context. The proof of this rule lies in an induction on the depth of C.

The proof that $\phi \equiv \phi'$ can also be verified by comparing the truth tables of both formulas. Here we give an explicit derivation. Remark that some steps of this derivation require using some rules from System 2 in the reverse direction: the first and second steps, for instance, require using $(R3)$ in reverse.

$$
\begin{aligned}
\phi &\equiv \mathrm{m}(\mathrm{m}(\mathrm{m}(x,y,z),u,v),y,z) \\
&\equiv \mathrm{m}(\mathrm{m}(x,\mathrm{m}(u,y,z),\mathrm{m}(v,y,z)),y,z) \\
&\equiv \mathrm{m}(\mathrm{m}(\mathrm{m}(u,y,z),y,z),\mathrm{m}(\mathrm{m}(v,y,z),y,z),x) \\
&\equiv \mathrm{m}(\mathrm{m}(u,y,z),\mathrm{m}(v,y,z),x) \\
&\equiv \mathrm{m}(\mathrm{m}(x,u,v),x,y) \\
&\equiv \phi'.
\end{aligned}
$$

Furthermore, ϕ' cannot be rewritten into a smaller formula. Indeed, there are strictly more than 3 essential variables in ϕ' : it is thus impossible to find a median formula of size 1 equivalent to it. Thus, ϕ' is a median normal form for ϕ, yet is unreachable using System 2.

Now, even though it is no longer complete, System 2 remains sound.

Proposition 1. *System 2 is sound but not complete.*

Proof. Soundness follows from the fact that System 1 is sound (Theorem 1). Incompleteness follows from Example 6: the rule $\phi \longrightarrow \phi'$, that is the rule

$$
\mathrm{m}(\mathrm{m}(\mathrm{m}(x,y,z),u,v),y,z) \longrightarrow \mathrm{m}(\mathrm{m}(x,u,v),x,y),
$$

cannot be derived from the axioms of System 2. $\qquad\square$

6 EXTENSION TO THE NON-MONOTONIC CASE

In this section, we extend our results to the non-monotonic case, i.e., we now allow negations to appear in the median formulas. This allows us to

represent all Boolean functions. Indeed, the set $\{\mathrm{m}, \neg\}$ is complete with regard to Boolean functions, that is, we can represent every Boolean function using medians and negations. Up to some adaptation of the equational specification to include the negation, most of the results and definitions given are similar to those of monotone Boolean functions. We thus recall most of the formalism of [2].

6.1 Equational specification

We supplement System 1 with two rules in order to deal with negated variables and negated terms.

System 3.

$$(B1)\ \mathrm{m}(x, y, z) = \mathrm{m}(x, z, y) = \mathrm{m}(z, x, y),$$
$$(B2)\ \mathrm{m}(x, x, y) = x,$$
$$(B3)\ \mathrm{m}(\mathrm{m}(x, u, v), \mathrm{m}(y, u, v), z) = \mathrm{m}(\mathrm{m}(x, y, z), u, v),$$
$$(B4)\ \mathrm{m}(\overline{y}, y, x) = x,$$
$$(B\neg)\ \overline{\mathrm{m}(x, y, z)} = \mathrm{m}(\overline{x}, \overline{y}, \overline{z}),$$

for all x, y, z, u, v in $\{0, 1\}$.

Remark that rule $(M4)$ of System 1, namely that $\mathrm{m}(\bot, \top, x) = x$, can be derived from rule $(B4)$ of System 3. The propagation rule $(B\neg)$ in particular allows us to only consider literals, that is, variables and negated variables, in median formulas. For instance, the formula $\overline{\mathrm{m}(\overline{x_1}, x_2, \overline{x_3})}$ can be equivalently rewritten into the formula $\mathrm{m}(x_1, \overline{x_2}, x_3)$.

Just as for monotone Boolean functions, in order to rewrite a median formula into any equivalent other, we need a sound and complete system with regard to Boolean functions. In fact, System 3 is both sound and complete. These properties stem the soundness and completeness of System 1 for monotone Boolean functions.

Theorem 4. *The algebra $< \mathbb{B}, \mathrm{m}, \neg, 0, 1 >$ together with the axioms of System 3 is sound and complete.*

6.2 Extension of MNFs and corresponding complexity

In this section, we adapt the notions introduced in Section 4 in order to manipulate Boolean functions. As the notions are exactly the same for both monotone and arbitrary Boolean functions, we keep the same terminology.

Definition 5. *We denote the set of* median terms *(median expressions, median formulas) by* $\mathbb{M}$. *It is the set of all formulas that are constructed using constants, variables, negated variables, and the median* m. *Given a median term ϕ of depth d, let $n_0, \ldots, n_d$ be nonnegative integers such that for all $i \in \{0\} \cup [d]$, n_i is the number of medians in ϕ at depth i. The* structural representation *of ϕ is the tuple*

$$S_\phi = (n_d, \ldots, n_0).$$

Similarly to the case of monotone functions, we can compare median terms given an ordering $\leq_S$: $\phi_1 \leq_S \phi_2$ if $S_{\phi_1} \leq_{lex} S_{\phi_2}$. As this is a well-order on a finite set of formulas, given a median term, we can consider its set of median normal forms.

Definition 6. *We say that a* median term ϕ is in *median normal form (MNF) if for every median term $\phi' \equiv \phi$, we have*

$$\phi \leq_S \phi'.$$

Given our previous results on monotone Boolean functions, a natural question arises. Let f be a monotone Boolean function and ϕ its median normal form. Does allowing $\neg$ in our language allow us to give a representation of f in median normal form smaller than ϕ?

It is straightforward that, given two minimal representations ϕ and $\phi_\neg$ of the same monotone function in which negations are not allowed and negations are allowed, respectively, then $|\phi_\neg| \leq |\phi|$, with equality when $\phi_\neg = \phi$.

Surprisingly, in certain cases the comparison can be strict. In other words, some monotone functions may have representations that are strictly smaller with negations than without.

Proposition 2. *There exists a monotone Boolean function f such that $|\phi_\neg| < |\phi|$, with $\phi_\neg$ and ϕ the minimal representations of f with and without negations, respectively.*

Proof. Consider the two median formulas

$$\phi = \mathrm{m}(y, \mathrm{m}(u, v, t), \mathrm{m}(x, z, \mathrm{m}(u, v, t))),$$
$$\phi_\neg = \mathrm{m}(x, \mathrm{m}(\overline{x}, y, z), \mathrm{m}(u, v, t)).$$

Remark that $|\phi| = 4$ and $|\phi_\neg| = 3$. First, we will prove that $\phi \equiv \phi_\neg$. One might verify by computing the truth tables of both formulas, or use the following derivation, that requires using the rule $(B3)$ in reverse and the propagation

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rule $(B\neg)$:

$$\phi_\neg = m(x, m(\overline{x}, y, z), m(u, v, t))$$
$$\equiv m(y, m(m(u, v, t), x, \overline{x}), m(x, z, m(u, v, t)))$$
$$\equiv m\, m(y, m(u, v, t), m(x, z, m(u, v, t)))$$
$$= \phi.$$

Second, we will prove that ϕ is the smallest representation equivalent to $\phi_\neg$, up to permutations and without negations. As the space of smaller formulas is finite, we have verified that no smaller formula equivalent to ϕ without negation exists by an automated exhaustive search. Here we give a formal proof of this fact. Suppose there exists a smaller formula ϕ' that does not contain negations and is strictly smaller than ϕ. The idea behind the proof is to exhibit a tuple (sometimes called an *assignment* in this proof) X such that $\phi'(X) \neq \phi_\neg(X)$.

Remark that $|\phi'| = 3$ because all variables are essential in ϕ: its essential arity is 6, but median formulas of size 2 have at most an essential arity of 5. Without loss of generality ϕ' may follow two different structures (cases (a) and (b) below). Furthermore, ϕ' may either contain two repeated variables, or one constant. In the rest of the proof we will produce assignments $\phi_\neg$ and ϕ' do not agree on. Consider the assignments:

- X: $x, y, u = 1$ and $v, t, z = 0$;

- Y: $u, y, z = 1$ and $v, t, x = 0$;

- Z: $v, u, t = 1$ and $x, y, z = 0$.

Each of these assignments are false points for $\phi_\neg$.

1. First, suppose that there are no constants in ϕ'.

   (a) Suppose that $\phi' = m(x_1, m(x_2, x_3, x_4), m(x_5, x_6, x_7))$. There is a repeated variable; if $x_2 = x_3$ or $x_2 = x_4$, then ϕ' can be simplified further into $m(x_1, x_2, m(x_5, x_6, x_7))$, of essential arity at most 5, with the majority rule $(B2)$. A similar reasoning ensures that, without loss of generality [‡], either $x_1 = x_2$ or $x_2 = x_5$.

      i. suppose that $x_1 = x_2$.

         A. if x is the variable that is repeated, then given the assignment X we have $\phi'(X) = 1$.

[‡] Because the median is symmetric.

B. if y is the variable that is repeated (or equivalently, z), then given the assignment X we also have $\phi'(X) = 1$.

C. if u is the variable that is repeated (or equivalently, v or t) then given the assignment X we also have $\phi'(X) = 1$.

ii. Suppose that $x_2 = x_5$.

A. if x is the variable that is repeated, then given the assignment X we have $\phi'(X) = 1$.

B. if y is the variable that is repeated (equivalently, z), then if $x = x_1$ then $\phi'(X) = 1$; else, if $u = x_1$ then $\phi'(X) = 1$; else, if $z = x_1$ then $\phi_\neg(Y) = 0$ and $\phi'(Y) = 1$.

C. if u is the variable that is repeated (equivalently, v or t) then if $x = x_1$ or $y = x_1$ then $\phi'(X) = 1$; else, if $d = x_1$ then $\phi_\neg(Z) = 0$ and $\phi'(Z) = 1$.

(b) Suppose that $\phi' = \mathrm{m}(\mathrm{m}(\mathrm{m}(x_1, x_2, x_3), x_4, x_5), x_6, x_7)$. The reasoning is similar to the one above. The possibilities for repeated variables are, without loss of generality, $x_1 = x_3$, $x_1 = x_5$, or $x_3 = x_5$.

i. Suppose that $x_1 = x_3$.

A. if x is repeated ($x = x_1$), X is a suitable assignment, that produces a clash: $\phi'(X) \neq \phi_\neg(X)$.

B. if y is repeated, then X is a suitable assignment.

C. if u is repeated, then X is a suitable assignment.

ii. Suppose that $x_1 = x_5$.

A. if x is repeated, then X is a suitable assignment.

B. if y is repeated, then if $x = x_3$ or $u = x_3$ then X is a suitable assignment; if $z = x_3$ then Y is a suitable assignment.

C. if u is repeated, then if $x = x_3$ or $y = x_3$ then X is a suitable assignment; else, if $z = x_3$ then Y is a suitable assignment.

iii. Suppose that $x_3 = x_5$.

A. if x is repeated, then X is a suitable assignment.

B. if y is repeated, then if $x = x_7$ or $u = x_7$ then X is a suitable assignment; else, if $z = x_7$ then Y is a suitable assignment.

C. if u is repeated, then X is a suitable assignment.

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2. Second, suppose that there is a constant in ϕ'. In particular no variable is repeated.

   (a) Suppose that $\phi' = \mathrm{m}(x_1, \mathrm{m}(x_2, x_3, x_4), \mathrm{m}(x_5, x_6, x_7))$, such that either $x_1, \ldots, x_7$ is a constant.

      i. If this constant is 1, then all tuples of weight at least 3 are true points of ϕ', but there are false points of $\phi_\neg$ of weight 3.

      ii. Suppose this constant is 0.

         A. If $x_1 = 0$, then $\phi' \equiv \mathrm{m}(x_2, x_3, x_4) \wedge \mathrm{m}(x_5, x_6, x_7)$. Any tuple of weight 3 is a false point for this function, but there are true points of $\phi_\neg$ of weight 3.

         B. Suppose that $x_2 = 0$ (equivalently for all remaining cases). If $x_1 = x$ then the assignment $x, y, u = 0$, $z, v, t = 1$ is a true point for $\phi_\neg$ but not for ϕ'; the same assignment is adequate for the cases $x_1 = y$ (equivalently z) and $x_1 = u$ (equivalently v, t).

   (b) Suppose that $\phi' = \mathrm{m}(\mathrm{m}(\mathrm{m}(x_1, x_2, x_3), x_4, x_5), x_6, x_7)$, such that either $x_1, \ldots, x_7$ is a constant.

      i. Suppose this constant is 1. Remark that there is no tuple X' of size 2 such that $\phi_\neg(X') = 1$; however, no matter where 1 occurs in ϕ' (either $x_7 = 1$ or $x_5 = 1$ or $x_3 = 1$) one can produce assignments X' of weight 2 such that $\phi'(X') = 1$. For instance, if $x_7 = 1$ then one can consider the assignment $x_4 = 1, x_5 = 1, x_i = 0$ for $i \neq 4, 5$.

      ii. Suppose this constant is 0. Remark that there is no tuple X' of size 4 such that $\phi_\neg(X') = 0$; however, no matter where 0 occurs in ϕ' (either $x_7 = 0$ or $x_5 = 0$ or $x_3 = 0$) one can produce assignments X' of weight 4 such that $\phi'(X') = 0$.

Thus, there is no median formula ϕ' strictly smaller than ϕ and such that $\phi \equiv \phi'$ and such that ϕ' does not contain negated variables. $\quad\square$

This fact raises further questions to be investigated in the future. For instance, the problem of finding a general description, that is, that holds for representations of both monotone and non-monotone functions, of median normal forms remains open.

Another question of interest is whether the corresponding decision problems SMALLMED$^{\mathbb{B}}$ and SMALLMED$^{\mathbb{B}}_S$, that are the problems SMALLMED and

SMALLMED_S, generalized to all Boolean functions (not just monotone), are in the same complexity classes as SMALLMED and SMALLMED_S.

Definition 7. *Consider the decision problem* $\text{SMALLMED}^{\mathbb{B}}$:
Input: a median term ϕ (that represents a Boolean function) and a decreasing sequence S
Output: succeeds if there exists a formula ψ equivalent to ϕ whose structural representation is strictly smaller than S. Fails if none exists.

Theorem 5. $\text{SMALLMED}^{\mathbb{B}}$ *is in the class* Σ_2^P.

Proof. The proof is the same as for MONOTONE SMALLMED; remark that Algorithm 1 also solves $\text{SMALLMED}^{\mathbb{B}}$. $\qquad\square$

Remark 5. *As in Remark 4, the definition of* $\text{SMALLMED}^{\mathbb{B}}$ *is independent from the objects that are represented, namely, Boolean functions.*

Deciding whether two formulas are equivalent using System 3 thus remains a complex problem. As an illustration of this fact, remark that to prove the rule

$$\text{m}(\text{m}(\text{m}(x,y,z),u,v),y,z) \longrightarrow \text{m}(\text{m}(x,u,v),x,y),$$

by rewriting the left-hand side into the right-hand side using System 3, one needs to apply a rule that increases the size of the formula; c.f., Proposition 1.

Remark 6. *Just as for monotone Boolean functions in Section 5.2, in order to simplify the search for a derivation between formulas, we may consider a term rewriting system instead of a full equational specification, extracted from System 3 by orienting its equations according to the decreasing structural ordering.*

System 4.

$$(R1)\, \text{m}(x,y,z) = \text{m}(x,z,y) = \text{m}(z,x,y),$$
$$(R2)\, \text{m}(x,x,y) \longrightarrow x,$$
$$(R3)\, \text{m}(\text{m}(x,u,v),\text{m}(y,u,v),z) \longrightarrow \text{m}(\text{m}(x,y,z),u,v),$$
$$(R4)\, \text{m}(\bot,\top,x) \longrightarrow x,$$
$$(R5)\, \text{m}(\overline{y},y,x) \longrightarrow x,$$
$$(R\neg)\, \overline{\text{m}(x,y,z)} \longrightarrow \text{m}(\overline{x},\overline{y},\overline{z}),$$

for all variables x,y,z,u,v in $\mathbb{B}$.

Even though System 4 is also no longer complete, like System 2, it remains sound.

7 CONCLUSION AND FUTURE WORK

In this paper, we discussed a median-based formalism to efficiently represent monotone Boolean functions as well as polynomial functions over distributive lattices. To this end, we propose median normal forms defined as being median expressions that are minimal with respect to a structural ordering of formulas.

We also formalized the problem of finding median formulas of smaller structural representation and investigated its computational complexity. This task turns out to be at most moderately intractable. In fact, we showed that the corresponding decision problem, that is, falls into Σ_2^P or co-NP according to whether the structural representation is given as part of the input. However, the question of determining corresponding complexity lower bounds remains open.

Furthermore, a natural generalization of the results to arbitrary Boolean functions, i.e. by allowing negated variables to appear in our formalism. A surprising result (Proposition 2) is that in certain cases allowing negated variables allows formulas that are strictly smaller than if negation hadn't been allowed. These and other complexity questions concerning decision problems that appear naturally in this median-based formalism are to be investigated in forthcoming collaborations.

Other connectives have been considered in [1] such as the Sheffer stroke $x \uparrow y \equiv \neg x \wedge \neg y$, or the generator of monotone constant-preserving clique functions, $u(x, y, z) \equiv (x \vee y) \wedge z$. These connectives share with the median the property of being *quasi-Sheffer*, that is, any Boolean function can be represented using a composition of one of these connectives, literals, and constants. We will investigate similar questions to those considered in this paper, such as the definition of normal forms with regard to these connectives, the definition of a sound and complete equational specification with the goal of simplifying formulas, or the complexity of related decision problems.

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