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Deriving proved equality tests in Coq-elpi:  
Stronger induction principles for containers in Coq

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Abstract

We describe a procedure to derive equality tests and their correctness proofs from inductive type declarations in Coq. Programs and proofs are derived compositionally, reusing code and proofs derived previously.

The key steps are two. First, we design appropriate induction principles for data types defined using parametric containers. Second, we develop a technique to work around the modularity limitations imposed by the purely syntactic termination check Coq performs on recursive proofs. The unary parametricity translation of inductive data types turns out to be the key to both steps.

Last but not least, we provide an implementation of the procedure for the Coq proof assistant based on the Elpi [6] extension language.

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1 Introduction

Modern typed programming languages come with the ability of generating boilerplate code automatically. Typically when a data type is declared a substantial amount of code is made available to the programmer at little cost, code such as an equality test, a printing function, generic visitors etc. For example the `derive` directive of Haskell or the `pxx_deriving OCaml` preprocessor provide these features for the respective programming language.

The situation is less than ideal in the Coq proof assistant. It is capable of synthesizing the recursor of a data type, that, following the Curry-Howard isomorphism, implements the induction principle associated to that data type. It supports all data types, containers such as lists included, but generates a quite weak principle when a data type uses a container.

Take for example the data type rose tree (where `U` stands for a universe such as `Prop` or `Type`):

```coq
Inductive rtree A : U :=  
| Leaf (a : A)  
| Node (l : list (rtree A)).
```

Its associated induction principle is the following one:

```coq
Lemma rtree_ind : ∀ A (P : rtree A → U),  
       (∀ a : A, P (Leaf A a)) →  
       (∀ l : list (rtree A), P (Node A l)) →  
       ∀ r : rtree A, P r.
```

Remark that the recursive step, line 3, lacks any induction hypotheses on (the elements of) `l` while one would expect `P` to hold on each and every subtree. Even a very basic recursive program such as an equality test cannot be proved correct using this induction principle. To be honest, the Coq user is not even supposed to write equality tests by hand, nor to prove them correct interactively. Coq provides two facilities to synthesize equality tests and their correctness proofs called `Scheme equality` and `decide equality`. The former is fully automatic...
but is unfortunately very limited, for example it does not support containers. The latter
requires human intervention and generates a single, large, term that mixes code and proofs.

As a consequence, users often need to manually write induction principles, equality tests
and their correctness proofs. This situation is very unfortunate because the need for the
automatic generation of boilerplate code such as equality tests is higher than ever in the Coq
ecosystem. All modern formal libraries structure their contents in a hierarchy of interfaces
and some machinery such as Type Classes [17] or Canonical Structures [8] are used to link
the abstract library to the concrete instances the user is working on. For example the first
interface one is required to implement in order to use the theorems in the Mathematical
Components library [9] on a type $T$ is the $\text{eqType}$ one, requiring a correct equality test on $T$.

In this paper we use the framework for meta programming based on Elpi [6, 18] developed
by the author and we focus on the derivation of equality tests. It turns out that generating
equality tests is easy, while their correctness proofs are hard to synthesize, for two reasons.
The first problem is that the standard induction principles generated by Coq, as shown
before, are too weak. In order to strengthen them one needs quite some extra boilerplate,
such as the derivation of the unary parametricity translation of the data types involved. The
second reason is that termination checking is purely syntactic in Coq: in order to check that
the induction hypothesis is applied to a smaller term, Coq may need to unfold all theorems
involved in the proof. This forces proofs to be transparent that, in turn, breaks modularity:
A statement is no more a contract, changing its proof may impact users.

In this paper we describe a derivation procedure for equality tests and their correctness
proofs where programs and proofs are both derived compositionally, reusing code and proofs
derived previously. This procedure also confines the termination check issue, allowing proofs
to be mostly opaque. More precisely the contributions of this paper are the following ones:

- A technique to confine the issue stemming from the purely syntactic termination check
  implemented by Coq out of the main proofs. In this paper we apply it to the correctness
  proof of equality tests, but the technique is applicable to all proofs that proceed by
  structural induction.

- A modular and structured process to derive proved equality tests and, en passant,
  stronger induction principles for inductive types defined using containers.

- An implementation based on the Elpi extension language for the Coq proof assistant.

By installing the coq-elpi package\(^1\) and issuing the command Elpi derive rtree one gets the
following terms synthesized out of the type declaration for rtree:

\begin{verbatim}
Definition eq_axiom T f x := \forall y, reflect (x = y) (f x y).

Definition rtree_eq : \forall A, (A \to A \to \text{bool}) \to rtree A \to rtree A \to \text{bool}.

Lemma rtree_eq_OK : \forall A (A_eq : A \to A \to \text{bool}), (\forall a, eq_axiom A A_eq a) \to
\forall t, eq_axiom (rtree A) (rtree_eq A A_eq) t.
\end{verbatim}

reflect is a predicate stating the equivalence between the proposition $(x = y)$ and the
boolean test $(f x y)$; rtree_eq is a (transparent) equality test and rtree_eq_OK is its (opaque)
correctness proof under the assumption that the equality test $A\_eq$ is correct.

The paper introduces the problem in section 2 by describing the shape of an equality
test and of its correctness proof and explaining the modularity problem that stems for the
termination checker of Coq. It then presents the main idea behind the modular derivation

\(^1\) See https://github.com/LPCIC/coq-elpi for the installation instructions
3 is called passing as the 
\texttt{rtree\_eq\_OK} nil ..
is applied to a term smaller than (\texttt{t} for the elements of type (\texttt{y}.
The problem: opaque proofs v.s. syntactic termination checking

Recursors, or induction principles, are not primitive notions in Coq. The language provides constructors for fix point and pattern matching that work on any inductive data the user can declare. For example in order to test two lists \texttt{11} and \texttt{12} for equality one typically takes in input an equality test \texttt{A\_eq} for the elements of type \texttt{A} and then performs the recursion:

\begin{verbatim}
Definition list\_eq A (A\_eq : A \to A \to bool) :=
fix rec (l1 l2 : list A) {struct l1} : bool :=
   match l1, l2 with
   | nil, nil => true
   | x :: xs, y :: ys => A\_eq x y && rec xs ys
   | _, _ => false
end.
\end{verbatim}

Coq accepts this definition because the recursive call is on \texttt{xs} that is a syntactically smaller term of the argument labelled as decreasing by the (\texttt{struct l1}) annotation.

We can define the equality test for \texttt{rtree} by reusing the equality test for lists:

\begin{verbatim}
Definition rtree\_eq B (B\_eq : B \to B \to bool) :=
fix rec (t1 t2 : rtree B) {struct t1} : bool :=
   match t1, t2 with
   | Leaf x, Leaf y => B\_eq x y
   | Node l1, Node l2 => list\_eq (rtree B) rec l1.l2
   | _, _ => false
end.
\end{verbatim}

Note that \texttt{list\_eq} is called passing as the \texttt{A\_eq} argument the fixpoint \texttt{rec} itself (line 12). In order to check that the latter definition is sound, Coq looks at the body of \texttt{list\_eq} to see weather its parameter \texttt{A\_eq} is applied to a term smaller than \texttt{t1}. Since \texttt{l1} is a subterm of \texttt{t1} and since \texttt{x} is a subterm of \texttt{l1}, then the recursive call (\texttt{rec x y}) at line 5 is legit.

The fact that checking \texttt{rtree\_eq} requires inspecting the body of \texttt{list\_eq} is not very annoying: we want both \texttt{list\_eq} and \texttt{rtree\_eq} to compute, hence their body matters to us.

On the contrary proof terms are typically hidden to the type checker once they have been validated, for both performance and modularity reasons. The desire is to make only the statement of theorems binding, and keep the freedom to clean, refactor, simplify proofs without breaking the rest of the formal development.

For example, lets assume that \texttt{list\_eq\_OK} is an opaque proof that \texttt{list\_eq} is correct.

\begin{verbatim}
Lemma list\_eq\_OK : \forall A (A\_eq : A \to A \to bool),
   (\forall a, eq\_axiom A A\_eq a) \to
   \forall l, eq\_axiom (list A) (list\_eq A A\_eq) l.
Proof. .. Qed. (* proof is opaque, hence hidden *)
\end{verbatim}

It seems desirable to use this lemma in order to prove the correctness of \texttt{rtree\_eq}, since it calls \texttt{list\_eq}.

\begin{verbatim}
Lemma rtree\_eq\_OK B B\_eq (HB: \forall b, eq\_axiom B B\_eq b) :
   \forall t, eq\_axiom (rtree B) (rtree\_eq B B\_eq) t :=
   fix IH (t1 t2 : rtree B) {struct t1} :=
      match v1, t2 with
      | Node l1, Node l2 => .. list\_eq\_OK (rtree B) (tree\_eq B B\_eq) IH l1 l2 ..
      | Leaf b1, Leaf b2 => .. HB b1 b2 ..
      | .. => ..
end.
\end{verbatim}
Unfortunately this term is rejected: we pass $\text{IH}$, the induction hypothesis, as the witness that $(\text{tree}\_eq \ B \ B\_eq)$ is a correct equality test (the argument at line 10 preceding $\text{IH}$) but Coq does not know how $\text{list}\_eq\_OK$ uses this argument, since its body is opaque.

The issue seems unfixable without changing Coq in order to use a more modular check for termination, for example based on sized types [1]. We propose a less ambitious but more practical approach here, that consists in putting the transparent terms that the termination checker is going to inspect outside of the main proof bodies so that they can be kept opaque.

The intuition is to “reify” the property the termination checker wants to enforce. It can be phrased as “x is a subterm of t and has the same type”. More in general we model “x is a subterm of t with property P”.

### 3 The idea: put unary parametricity translation to good use

Given an inductive type $T$ we name $\text{is}_T$ an inductive predicate describing the type of the inhabitants of $T$. This is the one for natural numbers:

```coq
Inductive is_nat : nat → U :=
| is_0 : is_nat 0
| is_S n (pn : is_nat n) : is_nat (S n).
```

The one for a container such as $\text{list}$ is more interesting:

```coq
Inductive is_list A (PA : A → U) : list A → U :=
| is_nil : is_list A PA nil
| is_cons a (pa : PA a) l (pl : is_list A PA l) : is_list A PA (a :: l).
```

Remark that all the elements of the list validate $PA$.

When a type $T$ is defined in terms of another type $C$, typically a container, the $\text{is}_C$ predicate shows up inside $\text{is}_T$. For example:

```coq
Inductive is_rtree A (PA : A → U) : rtree A → U :=
| is_Leaf a (pa : PA a) : is_rtree A PA (Leaf A n)
| is_Node l (pl : is_list (rtree A) (is_rtree A PA) l) : is_rtree A PA (Node A l).
```

Note how line 3 expresses the fact that all elements in the list $l$ validate $(\text{is}_\text{rtree} \ A \ PA)$.

Our intuition is that these predicates reify the notion of being of a certain type, structurally. What we typically write $(t : T)$ can now be also phrased as $(\text{is}_T \ t)$ as one would do in a framework other than type theory, such as a mono-sorted logic.

It turns out that the inductive predicate $\text{is}_T$ corresponds to the unary parametricity translation [21] of the type $T$. Keller and Lasson in [7] give us an algorithm to synthesize these predicates automatically. What we look for now is a way to synthesize a reasoning principle for a term $t$ when $(\text{is}_T \ t)$ holds.

### 3.1 Stronger induction principles for containers

Let’s have a look at the standard induction principle of lists.

```coq
Lemma list_ind A (P : list A → U) :
 P nil →
 (\forall a l, P l → P (a :: l)) →
 \forall l : list A, P l.
```

This principle is parametric on $A$: no knowledge on any term of type $A$ such as $a$ is ever available. We want to synthesize a more powerful principle that lets us choose an invariant for the subterms of type $A$ (the differences are underlined):
Lemma list_induction A (PA: A → U) (P: list A → U):

P nil →
(∀ a (pa : PA a) l, P l → P (a :: l)) →
∀ l, is_list A PA l → P l.

Note the extra premise (is_list A PA 1): The implementation of this induction principle goes by recursion on the term of this type and finds as an argument of the is_cons constructor the proof evidence (pa : PA a) it feeds to the second premise (line 3). Intuitively all terms of type (1list A) validate the property P, while all terms of type A validate the property PA.

More in general to each type we attach a property. For parameters we let the user choose (we take another parameter, PA here). For the type being analysed, list A here, we take the usual induction predicate P. For terms of other types we use their unary parametricity translation. Take for example the induction principle for rtree.

Lemma rtree_induction A PA (P : rtree A → U) :

(∀ a, PA a → P (Leaf A a)) →
∀ l, is_list (rtree A) P l → P (Node A l)) →
∀ t, is_rtree A PA t → P t.

Line 3 uses is_list to attach a property to l, and given that l has type (1list (rtree A)) the property for the type parameter (rtree A) is exactly P. Note that this induction principle gives us access to p, the property one is proving, on the subtrees contained in 1.

3.1.1 Synthesizing stronger induction principles

We postpone a detailed description of the synthesis to section 5.4, here we just sketch how to build the type on the induction principle.

It turns out that the types of the constructors of is_T give us a very good hint on the type of the induction principle. The type of the first premise

(∀ a, PA a → P (Leaf A a)) →

is exactly the type of the is_Leaf constructor

| is_Leaf a (pa : PA a) : is_rtree A PA (Leaf A n)

where (is_rtree A PA) is replaced by P. The same holds for the other premise: its type can be trivially obtained from the type of is_Node.

Our intuition is that the inductive predicate is_T provides the same information that typing provides. Induction principles give P on (smaller) terms of the same type, that would be terms for which is_T holds. Given their inductive nature, is_T predicates are able to propagate the desired property inside parametric containers.

3.2 Isolating the syntactic termination check problem

As one expects, it is possible to prove that is_T holds for terms of type T.

Definition nat_is_nat : ∀ n : nat, is_nat n :=
fix rec n : is_nat n :=
match n as i return (is_nat i) with
| 0 => is_0
| S p => is_S p (rec p)
end.

For containers (T A) we can prove (is_T A PA) when PA is trivial.
These facts are then to be used in order to satisfy the premise of our induction principles.

Going back to our goal, we can build correctness proofs of equality tests in two steps.

For example, for natural numbers we can generate two lemmas:

\begin{verbatim}
Lemma nat_eq_correct : \forall n, is_nat n \rightarrow eq_axiom nat nat_eq n :=
  nat_induction (eq_axiom nat nat_eq) P0 PS.
Lemma nat_eq_OK n : eq_axiom nat nat_eq n :=
  nat_eq_correct n (is_nat n).
\end{verbatim}

where P0 and PS (line 2) stand for the two proof terms corresponding to the base case and
the inductive step of the proof. We omit them here for brevity.

For containers such as (list A) we can link the pieces in a similar way (at line 3 we omit
the proofs for nil and cons as before).

\begin{verbatim}
Lemma list_eq_correct A A_eq : \forall l, is_list A (eq_axiom A A_eq) l \rightarrow
  eq_axiom list A (list_eq A A_eq) l :=
  list_induction A (eq_axiom A A_eq) (eq_axiom (list A) (list_eq A A_eq)) Nil Pcons.
Lemma list_eq_OK A A_eq (HA : \forall a, eq_axiom A A_eq a) l :
  eq_axiom (list A) (list_eq A A_eq) l :=
  list_eq_correct A A_eq l (list_is_list A (eq_axiom A A_eq) HA l).
\end{verbatim}

It is interesting to look at a data type that uses a container such as rtree: the induction
hypothesis P1 given by rtree_induction perfectly fits the premise of list_eq_correct (line 7).

\begin{verbatim}
Lemma rtree_eq_correct A A_eq : \forall t, is_rtree A (eq_axiom A A_eq) t \rightarrow
  eq_axiom (rtree A) (rtree_eq A A_eq)
  :=
  rtree_induction A (eq_axiom A A_eq) (eq_axiom (rtree A) (rtree_eq A_eq))
  PLeaf
  (fun l (P1 : is_list (rtree A) (eq_axiom (rtree A) (rtree_eq A_eq))) l =>
   .. list_eq_correct (rtree A) (rtree_eq A A_eq) l P1 ..).
Lemma rtree_eq_OK A A_eq (HA : \forall a, eq_axiom A A_eq a) t :
  eq_axiom (rtree A) (rtree_eq A A_eq) t :=
  rtree_eq_correct A A_eq t (rtree_is_rtree A (eq_axiom A A_eq) HA t).
\end{verbatim}

Type checking the terms above does not require any term to be transparent. Actually
they are applicative terms, there is no apparently recursive function involved.

Still there is no magic, we just swept the problem under the rug. In order to type check
the proof of rtree_is_rtree Coq needs to look at the proof term of list_is_list:

\begin{verbatim}
Definition rtree_is_rtree A PA (HPA : \forall a, PA a) :=
  fix IH t {struct t} : is_rtree A PA t :=
  match t with
  | Leaf a => is_Leaf A PA a (HPA a)
  | Node l => is_Node A PA l (list_is_list (rtree A) (is_rtree A) IH l)
end.
\end{verbatim}

As we explained in section 2 Coq would reject this term if the body of list_is_list was
opaque.

Even if we cannot make the problem disappear (without changing the way Coq checks
termination), we claim we confined the termination checking issue to the world of reified
type information. The transparent proofs of theorems such as T_is_T are separate from the
other, more relevant, proofs that can hence remain opaque as desired.
Elpi [6] is a dialect of λProlog [12], a higher order logic programming language. Elpi can be used as an extension language for Coq [18] in order to develop new commands in a programming language that has native support for bound variables.

Coq terms are represented in λ−tree syntax style [11] (sometimes also called Higher Order Abstract Syntax) reusing the binders of the programming language to represent the ones of Coq. For example, the term (fun x => fact x) is represented as (lam (λx, app ["fact", x])). We say that app and lam are object level term constructors standing for iterated (n-ary) application and unary lambda abstraction; "fact" is a constant and x is a variable bound by λx, that is the binder of the programming language. 2

Programs are organized in clauses that represent both a data base of known facts and a set of rules to derive new facts out of known ones. For example one could use a relation named eq-db to link a type to its equality test.

```
eq-db "nat" "nat_eq",
eq-db (app ["list", B]) (app ["list_eq", B, B_eq]) :- eq-db B B_eq.
```

The first clause is a fact stating that nat_eq is the equality test for type nat. The second clause is an inference one and reads: the equality test for (list B) is (list_eq B B_eq) if B_eq is the equality test for B.

The eq-db data base can be queried for an equality test for, say, (list nat) by writing the goal (eq-db (app ["list", "nat"])) F where F is a variable to be filled in. By chaining the two clauses Elpi answers (F = app ["list_eq", "nat", "nat_eq"]) that reads back in the Coq syntax as (list_eq nat nat_eq), the desired equality test for (list nat).

It is worth pointing out that in λProlog the set of clauses is dynamic: a program is allowed to add clauses inside a specific scope (typically the one of a binder) and the runtime collects them when the scope ends. As we will see, this feature is useful when a derivation takes place under an hypothetical context, e.g. when one assumes a parameter A and an equality test A_eq. No other feature of the Elpi language is relevant to this paper.

Finally, the integration of Elpi in Coq exposes to the extension language primitives to access the logical environment, e.g. to read an inductive data type declaration; to declare a new inductive type; to define a new constant; etc.

5 Anatomy of the derivation

The structure of the derivation is depicted in the following diagram. Each box represents a component deriving a complete term. An arrow from component A to component B tells that the terms generated by B are used by the terms generated by A. The interfaces between these components are indeed types: one can replace the work done by each component with a few hand written terms, if necessary.

---

2 Here we simplify a little the embedding and use strings to represent named terms, omitting their nodes: For example nat, an inductive type, is actually written (indt "Coq.Init.Datatypes.nat"), while fact, a defined constant, is written (const "Coq.Arith.Factorial.fact").
The eq component is in charge of synthesizing the program performing the equality test. The correctness proof generated by eqcorrect goes by induction on the first term of the two being compared and then goes on in a different branch for each constructor K. The property being proved by induction is expressed using eq_axiom that, as we will detail in section 5.6 is equivalent to a double implication. The bcongr component proves that the property is preserved by equal contexts, that is when the two terms are built using the same constructor. When they are not the program must return false and the equality be false as well: this is shown by eqK, that performs the case split on the second term. The no confusion property of constructor is key to this contextual reasoning. projK and isK generate utility functions that are then used by injection and discriminate to prove that constructors are injective and different. As we sketched in the previous sections the unary parametricity translation plays a key role in expressing the induction principle. The inductive predicate is_T for an inductive type T is generated by param1 while param1P shows that terms of type T validate is_T. functor shows that is_T is a functor when T has parameters. This property is both used to synthesize induction principles and also to combine the pieces together in the correctness proof. The eqOK component hides the is_T relation from the theorems proved by eqcorrect by using the lemmas T_is_T proved by param1P.

5.1 Equality test

Synthesizing the equality test for a type T proceeds as follows. First the test takes in input each type parameter A together with an equality test A_eq. Then the recursive function takes in input two terms of type T and inspects both via pattern matching. Outside the diagonal, where constructors are different, it says false. On the diagonal it composes the calls on the arguments of the constructors using boolean conjunction. The code called to compare two arguments depends on their type: If it is T then it is a recursive call; if it is a type parameter A then we use A_eq; if it is another type it uses the corresponding equality test.

Let us take for example the equality test for rose trees:

```coq
Definition rtree_eq A (A_eq : A → A → bool) :=
fix rec (t1 t2 : rtree A) {struct t1} : bool :=
  match t1, t2 with
  | Leaf a, Leaf b => A_eq a b
  | Node l, Node s => list_eq (rtree A) rec l s
  | _, _ => false
end.
```

Line 5 calls list_eq since the type of l and s is (list (rtree A)) and it passes to it rec since the type parameter of list is (rtree A).
Here is an excerpt of Elpi code used to synthesize the body of the branches:

```elpi
eq-db "A" "A_eq".
eq-db (app["rtree", "A"] ) "rec".
eq-db (app["list", B]) (app["list_eq", B, B_eq]) :- eq-db B B_eq.
```

The first clause says that \( A_{\text{eq}} \) is the equality test for type \( A \), and is used to build the branch at line 4. The third clause, chained with the second one, combines list_eq with rec building the branch at line 5. The first two clauses are present only during the derivation of the body of the fixpoint, under the context formed by the type parameter \( A \), its equality test \( A_{\text{eq}} \), and the recursive call rec itself. Once the derivation is complete both clauses are removed from the data base and the following one is permanently added.

```elpi
eq-db (app["rtree", B]) (app["rtree_eq", B, B_eq]) :- eq-db B B_eq.
```

5.2 Parametricity

The param1 component is able to generate the unary parametricity translation of types and terms following [7]. We already gave many examples in section 3. The param1P component synthesizes proofs that terms of type \( T \) validate is\(_{\text{T}}\) by a trivial structural recursion: constructor \( k \) is mapped to is\(_k\). When \( T \) is a container we assume the triviality of the property on the type parameter. For example:

```elpi
Definition rtree_is_rtree A (PA : A → U) : (∀x, PA x) → ∀t, is_rtree A PA t.
```

5.3 Functoriality

The functor component implements a double service. For non-indexed containers it synthesizes a simple map:

```elpi
Definition list_map A B : (A → B) → list A → list AB.
```

The derivation becomes more interesting when the container has indexes, e.g. when the container is a is\(_T\) inductive predicate. On indexed data types the derivation avoids to map the indexes and consequently all type variables occurring in the types of the indexes. For example, mapping the is\(_\text{list}\) inductive predicate gives:

```elpi
Lemma is_list_funct A P Q : (∀a, P a → Q a) → ∀l, is_list A P l → is_list A Q l.
```

This property corresponds to the functoriality of is\(_\text{list}\) over the property about the type parameter. Note that parameters of arity one, such as \( p \), are mapped point wise.

As we did for the eq-db data base of equality tests, we can store these maps as clauses and use the data base later on in the induction and eqcorrect derivations. Here is an excerpt of Elpi code for this data base, that we call funct-db:

```elpi
funct-db (app["is_list", A, P]) (app["is_list", A, Q]) (app["is_list_funct", A, P, Q, F]) :- funct-db P Q F.
```

Note that the terms involved are “point free”, i.e. the first two arguments are terms of arity one, while the third term is of arity two. The identity is written as follows:

```elpi
funct-db P Q (lam (λa, lam (λp, p))).
```

This means that when one has a term \( a \) and a term \( (p : P a) \), in order to obtain a term \( (q : Q a) \) he can query funct-db by asking Elpi to fill in \( M \) in (funct-db "P" "Q" M). If the answer is \( M = f \) then the desired term is obtained by passing \( a \) and \( p \) to \( f \), that is \( f \ a \ p : Q a \).
5.4 Induction

In order to derive the induction principle for type T we first derive its unary parametricity
translation is_T. The is_T inductive predicate has one constructor is_K for each constructor
K of the type T. The type of is_K relates to the type of K in the following way. For each
argument (a : A) of K, is_K takes two arguments: (a : A) and (pa : is_A a). Finally the
type of (is_K a1 pa1 .. an pa) is (is_T (K a1 .. an)).

The induction principle is synthesized by following these steps:
1. take in input each parameter A1 P1 .. An Pan of is_T.
2. take in input a predicate (P : T A1 .. An → U).
3. for each constructor is_K of type
   (∀ A1 PA1 .. An PaAn, is_T A1 PA1 .. An PaAn (K a1 .. am))
4. take in input assumption HK of type (∀ a1 is_T .. (PAn .. a1)
5. perform recursion on x and a case split. Then in each branch
   a. bind all arguments of is_K, namely
      (ai : A1) (pa1 : is_A1 ai) .. (an : An) (pan : is_An an)
   b. obtain qa1 by mapping the corresponding pai (as in funct-db, see below).
   c. return (HK ai qa1 .. an qan)

Let’s take for example the induction principle for rose trees:

```
Definition rtree_induction A PA P
  (HNode : ∀ a, PA a → P (Leaf A a))
  (∀ t, is_rtree A PA t → P t)
  :=
  fix IH (t: rtree A) (x: is_rtree A PA t) {struct x}: P t :=
  match x with
  | is_Leaf a pa => HLeaf a pa
  | is_Node l pl => (* pl: is_list (rtree A) (is_rtree A PA) l *)
  | HNode l (is_list_funct (rtree A) (is_rtree A PA) P IH l pl)
  end.
```

Note how, intuitively, the type of HLeaf can be obtained from the type of is_Leaf by
replacing (is_rtree A PA) with P.

Finally let us see how the second argument to HNode is synthesized. We take advantage of
the fact that Elpi is a logic programming language and we query the data base funct-db as
follows. First we temporarily register the fact that IH maps (is_rtree A PA) to P obtaining,
among others, the following clauses.

```
funct-db (app["is_rtree", "A", "PA"] ) "P" "IH".
funct-db (app["is_list","A","P"] ) (app["is_list","A","Q"] ) (app["is_list_funct","A","P","Q","F"] ) :=
funct-db P Q F.
```

Then we query funct-db as follows:

```
funct-db (app["is_list", app["rtree","A"], app["is_rtree","A","PA"]])
  (app["is_list", app["rtree","A"], "P"]) Q.
```

The answer (Q = app["is_list_funct",app["rtree","A"],app["is_rtree","A","PA"],"P","IH"])
is exactly the second term we need to pass to HNode (once applied to l and pl, line 10 above).

It is worth pointing out that, for the term to be accepted by the termination checker the
map over is_list must be transparent.
To sum up the unary parametricity translation gives us the type of the induction principle, up to a trivial substitution. The functoriality property of the inductive predicates obtained by parametricity gives us a way to prove the branches.

5.5 No confusion property

In order to prove that an equality test is correct one has to show the so called “no confusion” property, that is that constructors are injective and disjoint (see for example [10]).

The simplest form of the property of being disjoint is expressed on bool:

**Lemma** bool_discr : true = false → ∀T : U, T.

This lemma is proved by hand once and for all. What the isK component synthesizes is a per-constructor test to be used in order to reduce a discrimination problem on type T to a discrimination problem on bool. For the rose tree data type isK generates:

**Definition** is_Node A (t : rtree A) := match t with Node _ => true | _ => false end.

**Definition** is_Leaf A (t : rtree A) := match t with Leaf _ => true | _ => false end.

The discriminate components uses one more trivial fact, eq_f, in order to assemble these tests together with bool_discr.

**Lemma** eq_f T1 T2 (f : T1 → T2) : ∀a b, a = b → f a = f b.

From a term H of type (Node 1 = Leaf a) the discriminate procedure synthesizes:

(bool_discr (eq_f (rtree A) (rtree A) (is_Node A) H)) : ∀T : U, T

Note that the type of the term (eq_f .. H) is (is_Node A (Node 1) = is_Node A (Leaf a)) that is convertible to (true = false), the premise of bool_discr.

In order to prove the injectivity of constructors the projK component synthesizes a projector for each argument of each constructor. For the cons constructor of list we get:

**Definition** get_cons1 A (d1 : A) (d2 : list A) (l : list A) : A :=

match l with nil => d1 | x :: _ => x end.

**Definition** get_cons2 A (d1 : A) (d2 : list A) (l : list A) : list A :=

match l with nil => d2 | _ :: xs => xs end.

Each projector takes in input default values for each and every argument of the constructor. It is designed to be used by the injection procedure as follows. Given a term H of type (x :: xs = y :: ys) it synthesizes:

(eq_f (list A) A (get_cons1 A x xs) (x :: xs) (y :: ys) H) : x = y

(eq_f (list A) (list A) (get_cons2 A x xs) (x :: xs) (y :: ys) H) : xs = ys

These terms are easy to build given that the type of H contains the default values to be passed to the projectors. Note that the type of the second term is actually:

get_cons2 A x xs (cons x xs) = get_cons2 A x xs (cons y ys)

that is convertible to the desired type (xs = ys).

5.6 Congruence

In the definition of eq_axiom we use the reflect predicate [9]. It is a sort of if-and-only-if specialized to link a proposition and a boolean test. It is defined as follows:
In our case the shape of \( P \) is always an equation between two terms of an inductive type, i.e. constructors. When the same constructor occurs in both sides, as in \((k \; x_1 \ldots x_n = k \; y_1 \ldots y_2)\), the equality test discards \( k \) and proceeds on each \((x_1 = y_1)\). The \texttt{bcongr} component synthesizes lemmas helping to prove the correctness of this step. For example:

```
Lemma list_bcongr_cons A:
  \( \forall (x : A) \; y \; : \; A \; b \; : \; \text{reflect} \; (x = y) \; b \; \rightarrow \; \text{reflect} \; (x \; :: \; xs = y \; :: \; ys) \; (b \; \&\& \; c) \) :
  \[
  \begin{align*}
  \text{reflect} \; (x \; :: \; xs = y \; :: \; ys) \; (b \; \&\& \; c) := & \\
  \text{match} \; \text{hb} \; \&\& \; \text{hc} \; \text{with} \; & \\
  \text{end}. \end{align*}
  \]
```

The elimination of \( \text{hb} \) and \( \text{hc} \) substitutes \( b \) and \( c \) by either \texttt{true} or \texttt{false}. In the branch at line 6 the boolean expression is hence \( (\text{true} \; \&\& \; \text{true}) \) while the proposition is \( (x \; :: \; xs = x \; :: \; xs) \) given that the two equations \((x = y)\) and \((xs = ys)\) were eliminated as well.

The argument of \( e \) at line 9 is the term generated by the \texttt{injection} component. The branch at line 11, covering the case where the heads are equal but the tails different, is very close to lines 8 and 9 but for the fact that the projector for the second argument of \texttt{cons} is used, instead of the projection for the first one.

There are other ways one could have expressed these lemmas, for example by not mentioning the \texttt{cons} constructor explicitly but rather an abstract function \( k \) known to be injective on the first and second argument. Even if we find this presentation more appealing on paper, in practice we found no advantage and we hence opted for the current approach.

\texttt{bcongr} gives us lemmas to propagate equality and inequality only under the same constructor. \texttt{eqK} complements this work by proving \texttt{eq_axiom} also when the constructors differ.

Recall that the induction principle does a case split on one term, the first one of the two being compared. \texttt{eqK} generates a lemma for each constructor, to be used in the corresponding branch of the induction, that performs the case split on the second term being compared.

This is the lemma generated for \texttt{Node}:

```
Inductive reflect (P : U) : bool → U :=
| ReflectT (p : P) : reflect P true
| ReflectF (np : P → False) : reflect P false.
```

Note that these lemmas are not related to the equality test specific to the inductive type. Indeed they deal with the \texttt{reflect} predicate, but not with the \texttt{eq_axiom} predicate that we use every time we talk about equality tests.
Note that the code for the first branch is what *discriminate* synthesizes; while the code in the second branch is what *bcongr* generates.

### 5.7 Correctness

The *eqcorrect* component combines the induction principle generated by *induction* with the case split on the second term provided by *eqK*.

Let’s recall the type of the correctness lemma for *list_eq*, of the induction principle and then let’s analyse the proof of *rtree_eq_correct*:

```plaintext
Lemma list_eq_correct A (fa : A → A → bool) l1 :  
  is_list A (eq_axiom A fa) l1 →  
  eq_axiom (list A) (list_eq (rtree A) (rtree_eq A A_eq)) l1 →  
  eq_axiom (rtree A) (rtree_eq A A_eq) (Node A l1).
Definition rtree_induction A PA P  
  (HLeaf : ∀ y, PA y → P (Leaf A y))  
  (HNode : ∀ l, is_list A P l → P (Node A l1)) :  
  ∀ t, is_rtree A PA t → P t.
Lemma rtree_eq_axiom_Node A (f : A → A → bool) l1 :  
  eq_axiom (list (rtree A)) (list_eq (rtree A) (rtree_eq A A_eq)) l1 →  
  eq_axiom (rtree A) (rtree_eq A A_eq) (Node A l1).
```

The proof is a rather straightforward application of the induction principle to the property

```plaintext
eq_axiom (rtree A) (rtree_eq A fa)
```

Each branch is then proved by the corresponding lemma generated by *eqK* with only one caveat: one may need to adapt the induction hypothesis, *Pl* here, in order to make it fit the premise of the lemma generated by *eqK*. In this specific case the ”adaptor” is *list_eq_correct*.

```plaintext
Lemma rtree_eq_correct A (fa : A → A → bool) :=  
  rtree_induction A (eq_axiom A fa)  
  (rtree_eq A (rtree_eq A fa))  
  (eq_axiom (rtree A) (rtree_eq A fa))  
  (rtree_eq_aLeaf (rtree_eq A fa) l1 (list_eq (rtree A) (rtree_eq A fa) 1)  
  rtree_eq_axiom_Node A fa l (list_eq_correct (rtree A) (rtree_eq A fa) l Pl)).
```

Logic programming provides a natural way to synthesize the adaptor. We load in the data base all the correctness proofs synthesized so far, as follows:

```plaintext
funct-db (app["is_list", A, PA])  
  (app["eq_axiom", app["list", A], app["list_eq", A, A_eq]]) R :=  
  R = (app["list_eq_correct", A, A_eq],  
  funct-db PA (app["eq_axiom", A, A_eq]).
```

This clause simply gives an operational reading to the type of *list_eq_correct*: the conclusion is true if the premise is. The only cleverness is to separate the premise in two parts, being a
(list A) with property PA and have PA be a sufficient condition to prove that A_eq is correct.

In this way clauses compose better: Search peels off just one type constructor at a time.

Indeed we extend the `funct-db` predicate, instead of building a new one just for correctness
lemmas, because functoriality lemmas are sometimes needed in addition to the correctness ones. Take for example this simple data type of a histogram.

```coq
Inductive histogram := Columns (bars : list nat).

Lemma histogram_induction (P : histogram → Type) :
    (∀ l, is_list nat is_nat l → P (Columns l)) →
    ∀ h, is_histogram h → P h.
```

Now look at the lemma synthesized by `eqK` for the `Columns` constructor.

```coq
Lemma histogram_eq_axiom_Columns l :
    eq_axiom (list nat) (list_eq nat nat_eq) l →
    ∀ h, eq_axiom_at histogram histogram_eq (Columns l) h.
```

Note that the type of `Pl` is `(is_list nat is_nat)` and that it needs to be adapted to match
`(is_list (eq_axiom nat nat_eq))`. The correctness lemma for `nat_eq`, namely `nat_eq_correct`
of type `(∀ n, is_nat n → eq_axiom nat nat_eq n)`, cannot be used directly but must undergo
the `is_list_funct` functor.

### 5.8 eqOK

The last derivation hides the `is_T` predicate to the final user by combining the output of
`eq_correct` and `param1P`.

```coq
Lemma list_eq_correct A A_eq :
    (∀ l, is_list A (eq_axiom A A_eq) l → eq_axiom (list A) (list_eq A A_eq) l).

Lemma list_eq_OK A A_eq A_eq_OK l :
    eq_axiom (list A) (list_eq A A_eq) l :=
    list_eq_correct A A_eq l (list_is_list A (eq_axiom A A_eq) A_eq_OK).
```

Both lemmas are needed. The former composes well and is needed if one defines a type
using lists as a container. The latter is what the user needs in order to work with lists.

### 5.9 Assessment

The code is quite compact thanks to the fact that the programming language is very high
level and that its programming paradigm is a good fit for this application.

On the average each components is about 200 lines of code. Simpler derivations like
`projK`, `isK` or even `param1P` are under 100 lines.

Debugging this kind of code did not pose particular difficulties. The typical error results
in the generated term being ill-typed. In that case the Coq type checker could be used to
identify the culprit. Given how small the derivations are, it was simple to identify the lines
generating the offending subterm.

The time required to design and develop the entire procedure amounts to approximatively
six months, but spanned over more than one and a half year: most of the time has been
spent improving the integration of Elpi in Coq in response to the experience gathered on this
type. At the time of writing the Elpi integration in Coq does not support mutual inductive
types, universe polymorphic definitions and primitive projections.
All derivations support polynomial types. Some derivations also support index data, eg
\textit{eq} is able to synthesize an equality test for vectors. Most of the derivations for contextual
reasoning, such as \textit{eqK} and \textit{bcongr} do not support indexes.

\section{Related work}

Systems similar to Coq \cite{Matita}, e.g. Matita \cite{Matita}, Lean \cite{Lean}, Agda \cite{Agda} and Isabelle \cite{Isabelle} all generate
induction principles automatically, and some of them also the no confusion properties.
To our knowledge Isabelle is the only system that generates sensible induction principles
and proved equality tests when containers are involved. As described in \cite{Iso} the (co)datatype
package is built on top of Bounded Natural Functors \cite{BNF}, a notion that makes the con-
struction of (co)datatypes in Higher Order Logic compositional. Our starting point is very
different since Coq, and type theory in general, internalizes the definitional mechanism for
(co)datatypes. As a consequence a package like the one described in this paper cannot
change it but only work around its eventual limitations. In particular the way Coq checks
recursive functions for termination is a fixed, syntactic, non modular, criteria for which some
alternatives have been studied (see for example \cite{Termination1, Termination2}) but never implemented. The non
modular criteria applies to induction principles as well, since they are proved using recursion.
It is a strength of the construction described in this paper to recover some modularity and
hence be able to synthesize mechanically most of what \cite{Iso} is able to synthesize.

Most Interactive Theorem Provers come with simple forms of Prolog-like automation,
usually in the form of Type Classes. The user typically resorts to that in order to perform
some of the inductive reasoning one needs in order to synthesize code in a type directed way.
To our knowledge no ready-to-use package to synthesize equality tests and their proofs was
written this way.

Some systems, notably Lean, come with a whole round meta programming framework.
Still, to our knowledge, the primary application is the development of proof commands, not
program/proof synthesis, in spite of the stunning similarity.

Coq provides two mechanisms strictly related to this work. The \texttt{Scheme Equality} com-
mand generates for a type \(T\) the code for the equality test (\texttt{T_eqb}) and a proof that equality
is decidable on \(T\). The proof internally uses the equality test, but its type does not:

\[
T_{\text{eq dec}} : \forall x \ y : T, \{x = y\} + \{x \neq y\}
\]

By unfolding the proof term, that is transparent, it should be possible to recover the fact
that \texttt{T_eqb} is a correct equality test. Data types defined using containers are not supported.
The \texttt{decide equality} tactic requires the user to start a lemma with a statement as the one
depicted above. The tactic only performs one (case split) step and has to be iterated by
hand. It does not remember which equalities were proved decidable before, it is up to the
user to eventually share code. The proof term generated is, in a type theoretic sense, a
program even if its code mixes the comparison test with its correctness proof. This proof
is fully transparent, and inlines all the contextual reasoning steps such as injection and
discrimination. As a result the term is very large and computationally heavy when run
within Coq.

In the programming language world derivation is much more developed. The dominant
approach is to provide some meta programming facilities, e.g. by providing a syntax to the
declaration of types and then use the programming language itself to write derivations \cite{MetaProg}.
that run at compile time as compiler plugins. Our approach is similar in a sense, since we work at the meta level on the syntax of types (and terms), but it is also very different since we pick a different programming language for meta programming. In particular we choose a very high level one that makes our derivations very concise and hides uninteresting details such as the representation of bound variables. The derivation described in the paper is the result of many failed attempts and we believe that the high level nature of the programming language we chose played an important role in the exploratory phase.

7 Conclusion

We described a technique to derive stronger induction principles for Coq data types built using containers. We use the unary parametricity translation of a data type in order to fuel its induction principle, to thread an invariant on the contained when used as a container and finally to confine the modularity problems stemming from the termination check implemented in Coq. Finally we provide a Coq package deriving correct equality tests for polynomial inductive data types.

It is work in progress to extend the derivation to inductive types with decidable indexes. Preliminary work hints that indexes of base types such as nat pose no problem. On the contrary when indexes mention containers, that admit a decidable equality only if their contained does, the paramIP component gets substantially more complex. In particular some notions of Homotopy Type Theory come in to play. For example the notion of being provable on the entire domain such as $(\forall a : A, P a) \rightarrow (\forall t : T A, is_T A P t)$ seems to require to be strengthened using the notion of contractibility (that is, the property should hold and its proof be unique), in order for the construction to compose well.

We also look forward to let the user tune the derivation process by annotating the type declarations. For example the user may want to skip certain arguments when generating the equality test, such as the integer describing the length of a sub vector in the cons constructor. The resulting equality test surely requires some user intervention in order to be proved correct, but it features a better computational complexity.

Finally, adding other derivations to the package seems appealing. For example the interface next to eqType in the hierarchy used in the Mathematical Component library is the one of countable types, i.e. types in bijection with natural numbers. The interface requires, roughly, a serialization function to another countable type, a tedious task that could be made automatic.

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