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Variational and Hamiltonian Formulations of Geophysical Fluids using Split Exterior Calculus

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Abstract

Variational and Hamiltonian formulations for geophysical fluids have proven to be a very useful tool in understanding the physics of flows and developing new numerical discretizations, and represent an important aspect of the geometric structure of the equations for geophysical fluid flow. However, the majority of such formulations have been developed using vector calculus: scalars and vectors. Another key aspect of the geometric structure is understanding the equations in terms of differential forms, and in particular straight and twisted differential forms. This arguably began with the work of Enzo Tonti, who developed a classification of physical quantities into source and configuration variables; which are unambiguously associated with inner-oriented/straight (configuration) and outer-oriented/twisted (source) geometric entities/differential forms. This classification has proven fruitful in electromagnetics and classical mechanics. However, an extension of the idea to compressible fluids was lacking until the development of the split covariant equations by Werner Bauer. This work aims to unify these two aspects of the geometric structure for fluids, by developing variational and Hamiltonian formulations for geophysical fluids using split exterior calculus. A key aspect is that the Hamiltonian structure gives a natural representation of the topological-metric splitting in the split covariant equations through the Poisson brackets (purely topological equations) and the functional derivatives of the Hamiltonian (metric-dependent equations). These new formulations are illustrated with some specific examples of commonly studied geophysical fluids: the shallow water equations, thermal shallow water equations and the compressible Euler equations.

1 Introduction

Fundamental aspects of the geometric structure of the equations for geophysical fluids can be understood by studying the variational [19, 18, 8, 39, 21] or the Hamiltonian [27, 11,

[16, 25, 29] formulations. These approaches facilitate a unified representation of a diverse range of topics, such as: stability theorems and pseudo-energy/pseudo-momentum/finite-amplitude invariants for wave-mean flow interactions and associated linearizations [30], the development of novel numerical discretizations [28, 10, 40, 15, 13, 12, 3] and development of approximate models [39]. However, existing literature on variational approaches in geophysical fluids works mostly in terms of vector calculus (scalars and vectors), with some limited work using exterior calculus and standard differential forms [7].

Another fundamental aspect of the geometric structure is the use of different forms instead of scalars and vectors. The use of exterior calculus in fluids [44, 1, 2, 7] and other physical theories such as electromagnetics and solid mechanics [9, 5, 22, 14, 23, 45] has a long history. For the most part, these approaches use standard differential forms. However, the work of Enzo Tonti [37, 36] suggests that it is fruitful to use instead split exterior calculus, and to represent physical quantities in terms of straight and twisted differential forms rather than standard differential forms. This permits a natural classification of variables into source (twisted) and configuration (straight) quantities, that has shown to be robust for a wide range of physical theories. Using these ideas, a split covariant formulation of several geophysical fluids was developed in [2], that naturally separated the equations into topological equations and metric equations. However, the underlying variational and Hamiltonian structures were not explored.

This work combines these two threads, and develops a version of the Euler-Poincaré framework with associated Euler-Lagrange equations in terms of split exterior calculus, along the corresponding curl-form Hamiltonian formulation (termed the split Hamiltonian formulation). We treat a single component, single phase fully compressible fluid in a rigid domain ($n = 2$ and $n = 3$) with material boundaries that can be characterized by a mass density, a velocity and possibly a thermodynamic scalar. Lagrangians (and Hamiltonians) are shown to be the result of a metric pairing (inner product) between source (twisted) and configuration (straight) variables, or in other words, between twisted and straight differential forms. The Poisson brackets arise through the use of a new notion specific to split exterior calculus: the topological pairing, which pairs k -forms with the associated $n - k$ form of the opposite type (that arises from the action of the Hodge star). Unlike the metric pairing, this pairing requires only the wedge and integration, which are purely topological operators. The split Hamiltonian formulation is shown to reproduce the split covariant equations from [2], and gives a natural representation of the topological-metric splitting found there through the Poisson brackets (topological equations) and the functional derivatives of the Hamiltonian (metric equations).

This new variational formulation of geophysical fluids in terms of split exterior calculus provides a deeper understanding of the geometric structure underlying the equations of motion. It is also believed that the split Hamiltonian formulation will provide new insight into existing discretization schemes and facilitate the development of new approaches. Just as the standard exterior calculus formulation underlies single grid compatible discretizations [4, 7], it seems likely that the split exterior calculus formulation underlies primal-dual grid compatible discretizations [6, 24, 17, 33]. A important example of such a discretization is the TRiSK scheme ([35, 32, 34, 41, 43, 42, 13], which is widely used in existing atmospheric and ocean models [10, 31] despite its known shortcomings with respect to accuracy.

The remainder of this paper is structured as follows. Section 2 introduce split exterior

calculus in a concise way, while in Section 3 we include the new notion of a topological pairing. A review of variational and curl-form Hamiltonian formulations using vector calculus appears in Section 4. Then, in Section 5 the Lagrangian and curl-form Hamiltonian formulations for $n = 3$ using split exterior calculus are presented. In Section 6 the properties of these equations, such as energy conservation, Casimirs and the Kelvin Circulation Theorem are verified. The formulation is illustrated (by introducing specific Lagrangians and Hamiltonians) in Section 7 with some examples of commonly studied geophysical fluids: shallow water, thermal shallow water and compressible Euler. Finally, Section 8 draws some conclusions and discusses future direction of research. Appendix A discusses the use of some alternative forms of the thermodynamic scalar (which are simply a change of variables) and Appendix B contains the details of some simplifications that arise when $n = 2$ or there is no thermodynamic scalar. Appendix C contains a proof about the properties of the new topological pairing.

2 Exterior Calculus, Differential Geometry and Vector Proxies

In this section we introduce the required concepts of differential geometry in a concise way without proofs. For more details, we refer the readers to textbooks on differential geometry or to [2] for a concise overview. We will present the definitions and operators in dimension independent form, but provides examples in form of local coordinate representations in 2D/3D as this will be useful for our later discussions.

2.1 Topological operators on topological manifold

We start our introduction of differential geometry with a discussion about differentiable topological manifolds, differential forms (DF), and topological operators. For the following definitions within this subsection neither orientation nor metric is needed.

We consider a domain of interest given by a smooth n -dimensional closed compact orientable manifold \mathcal{M} . The pair (U, ϕ_U) consisting of patches $U \subset \mathcal{M}$ that cover \mathcal{M} and of invertible maps $\phi_U : U \rightarrow V \in \mathbb{R}^n$ defines local coordinate representations \mathbf{x} on the patch $U \subset \mathcal{M}$ such that the local coordinates of point $\mathbf{x} \in \mathcal{M}$ are given by $(x^1, \dots, x^n) = \phi_U(\mathbf{x})$.

The span of vectors that are tangent to $\mathbf{x} \in \mathcal{M}$ form the tangent space $T_{\mathbf{x}}\mathcal{M}$. A *vector field* \mathbf{u} on \mathcal{M} is a smooth mapping from each \mathbf{x} to $T_{\mathbf{x}}\mathcal{M}$; hence it is tangent to \mathcal{M} everywhere. The space of vector fields is denoted as $\mathcal{X}(\mathcal{M})$. Vector fields can be represented on U in the local basis $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ of $T_{\mathbf{x}}\mathcal{M}$ as $\sum_i u^i \frac{\partial}{\partial x^i}$.

The dual objects to tangent vectors are *cotangent vectors*; at every point \mathbf{x} they are linear maps from $T_{\mathbf{x}}\mathcal{M}$ to \mathbb{R} . Cotangent vectors span at \mathbf{x} the cotangent space $T_{\mathbf{x}}^*\mathcal{M}$. A differential 1-form ${}^1\omega$ is a smooth mapping from \mathbf{x} to the cotangent space $T_{\mathbf{x}}^*\mathcal{M}$. Hence, ${}^1\omega$ defines a smooth mapping from the vector field \mathbf{u} to a scalar function ${}^1\omega(\mathbf{u})$ with values ${}^1\omega(\mathbf{u})(\mathbf{x}) \in \mathbb{R}$ at point \mathbf{x} . The superscript in front of ω indicates the differential form's degree.

On the coordinate patch U , we define the local basis (dx^1, \dots, dx^n) for 1-forms dual to the local basis $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ for vector fields via $dx^i(\frac{\partial}{\partial x^j}) = \delta_{ij}$. In this basis, a 1-form

can be represented as ${}^1\omega = \sum_i \omega_i dx^i$ with coordinate function ω_i .

A differential k -form ${}^k\omega$, or simply k -form, is a smooth mapping that assigns to each point \mathbf{x} an anti-symmetric k -linear mapping $T_{\mathbf{x}}\mathcal{M} \times \cdots \times T_{\mathbf{x}}\mathcal{M} \rightarrow \mathbb{R}$ on the tangent space $T_{\mathbf{x}}\mathcal{M}$. We denote the space of all k -forms as $\Lambda^k(\mathcal{M})$, or simply Λ^k . The direct sum $\Lambda(\mathcal{M}) := \bigoplus_{k=0}^n \Lambda^k(\mathcal{M})$ is a graded algebra where the algebraic structure is given by the wedge product (or exterior product) \wedge .

The *wedge product* of a k -form and an l -form gives a $(k+l)$ -form. For ${}^k\alpha, {}^k\beta \in \Lambda^k$, ${}^l\gamma, {}^l\delta \in \Lambda^l$, and a, b scalars, this product has the following properties:

1. bilinearity:
 $(a {}^k\alpha + b {}^k\beta) \wedge {}^l\gamma = a({}^k\alpha \wedge {}^l\gamma) + b({}^k\beta \wedge {}^l\gamma), \quad {}^k\alpha \wedge (a {}^l\gamma + b {}^l\delta) = a({}^k\alpha \wedge {}^l\gamma) + b({}^k\alpha \wedge {}^l\delta);$
2. anticommutativity:
 ${}^k\alpha \wedge {}^l\gamma = (-1)^{kl} {}^l\gamma \wedge {}^k\alpha;$
3. associativity:
 $({}^k\alpha \wedge {}^l\beta) \wedge {}^m\gamma = {}^k\alpha \wedge ({}^l\beta \wedge {}^m\gamma).$

Noting that 0-forms are scalar functions f , their wedge products with k -forms are given by the product $f \wedge {}^k\omega = f {}^k\omega$.

Remark 1 Using this wedge product notation also for zero forms will allow us to clearly indicate mappings between straight and twisted k -forms (introduced in Section 3), via the twisted identity 0-form ${}^0\tilde{1}$. Hence, we will use this notation throughout the entire manuscript.

Remark 2 Using the wedge-product, differential k -forms can be constructed out of 1-forms. For instance, the wedge product of two 1-forms ${}^1\alpha$ and ${}^1\beta$ gives the 2-form ${}^1\alpha \wedge {}^1\beta$ that is defined via ${}^1\alpha \wedge {}^1\beta(X_1, X_2) = {}^1\alpha(X_1) {}^1\beta(X_2) - {}^1\alpha(X_2) {}^1\beta(X_1)$ for all pairs of vector fields $X_1, X_2 \in \mathcal{X}(\mathcal{M})$. In the same vein, any k -form can be constructed like that from 1-forms.

To give an example in 2D, the wedge product of two 1-forms may be written in the general form in local coordinates like ${}^1\alpha \wedge {}^1\beta = a dx^1 \wedge dx^2$ for some scalar function a .

Remark 3 Following the last remark, k -forms on n -dimensional manifolds can generally be written in local coordinates like

$${}^k\omega = \sum_{1 \leq c_1 < \dots < c_k \leq n} \omega_{c_1 \dots c_k} dx^{c_1} \wedge \dots \wedge dx^{c_k}$$

with compoment functions $\omega_{c_1 \dots c_k}$.

Finally, we define two very important topological operators acting on k -forms, the exterior derivative d and the interior product $i_{(\cdot)}$. The *exterior derivative* d is a map from k to $k+1$ -forms that satisfies:

1. for a function ${}^0f \in \Lambda^0$, its total differential in local coordinates is $d {}^0f = \sum_i \frac{\partial f}{\partial x^i} dx^i$
2. product rule (Leibniz rule): d is \mathbb{R} linear and for ${}^k\alpha \in \Lambda^k$ and ${}^l\gamma \in \Lambda^l$ there is
 $d({}^k\alpha \wedge {}^l\gamma) = (d {}^k\alpha) \wedge {}^l\gamma + (-1)^{(k)} {}^k\alpha \wedge (d {}^l\gamma)$
3. closure: $d(d {}^k\omega) = d^2 {}^k\omega = 0$ for any k -form ${}^k\omega$.

The inverse operation to d is the contraction of a k -form ${}^k\omega$ with vector field X , denoted as $i_X {}^k\omega$. It is a map from k to $(k - 1)$ forms defined by

$$i_X {}^k\omega(X_2, \dots, X_k) = {}^k\omega(X, X_2, \dots, X_k)$$

for all vectors $X_i \in \mathcal{X}(\mathcal{M})$, $i = 2, \dots, k$ while the contraction of a 0-form is zero. As d , the interior product (also denoted as contraction operator) is \mathbb{R} -linear and, for ${}^k\alpha \in \Lambda^k$ and ${}^l\gamma \in \Lambda^l$: $i_X({}^k\alpha \wedge {}^l\gamma) = (i_X {}^k\alpha) \wedge {}^l\gamma + (-1)^{(k)} {}^k\alpha \wedge (i_X {}^l\gamma)$. Moreover, we have $i_{f(X)} {}^k\alpha = f i_X {}^k\alpha$.

Example 4 For instance in 2D, the contraction of a 1-form ${}^1\omega$ with a vector field \mathbf{u} is the scalar function $i_{\mathbf{u}} {}^1\omega = {}^1\omega(\mathbf{u})(\mathbf{x}) \in \Lambda^0$. The contraction of a 2-form ${}^2\omega$ with \mathbf{u} is the 1-form $i_{\mathbf{u}} {}^2\omega$. From its definition by $i_{\mathbf{u}} {}^2\omega(\mathbf{v}) = {}^2\omega(\mathbf{u}, \mathbf{v})$ for any vector \mathbf{v} and using local coordinates such that ${}^2\omega = \omega(\mathbf{x})dx^1 \wedge dx^2$ with component function $\omega(\mathbf{x})$, there follows the corresponding local representation $i_{\mathbf{u}} {}^2\omega = i_{\mathbf{u}} \omega(\mathbf{x})dx^1 \wedge dx^2 = \omega(\mathbf{x})(u^1 dx^2 - u^2 dx^1)$.

We can define the Lie derivative L_X of a k -form ${}^k\omega$ with respect to the vector field X in terms of interior product and exterior derivative:

$$L_X {}^k\omega = i_X d {}^k\omega + d i_X {}^k\omega. \quad (1)$$

This formula is frequently referred to as *Cartan's formula*.

2.2 Orientated manifolds and inner and outer oriented DF

To define the notion of integrals of k -forms we require to fix an orientation. We denote the selected orientation with Or and the opposite one with $-Or$.

Given a k -form in local coordinates, ${}^k\omega = \omega(\mathbf{x})dx^1 \wedge \dots \wedge dx^k$ with component function $\omega(\mathbf{x})$, its integral over the k -dimensional submanifold $V^k \subset \mathcal{M}$ is given by

$$\int_{V^k} {}^k\omega = \int_{\phi_U(V^k)} \omega(\mathbf{x})dx^1 \dots dx^k.$$

This definition is independent of the choice of coordinates, due to the change-of-variables formula. *Stokes theorem* for k -form ${}^k\omega$ on an oriented manifold \mathcal{M} is given by

$$\int_{\mathcal{M}} d {}^k\omega = \int_{\partial\mathcal{M}} {}^k\omega.$$

The integral values depend on the ambient orientation as the sign of the volume form changes from Or to $-Or$. Using a twisted volume form instead (see (4)) allows us to define integrals that are invariant under changes of the ambient orientation.

2.3 Metric on \mathcal{M} and metric and/or orientation dependent operators

The following DF and operators depend on the orientation and may change sign when the ambient orientation changes. Such change can be avoided when using twisted DF and twisted operators.

2.3.1 Formulations that depend on ambient orientation

To continue, we equip the topological manifold \mathcal{M} , not necessarily oriented, with a metric $\mathbf{g}(\cdot, \cdot) = g_{ij} dx^i dx^j$ with coefficients $g_{ij} = \mathbf{g}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $i, j = 1, \dots, n$, a nondegenerate symmetric bilinear form on \mathcal{M} . At point $\mathbf{x} \in \mathcal{M}$, $\mathbf{g}|_{\mathbf{x}}$ is an inner product on $T_{\mathbf{x}}\mathcal{M}$. The pair $(\mathcal{M}, \mathbf{g})$ is called Riemannian manifold.

The metric \mathbf{g} induces a nondegenerate symmetric bilinear form $\mathbf{g}^{(k)} = \langle \cdot, \cdot \rangle$, a mapping from two k -forms to functions, by $\langle \cdot, \cdot \rangle : \Lambda^k \times \Lambda^k \rightarrow C^\infty$ with $\langle {}^k\omega, {}^k\eta \rangle = \langle {}^k\eta, {}^k\omega \rangle$ for all ${}^k\omega, {}^k\eta \in \Lambda^k$. To present it in local coordinates, we note that the inverse of the coefficient matrix associated to \mathbf{g} , $[g_{ij}]$, is given by $[g^{ij}] = [g_{ij}]^{-1}$. Then, for ${}^k\alpha = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and ${}^k\beta = \sum_{i_1 < \dots < i_k} \beta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$, we find

$$\mathbf{g}^{(k)}({}^k\alpha, {}^k\beta) = \langle {}^k\alpha, {}^k\beta \rangle = \sum_{i_1 < \dots < i_k} \alpha_{i_1, \dots, i_k} \beta^{i_1, \dots, i_k} \quad (2)$$

with contravariant coefficients $\beta^{i_1 \dots i_k} = \sum_{j_1, \dots, j_k} g^{i_1 j_1} \dots g^{i_k j_k} \beta_{j_1 \dots j_k}$.

\mathbf{g} induces further a uniquely defined volume form on \mathcal{M} . In local coordinates it can be written as $\mu = \sqrt{|\mathbf{g}|} dx^1 \wedge \dots \wedge dx^n$ with $\mu(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) = 1$ and where $\sqrt{|\mathbf{g}|} = \sqrt{|\det([g_{ij}])|}$. This allows us to define the *Hodge star operator* \star . Let \mathcal{M} be an oriented n -dimensional manifold, $\langle \cdot, \cdot \rangle$ be a nondegenerate symmetric bilinear form, and μ be the volume form. Then, there exists a unique isomorphism, called *Hodge star operator*, $\star : \Lambda^k \rightarrow \Lambda^{n-k}$ satisfying

$${}^k\alpha \wedge \star {}^k\beta = \langle {}^k\alpha, {}^k\beta \rangle \mu \quad \text{for } {}^k\alpha, {}^k\beta \in \Lambda^k, \quad (3)$$

and with properties:

1. ${}^k\alpha \wedge \star {}^k\beta = {}^k\beta \wedge \star {}^k\alpha = \langle {}^k\alpha, {}^k\beta \rangle \mu$,
2. $\star 1 = \mu^n, \star \mu = 1$,
3. $\star \star {}^k\alpha = (-1)^{k(n-k)} {}^k\alpha$,
4. $\langle {}^k\alpha, {}^k\beta \rangle = \langle \star {}^k\alpha, \star {}^k\beta \rangle$.

3 Split exterior calculus for formulations independent of ambient orientation

In this section we introduce concepts of differential geometry and exterior calculus in an ambient orientation independent manner by using straight and twisted differential forms (DF). Based on these ideas, we introduce two novel key concepts essential for this manuscript, namely the topological pairing of straight and twisted DFs as a metric independent version of an inner product of k -forms and the topological functional derivatives with respect to this topological pairing.

Definition of twisted objects. Given an orientation Or on \mathcal{M} with opposite orientation $-Or$, and given a straight differential form ${}^k\omega \in \Lambda^k$, we define *twisted differential forms* by

$${}^k\tilde{\omega} := \{ \{ {}^k\omega, Or \}, \{ -{}^k\omega, -Or \} \} \in \tilde{\Lambda}^k. \quad (4)$$

We denote the space of twisted k -forms as $\tilde{\Lambda}^k$. Using a combination of straight and twisted differential forms to describe a system of equations allows for formulations that do *not* depend on the choice of the ambient orientation of \mathcal{M} (cf. Bauer2016).

We also define an orientation independent Hodge star operator which we refer to as *twisted Hodge star* $\tilde{\star}$. It is defined by $\tilde{\star} := \{\{\star, Or\}, \{-\star, -Or\}\}$. When we use twisted quantities, the formulation is independent of the chosen orientation. Possible sign changes are taken care of by the twisted forms and the twisted operators.

The pairing of differential forms. An important tool for our derivation is the pairing of differential forms. The inner product \langle, \rangle in (2) is a metric dependent pairing of differential forms of the same type. Representing it in Definition 5 in terms of wedge product \wedge and the twisted Hodge star $\tilde{\star}$ reveals that it depends on the metric, inherited by the metric dependency of $\tilde{\star}$. In addition, in Definition 6 we define the *topological pairing* of differential forms of opposite type (e.g. of a straight k -form and a twisted $n - k$ -form), denoted with $\langle\langle, \rangle\rangle$, as a metric independent version.

Definition 5 *On an orientable Riemannian manifold $(\mathcal{M}, \mathbf{g})$ (not necessarily oriented), the metric \mathbf{g} with associated inner product \langle, \rangle induces metric pairings between either two straight or two twisted k -forms:*

$$\langle {}^k\alpha, {}^k\beta \rangle := \int_{\mathcal{M}} {}^k\alpha \wedge \tilde{\star} {}^k\beta = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \beta^{i_1 \dots i_k} \int_{\mathcal{M}} n_{\tilde{\mu}} \quad (5)$$

$$\langle {}^k\tilde{\alpha}, {}^k\tilde{\beta} \rangle := \int_{\mathcal{M}} {}^k\tilde{\alpha} \wedge \tilde{\star} ({}^k\tilde{\beta}) = \sum_{i_1 < \dots < i_k} \tilde{\alpha}_{i_1 \dots i_k} \tilde{\beta}^{i_1 \dots i_k} \int_{\mathcal{M}} n_{\tilde{\mu}} \quad (6)$$

where $n_{\tilde{\mu}}$ denotes the twisted volume form such that $\int_{\mathcal{M}} n_{\tilde{\mu}} = 1$. Note that the inner product is symmetric.

Equation (5) is an orientation-independent realization of (2) as it uses the twisted Hodge star. Equation (6) is equivalent to (5). This equivalence can be seen when considering the definition of twisted k -forms. In Or , we have simply ${}^k\alpha$ and ${}^k\beta$, hence the pairing in (5). In $-Or$, the two minus signs of $-{}^k\alpha$ and $-{}^k\beta$ cancel.

Definition 6 *On an orientable topological manifold \mathcal{M} (not necessarily oriented) with twisted volume form $n_{\tilde{\mu}}$, we define the topological pairing $\langle\langle, \rangle\rangle$ of straight (twisted) k -forms with twisted (straight) $(n - k)$ forms by*

$$\langle\langle {}^k\alpha, {}^{(n-k)}\tilde{\beta} \rangle\rangle := \int_{\mathcal{M}} {}^k\alpha \wedge {}^{(n-k)}\tilde{\beta} = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \tilde{\beta}^{(1 \leq \hat{i}_1 < \dots < \hat{i}_k \leq n)} \int_{\mathcal{M}} n_{\tilde{\mu}} \quad (7)$$

$$\langle\langle {}^k\tilde{\alpha}, {}^{(n-k)}\beta \rangle\rangle := \int_{\mathcal{M}} {}^k\tilde{\alpha} \wedge {}^{(n-k)}\beta = \sum_{i_1 < \dots < i_k} \tilde{\alpha}_{i_1 \dots i_k} \beta^{(1 \leq \hat{i}_1 < \dots < \hat{i}_k \leq n)} \int_{\mathcal{M}} n_{\tilde{\mu}} \quad (8)$$

where $\tilde{\beta}^{(1 \leq \hat{i}_1 < \dots < \hat{i}_k \leq n)}$ (or $\beta^{(1 \leq \hat{i}_1 < \dots < \hat{i}_k \leq n)}$) are the coefficient functions of the $(n - k)$ -form ${}^{(n-k)}\tilde{\beta}$ (or ${}^{(n-k)}\beta$) in which \hat{i}_k means to omit the index i_k .

Proposition 7 *The topological pairing has the following properties:*

1.) *The topological pairing is independent of the metric \mathbf{g} and of the ambient orientation;*

2.) *The topological pairing $\langle\langle^k\alpha, {}^{(n-k)}\tilde{\beta}\rangle\rangle$ is an inner product if the metric equation ${}^{(n-k)}\tilde{\beta} = \tilde{\star} {}^k\beta$ holds; then, the coefficient functions for ${}^{(n-k)}\tilde{\beta}$ follow from ${}^k\beta$ via*

$$\tilde{\beta}^{(1 \leq i_1 < \dots < i_k \leq n)} = \sum_{i_1 < \dots < i_k} \beta^{i_1 \dots i_k} \sqrt{|\mathbf{g}|} \operatorname{sign} \begin{pmatrix} 1 \dots n \\ i_1 \dots i_k \end{pmatrix}. \quad (9)$$

An analogous statement holds for $\langle\langle^k\tilde{\alpha}, {}^{(n-k)}\beta\rangle\rangle$.

3.) *In case the metric equation holds, the topological pairing is symmetric with respect to a duality exchange:*

$$\langle\langle^k\alpha, {}^{(n-k)}\tilde{\beta}\rangle\rangle = \langle\langle^k\beta, {}^{(n-k)}\tilde{\alpha}\rangle\rangle \quad (10)$$

where $\tilde{\star} {}^k\alpha = {}^{(n-k)}\tilde{\alpha}$ and $\tilde{\star} {}^{(n-k)}\tilde{\beta} = {}^k\beta$.

The proof for Proposition 7 can be found in Appendix C.

Functional derivatives. On the basis of the pairings of DF, we define two versions of functional derivatives: (i) with respect to the inner product \langle, \rangle and (ii) with respect to the topological pairing $\langle\langle, \rangle\rangle$.

Given functionals $\mathcal{F}[{}^k u] : \Lambda^k \rightarrow \mathbb{R}$ (or similarly for $\mathcal{F}[{}^k \tilde{u}] : \tilde{\Lambda}^k \rightarrow \mathbb{R}$) depending on straight k -forms, variations of the functional $\delta\mathcal{F}$ results from variations of ${}^k u$, defined by $\delta\mathcal{F} := \mathcal{F}[{}^k u + \delta {}^k u] - \mathcal{F}[{}^k u]$. Using for an arbitrary straight test function ${}^k \omega$ the representation $\delta {}^k u = \epsilon {}^k \omega$ permits us to evaluate $\delta\mathcal{F}$ in terms of a Taylor expansion in ϵ :

$$\mathcal{F}[{}^k u + \epsilon {}^k \omega] = \mathcal{F}[{}^k u] + \frac{d}{d\epsilon} \mathcal{F}[{}^k u + \epsilon {}^k \omega] \Big|_{\epsilon=0} \epsilon + \mathcal{O}(\epsilon^2) \quad (11)$$

By a suitable definition, we relate the derivatives of \mathcal{F} with respect to ϵ to functional derivatives (here in standard notation):

$$\frac{d}{d\epsilon} \mathcal{F}[{}^k u + \epsilon {}^k \omega] \Big|_{\epsilon=0} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\mathcal{F}[{}^k u + \epsilon {}^k \omega] - \mathcal{F}[{}^k u]) =: \int_{\mathcal{M}} n_\mu \frac{\delta \mathcal{F}}{\delta {}^k u} {}^k \omega =: \delta \mathcal{F} \quad (12)$$

This definition relates the derivative of \mathcal{F} with respect to ϵ with a linear functional with kernel $\frac{\delta \mathcal{F}}{\delta {}^k u}$ that acts on the test function ${}^k \omega(x)$. To guarantee that this definition exists, we assume to take only differentiable functionals. An analogous definition holds for $\delta \mathcal{F}[{}^k \tilde{u}]$.

In the following two definitions, we relate the definition of functional derivatives (12) in standard notation with the split exterior calculus notation.

Definition 8 *The standard (or metric) functional derivatives of $\mathcal{F}[{}^k u] : \Lambda^k \rightarrow \mathbb{R}$ (or $\mathcal{F}[{}^k \tilde{u}] : \tilde{\Lambda}^k \rightarrow \mathbb{R}$) with respect to the k -form ${}^k u$ (or ${}^k \tilde{u}$) and with respect to the inner product \langle, \rangle is defined by*

$$\delta \mathcal{F} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[{}^k u + \epsilon {}^k \omega] - F[{}^k u]) =: \langle {}^k \omega, \frac{\delta F}{\delta {}^k u} \rangle = \int_{\mathcal{M}} {}^k \omega \wedge \tilde{\star} \frac{\delta F}{\delta {}^k u} \quad \forall {}^k \omega \in \Lambda^k \quad (13)$$

for arbitrary test functions ${}^k \omega$ (or ${}^k \tilde{\omega}$). In particular, if ${}^k u \in \Lambda^k$ is a straight k -form, also $\frac{\delta F}{\delta {}^k u} \in \Lambda^k$ is a straight k -form, while for a twisted k -form ${}^k \tilde{u} \in \tilde{\Lambda}^k$, $\frac{\delta F}{\delta {}^k \tilde{u}} \in \tilde{\Lambda}^k$ is a twisted k -form.

Definition 9 The topological functional derivatives of $\mathcal{F}[{}^k u] : \Lambda^k \rightarrow \mathbb{R}$ (resp. $\mathcal{F}[{}^{k\tilde{u}}] : \tilde{\Lambda}^k \rightarrow \mathbb{R}$) with respect to the k -form ${}^k u$ (resp. ${}^{k\tilde{u}}$) and with respect to the topological pairing $\langle\langle, \rangle\rangle$ is defined by

$$\begin{aligned} \delta\mathcal{F} &:= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[{}^k u + \epsilon {}^k \omega] - F[{}^k u]) =: \langle\langle {}^k \omega, \frac{\widetilde{\delta F}}{\delta {}^k u} \rangle\rangle = \int_{\mathcal{M}} {}^k \omega \wedge \frac{\widetilde{\delta F}}{\delta {}^k u} \quad \forall {}^k \omega \in \Lambda^k \\ \text{resp. } \delta\mathcal{F} &:= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[{}^{k\tilde{u}} + \epsilon {}^{k\tilde{\omega}}] - F[{}^{k\tilde{u}}]) =: \langle\langle {}^{k\tilde{\omega}}, \frac{\widetilde{\delta F}}{\delta {}^{k\tilde{u}}} \rangle\rangle = \int_{\mathcal{M}} {}^{k\tilde{\omega}} \wedge \frac{\widetilde{\delta F}}{\delta {}^{k\tilde{u}}} \quad \forall {}^{k\tilde{\omega}} \in \tilde{\Lambda}^k \end{aligned} \quad (14)$$

for arbitrary test functions ${}^k \omega$ (resp. ${}^{k\tilde{\omega}}$) under the metric dependent condition

$$\frac{\widetilde{\delta F}}{\delta {}^k u} := \tilde{\star} \frac{\delta F}{\delta {}^k u} \quad \left(\text{resp.} \quad \frac{\widetilde{\delta F}}{\delta {}^{k\tilde{u}}} := \tilde{\star} \frac{\delta F}{\delta {}^{k\tilde{u}}} \right). \quad (15)$$

In particular, if ${}^k u \in \Lambda^k$ is a straight k -form, $\frac{\widetilde{\delta F}}{\delta {}^k u} \in \tilde{\Lambda}^{(n-k)}$ is a twisted $(n-k)$ -form, while for a twisted k -form ${}^{k\tilde{u}} \in \tilde{\Lambda}^k$, $\frac{\widetilde{\delta F}}{\delta {}^{k\tilde{u}}} \in \Lambda^{(n-k)}$ is a straight $(n-k)$ -form (the appearance of two tildes in $\frac{\widetilde{\delta F}}{\delta {}^{k\tilde{u}}}$ indicates that the twisting cancels to produce a straight form).

Note that the topological functional derivatives (14) of Definition 9 do not depend on the metric as they apply the topological pairing $\langle\langle, \rangle\rangle$ of straight (resp. twisted) k -forms with twisted (resp. straight) $(n-k)$ -forms. Only when enforcing the metric dependent equations (15), these functional derivatives coincide with those of Definition 8. This equivalence follows immediately when substituting (15) into (14) which allows us to recover the definition of standard functional derivatives in (13).

Further useful relations. Finally, we introduce operators and relations that are based on the above defined ones and will be useful later on for our derivations.

For instance, there exist an identification of vector proxy with 1-form, the flat \flat and sharp \sharp -operators. Applying $\langle\cdot, \cdot\rangle$ we find an identification between the 1-forms ω^1 and a vector field (vector proxy) by $\flat : \mathcal{X} \rightarrow \Lambda^1$, $\mathbf{u}^\flat(\mathbf{v})(\mathbf{x}) = \langle \mathbf{u}, \mathbf{v} \rangle(\mathbf{x})$ for all vector fields $\mathbf{v} \in \mathcal{X}(M)$. The flat of the vector field $\mathbf{u} = \sum_i u^i \frac{\partial}{\partial x^i}$ in local coordinates is hence given by $\mathbf{u}^\flat = \sum u_i dx^i$ with coefficients $u_i = \sum_j g_{ij} u^j$. The inverse operator is given by $\sharp : \Lambda^1 \rightarrow \mathcal{X}(M)$. We refer to both mappings \flat and \sharp as *Riemannian lift*.

This operator allows us to represent the interior product in terms of \wedge and $\tilde{\star}$.

Definition 10 (Hirani's formula) The interior product $i_X {}^k \omega$ of a k -form ${}^k \omega \in \Lambda^k$ with the vector field $X \in \mathcal{X}(\mathcal{M})$ can be written as

$$i_X {}^k \omega = (-1)^{k(n-k)} \tilde{\star} (\tilde{\star} {}^k \omega \wedge (X)^\flat). \quad (16)$$

4 Variational Formulations using Vector Calculus

This section presents a review of variational and Hamiltonian formulations using vector calculus for a single component, single phase fluid that can be described in terms of a (transport) velocity \mathbf{u} , a mass density D and a thermodynamic scalar s , in preparation for the development of these formulations using split exterior calculus. More details

on the traditional approach can be found in standard texts on the subject, such as [19, 18, 29, 27]. We assume a fixed domain $\Omega \subset \mathbb{R}^n$ (this is relaxed to general manifolds later) with boundary $\partial\Omega$, where the boundary is a material surface ($\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$). An extension to domains with time dependent boundaries (such as a free surface or elastic lid) is possible, and leads only to an additional surface term in the Lagrangian \mathcal{L} or Hamiltonian \mathcal{H} , but this is deferred to future work. As is standard in geophysical fluid dynamics, we use the scalar (s) rather than density ($S = Ds$) form of the thermodynamic variable in the Lagrangian formulation. In contrast, in the Hamiltonian formulation we use the density form of the thermodynamic variable. We consider only the case when $n = 3$, the case of $n = 2$ is discussed in Appendix B.

4.1 Variational Formulation

Given the Lagrangian functional $\mathcal{L}[\mathbf{u}, D, s]$ that characterizes the fluid, standard variational techniques (see, for example, [19, 20]) yield the corresponding Euler-Lagrange momentum equation in 3D:

$$\frac{\partial}{\partial t} \left(\frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) + \mathbf{L}_{\mathbf{u}} \left(\frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) - \nabla \left(\frac{\delta \mathcal{L}}{\delta D} \right) + \frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} \nabla s = 0 \quad (17)$$

where $\mathbf{L}_{\mathbf{u}}$ is the Lie derivative with respect to \mathbf{u} . Expanding the Lie derivative term using $\mathbf{L}_{\mathbf{u}} \mathbf{x} = \nabla \times \mathbf{x} \times \mathbf{u} + \nabla(\mathbf{u} \cdot \mathbf{x})$ gives finally the curl-form Euler-Lagrangian equations:

$$\frac{\partial}{\partial t} \left(\frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) + \nabla \times \left(\frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) \times \mathbf{u} + \nabla \left(\mathbf{u} \cdot \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} - \frac{\delta \mathcal{L}}{\delta D} \right) + \frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} \nabla s = 0 \quad (18)$$

This is supplemented with kinematic transport equations for D and s

$$\frac{\partial D}{\partial t} = \nabla \cdot (D \mathbf{u}) \quad (19)$$

$$\frac{\partial s}{\partial t} = -\mathbf{u} \cdot \nabla s \quad (20)$$

Combining (19) and (20) gives an equation for S as

$$\frac{\partial S}{\partial t} = -\nabla \cdot (S \mathbf{u}) \quad (21)$$

In tensor notation, \mathbf{u} is a contravariant vector, while \mathbf{v} , introduced in the following definitions, is a covariant vector. This is akin to the transition from vector fields on the tangent bundle to vector field proxies of 1-forms on the cotangent bundle. See [39] for more details.

Introducing

$$T := -\frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} \quad \mathbf{v} := \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \quad B := \mathbf{u} \cdot \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} - \frac{\delta \mathcal{L}}{\delta D} + \frac{s}{D} \frac{\delta \mathcal{L}}{\delta s} \quad \mathbf{F} := D \mathbf{u} \quad (22)$$

in order to simplify the equations, and following standard definitions in the literature

[38, 19, 20], the momentum and transport equations can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\nabla \times \mathbf{v}}{D} \times \mathbf{F} + \nabla B + s \nabla T = 0 \quad (23)$$

$$\frac{\partial D}{\partial t} + \nabla \cdot (\mathbf{F}) = 0 \quad (24)$$

$$\frac{\partial s}{\partial t} + \frac{\mathbf{F}}{D} \cdot \nabla s = 0 \quad (25)$$

$$\frac{\partial S}{\partial t} + \nabla \cdot (s \mathbf{F}) = 0 \quad (26)$$

A closed set of equations is given by equations (23) - (24) and either (25) or (26).

4.1.1 Rotation

For geophysical fluids, it is often beneficial to work in terms of a coordinate system that is undergoing solid body rotation. In \mathbb{R}^3 , a solid body rotation is described by a rotation vector $\mathbf{\Omega}$, which is a uniform constant vector. Therefore, $\nabla \cdot \mathbf{\Omega} = \nabla \times \mathbf{\Omega} = \mathbf{0}$. It is important to note that $\mathbf{\Omega}$ is a pseudo-vector that changes sign under a reversal of orientation. The velocity \mathbf{R} associated with the rotation is defined by $\nabla \times \mathbf{R} = 2\mathbf{\Omega}$. This definition for \mathbf{R} is fundamental and holds for more general manifolds than \mathbb{R}^3 . In \mathbb{R}^3 the standard choice is $\mathbf{R} = \mathbf{\Omega} \times \mathbf{r}$, where \mathbf{r} is the position vector. Rotation is handled by adding a term to the Lagrangian \mathcal{L} , as

$$\mathcal{L}' = \mathcal{L} + \langle D \mathbf{u}, \mathbf{R} \rangle \quad (27)$$

Therefore

$$\frac{\delta \mathcal{L}'}{\delta \mathbf{u}} = \frac{\delta \mathcal{L}}{\delta \mathbf{u}} + D \mathbf{R}. \quad (28)$$

According to (22), the latter implies that $\mathbf{v}' = \mathbf{v} + \mathbf{R}$, or in other words, when rotation is added $\mathbf{v}' = \frac{1}{D} \frac{\delta \mathcal{L}'}{\delta \mathbf{u}}$ represents the absolute velocity. More details on this can be found in [19, 20].

4.2 Kelvin Circulation Theorem

Integrating (17) over a closed curve $\gamma(t)$ and using the fundamental theorem of calculus, the Kelvin circulation theorem is obtained as

$$\frac{d}{dt} \oint_{\gamma(t)} \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \cdot d\mathbf{x} = - \oint_{\gamma(t)} \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{s}} \nabla \mathbf{s} \cdot d\mathbf{x} \quad (29)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{L}_{\mathbf{u}}$ is the total derivative.

4.3 Potential Vorticity

The potential vorticity is

$$q = \frac{\nabla \times \mathbf{v} \cdot \nabla s}{D} = \mathbf{Q} \cdot \nabla s. \quad (30)$$

In fact, the potential vorticity can be defined more generally by replacing ∇s by $\nabla \lambda(s)$, where $\lambda(s)$ is an arbitrary function, but this is not pursued further. Combining (23) - (25) gives the evolution equation for potential vorticity density

$$\frac{\partial(Dq)}{\partial t} + \nabla \cdot (qD\mathbf{u}) = 0 \quad (31)$$

and for potential vorticity

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0. \quad (32)$$

Note that potential vorticity is materially conserved ($\frac{dq}{dt} = 0$).

4.4 Hamiltonian Formulation

Here we will focus on the variant that predicts (\mathbf{v}, D, S) , some alternatives are discussed in Appendix A. Note that we have switched from the thermodynamic scalar s to the thermodynamic scalar density S . The fundamental objects in the Hamiltonian formulation are the Poisson bracket $\{\mathcal{A}, \mathcal{B}\}$, which is a bilinear, anti-symmetric operator on functionals $\mathcal{A}[\mathbf{v}, D, S]$ and $\mathcal{B}[\mathbf{v}, D, S]$ satisfying the Jacobi identity and Leibniz rule; and the Hamiltonian $\mathcal{H}[\mathbf{v}, D, S]$. The evolution of an arbitrary functional $\mathcal{F}[\mathbf{v}, D, S]$ is then given by

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \quad (33)$$

In geophysical fluids in Eulerian coordinates, the Poisson bracket is almost always singular (non-canonical). Therefore there exists a set of functionals $\mathcal{C}[\mathbf{v}, D, S]$, termed Casimirs, that lie in the null space of the Poisson bracket

$$\{\mathcal{C}, \mathcal{A}\} = 0 \quad \forall \mathcal{A} \quad (34)$$

More details can be found in [29] or any standard text on non-canonical Hamiltonian mechanics.

4.4.1 Hamiltonian

The Hamiltonian \mathcal{H} corresponding to \mathcal{L} is obtained via a Legendre transform as

$$\mathcal{H}[\mathbf{v}, D, S] = \int \left(\mathbf{u} \cdot \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) - \mathcal{L} = \int (D\mathbf{u} \cdot \mathbf{v}) - \mathcal{L}. \quad (35)$$

Hence, in the Hamiltonian formulation the fluid's velocity is described by the covariant vector field \mathbf{v} . Using the chain rule, the functional derivatives of \mathcal{H} are given by

$$\frac{\delta \mathcal{H}}{\delta \mathbf{v}} = D\mathbf{u} = \mathbf{F} \quad \frac{\delta \mathcal{H}}{\delta D} = \mathbf{u} \cdot \mathbf{v} - \frac{\delta \mathcal{L}}{\delta D} + \frac{s}{D} \frac{\delta \mathcal{L}}{\delta s} = B \quad \frac{\delta \mathcal{H}}{\delta S} = -\frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} = T \quad (36)$$

which agree with the definitions in (22).

4.4.2 Poisson Brackets

The Poisson bracket is

$$\{\mathcal{A}, \mathcal{B}\} = \{\mathcal{A}, \mathcal{B}\}_R + \{\mathcal{A}, \mathcal{B}\}_S + \{\mathcal{A}, \mathcal{B}\}_Q \quad (37)$$

where

$$\{\mathcal{A}, \mathcal{B}\}_R = \int_{\Omega} \left(-\frac{\delta \mathcal{A}}{\delta D} \nabla \cdot \frac{\delta \mathcal{B}}{\delta \mathbf{v}} + \frac{\delta \mathcal{B}}{\delta D} \nabla \cdot \frac{\delta \mathcal{A}}{\delta \mathbf{v}} \right) d\Omega, \quad (38)$$

$$\{\mathcal{A}, \mathcal{B}\}_S = \int_{\Omega} \left(-\frac{\delta \mathcal{A}}{\delta S} \nabla \cdot \left(s \frac{\delta \mathcal{B}}{\delta \mathbf{v}} \right) + \frac{\delta \mathcal{B}}{\delta S} \nabla \cdot \left(s \frac{\delta \mathcal{A}}{\delta \mathbf{v}} \right) \right) d\Omega, \quad (39)$$

$$\{\mathcal{A}, \mathcal{B}\}_Q = \int_{\Omega} -\frac{\delta \mathcal{A}}{\delta \mathbf{v}} \cdot \left(\mathbf{Q} \times \frac{\delta \mathcal{B}}{\delta \mathbf{v}} \right) d\Omega, \quad (40)$$

and $\mathbf{Q} = \frac{\nabla \times \mathbf{v}}{D}$. Equation (37) defines a Poisson bracket with respect to the variables (\mathbf{v}, D, S) , which is known as the curl-form Poisson bracket. It is also possible to define a bracket predicting (\mathbf{m}, D, S) where $\mathbf{m} = D \mathbf{v}$ is the momentum. This is known as the Lie-Poisson bracket, and examples are found in [8, 21] (amongst others). The Lie-Poisson bracket gives the flux-form momentum equations.

4.4.3 Casimirs

The Casimirs take the general form

$$\mathcal{C}[\mathbf{v}, D, S] = \int DF\left(\frac{S}{D}, q\right) \quad (41)$$

where $F(s, q)$ is an arbitrary function of the thermodynamic scalar s and the potential vorticity q . The functional derivatives of $\mathcal{C}[\mathbf{v}, D, S]$ are

$$\frac{\delta \mathcal{C}}{\delta \mathbf{v}} = \nabla \times \left(\frac{\partial F}{\partial q} \nabla s \right) \quad (42)$$

$$\frac{\delta \mathcal{C}}{\delta D} = F - q \frac{\partial F}{\partial q} - s \frac{\partial F}{\partial s} + \frac{s}{\rho} \nabla \cdot \left(\frac{\partial F}{\partial q} \nabla \times \mathbf{v} \right) \quad (43)$$

$$\frac{\delta \mathcal{C}}{\delta S} = \frac{\partial F}{\partial s} - \frac{1}{\rho} \nabla \cdot \left(\frac{\partial F}{\partial q} \nabla \times \mathbf{v} \right) \quad (44)$$

Important cases are $F = 1$ (total mass), $F = q$ (total potential vorticity) and $F = s$ (total thermodynamic scalar).

5 Variational Formulations using Split Exterior Calculus

Instead of vector calculus, we now wish to use split exterior calculus: to work in terms of straight and twisted different forms instead of scalars and vectors. Our intention is to translate the standard variational and curl-form Hamiltonian formulations from vector calculus into split exterior calculus. The resulting novel variational framework applies

for arbitrary manifolds with $n \leq 3$ and is independent of the orientation of the ambient space. Moreover, the usage of DF allows for a generalized methodology to derive various equations from a given Lagrangian. A principle result for the Hamiltonian formulation is the appearance of Poisson brackets that involve only topological operators, with all metric operators appearing in the Hamiltonian. This is the major advantage of the split exterior calculus formulation over the vector calculus formulation.

5.1 Choice of predicted variables

The transport velocity \mathbf{u} is described by a twisted $n - 1$ form ${}^{n-1}\tilde{\mathbf{u}}$, describing an $n - 1$ dimensional flux and defined by

$${}^{n-1}\tilde{\mathbf{u}} = \mathbf{i}_{\mathbf{u}} {}^n\tilde{\mu} = \tilde{\star}({}^0\mathbf{I} \wedge {}^1\mathbf{u}) = \tilde{\star} {}^1\mathbf{u} \quad (45)$$

with ${}^n\tilde{\mu}$ the twisted n -volume, and where ${}^1\mathbf{u} = \mathbf{u}^\flat$ is the straight 1-form associated with \mathbf{u} . In fact, ${}^1\mathbf{u}$ and ${}^{n-1}\tilde{\mathbf{u}}$ are the two vector proxies associated with the vector field \mathbf{u} . The mass density D becomes the twisted n -form ${}^n\tilde{D}$, the thermodynamic scalar s the straight 0-form 0s and the thermodynamic scalar density $S = Ds$ the twisted n -form ${}^n\tilde{S} = {}^n\tilde{D} \wedge {}^0s$. There is a corresponding mass density straight 0-form ${}^0D = \tilde{\star} {}^n\tilde{D}$. These variables follow precisely the classification used in the work of Tonti [37, 36], which associates each physical quantity to an oriented geometric entity. One way of viewing this work, and in particular the Hamiltonian formulation using split exterior calculus, is as a generalization of the Tonti diagram to compressible fluids with arbitrary equations of state. The material boundary condition $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$ becomes $\mathbf{i}_{\mathbf{u}} {}^n\tilde{\mu} = \tilde{\star}(\tilde{\star} {}^n\tilde{\mu} \wedge {}^1\mathbf{u}) = \tilde{\star}({}^0\mathbf{I} \wedge {}^1\mathbf{u}) = {}^{n-1}\tilde{\mathbf{u}} = 0$ on $\partial\Omega$.

5.2 Lagrangian Formulation

The Lagrangian is $\mathcal{L}[{}^{n-1}\tilde{\mathbf{u}}, {}^n\tilde{D}, {}^0s]$, and the resulting Euler-Lagrange momentum equation is

$$\frac{\partial}{\partial t} \left(\frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \right) + L_{\mathbf{u}} \left(\frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \right) - d \frac{\delta \mathcal{L}}{\delta {}^n\tilde{D}} + \frac{1}{{}^0D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} \wedge d {}^0s = 0 \quad (46)$$

Using Cartan's magic formula to expand the Lie derviative this can be written in curl-form as

$$\frac{\partial}{\partial t} \left(\frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \right) + \mathbf{i}_{\mathbf{u}} \left(d \frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \right) + d \mathbf{i}_{\mathbf{u}} \left(\frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \right) - d \frac{\delta \mathcal{L}}{\delta {}^n\tilde{D}} + \frac{1}{{}^0D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} \wedge d {}^0s = 0 \quad (47)$$

where the functional derivatives of \mathcal{L} are calculated with respect to the topological pairing of Definition 9. In particular, (46) and (47) describe the dynamics of a straight 1-form. This is supplemented with kinematic transport equations for ${}^n\tilde{D}$ and 0s

$$\frac{\partial {}^n\tilde{D}}{\partial t} = -L_{\mathbf{u}} {}^n\tilde{D} = -d({}^0D \wedge {}^{n-1}\tilde{\mathbf{u}}) \quad (48)$$

$$\frac{\partial {}^0s}{\partial t} = -L_{\mathbf{u}} {}^0s = -\tilde{\star} ({}^{n-1}\tilde{\mathbf{u}} \wedge d {}^0s). \quad (49)$$

As before, (48) and (49) can be combined to yield

$$\frac{\partial {}^n\tilde{S}}{\partial t} = -L_{\mathbf{u}} {}^n\tilde{S} = -d \left({}^0s \wedge {}^{n-1}\tilde{\mathbf{F}} \right) \quad (50)$$

Proposition 11 Equations (18) - (21) are equivalent to (47) - (50).

Proof: Start by recalling (17)

$$\frac{\partial}{\partial t} \mathbf{v} + \mathbf{L}_{\mathbf{u}} \mathbf{v} - \nabla \left(\frac{\delta \mathcal{L}}{\delta D} \right) + \frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} \nabla s = 0 \quad (51)$$

and noting that (46) can be rewritten as

$$\frac{\partial}{\partial t} {}^1\mathbf{v} + \mathbf{L}_{\mathbf{u}} {}^1\mathbf{v} - \mathrm{d} \frac{\delta \mathcal{L}}{\delta {}^n\tilde{D}} + \frac{1}{{}^0D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} \wedge \mathrm{d} {}^0s = 0 \quad (52)$$

Therefore, the proof will consist of showing that $\mathbf{v} \rightarrow {}^1\mathbf{v}$, $s \rightarrow {}^0s$, $D \rightarrow {}^n\tilde{D}$, $\mathbf{L}_{\mathbf{u}} \mathbf{v} \rightarrow \mathbf{L}_{\mathbf{u}} {}^1\mathbf{v}$, $\nabla \left(\frac{\delta \mathcal{L}}{\delta D} \right) \rightarrow \mathrm{d} \frac{\delta \mathcal{L}}{\delta {}^n\tilde{D}}$ and $\frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} \nabla s \rightarrow \frac{1}{{}^0D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} \wedge \mathrm{d} {}^0s$.

Similarly, the kinematic equations (19) - (21) are

$$\frac{\partial D}{\partial t} + \mathbf{L}_{\mathbf{u}} D = 0 \quad (53)$$

$$\frac{\partial s}{\partial t} + \mathbf{L}_{\mathbf{u}} s = 0 \quad (54)$$

$$\frac{\partial S}{\partial t} + \mathbf{L}_{\mathbf{u}} S = 0 \quad (55)$$

while (48) - (50) are

$$\frac{\partial {}^n\tilde{D}}{\partial t} + \mathbf{L}_{\mathbf{u}} {}^n\tilde{D} = 0 \quad (56)$$

$$\frac{\partial {}^0s}{\partial t} + \mathbf{L}_{\mathbf{u}} {}^0s = 0 \quad (57)$$

$$\frac{\partial {}^n\tilde{S}}{\partial t} + \mathbf{L}_{\mathbf{u}} {}^n\tilde{S} = 0 \quad (58)$$

To show that the kinematic equations are equivalent requires that $\mathbf{L}_{\mathbf{u}} D \rightarrow \mathbf{L}_{\mathbf{u}} {}^n\tilde{D}$, $\mathbf{L}_{\mathbf{u}} S \rightarrow \mathbf{L}_{\mathbf{u}} {}^n\tilde{S}$ and $\mathbf{L}_{\mathbf{u}} s \rightarrow \mathbf{L}_{\mathbf{u}} {}^0s$ (along with $s \rightarrow {}^0s$, $S \rightarrow {}^n\tilde{S}$, $D \rightarrow {}^n\tilde{D}$ from before). \square

Following the same steps as in the vector calculus case, we proceed by defining

$${}^0T := -\frac{1}{{}^0D} \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} \quad (59)$$

$${}^1\mathbf{v} := \frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \quad (60)$$

$${}^0B := \mathbf{i}_{\mathbf{u}} \left(\frac{1}{{}^0D} \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{\mathbf{u}}} \right) - \frac{\delta \mathcal{L}}{\delta {}^n\tilde{D}} + \frac{{}^0s}{{}^0D} \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} = \mathbf{i}_{\mathbf{u}} {}^1\mathbf{v} - \frac{\delta \mathcal{L}}{\delta {}^n\tilde{D}} - {}^0s \wedge {}^0T \quad (61)$$

$${}^{n-1}\tilde{\mathbf{F}} := {}^0D \wedge {}^{n-1}\tilde{\mathbf{u}} \quad (62)$$

which are equivalent definitions to (22) but now in terms of straight and twisted differential forms. Substituting them into (47) gives the following momentum and transport

equations

$$\frac{\partial {}^1\mathbf{v}}{\partial t} + \mathbf{i}_{\mathbf{u}} d {}^1\mathbf{v} + d {}^0\mathbf{B} + {}^0\mathbf{s} \wedge d {}^0\mathbf{T} = 0 \quad (63)$$

$$\frac{\partial {}^n\tilde{\mathbf{D}}}{\partial t} + d {}^{n-1}\tilde{\mathbf{F}} = 0 \quad (64)$$

$$\frac{\partial {}^0\mathbf{s}}{\partial t} + \tilde{\star} \left(\frac{1}{{}^0\mathbf{D}} \wedge {}^{n-1}\tilde{\mathbf{F}} \wedge d {}^0\mathbf{s} \right) = 0 \quad (65)$$

$$\frac{\partial {}^n\tilde{\mathbf{S}}}{\partial t} + d \left({}^0\mathbf{s} \wedge {}^{n-1}\tilde{\mathbf{F}} \right) = 0 \quad (66)$$

Given the equivalence of the EP equations in split (47) and in VC form (18), the latter equations in (63)–(66) are equivalent to vector invariant equations in (23)–(26).

5.2.1 PV Flux Term

The PV flux term

$$\mathbf{i}_{\mathbf{u}} d {}^1\mathbf{v} = \mathbf{i}_{\mathbf{u}} {}^2\eta \quad (67)$$

with ${}^2\eta = d {}^1\mathbf{v}$ can be written in several equivalent ways. Start by introducing ${}^2\mathbf{Q}$, defined by

$${}^2\eta = d {}^1\mathbf{v} = {}^2\mathbf{Q} \wedge {}^0\mathbf{D} \quad (68)$$

Taking $\tilde{\star}$ of this yields the definition for ${}^{n-2}\tilde{\mathbf{Q}}$ as

$$\tilde{\star} {}^2\eta = {}^{n-2}\tilde{\eta} = {}^{n-2}\tilde{\mathbf{Q}} \wedge {}^0\mathbf{D} \quad (69)$$

Writing $\mathbf{i}_{\mathbf{u}} {}^2\eta$ out using Hirani's formula (16) gives

$$\mathbf{i}_{\mathbf{u}} {}^2\eta = \tilde{\star}(\tilde{\star} {}^2\eta \wedge {}^1\mathbf{u}) \quad (70)$$

Substituting (69) into (70) gives

$$\mathbf{i}_{\mathbf{u}} {}^2\eta = \tilde{\star}({}^{n-2}\tilde{\mathbf{Q}} \wedge {}^1\mathbf{F}) = \tilde{\star}({}^{n-2}\tilde{\mathbf{Q}} \wedge \tilde{\star} {}^{n-1}\tilde{\mathbf{F}}) = \tilde{\star}(\tilde{\star} {}^2\mathbf{Q} \wedge {}^1\mathbf{F}) = \mathbf{i}_{\mathbf{F}} {}^2\mathbf{Q} \quad (71)$$

where we have used ${}^1\mathbf{F} = {}^0\mathbf{D} \wedge {}^1\mathbf{u}$, $\mathbf{F} = D \mathbf{u}$ and $(\mathbf{F})^b = {}^1\mathbf{F}$. The middle form $\tilde{\star}({}^{n-2}\tilde{\mathbf{Q}} \wedge \tilde{\star} {}^{n-1}\tilde{\mathbf{F}})$ will be particularly useful in developing the Hamiltonian formulation.

5.2.2 Rotation

Since $\boldsymbol{\Omega}$ is a pseudo-vector, the associated 1-form is twisted, and

$${}^1\tilde{\Omega} = \boldsymbol{\Omega}^b \quad (72)$$

From this, an associated straight 2-form can be defined as

$$\tilde{\star} {}^1\tilde{\Omega} =: {}^2\Omega \quad (73)$$

In fact, as shown in [2], the rotation straight 2-form ${}^2\Omega$ is the correct dimension-agnostic way to describe solid body rotation. Note that ${}^2\Omega$ is a closed form: $d {}^2\Omega = \delta {}^2\Omega = 0$. From ${}^2\Omega$, the rotational velocity straight 1-form ${}^1\mathbf{R}$ is defined through

$$d {}^1\mathbf{R} = {}^2\Omega \quad (74)$$

which implies that

$$\mathbf{i}_{\mathbf{u}} d^1 \mathbf{R} = \mathbf{i}_{\mathbf{u}} 2^2 \Omega \quad (75)$$

which is the standard representation of the Coriolis term. Rotation is introduced into the Lagrangian exactly the same way as in the vector calculus case:

$$\mathcal{L}' = \mathcal{L} + \langle {}^0 D \wedge {}^1 \mathbf{u}, {}^1 \mathbf{R} \rangle \quad (76)$$

$$\frac{\delta \mathcal{L}'}{\delta {}^{n-1} \tilde{\mathbf{u}}} = \frac{\delta \mathcal{L}}{\delta {}^{n-1} \tilde{\mathbf{u}}} + {}^0 D \wedge {}^1 \mathbf{R} \quad (77)$$

with ${}^1 \mathbf{v}' = {}^1 \mathbf{v} + {}^1 \mathbf{R}$, using (59).

5.3 Hamiltonian Formulation

Here we will focus on the variant that predicts $({}^1 \mathbf{v}, {}^n \tilde{D}, {}^n \tilde{S})$, some alternatives are discussed in Appendix A.

5.3.1 Hamiltonian

The Hamiltonian is obtained from $\mathcal{L}[{}^{n-1} \tilde{\mathbf{u}}, {}^n \tilde{D}, {}^0 \mathbf{s}]$ using a Legendre transform:

$$\mathcal{H}[{}^1 \mathbf{v}, {}^n \tilde{D}, {}^n \tilde{S}] = \langle {}^0 D, \mathbf{i}_{\mathbf{u}} {}^1 \mathbf{v} \rangle - \mathcal{L} \quad (78)$$

This is a transformation from the twisted $n - 1$ -form ${}^{n-1} \tilde{\mathbf{u}}$ to the straight 1-form ${}^1 \mathbf{v}$.

Proposition 12 Equation (78) has topological functional derivatives given by

$$\frac{\delta \mathcal{H}}{\delta {}^1 \mathbf{v}} = {}^0 D \wedge {}^{n-1} \tilde{\mathbf{u}} = {}^{n-1} \tilde{\mathbf{F}} \quad \frac{\delta \mathcal{H}}{\delta {}^n \tilde{D}} = \mathbf{i}_{\mathbf{u}} {}^1 \mathbf{v} - \frac{\delta \mathcal{L}}{\delta {}^n \tilde{D}} + \frac{{}^0 \mathbf{s}}{{}^0 D} \star \frac{\delta \mathcal{L}}{\delta {}^0 \mathbf{s}} = {}^0 \mathbf{B} \quad \frac{\delta \mathcal{H}}{\delta {}^n \tilde{S}} = -\frac{1}{{}^0 D} \star \frac{\delta \mathcal{L}}{\delta {}^0 \mathbf{s}} = {}^0 \mathbf{T} \quad (79)$$

Proof: Start by writing $\mathcal{H}[{}^1 \mathbf{v}, {}^n \tilde{D}, {}^n \tilde{S}]$

$$\mathcal{H}[{}^1 \mathbf{v}, {}^n \tilde{D}, {}^n \tilde{S}] = \int {}^0 D \wedge (-1)^{n-1} (\tilde{\star} {}^1 \mathbf{v} \wedge {}^1 \mathbf{u}) - L \quad (80)$$

where $\mathcal{L}[{}^{n-1} \tilde{\mathbf{u}}, {}^n \tilde{D}, {}^0 \mathbf{s}] = \int L[{}^{n-1} \tilde{\mathbf{u}}, {}^n \tilde{D}, {}^0 \mathbf{s}]$. Using the properties of the wedge product and the twisted hodge star this becomes

$$\mathcal{H}[{}^1 \mathbf{v}, {}^n \tilde{D}, {}^n \tilde{S}] = \int {}^0 D \wedge {}^1 \mathbf{v} \wedge {}^{n-1} \tilde{\mathbf{u}} - L[{}^{n-1} \tilde{\mathbf{u}}, {}^n \tilde{D}, {}^0 \mathbf{s}] \quad (81)$$

Now taking variations

$$\delta \mathcal{H} = \int (\delta {}^0 D \wedge {}^1 \mathbf{v} \wedge {}^{n-1} \tilde{\mathbf{u}} + {}^0 D \wedge \delta {}^1 \mathbf{v} \wedge {}^{n-1} \tilde{\mathbf{u}} + {}^0 D \wedge {}^1 \mathbf{v} \wedge \delta {}^{n-1} \tilde{\mathbf{u}}) - \delta L[{}^{n-1} \tilde{\mathbf{u}}, {}^n \tilde{D}, {}^0 \mathbf{s}] \quad (82)$$

Note that

$$\delta L[{}^{n-1} \tilde{\mathbf{u}}, {}^n \tilde{D}, {}^0 \mathbf{s}] = \frac{\delta \mathcal{L}}{\delta {}^n \tilde{D}} \wedge \delta {}^n \tilde{D} + \frac{\delta \mathcal{L}}{\delta {}^{n-1} \tilde{\mathbf{u}}} \wedge \delta {}^{n-1} \tilde{\mathbf{u}} + \frac{\delta \mathcal{L}}{\delta {}^0 \mathbf{s}} \wedge \delta {}^0 \mathbf{s} \quad (83)$$

and

$$\delta^0 s = \frac{1}{^0D} \wedge \tilde{\star} \delta^n \tilde{S} - \frac{1}{^0D} \wedge ^0 s \wedge \tilde{\star} \delta^n \tilde{D} \quad (84)$$

The latter comes from the definition of $^n \tilde{S} = ^n \tilde{D} \wedge ^0 s$. Substituting (83) and (84) into (82) and grouping terms gives

$$\begin{aligned} \delta \mathcal{H} = & \int \delta^n \tilde{D} \wedge \left(i_u^1 v - \frac{\delta \mathcal{L}}{\delta^n \tilde{D}} + \frac{^0 s}{^0 D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta^0 s} \right) + \\ & \delta^1 v \wedge (^0 D \wedge ^{n-1} \tilde{u}) + \\ & \delta^n \tilde{S} \wedge \left(-\frac{1}{^0 D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta^0 s} \right) + \\ & \delta^{n-1} \tilde{u} \wedge \left(^0 D \wedge ^1 v - \frac{\delta \mathcal{L}}{\delta^{n-1} \tilde{u}} \right) \end{aligned}$$

The last line is zero since $^0 D \wedge ^1 v = \frac{\delta \mathcal{L}}{\delta^{n-1} \tilde{u}}$, proving the statement. \square

5.3.2 Poisson Brackets

The Poisson brackets, written with respect to the topological pairing, are

$$\{\mathcal{A}, \mathcal{B}\}_R = -\langle \langle \frac{\delta \mathcal{A}}{\delta^n \tilde{D}}, d \frac{\delta \mathcal{B}}{\delta^1 v} \rangle \rangle - \langle \langle d \frac{\delta \mathcal{B}}{\delta^n \tilde{D}}, \frac{\delta \mathcal{A}}{\delta^1 v} \rangle \rangle \quad (85)$$

$$\{\mathcal{A}, \mathcal{B}\}_S = -\langle \langle \frac{\delta \mathcal{A}}{\delta^n \tilde{S}}, d(^0 s \wedge \frac{\delta \mathcal{B}}{\delta^1 v}) \rangle \rangle - \langle \langle ^0 s \wedge d \frac{\delta \mathcal{B}}{\delta^n \tilde{S}}, \frac{\delta \mathcal{A}}{\delta^1 v} \rangle \rangle \quad (86)$$

$$\{\mathcal{A}, \mathcal{B}\}_Q = -\langle \langle \tilde{\star} (^{n-2} \tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{B}}{\delta^1 v}), \frac{\delta \mathcal{A}}{\delta^1 v} \rangle \rangle \quad (87)$$

with $^{n-2} \tilde{Q} \wedge ^n \tilde{D} = ^2 \eta = d^1 v$, where $^2 \eta$ is the absolute vorticity straight 2-form. By inspection, these brackets are bilinear and satisfy the Leibniz rule.

Proposition 13 *The Poisson brackets (85) - (87) are anti-symmetric.*

Proof: Start with the Leibniz rule for d and \wedge :

$$\int_V d(^0 a \wedge ^{n-1} \tilde{b}) = \int_V d^0 a \wedge ^{n-1} \tilde{b} + ^0 a \wedge d^{n-1} \tilde{b} \quad (88)$$

Now use Stokes theorem

$$\int_{\partial V} ^0 a \wedge ^{n-1} \tilde{b} = \int d^0 a \wedge ^{n-1} \tilde{b} + ^0 a \wedge d^{n-1} \tilde{b} \quad (89)$$

In our case we will have $^{n-1} \tilde{b} = 0$ on ∂V , since we will use $^{n-1} \tilde{b} = \frac{\delta \mathcal{B}}{\delta^1 v}$. There we can write

$$\int d^0 a \wedge ^{n-1} \tilde{b} + ^0 a \wedge d^{n-1} \tilde{b} = 0 \quad (90)$$

Now consider the $\{\mathcal{A}, \mathcal{B}\}_R$ bracket. We have

$$\{\mathcal{A}, \mathcal{B}\}_R + \{\mathcal{B}, \mathcal{A}\}_R = -\langle \langle \frac{\delta \mathcal{A}}{\delta^n \tilde{D}}, d \frac{\delta \mathcal{B}}{\delta^1 v} \rangle \rangle - \langle \langle d \frac{\delta \mathcal{B}}{\delta^n \tilde{D}}, \frac{\delta \mathcal{A}}{\delta^1 v} \rangle \rangle - \langle \langle \frac{\delta \mathcal{B}}{\delta^n \tilde{D}}, d \frac{\delta \mathcal{A}}{\delta^1 v} \rangle \rangle - \langle \langle d \frac{\delta \mathcal{A}}{\delta^n \tilde{D}}, \frac{\delta \mathcal{B}}{\delta^1 v} \rangle \rangle \quad (91)$$

Using (90) with ${}^0a = \frac{\delta \mathcal{A}}{\delta {}^n\tilde{D}}$ and ${}^{n-1}\tilde{b} = \frac{\delta \mathcal{B}}{\delta {}^1V}$, the first and last terms combine to zero. Similarly, using (90) again with ${}^0a = \frac{\delta \mathcal{B}}{\delta {}^n\tilde{D}}$ and ${}^{n-1}\tilde{b} = \frac{\delta \mathcal{A}}{\delta {}^1V}$, the second and third terms combine to zero. Therefore

$$\{\mathcal{A}, \mathcal{B}\}_R + \{\mathcal{B}, \mathcal{A}\}_R = 0 \quad (92)$$

The same argument can be used with the $\{\mathcal{A}, \mathcal{B}\}_S$ bracket, except that now ${}^0a = \frac{\delta \mathcal{A}}{\delta {}^n\tilde{S}}$ and ${}^{n-1}\tilde{b} = {}^0s \wedge \frac{\delta \mathcal{A}}{\delta {}^1V}$ or ${}^0a = \frac{\delta \mathcal{B}}{\delta {}^n\tilde{S}}$ and ${}^{n-1}\tilde{b} = {}^0s \wedge \frac{\delta \mathcal{B}}{\delta {}^1V}$. Therefore

$$\{\mathcal{A}, \mathcal{B}\}_S + \{\mathcal{B}, \mathcal{A}\}_S = 0 \quad (93)$$

Finally, consider the $\{\mathcal{A}, \mathcal{B}\}_Q$ bracket. We have

$$\{\mathcal{B}, \mathcal{A}\}_Q = -\langle\langle \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^1V}), \frac{\delta \mathcal{B}}{\delta {}^1V} \rangle\rangle = -\int \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^1V}) \wedge \frac{\delta \mathcal{B}}{\delta {}^1V} \quad (94)$$

Using the properties of $\tilde{\star}$ and \wedge this can be written as

$$\{\mathcal{B}, \mathcal{A}\}_Q = -\int \tilde{\star} \frac{\delta \mathcal{B}}{\delta {}^1V} \wedge {}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^1V} = (-1)^{n-1} \int {}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{B}}{\delta {}^1V} \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^1V} = \quad (95)$$

$$(-1)^{n-1} \int \frac{\delta \mathcal{A}}{\delta {}^1V} \wedge \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{B}}{\delta {}^1V}) = \int \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{B}}{\delta {}^1V}) \wedge \frac{\delta \mathcal{A}}{\delta {}^1V} = -\{\mathcal{A}, \mathcal{B}\} \quad (96)$$

Thus (85) - (87) are anti-symmetric. \square

5.3.3 Equations of Motion

Inserting functional derivatives (79) into the Poisson brackets (85) - (87) gives the equations of motion as

$$\frac{\partial {}^1V}{\partial t} + \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} {}^{n-1}\tilde{F}) + d {}^0B + {}^0s \wedge d {}^0T = 0 \quad (97)$$

$$\frac{\partial {}^n\tilde{D}}{\partial t} + d {}^{n-1}\tilde{F} = 0 \quad (98)$$

$$\frac{\partial {}^n\tilde{S}}{\partial t} + d ({}^0s \wedge {}^{n-1}\tilde{F}) = 0 \quad (99)$$

Using the results in Appendix A, it is also possible to predict 0s or ${}^n\tilde{s}$ instead of ${}^n\tilde{S}$. Omitting the details, predicting $({}^1V, {}^n\tilde{D}, {}^0s)$ the equations of motion are

$$\frac{\partial {}^1V}{\partial t} + \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} {}^{n-1}\tilde{F}) + d {}^0B' - \frac{\tilde{\star} {}^n\tilde{T}'}{{}^0D} \wedge d {}^0s = 0 \quad (100)$$

$$\frac{\partial {}^n\tilde{D}}{\partial t} + d {}^{n-1}\tilde{F} = 0 \quad (101)$$

$$\frac{\partial {}^0s}{\partial t} + \tilde{\star} \left(\frac{1}{{}^0D} \wedge {}^{n-1}\tilde{F} \wedge d {}^0s \right) = 0 \quad (102)$$

while predicting $({}^1V, {}^n\tilde{D}, {}^n\tilde{s})$ they are

$$\frac{\partial {}^1V}{\partial t} + \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} {}^{n-1}\tilde{F}) + d {}^0B' - \frac{{}^0T'}{{}^0D} \wedge d {}^0s = 0 \quad (103)$$

$$\frac{\partial {}^n\tilde{D}}{\partial t} + d {}^{n-1}\tilde{F} = 0 \quad (104)$$

$$\frac{\partial {}^n\tilde{s}}{\partial t} + \frac{1}{{}^0D} \wedge {}^{n-1}\tilde{F} \wedge d {}^0s = 0 \quad (105)$$

These are the general equations of motion written in Hamiltonian form using split exterior calculus for a single component, single phase fluid in a general manifold \mathbb{M} (with $n = 2$ or $n = 3$) that can be characterized by a mass density ${}^n\tilde{D}$, a velocity 1v and a thermodynamic variable from the set $({}^n\tilde{S}, {}^n\tilde{s}, {}^0s)$. They are closed by making specific choices for ${}^n\tilde{D}$, 1v and 0s ; and specifying the Lagrangian (or equivalently, the Hamiltonian) in terms of these variables, which determines ${}^{n-1}\tilde{F}$, 0B , ${}^0B'$, 0T , ${}^0T'$ and ${}^n\tilde{T}'$. Some specific examples of this for common geophysical fluids (shallow water, thermal shallow water, compressible Euler) are given in Section 7.

6 Properties of Split Exterior Calculus Variational Formulation

Now we will investigate some basic properties about the general formulation: conservation of energy and Casimirs, Kelvin Circulation theorem and potential vorticity. The last three turn out to depend on dimension and/or the presence of a thermodynamic scalar. Therefore, we discuss only the $n = 3$ case in these subsections. A discussion of the simplifications and changes that arise when $n = 2$ and/or there is no thermodynamic scalar is found in Appendix B.

6.1 Energy conservation

Proposition 14 *The three sets of equations of motion in 5.3.3 preserve total energy.*

Proof: For all three sets, the conservation of energy follows from the antisymmetry of corresponding Poisson bracket such that in each case

$$\frac{d}{dt} \mathcal{H} = \{\mathcal{H}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{H}\} = 0, \quad (106)$$

and analogously for \mathcal{H}' and \mathcal{H}'' (from Appendix A). \square

6.2 Kelvin Circulation Theorem

Integrating (46) over $\gamma(t)$, the Kelvin circulation theorem is

$$\frac{d}{dt} \oint_{\gamma(t)} \frac{1}{{}^0D} \wedge \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{u}} = - \oint_{\gamma(t)} \frac{1}{{}^0D} \wedge \tilde{\star} \frac{\delta \mathcal{L}}{\delta {}^0s} d{}^0s \quad (107)$$

This is a natural definition, since both sides are straight 1-forms that can be integrated over curves. When there is no thermodynamic scalar, this simplifies somewhat (the right hand side is vanishes), as discussed in Appendix B.

6.3 Potential Vorticity

For $n = 3$, we define the potential vorticity straight 3-form ${}^n\mathbf{q}$, twisted 0-form ${}^0\tilde{\mathbf{q}}$ and straight 0-form ${}^0\mathbf{q}$ by

$${}^n\mathbf{q} = d {}^0\mathbf{s} \wedge {}^2\mathbf{Q} \quad (108)$$

$${}^0\tilde{\mathbf{q}} = \tilde{\star} {}^n\mathbf{q} = \tilde{\star}(d {}^0\mathbf{s} \wedge {}^2\mathbf{Q}) \quad (109)$$

$${}^0\mathbf{q} = {}^0\mathbf{I} \wedge {}^0\tilde{\mathbf{q}} \quad (110)$$

recalling that ${}^0\mathbf{D} \wedge {}^2\mathbf{Q} = {}^2\boldsymbol{\eta} = d {}^1\mathbf{v}$. Note that

$${}^0\mathbf{D} \wedge {}^n\mathbf{q} = {}^n\tilde{\mathbf{D}} \wedge {}^0\tilde{\mathbf{q}} \quad (111)$$

By combining (63) - (65), evolution equations for the potential vorticity density ${}^0\mathbf{D} \wedge {}^n\mathbf{q}$ straight n -form and twisted 0-form ${}^0\tilde{\mathbf{q}}$ are obtained as

$$\frac{\partial {}^0\mathbf{D} \wedge {}^n\mathbf{q}}{\partial t} + L_{\mathbf{u}}({}^0\mathbf{D} \wedge {}^n\mathbf{q}) = 0 \quad (112)$$

$$\frac{\partial {}^0\tilde{\mathbf{q}}}{\partial t} + L_{\mathbf{u}}({}^0\tilde{\mathbf{q}}) = 0 \quad (113)$$

The last equation is a statement of material conservation of ${}^0\tilde{\mathbf{q}}$. These definitions only make sense when $n = 3$, when $n = 2$ the potential vorticity takes a different form, and the governing equations depend on whether or not a thermodynamic scalar is present. This is discussed in Appendix B.

6.4 Casimirs

Since the Casimirs are a function of the Poisson bracket, and the Poisson bracket changes when dimension changes or if there is no thermodynamic scalar, there will be three sets of Casimirs: $n = 3$, $n = 2$ and $n = 2$ with no thermodynamic scalar. Here we will discuss only the Casimirs for $n = 3$, the other two cases can be found in Appendix B. The Casimirs satisfy

$$\{\mathcal{C}, \mathcal{A}\} = 0 \quad \forall \mathcal{A} \quad (114)$$

and they are of the form

$$\mathcal{C}[{}^1\mathbf{v}, {}^n\tilde{\mathbf{D}}, {}^n\tilde{\mathbf{S}}] = \langle {}^0\mathbf{D}, F({}^0\mathbf{s}, {}^0\mathbf{q}) \rangle \quad (115)$$

where $F({}^0\mathbf{s}, {}^0\mathbf{q})$ is an arbitrary function of ${}^0\mathbf{s}$ and ${}^0\mathbf{q}$. The functional derivatives of this are

$$\frac{\delta \mathcal{C}'}{\delta {}^1\mathbf{v}} = d(F_q \wedge {}^0\tilde{\mathbf{I}} \wedge d {}^0\mathbf{s}) \quad (116)$$

$$\frac{\delta \mathcal{C}'}{\delta {}^n\tilde{\mathbf{D}}} = F - {}^0\mathbf{q} \wedge F_q - {}^0\mathbf{s} \wedge F_s + {}^0\mathbf{s} \wedge \frac{1}{{}^0\mathbf{D}} \wedge {}^0\tilde{\mathbf{I}} \wedge \tilde{\star}(d F_q \wedge {}^2\boldsymbol{\eta}) \quad (117)$$

$$\frac{\delta \mathcal{C}'}{\delta {}^n\tilde{\mathbf{S}}} = F_s - \frac{1}{{}^0\mathbf{D}} \wedge {}^0\tilde{\mathbf{I}} \wedge \tilde{\star}(d F_q \wedge {}^2\boldsymbol{\eta}) \quad (118)$$

where $F_q = \frac{\partial F}{\partial {}^0\mathbf{q}}$ and $F_s = \frac{\partial F}{\partial {}^0\mathbf{s}}$. Important cases are $F = 1$ (total mass), $F = {}^0\mathbf{q}$ (total potential vorticity) and $F = {}^0\mathbf{s}$ (total thermodynamic scalar).

Proposition 15 Equation (115) is a Casimir of the brackets (85) - (87)

Proof:

□

7 Specific Examples

In this section we show how some commonly used equations sets in geophysical fluids (shallow water equations, thermal shallow water equations and compressible Euler equations) fit into the general formulation discussed above. This will include the split covariant equations from [2].

7.1 Shallow Water Equations ($n = 2$, no thermodynamic scalar)

For the rotating shallow water equations, we have $n = 2$, the relevant mass variable is the twisted fluid height 2-form ${}^2\tilde{h}$, and there is no thermodynamic scalar.

7.1.1 Hamiltonian

The Lagrangian $\mathcal{L}[{}^{n-1}\tilde{u}, {}^2\tilde{h}]$ for the rotating shallow water equations is formed as usual, it is the kinetic energy plus a rotation term minus the potential energy:

$$\mathcal{L}[{}^{n-1}\tilde{u}, {}^2\tilde{h}] = \langle {}^0h, {}^0K \rangle + \langle {}^0h \wedge {}^1u, {}^1R \rangle - \frac{1}{2} \langle {}^0g \wedge {}^2\tilde{h}, {}^2\tilde{h} \rangle - \langle {}^0g \wedge {}^2\tilde{h}, {}^2\tilde{t} \rangle \quad (119)$$

with kinetic energy ${}^0K = \frac{1}{2} i_u {}^1u = \tilde{\star} \frac{{}^1u \wedge \tilde{\star} {}^1u}{2}$ and height straight 0-form ${}^0h = \tilde{\star} {}^2\tilde{h}$ where 0g is the gravitational constant straight 0-form and ${}^2\tilde{t}$ is the topography twisted 2-form. This gives

$${}^1v = {}^1u + {}^1R \quad (120)$$

for the absolute velocity 1v . Now note that

$$i_u {}^1v = \tilde{\star} ({}^1v \wedge \tilde{\star} {}^1u) = 2 {}^0K + \tilde{\star} ({}^1R \wedge \tilde{\star} {}^1u) \quad (121)$$

Taking the Legendre transform (35) of (119) gives the Hamiltonian as

$$\mathcal{H}[{}^1v, {}^2\tilde{h}] = \langle {}^0h, 2 {}^0K + \tilde{\star} ({}^1R \wedge \tilde{\star} {}^1u) \rangle - \langle {}^0h, {}^0K \rangle - \langle {}^0h \wedge {}^1u, {}^1R \rangle + \frac{1}{2} \langle {}^0g \wedge {}^2\tilde{h}, {}^2\tilde{h} \rangle \quad (122)$$

The second part of the first term is equal to $\langle {}^0h \wedge {}^1u, {}^1R \rangle$, and thus the Hamiltonian is finally given by

$$\mathcal{H}[{}^1v, {}^2\tilde{h}] = \langle {}^0h, {}^0K \rangle + \frac{1}{2} \langle {}^0g \wedge {}^2\tilde{h}, {}^2\tilde{h} \rangle + \langle {}^0g \wedge {}^2\tilde{h}, {}^2\tilde{t} \rangle \quad (123)$$

It is interesting to note that the Hamiltonian (and also the Lagrangian) can be written as a topological pairing between straight and twisted forms as

$$\mathcal{H}[{}^1v, {}^2\tilde{h}] = \frac{1}{2} \langle \langle {}^1u, {}^{n-1}\tilde{F} \rangle \rangle + \frac{1}{2} \langle \langle {}^0g \wedge {}^0h, {}^2\tilde{h} \rangle \rangle + \langle \langle {}^0g \wedge {}^0h, {}^2\tilde{t} \rangle \rangle \quad (124)$$

The functional derivatives of $\mathcal{H}[{}^1v, {}^2\tilde{h}]$ are

$$\frac{\delta \mathcal{H}}{\delta {}^1v} = {}^{n-1}\tilde{F} = \tilde{\star} ({}^0h \wedge {}^1u) = {}^0h \wedge {}^{n-1}\tilde{u} \quad \frac{\delta \mathcal{H}}{\delta {}^2\tilde{h}} = {}^0K + {}^0g \wedge ({}^0h + {}^0t) \quad (125)$$

where ${}^0t = \tilde{\star} {}^2\tilde{t}$ is the topography straight 0-form.

7.1.2 Equations of Motion

Putting the functional derivatives (125) into the Poisson brackets (85) and (87) gives the equations of motion

$$\frac{\partial {}^1\mathbf{v}}{\partial t} + {}^{n-2}\tilde{\mathbf{Q}} \wedge {}^{n-1}\tilde{\mathbf{F}} + d {}^0\mathbf{K} + {}^0g \wedge d({}^0\mathbf{h} + {}^0\mathbf{t}) = 0 \quad (126)$$

$$\frac{\partial {}^2\tilde{\mathbf{h}}}{\partial t} + d {}^{n-1}\tilde{\mathbf{F}} = 0 \quad (127)$$

Note that we did not use (86), since there is no thermodynamic scalar. Equations (126) - (127) are equivalent to the split covariant equations from [2]. This is a demonstration that the split covariant equations can be reproduced by the curl-form Hamiltonian formulation, with the Poisson brackets generating the topological equations and the functional derivatives of the Hamiltonian generating the metric closure equations.

7.2 Thermal Shallow Water Equations

The thermal shallow water equations (also known as the Ripa equations) extend the rotating shallow water equations to the case of variable buoyancy [26, 12]. As in the shallow water equations, the relevant mass variable is the fluid height ${}^2\tilde{\mathbf{h}}$. However, now there is a thermodynamic scalar: the buoyancy ${}^0s = g\frac{\rho}{\bar{\rho}}$; where $\rho = \rho(x, y, t)$ is the horizontally varying density and $\bar{\rho}$ is the density used in the Boussinesq approximation. When $s = g$, the rotating shallow water equations are recovered. There are three choices for the representation of the thermodynamics: the buoyancy density twisted n -form ${}^n\tilde{\mathbf{S}}$, buoyancy straight 0-form 0s and buoyancy twisted n -form ${}^n\tilde{s}$. Here we will present the Lagrangian $\mathcal{L}[{}^{n-1}\tilde{\mathbf{u}}, {}^2\tilde{\mathbf{h}}, {}^0s]$, and the Hamiltonian $\mathcal{H}[{}^1\mathbf{v}, {}^2\tilde{\mathbf{h}}, {}^n\tilde{\mathbf{S}}]$; but show the equations of motion for all three choices. The material in Appendix A can be used to make the change of variables to get $\mathcal{H}'[{}^1\mathbf{v}, {}^2\tilde{\mathbf{h}}, {}^0s]$ or $\mathcal{H}''[{}^1\mathbf{v}, {}^2\tilde{\mathbf{h}}, {}^n\tilde{s}]$ if desired, and this is left as an exercise for the interested reader.

7.2.1 Hamiltonian

Again, the Lagrangian $\mathcal{L}[{}^{n-1}\tilde{\mathbf{u}}, {}^2\tilde{\mathbf{h}}, {}^0s]$ is the kinetic energy plus a rotation term minus the potential energy:

$$\mathcal{L}[{}^{n-1}\tilde{\mathbf{u}}, {}^2\tilde{\mathbf{h}}, {}^0s] = \langle {}^0\mathbf{h}, {}^0\mathbf{K} \rangle + \langle {}^0\mathbf{h} \wedge {}^1\mathbf{u}, {}^1\mathbf{R} \rangle - \frac{1}{2} \langle {}^0s \wedge {}^2\tilde{\mathbf{h}}, {}^2\tilde{\mathbf{h}} \rangle - \langle {}^0s \wedge {}^2\tilde{\mathbf{h}}, {}^2\tilde{\mathbf{t}} \rangle \quad (128)$$

The same exact manipulations as in the shallow water case (except with a different potential energy) yield the Hamiltonian $\mathcal{H}[{}^1\mathbf{v}, {}^2\tilde{\mathbf{h}}, {}^n\tilde{\mathbf{S}}]$

$$\mathcal{H}[{}^1\mathbf{v}, {}^2\tilde{\mathbf{h}}, {}^n\tilde{\mathbf{S}}] = \langle {}^0\mathbf{h}, {}^0\mathbf{K} \rangle + \frac{1}{2} \langle {}^0s \wedge {}^2\tilde{\mathbf{h}}, {}^2\tilde{\mathbf{h}} \rangle + \langle {}^0s \wedge {}^2\tilde{\mathbf{h}}, {}^2\tilde{\mathbf{t}} \rangle \quad (129)$$

This has functional derivatives

$$\frac{\delta \mathcal{H}}{\delta {}^1\mathbf{v}} = {}^{n-1}\tilde{\mathbf{F}} = \tilde{\star}({}^0\mathbf{h} \wedge {}^1\mathbf{u}) = {}^0\mathbf{h} \wedge {}^{n-1}\tilde{\mathbf{u}} \quad \frac{\delta \mathcal{H}'}{\delta {}^2\tilde{\mathbf{h}}} = {}^0\mathbf{K} + \frac{{}^0\mathbf{S}}{2} \quad \frac{\delta \mathcal{H}'}{\delta {}^n\tilde{\mathbf{S}}} = \frac{{}^0\mathbf{h}}{2} + {}^0\mathbf{t} \quad (130)$$

where ${}^0\mathbf{S} = \tilde{\star} {}^n\tilde{\mathbf{S}} = {}^0\mathbf{h} \wedge {}^0s$.

7.2.2 Equations of Motion

Putting the functional derivatives (130) into the Poisson brackets (85) - (87) (or the equivalent when 0s or ${}^n\tilde{s}$ are predicted) gives the equations of motion

$$\frac{\partial {}^1v}{\partial t} + {}^{n-2}\tilde{Q} \wedge {}^{n-1}\tilde{F} + d {}^0K + d \frac{{}^n\tilde{S}}{2} + {}^0s \wedge d(\frac{{}^2\tilde{h}}{2} + {}^2\tilde{t}) = 0 \quad (131)$$

$$\frac{\partial {}^2\tilde{h}}{\partial t} + d {}^{n-1}\tilde{F} = 0 \quad (132)$$

$$\frac{\partial {}^n\tilde{S}}{\partial t} + d({}^0s \wedge {}^{n-1}\tilde{F}) = 0 \quad (133)$$

$$\frac{\partial {}^0s}{\partial t} + \frac{1}{{}^0h} \wedge \tilde{\star} \left({}^{n-1}\tilde{F} \wedge d {}^0s \right) = 0 \quad (134)$$

$$\frac{\partial {}^n\tilde{s}}{\partial t} + \frac{1}{{}^0h} \wedge {}^{n-1}\tilde{F} \wedge d {}^0s = 0 \quad (135)$$

where only one of (133) - (135) is needed.

7.3 Compressible Euler Equations ($n = 2$ and $n = 3$)

For the compressible Euler equations, the relevant mass variable is the density twisted n -form ${}^n\tilde{\rho}$ (with associated straight 0-form ${}^0\rho = \tilde{\star} {}^n\tilde{\rho}$) and the thermodynamic scalar is the entropy straight 0-form 0s , with associated thermodynamic scalar twisted n -form ${}^n\tilde{s}$ and thermodynamic scalar density twisted n -form ${}^n\tilde{S}$. It would also be possible to use the potential temperature instead of the entropy, which has advantages for an ideal gas (see Section 7.3.3). As for the thermal shallow water equations, we will present the Lagrangian $\mathcal{L}[{}^{n-1}\tilde{u}, {}^2\tilde{h}, {}^0s]$ and Hamiltonian $\mathcal{H}[{}^1v, {}^2\tilde{h}, {}^n\tilde{S}]$, but show the equations of motion for all three choices of thermodynamic variable. These equations apply equally well to the $n = 2$ (commonly referred to as slice) and $n = 3$ cases, with some slight simplification in the PV flux term arising when $n = 2$. It should be noted that these slice equations assume that there is no variation in the out of slice direction for any variable, and that the out of slice velocity is zero. They are therefore somewhat different than the slice equations presented in [7, 8]. An extension of the general framework to the case of non-zero out of slice velocity and incorporating variability in the out of slice direction for the thermodynamic scalar will be the subject of future work.

7.3.1 Hamiltonian

The Lagrangian $\mathcal{L}[{}^{n-1}\tilde{u}, {}^n\tilde{\rho}, {}^0s]$ is the sum of the kinetic energy plus a rotation term minus the sum of the gravitational potential energy and the internal energy

$$\mathcal{L}[{}^{n-1}\tilde{u}, {}^n\tilde{\rho}, {}^0s] = \langle {}^0\rho, {}^0K \rangle + \langle {}^0\rho \wedge {}^1u, {}^1R \rangle - \langle {}^0\rho, {}^0\Phi \rangle - \langle {}^0\rho, {}^0U(\alpha, {}^0s) \rangle \quad (136)$$

where ${}^0U(\alpha, {}^0s)$ is the internal energy, ${}^0\Phi$ is the geopotential and ${}^0K = \tilde{\star} \frac{\tilde{\star} {}^1u \wedge {}^1u}{2}$. A choice of internal energy is equivalent to a choice of equation of state. These equations will hold for arbitrary choices of 0U and ${}^0\Phi$, which allows a wide range of geophysical

fluids to be treated. Again following the same procedure as in the shallow water case, the Hamiltonian is given by

$$\mathcal{H}[{}^1\mathbf{v}, {}^n\tilde{\rho}, {}^n\tilde{S}] = \langle {}^0\rho, {}^0K \rangle + \langle {}^0\rho, {}^0\Phi \rangle + \langle {}^0\rho, {}^0U(\alpha, {}^0s) \rangle \quad (137)$$

The functional derivatives of $\mathcal{H}[{}^1\mathbf{v}, {}^n\tilde{\rho}, {}^n\tilde{S}]$ are given by

$$\frac{\delta \mathcal{H}}{\delta {}^1\mathbf{v}} = {}^{n-1}\tilde{\mathbf{F}} = \tilde{\star}({}^0\rho \wedge {}^1\mathbf{u}) = {}^0\rho \wedge {}^{n-1}\tilde{\mathbf{u}} \quad \frac{\delta \mathcal{H}}{\delta {}^n\tilde{\rho}} = {}^0K + {}^0\Phi + {}^0U - {}^0p {}^0\alpha + {}^0s {}^0T \quad \frac{\delta \mathcal{H}}{\delta {}^n\tilde{S}} = {}^0T \quad (138)$$

where we have the temperature straight 0-form ${}^0T = \frac{\partial {}^0U}{\partial {}^0s}$ and pressure straight 0-form ${}^0p = -\frac{\partial {}^0U}{\partial {}^0\alpha}$.

7.3.2 Equations of Motion

Again putting the functional derivatives (138) into the Poisson brackets (85) - (87) (or the equivalent when 0s or ${}^n\tilde{s}$ are predicted) gives the equations of motion

$$\frac{\partial {}^1\mathbf{v}}{\partial t} + \tilde{\star}({}^{n-2}\tilde{\mathbf{Q}} \wedge \tilde{\star} {}^{n-1}\tilde{\mathbf{F}}) + d {}^0K + d {}^0\Phi + \frac{1}{{}^0\rho} \wedge d {}^0p = 0 \quad (139)$$

$$\frac{\partial {}^n\tilde{\rho}}{\partial t} + d {}^{n-1}\tilde{\mathbf{F}} = 0 \quad (140)$$

$$\frac{\partial {}^n\tilde{S}}{\partial t} + d({}^0s \wedge {}^{n-1}\tilde{\mathbf{F}}) = 0 \quad (141)$$

$$\frac{\partial {}^0s}{\partial t} + \frac{1}{{}^0h} \wedge \tilde{\star} ({}^{n-1}\tilde{\mathbf{F}} \wedge d {}^0s) = 0 \quad (142)$$

$$\frac{\partial {}^n\tilde{s}}{\partial t} + \frac{1}{{}^0h} \wedge {}^{n-1}\tilde{\mathbf{F}} \wedge d {}^0s = 0 \quad (143)$$

Here we have used the fact that

$${}^0s \wedge d {}^0T + d({}^0U - {}^0p {}^0\alpha + {}^0s {}^0T) = \frac{1}{{}^0\rho} \wedge d {}^0p \quad (144)$$

by the fundamental thermodynamic relationship

$$d {}^0U = - {}^0p \wedge d {}^0\alpha + {}^0s \wedge d {}^0T \quad (145)$$

These are equivalent to split covariant equations from [2], although no thermodynamic equation was presented there. This is another demonstration that the split covariant equations can be reproduced by the curl-form Hamiltonian formulation, with the Poisson brackets generating the topological equations and the functional derivatives of the Hamiltonian generating the metric closure equations.

7.3.3 Predicting ${}^n\tilde{\Theta}$ instead of ${}^n\tilde{S}$

In geophysical fluid dynamics, potential temperature is often used instead of entropy, especially in the case of an ideal gas. If the thermodynamic scalar is potential temperature ${}^0\theta$ instead of entropy 0s , the fundamental thermodynamic relationship becomes

$$d {}^0U = - {}^0p \wedge d {}^0\alpha + {}^0\theta \wedge d {}^0\pi \quad (146)$$

where the Exner pressure straight 0-form ${}^0\pi = \frac{\partial {}^0U}{\partial {}^0\theta}$ and we again have

$${}^0\theta \wedge d {}^0\pi + d({}^0U - {}^0p {}^0\alpha + {}^0\theta {}^0\pi) = \frac{1}{{}^0\rho} \wedge d {}^0p \quad (147)$$

In fact, the formulation remains the same, with 0s replaced by ${}^0\theta$, ${}^n\tilde{S}$ by ${}^n\tilde{\Theta} = {}^n\tilde{D} \wedge {}^0\theta$ and 0T by ${}^0\pi$. However in the case of an ideal gas, ${}^0U = {}^0p {}^0\alpha - {}^0\theta {}^0\pi$, and the thermodynamic contribution to $\frac{\delta \mathcal{H}}{\delta {}^n\tilde{\rho}}$ drops out:

$$\frac{\delta \mathcal{H}}{\delta {}^n\tilde{\rho}} = {}^0B = {}^0K + {}^0\Phi + {}^0U - {}^0p {}^0\alpha + {}^0\theta {}^0\pi = {}^0K + {}^0\Phi \quad (148)$$

For more general equations of state, or other prognostic variables, however, the thermodynamic contribution to $\frac{\delta \mathcal{H}}{\delta {}^n\tilde{\rho}}$ remains and there seems to be little advantage to using potential temperature instead of entropy.

8 Conclusions and Outlook

This paper has presented variational and Hamiltonian formulations for geophysical fluids based on split exterior calculus. These formulations have been illustrated through the selection of Lagrangians that give the shallow water equation, thermal shallow water equations and compressible Euler equations. This has provided additional insight into the differential geometric structure underlying the equations of motion, including the splitting between topological and metric parts of the equations. In fact, the Poisson brackets are composed of purely topological operators, while all of the metric information resides in the Hamiltonian. This splitting exactly reproduces the existing split covariant equations from [2]. It remains to develop the corresponding Lie-Poisson bracket associated with the split Euler-Lagrange equation (47). Additional future work further developing the formulation could consist of an extension to: multicomponent, multiphase fluids; to fluids with irreversible processes; to domains with moving boundaries (such a free surface); to non-Eulerian vertical coordinates, to new Lagrangians (such as those for the Green-Naghdi equations); to slice equations with out-of-slice velocity and thermodynamic scalars; to various standard approximations (traditional, shallow atmosphere, quasi-hydrostatic, etc.); and to semi-compressible fluids (anelastic, pseudo-incompressible, Boussinesq, semi-hydrostatic).

The immediate application of this novel formulation is anticipated to be the development of new numerical methods, and a deeper understanding of existing methods. The new formulation is particularly interesting from a numerical modeling point of view, due to its relative simplicity: the only operators that appear are $\tilde{\star}$, \wedge and d , along the metric and topological pairing. Development of a discrete exterior calculus (or primal-dual discretization) that preserve a subset of the key properties of these operators should enable numerical discretizations that preserve important aspects of the Hamiltonian structure (such as anti-symmetry of the Poisson bracket and a subset of its Casimirs); and lead to discrete equations that have many of the same properties as the continuous ones. There is strong evidence that the TRiSK scheme, although inconsistent, is in fact a realization of such a discrete exterior calculus. This will be the subject of future work, with a focus on obtaining a consistent discretization scheme applicable in $n = 2$ and $n = 3$ for general, non-orthogonal grids (emphasizing the cubed-sphere and icosahedral grids).

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A Alternative Prognostic Variables: s , 0_S and $^n_{\tilde{S}}$

A.1 Vector Calculus

In the Hamiltonian formulation using vector calculus, it is also possible to predict the thermodynamic scalar s , rather than the thermodynamic scalar density S . This is simply a change of variables from the original set of (\mathbf{v}, D, S) .

Chain Rule Thus for an arbitrary functional we have $\mathcal{A}'[\mathbf{v}, D, s] = \mathcal{A}[\mathbf{v}, D, S]$ and the chain rule gives

$$\frac{\delta \mathcal{A}'}{\delta \mathbf{v}} = \frac{\delta \mathcal{A}}{\delta \mathbf{v}} \quad \frac{\delta \mathcal{A}'}{\delta D} = \frac{\delta \mathcal{A}}{\delta D} + s \frac{\delta \mathcal{A}}{\delta S} \quad \frac{\delta \mathcal{A}'}{\delta s} = D \frac{\delta \mathcal{A}}{\delta S} \quad (149)$$

The proof of this is straightforward, and is left for the interested reader.

Functional Derivatives: $\frac{\delta \mathcal{H}'}{\delta x}$ and $\frac{\delta \mathcal{C}'}{\delta x}$ Therefore

$$\frac{\delta \mathcal{H}'}{\delta \mathbf{v}} = \mathbf{F} \quad \frac{\delta \mathcal{H}'}{\delta D} = B' = B + sT \quad \frac{\delta \mathcal{H}'}{\delta s} = T' = DT \quad (150)$$

and

$$\frac{\delta \mathcal{C}'}{\delta \mathbf{v}} = \nabla \times \left(\frac{\partial F}{\partial q} \nabla s \right) \quad \frac{\delta \mathcal{C}'}{\delta D} = F - q \frac{\partial F}{\partial q} \quad \frac{\delta \mathcal{C}'}{\delta s} = D \frac{\partial F}{\partial s} - \nabla \cdot \left(\frac{\partial F}{\partial q} \nabla \times \mathbf{v} \right) \quad (151)$$

Poisson Bracket: $\{\mathcal{A}', \mathcal{B}'\}$ Using also the chain rule (149) in (38) - (40), the new Poisson brackets are

$$\{\mathcal{A}', \mathcal{B}'\}_R = \int_{\Omega} \left(-\frac{\delta \mathcal{A}'}{\delta D} \nabla \cdot \frac{\delta \mathcal{B}'}{\delta \mathbf{v}} + \frac{\delta \mathcal{B}'}{\delta D} \nabla \cdot \frac{\delta \mathcal{A}'}{\delta \mathbf{v}} \right) d\Omega \quad (152)$$

$$\{\mathcal{A}', \mathcal{B}'\}_s = \int_{\Omega} \frac{\nabla s}{D} \cdot \left(\frac{\delta \mathcal{A}'}{\delta \mathbf{v}} \frac{\delta \mathcal{B}'}{\delta s} - \frac{\delta \mathcal{B}'}{\delta \mathbf{v}} \frac{\delta \mathcal{A}'}{\delta s} \right) d\Omega \quad (153)$$

$$\{\mathcal{A}', \mathcal{B}'\}_Q = \int_{\Omega} -\frac{\delta \mathcal{A}'}{\delta \mathbf{v}} \cdot \left(\mathbf{Q} \times \frac{\delta \mathcal{B}'}{\delta \mathbf{v}} \right) d\Omega \quad (154)$$

The proof of this is again left for the interested reader. Note that the $\{\mathcal{A}', \mathcal{B}'\}_R$ and $\{\mathcal{A}', \mathcal{B}'\}_Q$ brackets have the same form as before, just with different arguments.

Equations of Motion Inserting the functional derivatives (150) into the Poisson brackets (152) - (154), the equations of motion are then

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{Q} \times \mathbf{F} + \nabla B' - \frac{T'}{D} \nabla s = 0 \quad (155)$$

$$\frac{\partial D}{\partial t} + \nabla \cdot \mathbf{F} = 0 \quad (156)$$

$$\frac{\partial s}{\partial t} + \frac{\mathbf{F}}{D} \cdot \nabla s = 0 \quad (157)$$

A.2 Split Exterior Calculus

In the Hamiltonian formulation using split exterior calculus, a change of variables from thermodynamic scalar density twisted n -form ${}^n\tilde{S}$ to either thermodynamic scalar straight 0-form 0s or thermodynamic scalar twisted n -form ${}^n\tilde{s} = \tilde{\star} {}^0s$ can be made. This is the split exterior calculus analogue of predicting s instead of S .

A.2.1 Predicting 0s

Chain Rule Now we have $\mathcal{A}'[{}^1v, {}^n\tilde{D}, {}^0s] = \mathcal{A}[{}^1v, {}^n\tilde{D}, {}^n\tilde{S}]$ for any functional \mathcal{A} and the chain rule for functional derivatives gives

$$\frac{\delta \mathcal{A}'}{\delta {}^1v} = \frac{\delta \mathcal{A}}{\delta {}^1v} \quad \frac{\delta \mathcal{A}'}{\delta {}^n\tilde{D}} = \frac{\delta \mathcal{A}}{\delta {}^n\tilde{D}} + {}^0s \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^n\tilde{S}} \quad \frac{\delta \mathcal{A}'}{\delta {}^0s} = {}^n\tilde{T}' = {}^0D \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^n\tilde{S}} \quad (158)$$

Functional Derivatives: $\frac{\delta \mathcal{H}'}{\delta x}$ and $\frac{\delta \mathcal{C}'}{\delta x}$ The functional derivatives of $\mathcal{H}'[{}^1v, {}^n\tilde{D}, {}^0s]$ are

$$\frac{\delta \mathcal{H}'}{\delta {}^1v} = {}^{n-1}\tilde{F} \quad \frac{\delta \mathcal{H}'}{\delta {}^n\tilde{D}} = {}^0B + {}^0s \wedge {}^0T =: {}^0B' \quad \frac{\delta \mathcal{H}'}{\delta {}^0s} = {}^0D \wedge \tilde{\star} {}^0T = {}^0D \wedge {}^n\tilde{T} := {}^n\tilde{T}' \quad (159)$$

where ${}^n\tilde{T} = \tilde{\star} {}^0T$. The functional derivatives of $\mathcal{C}'[{}^1v, {}^n\tilde{D}, {}^0s]$ are

$$\frac{\delta \mathcal{C}'}{\delta {}^1v} = d(F_q \wedge {}^0\tilde{I} \wedge d {}^0s) \quad \frac{\delta \mathcal{C}'}{\delta {}^n\tilde{D}} = F - {}^0q \wedge F_q \quad \frac{\delta \mathcal{C}'}{\delta {}^0s} = {}^n\tilde{D} \wedge F_s - {}^0\tilde{I} \wedge d F_q \wedge {}^2\eta \quad (160)$$

Poisson Bracket: $\{\mathcal{A}', \mathcal{B}'\}$ Using also the chain rule (158) in (85) - (87), the new Poisson brackets are

$$\{\mathcal{A}', \mathcal{B}'\}_R = -\langle\langle \frac{\delta \mathcal{A}'}{\delta {}^n\tilde{D}}, d \frac{\delta \mathcal{B}'}{\delta {}^1v} \rangle\rangle - \langle\langle d \frac{\delta \mathcal{B}'}{\delta {}^n\tilde{D}}, \frac{\delta \mathcal{A}'}{\delta {}^1v} \rangle\rangle \quad (161)$$

$$\{\mathcal{A}', \mathcal{B}'\}_s = -\langle\langle \tilde{\star} \left(\frac{1}{{}^0D} \wedge \frac{\delta \mathcal{B}'}{\delta {}^1v} \wedge d {}^0s \right), \frac{\delta \mathcal{A}'}{\delta {}^0s} \rangle\rangle + \langle\langle \frac{1}{{}^0D} \wedge \tilde{\star} \frac{\delta \mathcal{B}'}{\delta {}^0s} \wedge d {}^0s, \frac{\delta \mathcal{A}'}{\delta {}^1v} \rangle\rangle \quad (162)$$

$$\{\mathcal{A}', \mathcal{B}'\}_Q = -\langle\langle \tilde{\star} ({}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{B}'}{\delta {}^1v}), \frac{\delta \mathcal{A}'}{\delta {}^1v} \rangle\rangle \quad (163)$$

where again the $\{\mathcal{A}', \mathcal{B}'\}_R$ and $\{\mathcal{A}', \mathcal{B}'\}_Q$ brackets have the same form as before, just with different arguments.

Equations of Motion The equations of motion are found in Section 5.3.3.

A.2.2 Predicting ${}^n\tilde{s}$

Chain Rule Predicting ${}^n\tilde{s}$ instead of ${}^n\tilde{S}$ means $\mathcal{A}''[{}^1v, {}^n\tilde{D}, {}^n\tilde{s}] = \mathcal{A}[{}^1v, {}^n\tilde{D}, {}^n\tilde{S}]$ for any functional \mathcal{A} and the chain rule for functional derivatives gives

$$\frac{\delta \mathcal{A}''}{\delta {}^1v} = \frac{\delta \mathcal{A}}{\delta {}^1v} \quad \frac{\delta \mathcal{A}''}{\delta {}^n\tilde{D}} = \frac{\delta \mathcal{A}}{\delta {}^n\tilde{D}} + {}^0s \wedge \tilde{\star} \frac{\delta \mathcal{A}}{\delta {}^n\tilde{S}} \quad \frac{\delta \mathcal{A}''}{\delta {}^0s} = {}^0D \wedge \frac{\delta \mathcal{A}}{\delta {}^n\tilde{S}} \quad (164)$$

Functional Derivatives: $\frac{\delta \mathcal{H}''}{\delta x}$ and $\frac{\delta \mathcal{C}''}{\delta x}$ The functional derivatives of $\mathcal{H}''[{}^1v, {}^n\tilde{D}, {}^0s]$ are

$$\frac{\delta \mathcal{H}''}{\delta {}^1v} = {}^{n-1}\tilde{F} \quad \frac{\delta \mathcal{H}''}{\delta {}^n\tilde{D}} = {}^0B - {}^0s \wedge {}^0T := {}^0B' \quad \frac{\delta \mathcal{H}''}{\delta {}^n\tilde{S}} = {}^0D \wedge {}^0T := {}^0T' \quad (165)$$

where ${}^n\tilde{T} = \tilde{\star} {}^0T$. The functional derivatives of $\mathcal{C}''[{}^1v, {}^n\tilde{D}, {}^n\tilde{s}]$ are

$$\frac{\delta \mathcal{C}'}{\delta {}^1v} = d(F_q \wedge {}^0\tilde{I} \wedge d {}^0s) \quad \frac{\delta \mathcal{C}'}{\delta {}^n\tilde{D}} = F - {}^0q \wedge F_q \quad \frac{\delta \mathcal{C}'}{\delta {}^n\tilde{S}} = {}^0D \wedge F_s - {}^0\tilde{I} \wedge \tilde{\star}(d F_q \wedge {}^2\gamma) \quad (166)$$

Poisson Bracket: $\{\mathcal{A}'', \mathcal{B}''\}$ Again using the chain rule (164) in (85) - (87), the new Poisson brackets are

$$\{\mathcal{A}'', \mathcal{B}''\}_R = -\langle\langle \frac{\delta \mathcal{A}''}{\delta {}^n\tilde{D}}, d \frac{\delta \mathcal{B}''}{\delta {}^1v} \rangle\rangle - \langle\langle d \frac{\delta \mathcal{B}''}{\delta {}^n\tilde{D}}, \frac{\delta \mathcal{A}''}{\delta {}^1v} \rangle\rangle \quad (167)$$

$$\{\mathcal{A}'', \mathcal{B}''\}_s = -\langle\langle \frac{\delta \mathcal{A}''}{\delta {}^n\tilde{S}}, \frac{1}{{}^0D} \wedge \frac{\delta \mathcal{B}''}{\delta {}^1v} \wedge d {}^0s \rangle\rangle + \langle\langle \frac{1}{{}^0D} \wedge \frac{\delta \mathcal{B}''}{\delta {}^n\tilde{S}} \wedge d {}^0s, \frac{\delta \mathcal{A}''}{\delta {}^1v} \rangle\rangle \quad (168)$$

$$\{\mathcal{A}'', \mathcal{B}''\}_Q = -\langle\langle \tilde{\star}({}^{n-2}\tilde{Q} \wedge \tilde{\star} \frac{\delta \mathcal{B}''}{\delta {}^1v}), \frac{\delta \mathcal{A}''}{\delta {}^1v} \rangle\rangle \quad (169)$$

where again the $\{\mathcal{A}'', \mathcal{B}''\}_R$ and $\{\mathcal{A}'', \mathcal{B}''\}_Q$ brackets have the same form as before, just with different arguments.

Equations of Motion The equations of motion are found in Section 5.3.3.

B Simplifications when $n = 2$ and if there is no thermodynamic scalar

Some simplifications and changes arise for certain aspects of the variational formulation when $n = 2$ and if there is no thermodynamic scalar. These changes are discussed first using vector calculus, and then shown in split exterior calculus. We do not treat the case of $n = 3$ without a thermodynamic scalar, since it rarely arises in practice.

B.1 Vector Calculus

When $n = 2$ the Lie derivative for a vector field \mathbf{x} is $L_{\mathbf{u}} \mathbf{x} = (\nabla^T \cdot \mathbf{x}) \mathbf{u}^T + \nabla(\mathbf{u} \cdot \mathbf{x})$, and therefore the Euler-Poincaré equation in curl-form is

$$\frac{\partial}{\partial t} \left(\frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) + \nabla^T \cdot \left(\frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \right) \mathbf{u}^T + \nabla \left(\mathbf{u} \cdot \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} - \frac{\delta \mathcal{L}}{\delta D} \right) + \frac{1}{D} \frac{\delta \mathcal{L}}{\delta s} \nabla s = 0 \quad (170)$$

Here we have used the skew-gradient $\nabla^T = \hat{k} \times \nabla$ and 2D curl $\nabla^T \cdot = \hat{k} \cdot \nabla \times$, where \hat{k} is local vertical (these operators have intrinsic definitions valid on any orientable manifold).. Only the \mathbf{v} equation of motion changes, it becomes

$$\frac{\partial \mathbf{v}}{\partial t} + Q \mathbf{F}^T + \nabla B + s \nabla T = 0 \quad (171)$$

where $Q = \frac{\nabla^T \cdot \mathbf{v}}{D}$. This arises from a new $\{\mathcal{A}, \mathcal{B}\}_Q$ bracket, which is

$$\{\mathcal{A}, \mathcal{B}\}_Q = \int_{\Omega} -\frac{\delta \mathcal{A}}{\delta \mathbf{v}} \cdot \left(Q \frac{\delta \mathcal{B}^T}{\delta \mathbf{v}} \right) d\Omega \quad (172)$$

If there is no thermodynamic scalar, then the last term in (170) and (171) is dropped, and there is no equation of motion for s or S . This is equivalent to dropping the $\{\mathcal{A}, \mathcal{B}\}_s$ or $\{\mathcal{A}', \mathcal{B}'\}_s$ bracket.

Rotation Rotation is described by pseudo-scalar Ω , which can be associated with a pseudo-vector $\Omega \hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ points in the (local) vertical direction on the manifold. Then the rotational velocity follows as before, and its (two-dimensional) curl is simply $f = 2\Omega$. Rotation is introduced into the Lagrangian in the same way as before.

B.1.1 Kelvin Circulation Theorem

The Euler-Poincaré equation can be integrated along a curve $\gamma(t)$ as before to get the Kelvin circulation theorem. When there is no thermodynamic scalar, it simplifies to

$$\frac{d}{dt} \oint_{\gamma(t)} \frac{1}{D} \frac{\delta \mathcal{L}}{\delta \mathbf{u}} \cdot d\mathbf{x} = 0 \quad (173)$$

B.1.2 Potential Vorticity

For the case of $n = 2$, potential vorticity is defined as

$$q = \frac{\nabla^T \cdot \mathbf{v}}{D} \quad (174)$$

Note for $n = 2$, we have $q = Q$. The corresponding evolution equations are

$$\frac{\partial(Dq)}{\partial t} + \nabla \cdot (qD \mathbf{u}) - \nabla^T \cdot (T \nabla s) = 0 \quad (175)$$

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q - \frac{1}{D} \nabla^T \cdot (T \nabla s) = 0 \quad (176)$$

Therefore, potential vorticity is only materially conserved if there is no thermodynamic scalar s . However, mass-weighted potential vorticity is still conserved, modulo boundary effects.

B.1.3 Casimirs

The brackets change when going from $n = 3$ to $n = 2$, and also when dropping the thermodynamic scalar. Therefore, each case will have a separate set of Casimirs. These are detailed below.

Without a thermodynamic scalar When $n = 2$ without a thermodynamic scalar, the Casimirs are of the form

$$\mathcal{C}[\mathbf{v}, D] = \int DF(q) \quad (177)$$

where $F(q)$ is an arbitrary function of potential vorticity q . The functional derivatives of $\mathcal{C}[\mathbf{v}, D]$ are

$$\frac{\delta \mathcal{C}}{\delta \mathbf{v}} = -\nabla^T F' \quad \frac{\delta \mathcal{C}}{\delta D} = F - qF' \quad (178)$$

where $F' = \frac{dF}{dq}$. Important cases are $F = 1$ (total mass), $F = q$ (total potential vorticity) and $F = \frac{q^2}{2}$ (potential enstrophy).

With thermodynamic scalar When $n = 2$ with a thermodynamic scalar, the Casimirs are of the form

$$\mathcal{C}[\mathbf{v}, D, S] = \int DqF(s) + DG(s) \quad (179)$$

where $F(s)$ and $G(s)$ are arbitrary functions of the thermodynamic scalar s . The functional derivatives of $\mathcal{C}[\mathbf{v}, D, s]$ are

$$\frac{\delta \mathcal{C}}{\delta \mathbf{v}} = -\nabla^T F' \quad \frac{\delta \mathcal{C}}{\delta D} = G - sQF' - sG' \quad \frac{\delta \mathcal{C}}{\delta S} = QF' + G' \quad (180)$$

where $F' = \frac{dF}{ds}$ and $G' = \frac{dG}{ds}$. Alternatively, if s is predicted instead, we have $\mathcal{C}'[\mathbf{v}, D, s] = \mathcal{C}[\mathbf{v}, D, S]$ and

$$\frac{\delta \mathcal{C}'}{\delta \mathbf{v}} = -\nabla^T F' \quad \frac{\delta \mathcal{C}'}{\delta D} = G \quad \frac{\delta \mathcal{C}'}{\delta s} = DqF' + DG' \quad (181)$$

Important cases are $F = 0, G = 1$ (total mass), $F = 1, G = 0$ (total potential vorticity) and $F = 0, G = s$ (total thermodynamic scalar).

B.2 Split Exterior Calculus

One advantage of using split exterior calculus is that the Lie derivative is dimension-independent. Therefore the curl-form Euler-Poincare equations do not change. Only the PV flux term slightly simplifies, since ${}^{n-2}\tilde{Q}$ is a 0-form. It becomes

$$\mathbf{i}_u {}^2\eta = \tilde{\star}({}^{n-2}\tilde{Q} \wedge {}^1F) = {}^{n-2}\tilde{Q} \wedge {}^{n-1}\tilde{F} \quad (182)$$

In fact, when $n = 2$ then ${}^{n-2}\tilde{Q}$ is the potential vorticity. This is not the case for $n = 3$. This leads to a simplified $\{\mathcal{A}, \mathcal{B}\}_Q$ bracket

$$\{\mathcal{A}, \mathcal{B}\}_Q = -\langle\langle \frac{\delta \mathcal{A}}{\delta {}^1\mathbf{v}}, {}^{n-2}\tilde{Q} \wedge \frac{\delta \mathcal{B}}{\delta {}^1\mathbf{v}} \rangle\rangle. \quad (183)$$

If there is no thermodynamic scalar, then the last term in (46) and (47) is dropped, and there is no evolution equation for ${}^n\tilde{S}$, 0s or ${}^n\tilde{s}$. This is equivalent to dropping the $\{\mathcal{A}, \mathcal{B}\}_S$, $\{\mathcal{A}', \mathcal{B}'\}_s$ or $\{\mathcal{A}'', \mathcal{B}''\}_s$ bracket.

Rotation When $n = 2$ instead of ${}^1\tilde{\Omega}$ we have ${}^0\tilde{\Omega}$ since rotation can be described by a pseudo-scalar. However, this still gives ${}^2\Omega = \tilde{\star}{}^0\tilde{\Omega}$, and $d{}^1R = 2{}^2\Omega$. Rotation is again introduced into the Lagrangian in the same way as before.

B.2.1 Kelvin Circulation Theorem

When there is no thermodynamic scalar, this simplifies to

$$\frac{d}{dt} \oint_{\gamma(t)} \frac{1}{{}^0D} \wedge \frac{\delta \mathcal{L}}{\delta {}^{n-1}\tilde{u}} = 0 \quad (184)$$

B.2.2 Potential Vorticity

When $n = 2$, the potential vorticity differential forms are defined as

$${}^nq = {}^2Q \quad (185)$$

$${}^0\tilde{q} = \tilde{\star}{}^nq = \tilde{\star}{}^2Q = {}^{n-2}\tilde{Q} \quad (186)$$

$${}^0q = {}^0I \wedge {}^0\tilde{q} \quad (187)$$

The corresponding evolution equations are

$$\frac{\partial {}^0D \wedge {}^nq}{\partial t} + L_u({}^0D \wedge {}^nq) - d{}^0T \wedge d{}^0s = 0 \quad (188)$$

$$\frac{\partial {}^0\tilde{q}}{\partial t} + L_u({}^0\tilde{q}) - \frac{1}{{}^0D} \wedge \tilde{\star}(d{}^0T \wedge d{}^0s) = 0 \quad (189)$$

As before, now ${}^0\tilde{q}$ is only materially conserved if there is no thermodynamic scalar and the last term in (188) and (189) drops out.

B.2.3 Casimirs

As before, there are two sets of Casimirs: one for the case of a fluid without a thermodynamic scalar and one for the case with a thermodynamic scalar.

Without thermodynamic scalar The Casimirs $\mathcal{C}[{}^1v, {}^n\tilde{D}]$ when there is not thermodynamic scalar are

$$\mathcal{C}[{}^1v, {}^n\tilde{D}] = \langle {}^0D, F({}^0q) \rangle \quad (190)$$

where $F({}^0q)$ is an arbitrary function of potential vorticity straight 0-form 0q . The functional derivatives of $\mathcal{C}[{}^1v, {}^n\tilde{D}]$ are

$$\frac{\delta \mathcal{C}}{\delta {}^1v} = d({}^0\tilde{I} \wedge F') \quad \frac{\delta \mathcal{C}}{\delta {}^n\tilde{D}} = F - {}^0q \wedge F' \quad (191)$$

where $F' = \frac{dF}{d{}^0q}$. Important cases are $F = 1$ (total mass), $F = {}^0q$ (total potential vorticity) and $F = \frac{{}^0q \wedge {}^0q}{2}$ (potential enstrophy).

With thermodynamic scalar The Casimirs $\mathcal{C}[{}^1\mathbf{v}, {}^n\tilde{\mathbf{D}}, {}^n\tilde{\mathbf{S}}]$ when there is a thermodynamic scalar are

$$\mathcal{C}[{}^1\mathbf{v}, {}^n\tilde{\mathbf{D}}, {}^n\tilde{\mathbf{S}}] = \langle {}^0\mathbf{D} \wedge {}^0\mathbf{q}, F({}^0\mathbf{s}) \rangle + \langle {}^0\mathbf{D}, G({}^0\mathbf{s}) \rangle \quad (192)$$

where $F({}^0\mathbf{s})$ and $G({}^0\mathbf{s})$ are arbitrary functions of ${}^0\mathbf{s}$. The functional derivatives of $\mathcal{C}[{}^1\mathbf{v}, {}^n\tilde{\mathbf{D}}, {}^n\tilde{\mathbf{S}}]$ are

$$\frac{\delta \mathcal{C}}{\delta {}^1\mathbf{v}} = d({}^0\tilde{\mathbf{I}} \wedge F) \quad \frac{\delta \mathcal{C}}{\delta {}^n\tilde{\mathbf{D}}} = G - {}^0\mathbf{s} \wedge {}^0\mathbf{q} \wedge F' - {}^0\mathbf{s} \wedge G' \quad \frac{\delta \mathcal{C}}{\delta {}^n\tilde{\mathbf{S}}} = {}^0\mathbf{q} \wedge F' + G' \quad (193)$$

where $F' = \frac{dF}{d{}^0\mathbf{s}}$ and $G' = \frac{dG}{d{}^0\mathbf{s}}$. If ${}^0\mathbf{s}$ is predicted instead, we have

$$\frac{\delta \mathcal{C}'}{\delta {}^1\mathbf{v}} = d({}^0\tilde{\mathbf{I}} \wedge F) \quad \frac{\delta \mathcal{C}'}{\delta {}^n\tilde{\mathbf{D}}} = G \quad \frac{\delta \mathcal{C}'}{\delta {}^0\mathbf{s}} = {}^n\tilde{\mathbf{D}} \wedge {}^0\mathbf{q} \wedge F' + {}^n\tilde{\mathbf{D}} \wedge G' \quad (194)$$

Finally, if ${}^n\tilde{\mathbf{s}}$ is predicted, we have

$$\frac{\delta \mathcal{C}''}{\delta {}^1\mathbf{v}} = d({}^0\tilde{\mathbf{I}} \wedge F) \quad \frac{\delta \mathcal{C}''}{\delta {}^n\tilde{\mathbf{D}}} = G \quad \frac{\delta \mathcal{C}''}{\delta {}^n\tilde{\mathbf{S}}} = \tilde{\star}({}^n\tilde{\mathbf{D}} \wedge {}^0\mathbf{q} \wedge F' + {}^n\tilde{\mathbf{D}} \wedge G') = {}^0\mathbf{D} \wedge {}^0\mathbf{q} \wedge F' + {}^0\mathbf{D} \wedge G' \quad (195)$$

Important cases are $F = 0, G = 1$ (total mass), $F = 1, G = 0$ (total potential vorticity) and $F = 0, G = {}^0\mathbf{s}$ (total thermodynamic scalar).

C Proof of the properties of the topological pairing

Here is the proof of Proposition 7.

Proof: The proof to claim 1.) of Proposition 7 for both topological pairings follows simply by the definitions of straight and twisted DF, of the wedge product, and of an integral over DF as metric-free (topological) operations.

In what follows, we only present the proof for pairing (7), but the proof to (8) is equivalent. The first part of statement 2.) follows immediately when we substitute in (7) the metric equation $\tilde{\star} {}^k\beta$ for the $(n-k)$ -form ${}^{(n-k)}\tilde{\beta}$, because this gives immediately the inner product from Definition 5

The relationship (9) for the coefficient functions can be seen when applying $\tilde{\star}$ on the local representation of ${}^k\beta = \beta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$. To this end, we first state a result of Abraham and Marsden CITE:

$$\tilde{\star} (dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \sqrt{|\mathbf{g}|} \sum_{j_1, \dots, j_k} \sum_{j_{k+1}, \dots, j_n} \text{sign} \left(\begin{matrix} 1 \dots n \\ j_1 \dots j_n \end{matrix} \right) g^{i_1 j_1} \dots g^{i_k j_k} dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n},$$

in which the sum is over all shuffles $j_1, \dots, j_k, j_{k+1}, \dots, j_n$ with, in *Or*, positive sign for even and negative sign for odd permutations with respect to the initial ordering of $1, \dots, n$. In *-Or*, we have the inverse sign convention.

Next, we apply $\tilde{\star}$ on the k -form ${}^k\beta = \sum_{i_1 < \dots < i_k} \beta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ to obtain the $(n-k)$ -form:

$$\begin{aligned} \tilde{\star} {}^k\beta &= \sum_{i_1 < \dots < i_k} \beta_{i_1 \dots i_k} \tilde{\star} (dx^{i_1} \wedge \dots \wedge dx^{i_k}) \\ &= \sum_{i_1 < \dots < i_k} \beta_{i_1 \dots i_k} \sqrt{|\mathbf{g}|} \sum_{j_1, \dots, j_k} \sum_{j_{k+1}, \dots, j_n} \text{sign} \left(\begin{matrix} 1 \dots n \\ j_1 \dots j_n \end{matrix} \right) g^{i_1 j_1} \dots g^{i_k j_k} dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n}. \end{aligned} \quad (196)$$

\mathbf{g} is symmetric, hence $g^{i_l j_l} = g^{j_l i_l} \forall l \in \{1, \dots, k\}$. Therefore, we can sum over all i_1, \dots, i_k while imposing the ordering on the indices j_l such that $j_1 < \dots < j_k$, which imposes, in turn, the ordering $j_{k+1} < \dots < j_n$ on the complementary indices. Then,

$$\begin{aligned} \tilde{\star}^k \beta &= \sum_{j_1 < \dots < j_k} \sum_{i_1, \dots, i_k} \beta_{i_1 \dots i_k} g^{j_1 i_1} \dots g^{j_k i_k} \sqrt{|\mathbf{g}|} \text{sign} \left(\begin{matrix} 1 \dots n \\ j_1 \dots j_n \end{matrix} \right) dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n} \\ &= \sum_{j_1 < \dots < j_k} \beta^{j_1 \dots j_k} \sqrt{|\mathbf{g}|} \text{sign} \left(\begin{matrix} 1 \dots n \\ j_1 \dots j_n \end{matrix} \right) dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n}. \end{aligned} \quad (197)$$

using the contravariant coefficients $\beta^{j_1 \dots j_k}$ according to (2). This allows us to find the coordinate function $(\tilde{\star}^k \beta)_{j_{k+1} \dots j_n}$ of the $(n-k)$ form $\tilde{\star}^k \beta = (\tilde{\star}^k \beta)_{j_{k+1} \dots j_n} dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n}$ by the formula

$$(\tilde{\star}^k \beta)_{j_{k+1} \dots j_n} = \sum_{j_1 < \dots < j_k} \beta^{j_1 \dots j_k} \sqrt{|\mathbf{g}|} \text{sign} \left(\begin{matrix} 1 \dots n \\ j_1 \dots j_n \end{matrix} \right). \quad (198)$$

i.e. the coefficient function for $j_{k+1} \dots j_n$ is given by the sum over all complementary indices $j_1 < \dots < j_k$.

With this, the topological pairing of ${}^k \alpha = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ with $\tilde{\star}^k \beta$ gives

$$\begin{aligned} \langle \langle \alpha, \tilde{\star}^k \beta \rangle \rangle &= \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \sum_{j_1 < \dots < j_k} \beta^{j_1 \dots j_k} \sqrt{|\mathbf{g}|} \text{sign} \left(\begin{matrix} 1 \dots n \\ j_1 \dots j_n \end{matrix} \right) dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge (dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n}) \\ &= \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \beta^{i_1 \dots i_k} \int_{\mathcal{M}} {}^n \tilde{\mu} \end{aligned}$$

because only terms for which $i_l = j_l \forall l \in \{1, \dots, n\}$ are nonzero and because reordering dx_k^i in consecutive order, i.e. $1 < \dots < n$, gives ${}^n \mu = \sqrt{|\mathbf{g}|} (+1) dx^1 \wedge \dots \wedge dx^n$ in Or and $-{}^n \mu$ in $-Or$, hence ${}^n \tilde{\mu}$.

Finally, statement 3.) follows immediately from statement 2.). \square

Example 16 We consider the topological pairing of the twisted 1-form ${}^1 \tilde{\alpha} = \tilde{\alpha}_1 dx^1 + \tilde{\alpha}_2 dx^2 + \tilde{\alpha}_3 dx^3$ and the straight 2-form ${}^2 \beta = \beta_{12} dx^1 \wedge dx^2 + \beta_{23} dx^2 \wedge dx^3 + \beta_{31} dx^3 \wedge dx^1$ in a 3D Euclidean space ($g_{ij} = g^{ij} = \delta_{ij}, i, j = 1, 2, 3$, with $\sqrt{|\mathbf{g}|} = 1$). Then,

$$\begin{aligned} \langle \langle {}^1 \tilde{\alpha}, {}^2 \beta \rangle \rangle &= (\tilde{\alpha}_1 dx^1 + \tilde{\alpha}_2 dx^2 + \tilde{\alpha}_3 dx^3) \wedge (\beta_{12} dx^1 \wedge dx^2 + \beta_{31} dx^3 \wedge dx^1 + \beta_{23} dx^2 \wedge dx^3) \\ &= (\tilde{\alpha}_1 \beta_{23} + \tilde{\alpha}_2 \beta_{31} + \tilde{\alpha}_3 \beta_{12}) {}^3 \tilde{\mu} \end{aligned} \quad (199)$$

with ${}^3 \tilde{\mu} = (\{+dx^1 \wedge dx^2 \wedge dx^3, Or\}, \{-dx^1 \wedge dx^2 \wedge dx^3, -Or\})$. No notion of metric is involved in this pairing.

If we assume the metric equations to hold, i.e. $\tilde{\star}^2 \beta = {}^1 \tilde{\beta}$ for ${}^1 \tilde{\beta} = \sum_{i=1}^3 \tilde{\beta}_i dx^i$, the relations between the coefficients are given by (198):

$$\beta_{23} = \sum_{j_1=1} \tilde{\beta}_1 g^{11} \sqrt{|\mathbf{g}|} \text{sign} \left(\begin{matrix} 1 & 2 & 3 \\ j_1=1 & j_2=2 & j_3=3 \end{matrix} \right) = \tilde{\beta}_1,$$

$$\beta_{13} = \sum_{j_1=2} \tilde{\beta}_2 g^{22} \sqrt{|\mathbf{g}|} \operatorname{sign} \left(\begin{array}{ccc} 1 & 2 & 3 \\ j_1 = 2 & j_2 = 1 & j_n = 3 \end{array} \right) = -\tilde{\beta}_2,$$

$$\beta_{12} = \sum_{j_1=3} \tilde{\beta}_3 g^{33} \sqrt{|\mathbf{g}|} \operatorname{sign} \left(\begin{array}{ccc} 1 & 2 & 3 \\ j_1 = 3 & j_2 = 1 & j_n = 2 \end{array} \right) = \tilde{\beta}_3.$$

Then,

$$\langle {}^1\tilde{\alpha}, {}^1\tilde{\beta} \rangle = \left(\tilde{\alpha}_1 \tilde{\beta}_1 + \tilde{\alpha}_2 \tilde{\beta}_2 + \tilde{\alpha}_3 \tilde{\beta}_3 \right) {}^3\tilde{\mu} = (\tilde{\alpha}_1 \beta_{23} - \tilde{\alpha}_2 \beta_{13} + \tilde{\alpha}_3 \beta_{12}) {}^3\tilde{\mu} = \langle \langle {}^1\tilde{\alpha}, {}^2\beta \rangle \rangle$$

in which the last equality holds because $\beta_{13} = -\beta_{31}$. Hence, the topological pairing $\langle \langle, \rangle \rangle$ takes the role of the inner product \langle, \rangle in case the metric equation $\star^2 \beta = {}^1\tilde{\beta}$ holds.