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Fast Approximation of EEG Forward Problem and Application to Tissue Conductivity Estimation

Kostiantyn Maksymenko, Maureen Clerc, and Théodore Papadopoulo

Abstract—Bioelectric source analysis in the human brain from scalp electroencephalography (EEG) signals is sensitive to the conductivity of the different head tissues. Conductivity values are subject dependent, so non-invasive methods for conductivity estimation are necessary to fine tune the EEG models. To do so, the EEG forward problem solution (so-called lead field matrix) must be computed for a large number of conductivity configurations.

Computing one lead field requires a matrix inversion which is computationally intensive for realistic head models. Thus, the required time for computing a large number of lead fields can become impractical. In this work, we propose to approximate the lead field matrix for a set of conductivity configurations, using the exact solution only for a small set of basis points in the conductivity space. Our approach accelerates the computing time, while controlling the approximation error.

Our method is tested for brain and skull conductivity estimation, with simulated and measured EEG data, corresponding to evoked somato-sensory potentials. This test demonstrates that the used approximation does not introduce any bias and runs significantly faster than if exact lead field were to be computed.

Index Terms—EEG forward problem, EEG inverse problem, conductivity estimation, lead field matrix approximation.

I. INTRODUCTION

THE inverse problem of source reconstruction in electroencephalography (EEG) aims at finding the source distribution that best explains the electric potentials measured noninvasively from electrodes on the scalp surface. A model of the head's electromagnetic transmission plays a central role in accurate source localization. This model is built from a specification of both the geometry and the conductivity distribution of the modeled tissue compartments (scalp, skull, cerebrospinal fluid, brain gray and white matter, etc.). Because it is impractical to directly measure the head tissues conductivities in vivo for a specific subject, default values are often used. One problem is that the brain-to-skull conductivity ratio reported in the literature varies from 4 to 80 [1]–[3]. As the skull conductivity greatly influences the solution of the forward problem [4], localizing brain sources using uncertain conductivity values leads to important errors [5]–[8]. Taking into account the composite structure of the human skull could improve the accuracy of EEG source analysis [9]. In this case, the conductivity of each skull tissue should be estimated simultaneously. A possible solution is to estimate the tissue

conductivities and cortical activity simultaneously [10]–[13]. Doing this requires to solve EEG forward problem for possibly many different conductivity configurations. A lead field is a linear operator relating the brain electrical activity to the potentials on EEG electrodes. Computing one lead field requires a matrix inversion which is computationally intensive for realistic head geometry represented with meshes with a large number of vertices. Thus, the required time for computing a large number of solutions for different conductivities quickly becomes impractical.

One way to deal with this problem is to approximate lead fields using a relatively small set of precomputed solutions. The reduced basis method approximates the solution of parametrized PDEs [14] by reducing the number of basis functions for Galerkin projection based on a set of already computed exact solutions. Our method is inspired by the reduced basis method, but adapted to the particular structure of EEG forward problem and the nature of its parameter space. In an another method presented in [10], only the skull conductivity varies and each element of the lead field matrix is approximated using polynomial interpolation based on a set of precomputed values. With this approach, the complexity of the approximation would increase fast with the number of unknown conductivities.

In this work, we propose a fast lead field approximation method which is robust to the head model complexity and to the number of unknown conductivities.

This work is organized as follows. In section II, we recall the EEG forward problem and two numerical methods to solve it. In section III, we present our fast lead field approximation method. Finally, we evaluate the performance of our algorithm on simulated and measured EEG data in section IV.

II. FORWARD PROBLEM

The useful frequency range for electrophysiological signals in MEG and EEG is typically below 1 kHz, and most studies deal with frequencies between 0.1 and 100 Hz. Consequently, the physics of MEG and EEG can be described by the quasi-static approximation of Maxwell's equations [15]. In a conductive environment, it yields the fundamental Poisson equation with boundary condition :

$$\begin{cases} \nabla \cdot (\sigma \nabla V) = \nabla \cdot J & \text{in } \Omega \\ \sigma \frac{\partial V}{\partial \mathbf{n}} = \sigma \nabla V \cdot \mathbf{n} = 0 & \text{on } \partial \Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^3$ is the head domain, $\partial \Omega$ its boundary with outward pointing normal vector \mathbf{n} , $\sigma [S/m]$ is the conductivity,

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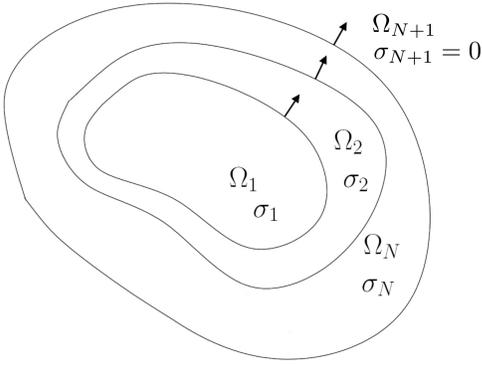


Fig. 1: Without loss of generality the head is modeled as a set of piecewise constant regions $\Omega_1, \dots, \Omega_N$ with constant isotropic conductivities $\sigma_1, \dots, \sigma_N$. From [16].

$J[A/m^2]$ is the primary current source density, supposed to be known in the forward problem, $V [V]$ is the unknown electric potential. It is usually assumed that the head is composed of several subdomains Ω_n with different electrical conductivities σ_n . We assume a piecewise constant head model (Fig. 1). Conductivity is supposed isotropic and constant in each domain Ω_N . The different tissue conductivities are denoted $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$.

A. Numerical approximation of the forward problem.

Equation (1) has no analytic solution for realistic (nonspherical) head geometries but several numerical methods can be used to approximate its solution. We will focus on the finite element method (FEM) [17], [18] and the boundary element method (BEM) [16], [19], which are both based on Galerkin projection. The method presented in this paper applies both to FEM and BEM but will be illustrated on the symmetric BEM [16].

For both the FEM and BEM approaches, the initial continuous problem (1) is discretized to a linear system of type:

$$H\mathbf{v} = \mathbf{d}. \quad (2)$$

Let N_V be the number of unknowns in the head model. Vector $\mathbf{v} \in \mathbb{R}^{N_V}$ represents unknowns of the model, which are values of the potential on the mesh nodes for FEM and standard BEM, and potentials and their normal derivatives on the meshes for symmetric BEM. The $N_V \times N_V$ matrix H , called "head matrix", can be computed once the head geometry, its conductivity values and finite element basis functions are fixed. Vector \mathbf{d} depends on source configuration. Notice that $H = H_{\boldsymbol{\sigma}}$ depends on conductivity. But an important property of matrix $H_{\boldsymbol{\sigma}}$ is that it can be represented as a linear combination of conductivity independent matrices:

$$H_{\boldsymbol{\sigma}} = \sum_{i=1}^{N_H} \gamma_i(\boldsymbol{\sigma}) \bar{H}_i, \quad (3)$$

where $\gamma_i(\boldsymbol{\sigma})$ are scalar functions, N_H represents the number of such conductivity-independent components \bar{H}_i . For FEM $\gamma_i(\boldsymbol{\sigma}) = \sigma_i$. For symmetric BEM multipliers $\gamma_i(\boldsymbol{\sigma})$ have more

complex structure, for example $\{-\sigma_i, \sigma_i^{-1}, \sigma_i + \sigma_j, \sigma_i^{-1} + \sigma_j^{-1}, \dots\}$.

B. Source model

The most common source model to represent electrical activity in the brain is a "current dipole". It represents an oriented source of current located at a single position \mathbf{r}_0 , with dipolar moment \mathbf{q} , and it is denoted by $J(\mathbf{r}) = \mathbf{q}\delta(\mathbf{r} - \mathbf{r}_0)$, where $\delta(\cdot)$ is a Dirac delta distribution.

The source space can be seen as a finite set of N_S dipoles with known positions. Without loss of generality, we can also assume that the dipole orientation is known. Let x_i represent the amplitude of i -th source and \bar{J}_i the i -th source with unit amplitude:

$$J(\mathbf{r}) = \sum_{i=1}^{N_S} x_i \bar{J}_i(\mathbf{r}).$$

The source term \mathbf{d} in (2) is linear with respect to J , therefore:

$$\mathbf{d}(J) = \sum_{i=1}^{N_S} x_i \mathbf{d}(\bar{J}_i(\mathbf{r})) = \sum_{i=1}^{N_S} x_i \mathbf{d}_i = D\mathbf{x},$$

where i -th column of $N_V \times N_S$ matrix D corresponds to the i -th unit source and $\mathbf{x} = (x_1, \dots, x_{N_S})$ is the vector of source amplitudes. Let us notice that in the case of FEM, matrix D does not depend on conductivities. For BEM, it does, so we will note it $D_{\boldsymbol{\sigma}}$ for generality. Moreover, $D_{\boldsymbol{\sigma}}$ can be represented as follows:

$$D_{\boldsymbol{\sigma}} = \sum_{i=1}^{N_D} \lambda_i(\boldsymbol{\sigma}) \bar{D}_i, \quad (4)$$

where matrices \bar{D}_i are independent of $\boldsymbol{\sigma}$ and $\lambda_i(\boldsymbol{\sigma})$ are scalars. In the case of symmetric BEM $\lambda_i(\boldsymbol{\sigma})$ are $\{1, \sigma_i, \dots\}$.

The linear system to solve then becomes:

$$H_{\boldsymbol{\sigma}}\mathbf{v} = D_{\boldsymbol{\sigma}}\mathbf{x}. \quad (5)$$

C. Lead field matrix

In the context of EEG, the *lead field* is a linear operator that maps source activation to potentials at sensor locations:

$$\mathbf{v}_{eeg} = L\mathbf{x}.$$

Computing \mathbf{v}_{eeg} requires applying to V a matrix S which selects or interpolates potentials only at electrode positions: $\mathbf{v}_{eeg} = S\mathbf{v}$. Let N_E be the number of EEG electrodes.

Using (5) and the selection matrix S :

$$\mathbf{v}_{eeg} = SH_{\boldsymbol{\sigma}}^{-1}D_{\boldsymbol{\sigma}}\mathbf{x}.$$

Let us remark that since the electric potential is only defined up to a constant, head matrix H is not full rank and has a one-dimensional kernel. So the inverse notation H^{-1} actually implies a deflation which is usually applied to this type of situation.

The $N_E \times N_S$ lead field matrix can thus be computed as:

$$L = SH_{\boldsymbol{\sigma}}^{-1}D_{\boldsymbol{\sigma}}. \quad (6)$$

Each column of the lead field matrix represents the contribution of the corresponding unit norm source on the EEG electrodes.

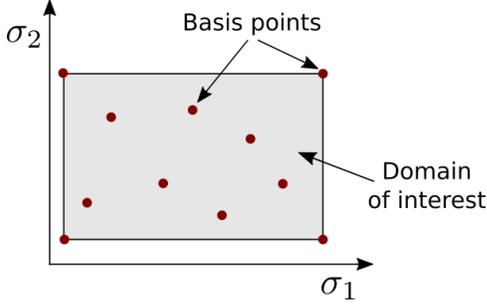


Fig. 2: Schematic representation of the domain of interest and basis points in conductivity space.

III. FAST LEAD FIELD APPROXIMATION METHOD

Once $\bar{H}_i, i = 1 \dots N_H$ and $\bar{D}_i, i = 1 \dots N_D$ have been computed, the numerical complexity of lead field computation is essentially due to the inversion of the head matrix H (see (6)). For realistic head models implying a large number of unknowns the time required for computing lead fields for a large set of conductivity configurations becomes impractical. In this context, it can be worthwhile to compute not the *exact* lead fields for all needed conductivity configurations, but an *approximation* thereof.

The general idea of our method is the following. First, we choose a domain of interest in conductivity space, i.e. a range of values for the head tissue conductivities. We then select *basis points* within this domain, for which *exact* forward solutions will be computed (Fig. 2). Based on these exact solutions a lead field *approximation* will be computed for any other conductivity configuration.

In this section, we develop an algorithm which:

- selects basis points within the domain of interest;
- based on these basis points, computes a fast approximation of the lead field matrix with controlled error.

A. Approximation problem formulation

Assume that the approximation of the lead field matrix $L(\sigma)$ is parametrized with a vector $\alpha \in \mathbb{R}^n$ where n is the number of basis points: We denote it $L_n(\alpha, \sigma)$. As the lead field is a linear operator, we consider the following approximation error, using the Frobenius matrix norm $\|L\|_F = (\sum l_{ij}^2)^{\frac{1}{2}}$:

$$E(L(\sigma), L_n(\alpha, \sigma)) = \frac{\|L(\sigma) - L_n(\alpha, \sigma)\|_F}{\|L(\sigma)\|_F}. \quad (7)$$

This error involves matrix L which is not known a priori. We therefore introduce the definition of an upper bound for the approximation error. The idea is to bound error E , that we can not directly compute, with an error function E_n , which can be easily computed, converges to zero with increasing number of basis points and thus ensures the convergence of E to zero.

Definition 1: For $n \in \mathbb{N}$, function $E_n(\alpha, \sigma)$ is an *upper bound approximation* of $E(L(\sigma), L_n(\alpha, \sigma))$ if:

$$\exists C > 0 \text{ such that } \forall \alpha \in \mathbb{R}^n, \\ E(L(\sigma), L_n(\alpha, \sigma)) \leq C \cdot E_n(\alpha, \sigma).$$

Let us remark that $C > 0$ is a constant which does not depend on α but may depend on σ .

Taking into account that E is positive, it can be directly derived from definition 1 that if $E_n \xrightarrow[n \rightarrow \infty]{} 0$ then $E(L, L_n) \xrightarrow[n \rightarrow \infty]{} 0$.

Definition 2: Matrix $L_n(\alpha_n^*(\sigma), \sigma)$ is a *valid approximation* of $L(\sigma)$ with *optimal approximation parameter* $\alpha_n^*(\sigma)$ if both following conditions hold:

- $\alpha_n^*(\sigma) = \operatorname{argmin}_{\alpha} E_n(\alpha, \sigma)$,
- $E_n(\alpha_n^*(\sigma), \sigma) \xrightarrow[n \rightarrow \infty]{} 0$.

B. Choice of the upper bound approximation

As mentioned previously, the point of the approximation is to avoid inversion of the matrix H_{σ} . One possible way to parametrize the approximation matrix $L_n(\alpha, \sigma)$ is to represent the inverse of the head matrix H_{σ}^{-1} as a linear combination of precomputed inversions $H_{\sigma_i}^{-1}$ at n basis points:

$$H_{\sigma}^{-1} \approx \sum_{i=1}^n \alpha_i H_{\sigma_i}^{-1}. \quad (8)$$

Based on (6), the proposed lead field approximation takes the form:

$$L_n(\alpha, \sigma) = \sum_{i=1}^n \alpha_i S H_{\sigma_i}^{-1} D_{\sigma}.$$

Using the linear decomposition of D_{σ} (4), we represent the lead field approximation as a linear combination of matrices which are independent of conductivity:

$$L_n(\alpha, \sigma) = \sum_{i=1}^n \sum_{j=1}^{N_D} \alpha_i \lambda_j(\sigma) S H_{\sigma_i}^{-1} \bar{D}_j = \\ \sum_{i=1}^n \sum_{j=1}^{N_D} \beta_{ij}(\sigma, \alpha) \bar{L}_{ij}, \quad (9)$$

where $\bar{L}_{ij} = S H_{\sigma_i}^{-1} \bar{D}_j$ and $\beta_{ij}(\sigma, \alpha) = \alpha_i \lambda_j(\sigma)$.

As λ_j are known, the question is how to compute optimal parameters α . Based on the property of an inverse matrix: $H_{\sigma}^{-1} \cdot H_{\sigma} = I$ and on (8), we could seek α by minimizing the following expression:

$$\alpha_n^*(\sigma) = \operatorname{argmin}_{\alpha} \left\| I - \sum_{i=1}^n \alpha_i H_{\sigma_i}^{-1} H_{\sigma} \right\|_F.$$

where I is the identity matrix. But in the context of the EEG forward problem, it is more relevant to consider matrix $S H_{\sigma}^{-1}$, which has smaller dimension than H_{σ}^{-1} and which contains only the relevant part of H_{σ}^{-1} . Let us introduce following optimization problem:

$$\alpha_n^*(\sigma) = \operatorname{argmin}_{\alpha} E_n(\alpha, \sigma) = \\ \operatorname{argmin}_{\alpha} \left\| S - \sum_{i=1}^n \alpha_i S H_{\sigma_i}^{-1} H_{\sigma} \right\|_F. \quad (10)$$

The function $E_n(\alpha, \sigma)$ is a candidate to be an upper bound approximation. Proposition 1 shows that it is indeed the case under a certain condition.

Proposition 1: If matrices $\{SH_{\sigma_i}^{-1}H_{\sigma}\}_{i=1}^n$ are linearly independent, then

$E_n(\alpha, \sigma) = \|S - \sum \alpha_i SH_{\sigma_i}^{-1}H_{\sigma}\|_F$ is an upper bound approximation of error $E(L, L_n)$ (7) and $\alpha_n^*(\sigma)$ is an optimal approximation parameter.

Proof:

$$\begin{aligned} \|L(\sigma) - L_n(\alpha, \sigma)\|_F &= \\ \left\| SH_{\sigma}^{-1}D_{\sigma} - \sum_i \alpha_i SH_{\sigma_i}^{-1}D_{\sigma} \right\|_F &= \\ \left\| \left(S - \sum_i \alpha_i SH_{\sigma_i}^{-1}H_{\sigma} \right) H_{\sigma}^{-1}D_{\sigma} \right\|_F &\leq \\ \left\| S - \sum_i \alpha_i SH_{\sigma_i}^{-1}H_{\sigma} \right\|_F \|H_{\sigma}^{-1}D_{\sigma}\|_F &= \\ \|H_{\sigma}^{-1}D_{\sigma}\|_F E_n(\alpha, \sigma). \end{aligned}$$

Now we can see that E_n is an upper bound of error (7):

$$E(L, L_n) = \frac{\|L - L_n\|_F}{\|L\|_F} \leq \frac{\|H_{\sigma}^{-1}D_{\sigma}\|_F}{\|L\|_F} E_n(\alpha, \sigma) = C \cdot E_n(\alpha, \sigma).$$

Let $v(M)$ represent the vectorization of matrix M . Using the fact that $\|M\|_F = \|v(M)\|_2$, problem (10) can be reformulated as:

$$\alpha_n^*(\sigma) = \underset{\alpha}{\operatorname{argmin}} \left\| S - \sum_i \alpha_i SH_{\sigma_i}^{-1}H_{\sigma} \right\|_F = \underset{\alpha}{\operatorname{argmin}} \left\| v(S) - \sum_i \alpha_i v(SH_{\sigma_i}^{-1}H_{\sigma}) \right\|_2,$$

which is a linear projection problem. It means that if $\{v(SH_{\sigma_i}^{-1}H_{\sigma})\}_{i=1}^n$ is a set of linearly independent vectors for $n = \dim(v(S))$, then $E_n(\alpha_n^*(\sigma), \sigma) = 0$. This directly implies that $E_n(\alpha_n^*(\sigma), \sigma) \xrightarrow{n \rightarrow \infty} 0$ so both conditions of Definition 2 are verified. ■

A linear dependence of matrices $\{SH_{\sigma_i}^{-1}H_{\sigma}\}_{i=1}^n$ would mean that lead fields corresponding to some basis points could be represented as a linear combination of other precomputed lead fields. So the linear independence condition of Proposition 1 depends on selection of basis points. This will be discussed in detail in the next section.

We now express the problem (10) as a simple least squares problem. Using the linear decomposition of H_{σ} as a sum of conductivity-independent matrices (3), we get:

$$SH_{\sigma_i}^{-1}H_{\sigma} = \sum_{j=1}^{N_H} \gamma_j(\sigma) SH_{\sigma_i}^{-1}\bar{H}_j.$$

We can then vectorize matrices $SH_{\sigma_i}^{-1}\bar{H}_j$ and "store" them as columns of a new matrix K . Let $K_{\cdot, (i-1)N_H+j} = v(SH_{\sigma_i}^{-1}\bar{H}_j)$ denote the $((i-1)N_H+j)$ -th column of K . The matrix K will thus have $n \cdot N_H$ columns and $N_E \cdot N_V$ rows.

Let us also denote $\Gamma_{\sigma} = I \otimes \gamma(\sigma)$, where $\gamma(\sigma) = (\gamma_1(\sigma), \dots, \gamma_{N_H}(\sigma))$ and I is an identity matrix of size n :

$$\Gamma_{\sigma} = \begin{bmatrix} \gamma_1(\sigma) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N_H}(\sigma) & 0 & \dots & 0 \\ 0 & \gamma_1(\sigma) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_{N_H}(\sigma) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_1(\sigma) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{N_H}(\sigma) \end{bmatrix}$$

The following proposition allows to compute $\alpha_n^*(\sigma)$:

Proposition 2: The solution $\alpha_n^*(\sigma)$ of problem (10) is given by the solution of the system:

$$(\Gamma_{\sigma}^T K^T K \Gamma_{\sigma}) \alpha_n^*(\sigma) = \Gamma_{\sigma}^T K^T v(S). \quad (11)$$

Proof: Using the decomposition of matrix H_{σ} (3), we get:

$$\begin{aligned} \sum_{i=1}^n \alpha_i SH_{\sigma_i}^{-1}H_{\sigma} &= \sum_{i=1}^n \alpha_i SH_{\sigma_i}^{-1} \left(\sum_{j=1}^{N_H} \gamma_j(\sigma) \bar{H}_j \right) = \\ &= \sum_{i=1}^n \alpha_i \sum_{j=1}^{N_H} \gamma_j(\sigma) SH_{\sigma_i}^{-1} \bar{H}_j. \end{aligned}$$

Thus (10) becomes:

$$\begin{aligned} \alpha_n^*(\sigma) &= \underset{\alpha}{\operatorname{argmin}} \left\| S - \sum_i \alpha_i SH_{\sigma_i}^{-1}H_{\sigma} \right\|_F = \\ \underset{\alpha}{\operatorname{argmin}} \left\| v(S) - \sum_{i=1}^n \alpha_i \sum_{j=1}^{N_H} \gamma_j(\sigma) v(SH_{\sigma_i}^{-1}\bar{H}_j) \right\|_2 &= \\ \underset{\alpha}{\operatorname{argmin}} \left\| v(S) - \sum_{i=1}^n \alpha_i \sum_{j=1}^{N_H} \gamma_j(\sigma) K_{\cdot, (i-1)N_H+j} \right\|_2 &= \\ \underset{\alpha}{\operatorname{argmin}} \|v(S) - K \Gamma_{\sigma} \alpha\|_2. \end{aligned}$$

The solution of this well known projection problem is given by (11). ■

Remarks:

- 1) The size of matrix $\Gamma_{\sigma}^T K^T K \Gamma_{\sigma}$ is $n \times n$. The complexity of problem (11) therefore does not depend on the size of the head model but only on the number of basis points.
- 2) Matrices $K^T K$ and $K^T v(S)$ are independent of σ so they can be precomputed. Moreover, adding new basis point amounts to adding new elements to these matrices - they do not have to be fully recomputed.

To summarize, given a set of n basis points, we use (11) to compute the optimal approximation parameter $\alpha_n^*(\sigma)$ and then use (9) to compute lead field approximation $L_n(\alpha_n^*(\sigma), \sigma)$.

C. Choice of basis points

The method introduced in section III-B supposes to dispose of a set of basis points with precomputed matrices $K^T K$ and $K^T v(s)$. This section answers the question of how to select them. To work with not continuous but discretized domain of interest, we start by sampling it. We denote Σ the set of basis points in conductivity space, which we denote σ_i^* . The idea is to start with a small number of basis points and to add new points in an iterative manner. We propose Algorithm 1.

Algorithm 1 Basis points selection

- 1: Sample the domain of interest with M points, $\{\sigma_m\}_{m=1}^M$
 - 2: Select initial p basis points. $n = p$ \triangleright Start with p arbitrary chosen basis points
 - 3: $\Sigma = \{\sigma_1^*, \dots, \sigma_p^*\}$
 - 4: Compute matrices $K^T K$ and $K^T v(S)$
 - 5: **repeat**
 - 6: $n \leftarrow n + 1$
 - 7: **for** $m = 1$ to M **do**
 - 8: Compute $\alpha_n^*(\sigma_m)$ with (11)
 - 9: Compute $E_n(\alpha_n^*(\sigma_m), \sigma_m)$
 - 10: $\sigma_n^* \leftarrow \operatorname{argmax}_{\sigma_m} E_n(\alpha_n^*(\sigma_m), \sigma_m)$ \triangleright σ_n^* is a new basis point to add
 - 11: $\Sigma \leftarrow \Sigma \cup \sigma_n^*$
 - 12: Update matrices $K^T K$ and $K^T v(S)$
 - 13: **until** convergence \triangleright See Section IV-B for discussion of convergence criteria
-

Algorithm 1 starts by sampling the domain of interest and choosing p initial basis points from these samples. One possible choice is to use a dense uniform grid for sampling and to take boundary vertices for initial basis points. Then, for each sample, we compute a valid upper bound error E_n . We choose the sample which results in highest upper bound error as a new basis point and update matrices $K^T K$ and $K^T v(S)$ for this conductivity configuration. This means that the new basis point is the conductivity sample with the highest error when approximated with the current basis. We iterate this procedure until we reach some convergence criterion. Numerical results show (see Section IV-B) that the valid upper bound error decreases with the same speed as the approximation error when increasing the number of basis points. Controlling the upper bound error is a reasonable empirical choice for convergence criterion.

This algorithm has interesting properties which make it more efficient than just a random selection of basis points.

Algorithm 1 controls, in a strong way, the valid upper bound error:

$$\forall i \ E_{n+1}(\alpha_{n+1}^*(\sigma_i), \sigma_i) \leq E_n(\alpha_n^*(\sigma_i), \sigma_i) .$$

This is true because the projection space at step n is a subspace of the one at step $n + 1$.

To ensure that the system 11 has a unique solution we need the matrix $K\Gamma_\sigma$ to be full rank. Let us analyze two possible situations:

- 1) Let Algorithm 1 verify $\max_{\sigma_m} E_n(\alpha_n^*(\sigma_m), \sigma_m) > 0 \ \forall n \leq M$. It means that each iteration adds a basis

point which has strictly positive projection error, which means that it is linearly independent of current basis. This implies that the updated basis stays full rank.

- 2) If $\exists n < M$ such that $\max_{\sigma_m} E_n(\alpha_n^*(\sigma_m), \sigma_m) = 0$, it means that lead fields for every sample from conductivity space can be *exactly* represented by linear combination of basis elements. In practice, this naturally means that we don't need to continue adding basis points, unless we introduce new conductivity samples.

All these observations show that Algorithm 1 guarantees that we will verify necessary condition for Proposition 1 and that the system 11 has a unique solution.

Remark: The fact that projection matrix $K\Gamma_\sigma$ is always full rank, ensures that matrix $\Gamma_\sigma^T K^T K \Gamma_\sigma$ is always invertible, thus $\alpha_n^*(\sigma)$ is continuous with respect to conductivity. So is the upper bound approximation $E_n(\alpha_n^*(\sigma), \sigma)$. This guarantees that this error is bounded on any compact conductivity domain. And we still need the condition $\sigma > 0$ to ensure the existence of the matrix H_σ for symmetric BEM.

Once the basis points are chosen and intermediate matrices are computed, a new lead field matrix corresponding to any conductivity σ in the domain can be approximated with Algorithm 2.

Algorithm 2 Lead field approximation

- 1: For given σ , $K^T K$, $K^T S$, $\{\bar{L}_{ij}\}$:
 - 2: Compute $\alpha_n^*(\sigma)$ with (11) \triangleright Optimal approximation parameter
 - 3: Compute $L_n(\alpha_n^*(\sigma), \sigma)$ with (9) \triangleright Lead field approximation
-

IV. NUMERICAL RESULTS

A. Data description

For numerical implementation of the proposed method, we use the data from [20], which includes anatomical data as well as EEG data recorded with Yokogawa/KIT, and processed with Brainstorm [21]. We use a realistic head model with three layers - brain, skull and scalp. Each surface is represented as a triangular mesh with 1082 vertices. The source space contains 15002 dipoles with fixed orientations (normal to surface).

We use the symmetric BEM implementation from Open-MEEG [16], [22]. The size of head matrix H is 7566×7566 . Matrix D has size 7566×15002 .

We use a model with 41 EEG electrodes, so the size of the selection matrix S is 41×7566 and the lead field is a 41×15002 matrix.

The conductivity of the scalp is taken equal to 1, while the brain and skull conductivities (σ_1 and σ_2 respectively) are variable and form the parameter space (a subspace of \mathbb{R}^2). We are interested in a subset of this space represented by a rectangle which covers values $(\sigma_1, \sigma_2) \in [0.5, 2] \times [10^{-4}, 10^{-1}]$. This rectangle is uniformly sampled with $25 \times 25 = 625$ points.

B. Approximation error convergence

For each of the 625 sample points, the exact lead field is computed in order to evaluate the *approximation error* (7).

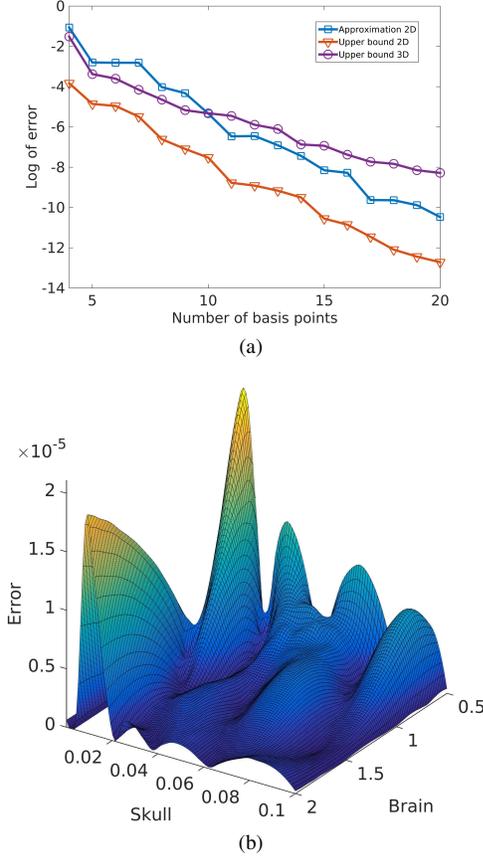


Fig. 3: (a) Logarithm of maximal (over conductivity samples) upper bound and approximation errors for two and three unknown conductivities as a function of number of basis points; (b) Approximation error for 22 basis points;

This is just done for purposes of validation and not necessary for the algorithm. We then compare it with its *upper bound error*, computed with (10).

We follow Algorithm 1 initializing the basis point set with the four corners of the parameter space, and new points are added one by one.

As shown in Fig. 3b, the approximation error (7) over the conductivity sample points is bounded by $2 \cdot 10^{-5}$ with 22 basis points. Fig. 3a shows that both approximation and upper bound errors (“Approximation 2D” and “Upper bound 2D”) decrease exponentially fast with the number of basis points. Moreover, they decrease with the same speed. This is an important property because in practice, we do not want to measure the true approximation error as it is very costly, but Fig. 3a shows that the decrease of the upper bound error can be used as a convergence criterion for Algorithm 1.

We also tested our algorithm for 3d conductivity region, considering scalp conductivity as variable. The “Upper bound 3D” in Fig. 3a represents the decreasing of upper bound error with the number of basis points. We can notice that it is still exponentially fast, even if the absolute value of a slope is smaller.

Remark. In the case of symmetric BEM, head matrix H is not homogeneous with respect to conductivities (even if the

lead field is). So, considering scalp conductivity as unknown does increase the complexity of the projection problem (10).

We also evaluated the upper bound error on the points which did not belong to the sampling we used in algorithm, and was not significantly bigger than the one on sample points. This shows the good continuity properties of our method, mentioned previously.

C. Approximation time

The time required for computing 625 exact lead field matrices is *4 hours*, which results in 23 seconds per matrix (2 physical cores @ 2.60GHz, 16Gb RAM).

Precomputing of all required matrices on *20 basis points* for Algorithm 2 is achieved in *14 minutes*.

Once basis points are computed it takes *58 seconds* to approximate 625 lead fields, which results in *0.09 second per matrix*.

As mentioned in previous section, the approximation time does not depend on the complexity of the head model as it only requires to solve a least-squares problem on the number of basis points. For more complex head models, the precomputing time will increase, but the approximation time will remain equal to 0.09 sec for 20 basis points.

D. Conductivity estimation with simulated sources

We now examine convergence properties in a realistic application of conductivity estimation.

Using the same head model as above, we simulate a dataset \mathbf{y} which corresponds to a single dipole source for a reference conductivity (ground truth). Then, we use a simple dipole fitting approach to solve the inverse problem. For each of 625 conductivity samples in the domain, we approximate the lead field matrix and choose a source as a column of this matrix which fits measurement the best in terms of Euclidean norm. Let $M_{\cdot,j}$ represent the j -th column of a matrix M . For each conductivity σ the data fitting error is defined as follows:

$$R(\sigma) = \min_{j,a} \|\mathbf{y} - a \cdot L(\sigma)_{\cdot,j}\|_2,$$

where $L(\sigma)_{\cdot,j}$ denotes j -th source’s lead field, and a - its optimal amplitude. $R(\sigma)$ represents the best fitting error of a single dipole corresponding to the measurement \mathbf{y} and conductivity σ .

Computing $R(\sigma_i)$ for each of 625 conductivity samples, we obtain a *data fitting error map* on the conductivity domain. We estimated the conductivity as one which minimizes this data fitting error.

For our simulation, we took the source whose lead field had the slowest convergence of the approximation error, i. e. the “worst” source in terms of lead field approximation. Using the exact lead fields, we obtain a data fitting error map (Fig. 4) whose minimum lies at the simulated conductivity point (1.25 for brain and 0.05 for skull).

The goal here is to show that approximated lead fields are of sufficient quality for conductivity estimation. To do so, we compute lead field approximation for different number of basis points with our method. It is visible from Fig. 7a that the

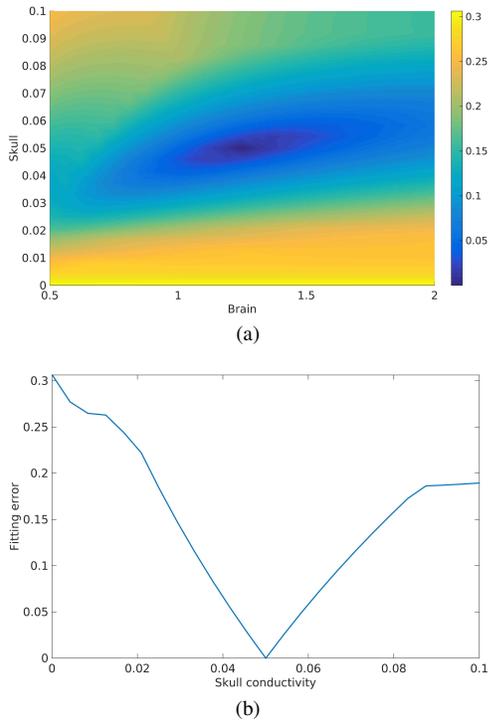


Fig. 4: Data fitting error map using exact lead fields: (a) full domain of interest. The minimum of error map corresponds to simulated conductivity configuration (1.25 for brain and 0.05 for skull); (b) Fitting error with brain conductivity equal to 1.25 (one column of map a).

”shape” of the data fitting error map converges very fast to the exact one. *10 basis points* are sufficient to correctly estimate the simulated conductivity.

E. Conductivity estimation with real data

We use the same subject (head model) and conductivity domain of interest as in the previous section. Real EEG data was taken from a median nerve stimulation experiment. The right median nerve was percutaneously stimulated using monophasic square-wave impulses with a duration of 0.3 ms at 2.8 Hz. We filtered and averaged the data, and removed heartbeats and eye movements artifacts using Brainstorm software [21]. For more experiment and pre-processing details see [20].

We are interested in the N20-P20 somatosensory averaged evoked potentials originating from Brodmann’s area 3b situated in the posterior bank of the Rolandic fissure [23, p. 1076]. We can see a remarkable activity peaking around 20 ms which has a dipolar topography on the left hemisphere (Fig. 5).

We analyzed 5 time samples ($\{\mathbf{y}(t)\}_{t=1}^5$) of a time window from 0.0185s to 0.0205s. The data fitting map is computed as follows:

$$R(\boldsymbol{\sigma}) = \min_j \sum_{t=1}^5 \min_a \|\mathbf{y}(t) - a \cdot L(\boldsymbol{\sigma})_{:,j}\|_2 .$$

The data fitting error, computed using the exact lead fields, is shown in Fig. 6. It is normalized to its minimum value,

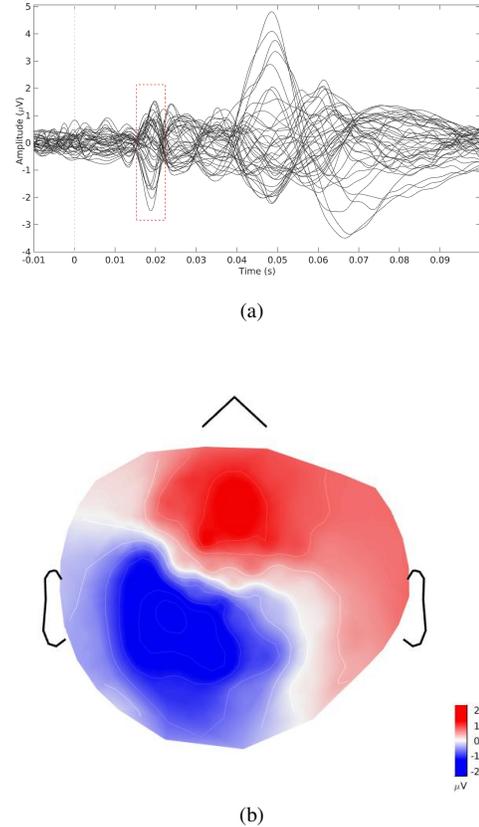


Fig. 5: Signal of interest: (a) Averaged time series per EEG channel; (b) Topography at 19.5 ms.

i.e. $\forall i : R(\boldsymbol{\sigma}_i) = \frac{R(\boldsymbol{\sigma}_i)}{\min_k R(\boldsymbol{\sigma}_k)}$. Many factors contribute to the fitting error: additive noise, wrong conductivity model, simplified head and source models, inverse problem assumption, etc. The normalized data fitting error map can be interpreted as follows. Considering the error resulting from all these factors, except conductivity, to be equal to 1, the normalized error shows the relative impact of ”incorrect” conductivity estimation compared to other factors of error.

Because of the different sources of noise, the impact of the brain conductivity becomes less important and we can not significantly find its optimal value. But, for each fixed brain conductivity, we can define the skull conductivity which minimizes the error: this skull conductivity value lies between 0.01 S/m and 0.02 S/m. We can also notice that conductivity contributes up to 10% relatively to the other sources of error, which shows the importance of the correct estimation of this parameter.

As can be seen in Fig. 7b, it is enough to use 11 basis points are sufficient to obtain an error map similar to the one obtained using exact lead fields.

V. CONCLUSION

In this work, we introduced a method for fast approximation of the EEG forward problem solution for different conductivities. Computing the exact lead field for many conductivity configurations is time consuming, but our approach only needs to compute a relatively small number of exact solutions. All

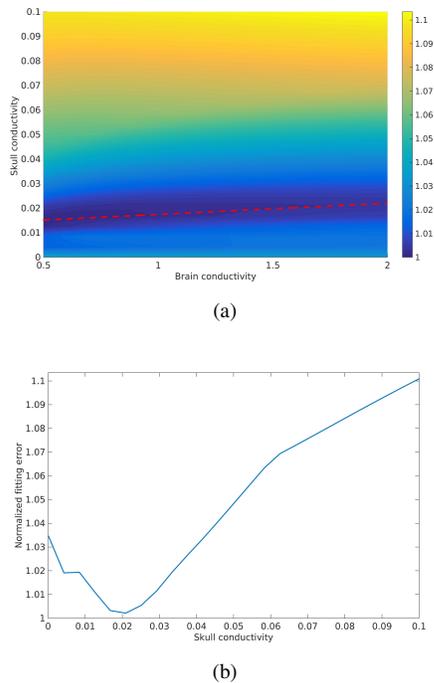


Fig. 6: Data fitting error map using exact lead fields: (a) full domain of interest. Red line represents the optimal skull conductivity for each brain conductivity; (b) Data fitting error with brain conductivity equal to 1.

other lead fields can be approximated in a fast way with controlled approximation error. This accelerated computing time allows to explore conductivity space, which is crucial for non-invasive head tissue conductivity estimation from EEG data.

Our method provides both the lead field approximation with given basis points as well as the way to choose these points. It is done in a way to guarantee the monotonic convergence of approximation error. Moreover, we showed that the complexity of this approximation does not depend on the number of vertices in a head model.

Besides the theoretical properties of our algorithm, we studied its empirical performance. Realistic simulations showed the exponential decrease of approximation error with respect to the number of basis points. We also showed the usefulness of the method in realistic context of EEG inverse problem with both simulated and real EEG data. As expected, a relatively small number of precomputed basis points provide results which are similar but remarkably faster to using exact matrices.

The main motivation of this work was to propose a tool which would boost computing of lead fields, which is necessary for simultaneous estimation of brain sources' activities and head tissues conductivity. We presented a simple approach of solving this inverse problem, based on a single dipole data fitting, to show that our method can be efficiently used for this kind of problems. We plan to continue studying the EEG inverse problem methods which consider conductivity as unknown but with more complex source models (e.g. multiple

dipoles), with more unknown conductivities, e.g. composite skull structure [9], and different conductivity space sampling approaches, e.g. using gradient descent instead of uniformly sampled domain. We believe that our method will significantly improve the practical aspects of such studies.

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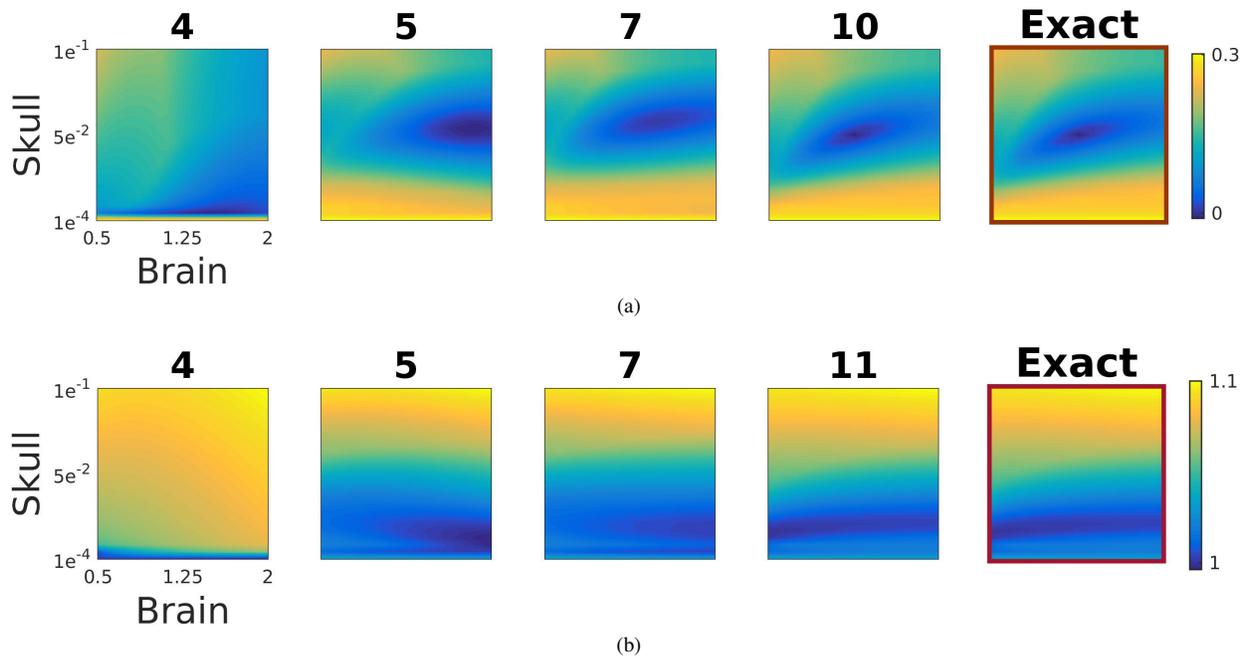


Fig. 7: Data fitting error map using approximated vs. exact lead fields: (a) for simulated data, using 4, 5, 7, 10 basis points. 10 points are sufficient to obtain an error map similar to the one obtained using exact lead fields; (b) real data, using 4, 5, 7, 11 basis points. 11 points are sufficient.

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