Robust stability under relaxed persistent excitation conditions

Denis Efimov, Nikita Barabanov, Roméo Ortega

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Abstract—**For linear time-varying systems with a persistently excited state matrix it is well-known that input-to-state stability is recovered. In this note a relaxed condition on persistence of excitation is studied together with related robust stability notions (input-to-state stability and integral input-to-state stability). The results are illustrated by simulations for scalar systems.**

I. INTRODUCTION

The problems of stability and robust stability are one of the main issues studied in the theory of control. There are some special systems related to benchmark models or important problem statements whose stability has been analyzed in detail. One of such a problem deals with a time-varying linear dynamical system of the following form:

$$\dot{p}(t) = -\gamma R(t)R^\top(t)p(t) + b(t),\ t \geq t_0, \quad (1)$$
$$p(t_0) \in \mathbb{R}^l,\ t_0 \in \mathbb{R},$$

where $p(t) \in \mathbb{R}^l$ is the state, the functions $R : \mathbb{R} \to \mathbb{R}^{l \times k}$ and $b : \mathbb{R} \to \mathbb{R}^l$ are (Lebesgue) measurable and essentially bounded; $\gamma > 0$. This problem is related with adaptive estimation and identification of linear regression models [1]. The peculiarity of this system consists in the requirement imposed on the matrix function R, which is assumed usually to be *persistently excited*. It is a well-known fact [2], that if R is persistently excited, then (1) is globally exponentially stable for $b(t) \equiv 0$, $t \geq t_0$ and input-to-state stable (ISS) for any essentially bounded b.

Recently, several relaxed excitation notions have been proposed for R in [3], [4] implying global (not uniform) asymptotic stability of (1). The goal of this note is to compare them and to establish relations with ISS and integral ISS (iISS) properties introduced in [5], [6].

The paper outline is as follows. Preliminaries (persistence of excitation and stability notions) are introduced in Section

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II. The relaxed excitation notions from [3], [4] are discussed in Section III. Relations with ISS and iISS properties with respect to different perturbations (exogenous disturbances or measurement noises) are considered in Section IV. The scalar case and some results of simulation are presented in Section V. Some concluding remarks are given in Section VI.

II. PRELIMINARIES

Denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ is the set of natural numbers. Let $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$ (for $n \in \mathbb{N}$) or the corresponding induced matrix norm. For a (Lebesgue) measurable and essentially bounded function $b : \mathbb{R} \to \mathbb{R}^l$ denote $\|b\|_\infty = \sup_{t \in \mathbb{R}} |b(t)|$.

Denote by $\mathcal{K}^+$ the class of continuous functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha \in \mathcal{K}^+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha \in \mathcal{K}^+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it increases to infinity. A continuous function $\mu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\mu(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t \to +\infty} \mu(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$.

A. Persistence of excitation condition

A (Lebesgue) measurable and square integrable matrix function $R : \mathbb{R} \to \mathbb{R}^{l \times k}$ with the dimension $l \times k$ verifies the (ℓ, ϑ)–Persistence of Excitation (PE) condition, if there exist constants $\ell > 0$ and $\vartheta > 0$ such that

$$\int_t^{t+\ell} R(s)R^\top(s)ds \geq \vartheta I_l$$

for any $t \in \mathbb{R}$, where I_l denotes the identity matrix of dimension $l \times l$.

Lemma 1. *[7] Assume that in the system (1) the function R is continuous, bounded by $\rho = \sup_{t \in \mathbb{R}} |R(t)| < +\infty$, and (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$. Then, for any $t_0 \in \mathbb{R}$ and any initial condition $p(t_0) \in \mathbb{R}^l$:*

$$|p(t)| \leq \sqrt{\zeta}\rho[e^{-0.5\gamma\zeta^{-1}(t-t_0)}|p(t_0)| + \gamma^{-1}\zeta\|b\|_\infty] \quad \forall t \geq t_0$$

for $\zeta = \gamma\eta^{-1}e^{2\eta\ell}$ and $\eta = -0.5\ell^{-1}\ln(1 - \frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^4})$.

Note that from the definition of PE property, in the conditions of the above lemma,

$$\vartheta \leq \ell\rho^2,$$

then $\frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^4} \in (0, 0.5]$ for any $\gamma > 0$, $\ell > 0$ and $\vartheta > 0$. Therefore, η and ζ belong to $(0, +\infty)$.

B. Robust stability properties

The system (1) is called nonuniform ISS [8] if there exist $\mu \in \mathcal{KL}$, $\sigma \in \mathcal{K}^+$ and $\nu \in \mathcal{K}$ such that for all $t_0 \in \mathbb{R}$, all $p(t_0) \in \mathbb{R}^l$ and (Lebesgue) measurable and essentially bounded $b : \mathbb{R} \to \mathbb{R}^l$:

$$|p(t)| \leq \mu(\sigma(t_0)|p(t_0)|, t - t_0) + \nu(\|b\|_\infty) \quad \forall t \geq t_0.$$

The system (1) is called nonuniform iISS [8] if there exist $\mu \in \mathcal{KL}$, $\sigma \in \mathcal{K}^+$, $\alpha \in \mathcal{K}_\infty$ and $\nu \in \mathcal{K}$ such that for all $t_0 \in \mathbb{R}$, all $p(t_0) \in \mathbb{R}^l$ and (Lebesgue) measurable and integrable $b : \mathbb{R} \to \mathbb{R}^l$:

$$\alpha(|p(t)|) \leq \mu(\sigma(t_0)|p(t_0)|, t - t_0) + \int_{t_0}^t \nu(|b(s)|)ds \quad \forall t \geq t_0.$$

The system (1) is called uniform ISS or iISS if $\sigma(s) = const$ for all $s \in \mathbb{R}$.

III. RELAXED CONVERGENCE CONDITIONS FOR THE UNFORCED CASE

First, lest us consider the conditions of asymptotic stability of (1) for the case $b(t) \equiv 0$ for all $t \in \mathbb{R}$. Then, to relax the PE condition we will assume that the parameters ℓ and ϑ are time-dependent:

Assumption 1. *Assume that for any $t_0 \in \mathbb{R}$ there exist sequences of positive numbers $\{t_k\}_{k=0}^{+\infty}$, $\{\ell_k\}_{k=0}^{+\infty}$ and $\{\vartheta_k\}_{k=0}^{+\infty}$ such that for all $k \geq 0$:*

$$t_{k+1} \geq t_k + \ell_k,$$
$$\int_{t_k}^{t_k+\ell_k} R(s)R^\top(s)ds \geq \vartheta_k I_l.$$

There are several results relating this assumption and asymptotic stability of (1):

Proposition 1. *[4] Let Assumption 1 be satisfied and*

$$\sum_{k=0}^{+\infty} \frac{\vartheta_k}{\gamma^{-1} + \gamma \left(\int_{t_k}^{t_k+\ell_k} |R(t)|^2 dt\right)^2} = +\infty. \quad (2)$$

Then the system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is globally asymptotically stable at the origin.

In [3] another condition of global asymptotic stability (GAS) for (1) has been established, which is based on the result given below that contains a small extension together with a proof:

Proposition 2. *Let $b(t) \equiv 0$ for all $t \in \mathbb{R}$ and $\Phi(t, t_0)$ be the transition matrix of (1) satisfying*

$$\frac{\partial}{\partial t}\Phi(t, t_0) = -\gamma R(t)R^\top(t)\Phi(t, t_0), \ t \geq t_0, \ \Phi(t_0, t_0) = I_l,$$

then for all $t \geq t_0$:

$$e^{-2\gamma \int_{t_0}^t |R(s)|^2 ds} I_l \leq \Phi^\top(t, t_0)\Phi(t, t_0)$$
$$\leq I_l - \frac{\int_{t_0}^t R(s)R^\top(s)ds}{\gamma^{-1} + \gamma \left(\int_{t_0}^t |R(s)|^2 ds\right)^2}.$$

Proof. For any $t_0 \in \mathbb{R}$ and $p(t_0) \in \mathbb{R}^l$ denote $p(t) = \Phi(t, t_0)p(t_0)$ (the solution of (1) with initial condition $p(t_0)$ at initial time t_0) and consider an auxiliary Lyapunov function $W(p) = p^\top p$, whose derivative for the system (1) has the form:

$$\frac{d}{dt}W(p(t)) = -2\gamma p^\top(t)R(t)R^\top(t)p(t) \quad (3)$$
$$= -2\gamma|R^\top(t)p(t)|^2,$$

then

$$\frac{d}{dt}W(p(t)) \geq -2\gamma|R(t)|^2 W(p(t))$$

implying

$$W(p(t)) \geq e^{-2\gamma \int_{t_0}^t |R(r)|^2 dr} W(p(t_0)),$$

which can be interpreted as

$$p^\top(t_0)\Phi^\top(t, t_0)\Phi(t, t_0)p(t_0) \geq p^\top(t_0)e^{-2\gamma \int_{t_0}^t |R(r)|^2 dr}p(t_0)$$

providing the required lower estimate on $\Phi^\top(t, t_0)\Phi(t, t_0)$ due to an arbitrary choice of $p(t_0)$. From another side, since $0.5|a|^2 \leq |b|^2 + |a - b|^2$ for any $a, b \in \mathbb{R}^l$, for $a = R^\top(t)p(t_0)$ and $b = R^\top(t)p(t)$ we obtain:

$$0.5|R^\top(t)p(t_0)|^2 \leq |R^\top(t)p(t)|^2 + |R^\top(t)(p(t_0) - p(t))|^2. \quad (4)$$

Note that

$$|R^\top(t)(p(t) - p(t_0))|^2 = |R^\top(t)\int_{t_0}^t \dot{p}(s)ds|^2$$
$$= \gamma^2 \int_{t_0}^t p^\top(s)R(s)R^\top(s)ds R(t)R^\top(t)\int_{t_0}^t R(s)R^\top(s)p(s)ds$$
$$\leq \gamma^2|R(t)|^2 \int_{t_0}^t p^\top(s)R(s)R^\top(s)ds \int_{t_0}^t R(s)R^\top(s)p(s)ds$$
$$\leq \gamma^2|R(t)|^2 \left(\int_{t_0}^t |R(s)||R^\top(s)p(s)|ds\right)^2$$
$$\leq \gamma^2|R(t)|^2 \int_{t_0}^t |R(s)|^2 ds \int_{t_0}^t |R^\top(s)p(s)|^2 ds,$$

where the Cauchy-Schwarz inequality has been used on the last step. Integrating the obtained inequality we conclude that

$$\int_{t_0}^t |R^\top(s)(p(s) - p(t_0))|^2 ds \leq \int_{t_0}^t \gamma^2|R(r)|^2 \int_{t_0}^r |R(s)|^2 ds$$
$$\times \int_{t_0}^r |R^\top(s)p(s)|^2 ds dr$$
$$\leq \gamma^2 \left(\int_{t_0}^t |R(s)|^2 ds\right)^2 \int_{t_0}^t |R^\top(s)p(s)|^2 ds.$$

Next, integrating (4) the following relation is derived:

$$0.5 \int_{t_0}^t |R^\top(s)p(t_0)|^2 ds \leq \int_{t_0}^t |R^\top(s)p(s)|^2 ds$$
$$+ \int_{t_0}^t |R^\top(s)(p(s) - p(t_0))|^2 ds$$
$$\leq \left[1 + \gamma^2 \left(\int_{t_0}^t |R(s)|^2 ds\right)^2\right] \int_{t_0}^t |R^\top(s)p(s)|^2 ds.$$

Finally, integrating (3) we obtain:

$$W(p(t)) - W(p(t_0)) = -2\gamma \int_{t_0}^t |R^\top(s)p(s)|^2 ds$$

$$\leq -\frac{\gamma \int_{t_0}^t |R^\top(s)p(t_0)|^2 ds}{1 + \gamma^2 \left(\int_{t_0}^t |R(s)|^2 ds\right)^2},$$

or equivalently,

$$p^\top(t_0)\Phi^\top(t,t_0)\Phi(t,t_0)p(t_0)$$

$$\leq p^\top(t_0)\left(I_l - \frac{\gamma \int_{t_0}^t R(s)R^\top(s)ds}{1 + \gamma^2 \left(\int_{t_0}^t |R(s)|^2 ds\right)^2}\right) p(t_0),$$

and since this inequality is satisfied for any $p(t_0) \in \mathbb{R}^l$, the desired upper estimated is substantiated. $\qquad\square$

This proposition provides generic properties of the transition matrix of (1), from which the stability conditions can be deduced:

Corollary 1. *Let Assumption 1 be satisfied and*

$$\prod_{k=0}^{+\infty}\left(1 - \frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2}\right) = 0 \quad (5)$$

Then the system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is globally asymptotically stable at the origin.

Further, almost all the proofs are omitted for brevity of the presentation.

Remark 1. Since $0 < \vartheta_k \leq \int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt$, $\sup_{x\geq 0}\frac{x}{1+x^2} = 0.5$ and $\ln(1-s) \leq -s$ for $0 < s < 1$, by applying $\ln(\cdot)$ to both sides of (5) we conclude:

$$-\infty = \sum_{k=0}^{+\infty}\ln\left(1 - \frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2}\right)$$

$$\leq -\sum_{k=0}^{+\infty}\frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2}$$

and

$$\sum_{k=0}^{+\infty}\frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2} \leq +\infty,$$

which implies (2). And *vice versa*, the estimate (2) also implies (5), since using the inequality $1 - x \leq e^{-x}$ that

is satisfied for all $x \in \mathbb{R}$, and by applying $e^{-(\cdot)}$ to both sides of (2) we obtain:

$$0 = e^{-\sum_{k=0}^{+\infty}\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2}}$$

$$= \prod_{k=0}^{+\infty} e^{-\frac{\vartheta_k}{\gamma^{-1}+\gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2}}$$

$$\geq \prod_{k=0}^{+\infty}\left(1 - \frac{\vartheta_k}{\gamma^{-1} + \gamma\left(\int_{t_k}^{t_k+\ell_k}|R(t)|^2 dt\right)^2}\right) \geq 0.$$

Thus, the conditions (2) and (5) are equivalent.

IV. ROBUSTNESS WITH RESPECT TO EXTERNAL INPUTS

Now, let us return to stability analysis of the perturbed system (1) with $b(t) \neq 0$.

Proposition 3. *Consider the system (1), then for any $t_0 \in \mathbb{R}$ and any $p(t_0) \in \mathbb{R}^l$*

$$|p(t)| \leq \sqrt{\left\|I_l - \frac{\int_{t_0}^t R(s)R^\top(s)ds}{\gamma^{-1} + \gamma\left(\int_{t_0}^t |R(s)|^2 ds\right)^2}\right\|}$$

$$\times \left(|p(t_0)| + e^{\gamma\int_{t_0}^t |R(s)|^2 ds}\int_{t_0}^t |b(s)|ds\right)$$

for all $t \geq t_0$.

The latter result provides a generic auxiliary estimate on solutions of (1), which is used next to derive the stability conditions. To this end introduce a sequence

$$\rho_k = \int_{t_k}^{t_{k+1}}|R(t)|^2 dt \quad \forall k \geq 0.$$

Theorem 1. *Consider the system (1) satisfying Assumption 1 with an additional constraint:*

$$t_{k+1} = t_k + \ell_k$$

for all $k \geq 0$, and assume that the property (5) is valid.
If

$$\sup_{k\geq 0}\left\{\ell_k e^{\gamma\rho_k} + \sum_{i=0}^k\prod_{j=i}^k\sqrt{1 - \frac{\vartheta_j}{\gamma^{-1} + \gamma\rho_j^2}}\,\ell_i e^{\gamma\rho_i}\right\}$$

$$\leq \eta_{t_0} < +\infty,$$

then for each $t_0 \in \mathbb{R}$ there exists a function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$:

$$|p(t)| \leq \varrho_{t_0}(|p(t_0)|, t - t_0) + \eta_{t_0}\|b\|_\infty \quad \forall t \geq t_0. \quad (6)$$

If

$$\sup_{k\geq 0} e^{\gamma\rho_k} \leq \beta_{t_0} < +\infty,$$

then for each $t_0 \in \mathbb{R}$ there exists a function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$:

$$|p(t)| \leq \varrho_{t_0}(|p(t_0)|, t - t_0) + \beta_{t_0}\int_{t_0}^t |b(t)|dt \quad \forall t \geq t_0. \quad (7)$$

Thus, the conditions of non-uniform ISS property (6) are rather sophisticated and discussed below, while it is worth to stress here that the swap "conditions" of non-uniform iISS property (7) are very weak and satisfied, for example, if the system (1) is asymptotically stable for $b = 0$ and

$$\int_{-\infty}^{+\infty} |R(t)|^2 dt < +\infty$$

or

$$\sup_{k \geq 0} \ell_k < +\infty, \ \operatorname{ess\,sup}_{t \geq 0} |R(t)| < +\infty.$$

A. Relation with PE conditions

Note that dependence of η_{t_0} on the initial time $t_0 \in \mathbb{R}$ is important in the formulation of Theorem 1, since if a uniform result would be obtained, *i.e.* there exist $\bar{\eta} \in \mathbb{R}_+$ such that $\sup_{t_0 \in \mathbb{R}} \eta_{t_0} \leq \bar{\eta}$, then it implies global exponentially stability (GES) of (1) (and, consequently, fulfillment of PE condition [9]), as it is shown in the following lemma:

Lemma 2. *Assume that for the system (1) there exist constants $C_1 \geq 1$ and $C_2 > 0$ such that for all $t_0 \in \mathbb{R}$ and all initial conditions $p(t_0) \in \mathbb{R}^l$ we have*

$$|p(t)| \leq C_1 |p(t_0)| + C_2 \|b\|_\infty \quad \forall t \geq t_0.$$

Then the system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is GES.

Proof. Pick up a number $\epsilon \in (0, 1/C_2)$ and set $C_3 = \frac{2C_1}{1 - \epsilon C_2}$. Consider a solution $p(t)$ of the system (1) with $|p(t_0)| = 1$ and $b(t) \equiv 0$ for all $t \geq t_0$. It is sufficient to prove that function $y(t) = e^{\epsilon(t - t_0)} p(t)$ is bounded. Function $y(t)$ satisfies equation

$$\dot{y}(t) = -\gamma R(t) R(t)^T y(t) + \epsilon y(t).$$

We will prove that $|y(t)| < C_3$ for all $t \geq t_0$. Recall that $C_1 \geq 1$ and, therefore, $C_3 > 1$. Since $|y(t_0)| = 1$, we have $|y(t)| < C_3$ for all sufficiently small $t \geq t_0$. Assume, on the contrary to the claim existence of $T > t_0$ such that $|y(T)| = C_3$. Choose the first T satisfying this property. Denote $b(t) = \epsilon y(t)$ on $[t_0, T]$. Then according to the assumption

$$|y(t)| \leq C_1 + C_2 \epsilon \sup_{t \in [t_0, T]} |y(t)|$$

for all $t \in [t_0, T]$. Therefore,

$$|y(t)| \leq C_1 + C_2 \epsilon C_3 = C_1 \left(1 + \frac{2\epsilon C_2}{1 - \epsilon C_2}\right) = C_1 \frac{1 + \epsilon C_2}{1 - \epsilon C_2} < C_3$$

for all $t \in [t_0, T]$. The contradiction at $t = T$ shows that $|y(t)| < C_3$ for all $t \geq t_0$, which completes the proof. $\square$

Corollary 2. *If system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is not GES, then the assumption of Lemma 2 is not satisfied, and therefore the system (1) is not ISS.*

The proof of Theorem 1 shows that for every $t_0 \in \mathbb{R}$ function $\rho_{t_0}(s, t - t_0)$ in (6) may be chosen as $\rho_{t_0}(s, t - t_0) = sk_{t_0}(t - t_0)$ with a decreasing function $k_{t_0} \colon \mathbb{R}_+ \to \mathbb{R}_+$. In

such a case the following non-uniform counterpart of Lemma 2 can be formulated:

Lemma 3. *Assume for each $t_0 \in \mathbb{R}$ there exist a number $\eta_{t_0} > 0$ and a continuous function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$ in the system (1) we have (6). Then for every $t_0 \in \mathbb{R}$ system (1) with $b(t) \equiv 0$ for all $t \in \mathbb{R}$ is GES on $[t_0, \infty)$.*

Notice that the lemma implies that for every $t_0 \in \mathbb{R}$ function R satisfies the PE condition on interval $[t_0, \infty)$, that is, there exist positive real numbers m_{t_0} and θ_{t_0} such that

$$\int_t^{t + m_{t_0}} R(s) R^\top(s) ds \geq \theta_{t_0} I_l$$

for all $t \geq t_0$. This condition is stronger than (5).

Remark 2. Notice that in Theorem 1 and this lemma, the stability conditions of the system (1) are dependent on $t_0 \in \mathbb{R}$ (the function ϱ_{t_0} and the parameter η_{t_0}). Hence, in both cases we have stability properties which are, in general, non uniform with respect to $t_0 \in \mathbb{R}$, and there is no contradiction with Lemma 2.

B. Robustness with respect to measurement noise

The conditions of ISS obtained for (1) in Theorem 1 are rather restrictive and related with PE property of the matrix R, in the case when b in (1) plays the role of exogenous disturbance. These conditions can be significantly relaxed considering the case with b representing a measurement noise:

$$\dot{p}(t) = -\gamma R(t) R^\top(t) \left(p(t) + b(t)\right), \ t_0 \in \mathbb{R}, \ \gamma > 0, \quad (8)$$

where again $p(t) \in \mathbb{R}^l$ is the state, the functions $R : \mathbb{R} \to \mathbb{R}^{l \times k}$ and $b : \mathbb{R} \to \mathbb{R}^l$ are (Lebesgue) measurable and essentially bounded. For all $k \geq 0$ denote a sequence

$$\phi_k = \operatorname{ess\,sup}_{t \in [t_k, t_{k+1})} |R(t)|^2,$$

then $\rho_k \leq \ell_k \phi_k$.

Corollary 3. *Consider the system (8) satisfying Assumption 1 with an additional constraint:*

$$t_{k+1} = t_k + \ell_k$$

for all $k \geq 0$, and assume that the property (5) is valid.
If

$$\gamma \sup_{k \geq 0} \left\{ \ell_k \phi_k e^{\gamma \rho_k} + \sum_{i=0}^{k} \prod_{j=i}^{k} \sqrt{1 - \frac{\vartheta_j}{\gamma^{-1} + \gamma \rho_j^2}} \ell_i \phi_i e^{\gamma \rho_i} \right\}$$
$$\leq \eta_{t_0} < +\infty$$

then for each $t_0 \in \mathbb{R}$ there exists a function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$ the estimate (6) is satisfied.
If

$$\gamma \sup_{k \geq 0} \phi_k e^{\gamma \rho_k} \leq \beta_{t_0} < +\infty,$$

then for each $t_0 \in \mathbb{R}$ there exists a function $\varrho_{t_0} \in \mathcal{KL}$ such that for all $p(t_0) \in \mathbb{R}^l$ the estimate (7) is satisfied.

Recall that the conclusion of Lemma 2 cannot be applied here for the ISS case, since the system (8) in not in the form of (1).

V. SCALAR CASE

To illustrate the proposed in Theorem 1 result let us analyze the case with $l = 1$.

First, consider the system (1), then solutions can be calculated analytically for all $t \geq t_0$:

$$p(t) = e^{-\gamma \int_{t_0}^t R^2(\tau)d\tau} p(t_0) + \int_{t_0}^t e^{-\gamma \int_s^t R^2(\tau)d\tau} b(s)ds$$

and

$$|p(t)| \leq e^{-\gamma \int_{t_0}^t R^2(\tau)d\tau} |p(t_0)| + \int_{t_0}^t |b(s)|ds.$$

Under conditions of Theorem 1 we obtain for all $k \geq 0$:

$$|p(t_{k+1})| \leq e^{-\gamma \vartheta_k} |p(t_k)| + \int_{t_k}^{t_{k+1}} |b(s)|ds,$$

or for an essentially bounded input b:

$$|p(t_{k+1})| \leq e^{-\gamma \vartheta_k} |p(t_k)| + \ell_k \|b\|.$$

Recursively applying the above inequalities we get for any $K \geq 0$:

$$|p(t_K)| \leq e^{-\gamma \sum_{k=0}^{K-1} \vartheta_k} |p(t_0)| + \int_{t_0}^{t_K} |b(s)|ds$$

or

$$|p(t_K)| \leq e^{-\gamma \sum_{k=0}^{K-1} \vartheta_k} |p(t_0)| + \sum_{k=0}^{K-1} \ell_k e^{-\gamma \sum_{s=k+1}^{K-1} \vartheta_s} \|b\|,$$

$$\tag{9}$$

which results in the following condition of the system convergence for the case $b = 0$:

$$\sum_{k=0}^{+\infty} \vartheta_k = +\infty,$$

and the asymptotic gain property is satisfied for any $t_0 \in \mathbb{R}$ if

$$\sup_{k \geq 1} \sum_{i=0}^{k-1} \ell_i e^{-\gamma \sum_{s=i+1}^{k-1} \vartheta_s} \leq \eta(t_0) < +\infty.$$

These estimates imply that a kind of iISS property is *always* presented in this case for asymptotically stable systems (1).

Example 1. Select

$$R(t) = \frac{e^t \cos(t)}{1 + e^t}, \ b(t) = \sin(t), \ \gamma = 1, \ p(t_0) = 1$$

and impose

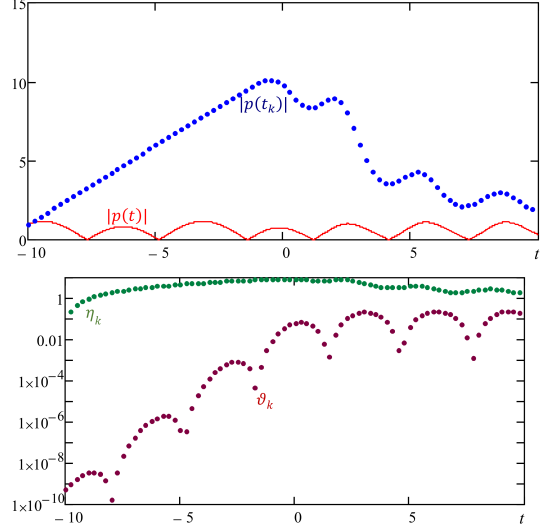$$\ell_k = \bar{\ell} = 0.25 \quad \forall k \geq 0,$$

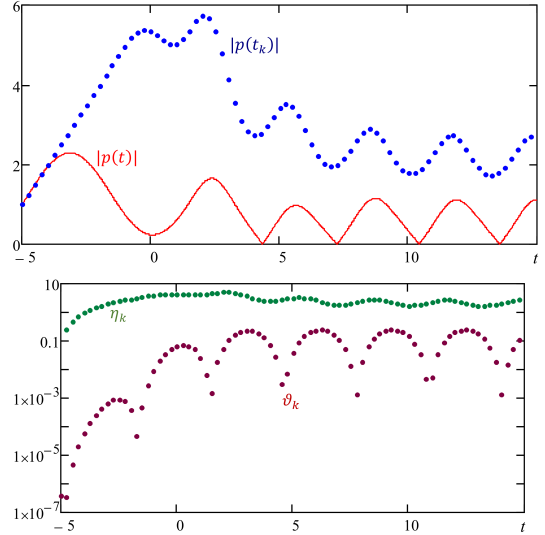Figure 1. The results of simulation for (1) with $t_0 = -10$

Figure 2. The results of simulation for (1) with $t_0 = -5$

then the plot of $|p(t)|$ and the estimate on $|p(t_k)|$ from (9), together with

$$\vartheta_k = \int_{t_k}^{t_{k+1}} R^2(t)dt, \ \eta_k = \sum_{i=0}^{k-1} \ell_i e^{-\gamma \sum_{s=i+1}^{k-1} \vartheta_s}$$

are shown in figures 1 and 2 for $t_0 = -10$ and $t_0 = -5$, respectively. These plots demonstrate dependence of η on t_0, and that the system is not uniformly GES (it is GAS) and possesses certain robustness against external inputs, as it has been established in Theorem 1.

Second, consider the system (8) and for simplicity assume that

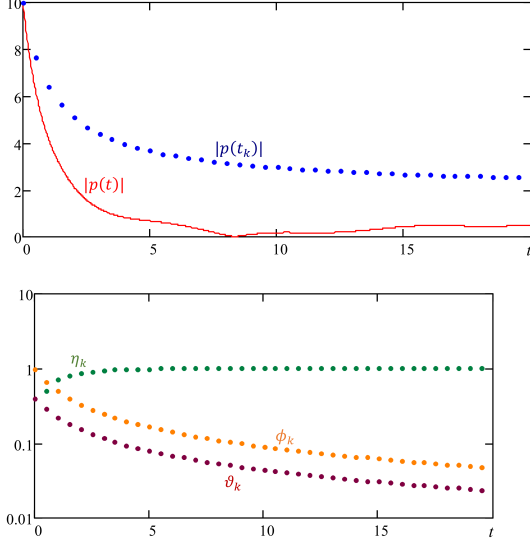$$\sup_{k \geq 0} \phi_k \leq \bar{\phi} < +\infty,$$

Figure 3. The results of simulation for (8) with $t_0 = 0$

VI. CONCLUSION

For a time-varying linear system under a relaxed notion of PE, several relations with ISS and iISS properties are established. It is shown that iISS is rather natural and it is satisfied almost always in this class of systems, and especially with respect to measurement noises. All proposed conditions are sufficient, and investigation of necessary counterpart can be a direction of future research.

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then for any $K \geq 0$:

$$|p(t_K)| \leq e^{-\gamma \sum_{k=0}^{K-1} \vartheta_k} |p(t_0)| + \gamma\bar{\phi} \int_{t_0}^{t_K} |b(s)|ds$$

or

$$|p(t_K)| \leq e^{-\gamma \sum_{k=0}^{K-1} \vartheta_k} |p(t_0)| + \gamma \sum_{k=0}^{K-1} \phi_k \ell_k e^{-\gamma \sum_{s=k+1}^{K-1} \vartheta_s} \|b\|, \tag{10}$$

and the ISS property can be observed under the same restrictions as for the case of (1):

$$\sum_{k=0}^{+\infty} \vartheta_k = +\infty, \ \sup_{k\geq 1} \sum_{i=0}^{k-1} \phi_i \ell_i e^{-\gamma \sum_{s=i+1}^{k-1} \vartheta_s} < +\infty,$$

while iISS property is naturally yielded for any asymptotically stable system (8).

Example 2. Select $t_0 \geq 0$,

$$R(t) = \frac{1}{\sqrt{1+t}}, \ b(t) = 1 + \sin(t), \ \gamma = 1, \ p(t_0) = 10,$$

then the system is not exponentially stable (*i.e.* $p(t) = p(t_0)\frac{1+t_0}{1+t}$ for $t \geq t_0 \geq 0$ with $b(t) = 0$), and impose

$$\ell_k = \bar{\ell} = 0.5 \quad \forall k \geq 0.$$

The plot of $|p(t)|$ and the estimate on $|p(t_k)|$ from (10), together with

$$\vartheta_k = \int_{t_k}^{t_{k+1}} R^2(t)dt, \ \phi_k = \sup_{t\in[t_k,t_{k+1})} |R(t)|^2,$$

$$\eta_k = \sum_{i=0}^{k-1} \phi_i \ell_i e^{-\gamma \sum_{s=i+1}^{k-1} \vartheta_s}$$

are shown in Fig. 3 for $t_0 = 0$.