Adaptive Estimation for Uncertain Nonlinear Systems: A Sliding-Mode Observer Approach

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Abstract—**This paper deals with the problem of adaptive estimation, *i.e.* the simultaneous estimation of the state and parameters, for a class of uncertain nonlinear systems. A nonlinear adaptive sliding-mode observer is proposed based on a nonlinear parameter estimation algorithm. The nonlinear parameter estimation algorithm provides a rate of convergence faster than exponential while the sliding-mode observer ensures ultimate boundness for the state estimation error attenuating the effects of the external disturbances. Linear matrix inequalities (LMIs) are provided for the synthesis of the adaptive observer and some simulation results show the feasibility of the proposed approach.**

Index Terms—**Adaptive Observer, Nonlinear Systems, Sliding-Modes**

I. Introduction

DISTURBANCES and uncertainties widely exist in almost all physical systems in the real world, in the form of external perturbations, unknown system dynamics and/or unknown parameters. One of the main problems for the robust control design is the existence of such disturbances and the fact that the whole state is not always available for measuring. A useful tool to deal with this problem has been studied during the last decades, *i.e.* the adaptive control theory. This theory has received a great deal of attention due to the need of adaptive control design. This area has grown to turn into one of the widest in terms of algorithms, analytical tools and techniques for design (see, for instance [1]). Particularly, the design of observers estimating simultaneously the whole state and the parameters of the system by some on-line adaptation law is an important problem in the adaptive control area (see, *e.g.* [2]).

In this framework, a class of adaptive extended state observers is proposed in [3] that significantly increases the applications of extended state observers to nonlinear disturbed systems. In [4] the problem of adaptive observer synthesis for Lipschitz nonlinear systems is addressed. The sufficient conditions are given in terms of LMIs that ensure the state estimation error convergence to zero with known parameters. Also, an adaptive observer form is presented which is used to achieve the adaptive estimation under additional constraints. In [5] an adaptive observer for a class of parabolic PDEs is proposed, the observer is designed to provide online estimates of the system state and unknown parameters based on sampled data and also sufficient conditions for the observer to be exponentially convergent are established. On the other hand, a hybrid adaptive observer is designed in [6] and it is shown that the state estimation error is exponentially convergent if the sampling period is small enough and a persistent excitation condition holds. In this vein, in [7] an adaptive observer was proposed to exponentially estimate the state and the unknown parameters under a persistent excitation condition for uniformly observable Multiple-Input-Multiple-Output (MIMO) nonlinear systems. The authors also proposed an adaptive observer for a class of uniformly observable MIMO nonlinear systems with general nonlinear parametrization which provides exponential convergence to zero of the adaptive estimation error. Adaptive observers, using delays are proposed by [8]. The authors provided a method for redesigning adaptive observers for nonlinear systems that increases the computational effort but provides better parameter estimations and some robustness properties. In the context of nonlinear parametrization, very few results are available in the literature. For instance, in [9] an adaptive control is proposed for a class of nonlinear systems with a triangular structure and nonlinear parametrization. This approach provides boundedness for regulation and tracking tasks. In [10] adaptive observers are proposed for a class of systems with nonlinearities in the parameters. These observers can reconstruct asymptotically the unknown state and the parameters.

In the sliding-mode area, different adaptive observers have been proposed. In [11] an adaptive sliding-mode observer is designed for the selective catalytic reduction system in diesel-engine after-treatment systems achieving better performance with respect to a Luenberger-like observer. In the fault detection problem, an adaptive sliding-mode observer is proposed by [12] to solve the problem of sensor fault diagnosis in an industrial gas turbine. Such an adaptive observer provides simultaneous fault detection and identifi-

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cation despite the presence of noise. In [13] an adaptive-gain second-order sliding-mode observer is developed for estimating the system states and the uncertain parameters. The performance of the proposed observer is illustrated in a fuel cell air-feed system using a hardware-in-loop emulator. An adaptive sliding-mode observer is proposed by [14] for a class of nonlinear systems with unknown parameters and faults. Based on the main properties of the sliding-mode observers, an asymptotic fault reconstruction is given taking into account that the relative degree of the output, with respect to the fault, is equal to one. It is important to highlight that most of the proposed adaptive observers are based on linear parameter estimation algorithms. Additionally, most of these works do not consider external disturbances and the convergence rate of the state and parameter estimation errors is asymptotic or exponential. In this context, in [15] it was shown that improvements of convergence rate, for the parameter estimation error, cannot be achieved by simply increasing the observer gains but changing the structure of the adaptive observers.

In this paper a nonlinear adaptive sliding-mode observer is proposed, based on a nonlinear parameter identification algorithm, for uncertain nonlinear systems. The proposed nonlinear adaptive observer is an extension of the one presented by [16]. This extension tries to increase the class of nonlinear systems for which the adaptive observer can be applied allowing to have a parameter distribution matrix and nonlinear terms depending on the full state and the input. In [16], such functions were considered just depending on the measurable output and input. The proposed parameter identification algorithm provides an ultimate bound for the state and parameter estimation error and attenuating the effects of the external disturbances. The synthesis of the observer is given in terms of LMIs. The proofs are developed based on the Lyapunov function approach and the input-to-state stability theory. Some simulation results illustrate the efficiency of the proposed nonlinear adaptive sliding-mode observer.

This paper is organized as follows. The problem statement is given in Section II. The preliminaries are discussed in SectionIII . The adaptive observer and the convergence properties are presented in Section IV. Simulation results are shown in Section V and conclusions in Section VI.

Notation: The Euclidean norm of a vector $q \in \mathbb{R}^n$ is denoted by $\|q\|$. For a matrix $Q \in \mathbb{R}^{m\times n}$, denote its smallest singular value $\sigma_{\min}(Q) = \sqrt{\lambda_{\min}(Q^TQ)}$ and its induced norm as $\|Q\| := \sqrt{\lambda_{\max}(Q^TQ)} = \sigma_{\max}(Q)$, where $\lambda_{\max}$ is the maximum eigenvalue and $\lambda_{\min}$ is the minimum one, $\sigma_{\max}$ is the largest singular value. For a Lebesgue measurable function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, define the norm $\|u\|_{(t_0,t_1)} := ess\sup_{t\in(t_0,t_1)} \|u(t)\|$, then $\|u\|_\infty := \|u\|_{(0,+\infty)}$ and the set of functions u with the property $\|u\|_\infty < +\infty$ is denoted as $\mathcal{L}_\infty$. For a matrix $Q : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times n}$, denote $\|Q\|_\infty := \|Q\|_{(0,+\infty)}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if, for each fixed s, $\beta(r, s) \in \mathcal{K}$ with respect to r, and for each fixed r, $\beta(r, s)$ is decreasing to zero with respect to s. The term $\nabla V(x)f(x)$ denotes the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at any point x.

II. Problem Statement

Consider the following class of uncertain nonlinear systems,

$$\dot{x} = Ax + \phi(x, u) + G(t, x, u)\theta + Dw(t), \quad (1)$$
$$y = Cx, \quad (2)$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the measurable output vector, $u \in \mathbb{R}^m$ is the control input vector, $\theta \in \mathbb{R}^q$ is a vector of unknown constant parameters, and $w \in \mathbb{R}^l$ is a vector of external disturbances. The matrices *A, C* and *D* are known, they have corresponding dimensions, and the pair *(A,C)* is detectable. The functions $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $G : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times q}$ are also known and they ensure uniqueness and existence of solutions for system (1) for all admissible disturbances. It is important to highlight that in [16] the functions $\phi(x, u)$ and $G(t, x, u)$ were considered depending only on the measurable output y.

The aim of this paper is to provide estimations of the state and parameter vectors, *i.e.* x and θ, respectively; only using the information of the output y and attenuating as much as possible the effects of the external disturbances w.

Let us consider that system (1)-(2) satisfies following assumptions.

Assumption 1. $||x||_\infty < +\infty$, $||u||_\infty < +\infty$, $||\omega||_\infty < +\infty$, *and* $||G(t, x(t), u(t)||_\infty < +\infty$ *for all* $t \geq 0$.

Assumption 2. *The functions* $\phi(u, x)$ *and* $G(t, x, u)$ *are globally Lipschitz with respect to x; and the parameter vector* θ *is bounded,* i.e. *for all* $t \geq 0$*, all* $x, \hat{x} \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^m$*, the following conditions are satisfied*

$$||\phi(x, u) - \phi(\hat{x}, u)|| \leq L_\phi||x - \hat{x}||,$$
$$||G(t, x, u) - G(t, \hat{x}, u)|| \leq L_g||x - \hat{x}||,$$
$$||\theta|| \leq \theta^+,$$

with L_ϕ*,* L_ϕ*,* $\theta^+ > 0$ *being some known positive constants.*

III. Preliminaries

Consider the following nonlinear system,

$$\dot{x} = f(x, w), \quad (3)$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^l$ is the external disturbances, and $f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ is a locally Lipschitz function. For an initial condition $x_0 \in \mathbb{R}^n$ and an external disturbance $w \in \mathcal{L}_\infty$, denote the solution by $x(t, x_0, w)$ for any $t \geq 0$ for which the solution exists.

The following stability properties for system (3) are introduced (for more details see [16], [17], [18], [19] and [20]).

Definition 1. [17]. *The system* (3) *is said to be* Input-to-State Stable (ISS) *if for any* $w \in \mathcal{L}_\infty$ *and any* $x_0 \in \mathbb{R}^n$ *there exist some functions* $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ *such that*

$$||x(t, x_0, w)|| \leq \beta(||x_0||, t) + \gamma(||w||_\infty), \forall t \geq 0.$$

Lemma 1. [16]. *Let* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be a smooth function. If there exist some positive constants* $\psi_1, \psi_2, \psi_3, \psi_4 > 0$ *and* $\gamma \in (0, 1]$ *such that*

$$\psi_1||x||^2 \leq V(x) \leq \psi_2||x||^2,$$
$$\nabla V(x)f(x, \omega) \leq -\psi_3 V^\gamma(x), \forall ||x|| \geq \mu := \psi_4||\omega||_\infty,$$

then system (3) *is* ISS *with respect to the input* ω*. Moreover, the following bounds are satisfied:*

$$||x(t, x_o, \omega)|| \leq \begin{cases} \frac{\psi(t)}{\psi_1}, \ \forall\gamma \in (0, 1), \ \forall t \leq T(x_0), \\ e^{-\frac{\psi_3}{2}t}\sqrt{\frac{\psi_2}{\psi_1}}||x_0||, \ \gamma = 1, \ \forall t \leq T(x_0), \end{cases}$$
$$||x(t, x_o, \omega)|| \leq \sqrt{\frac{\psi_2}{\psi_1}}\mu, \ \forall t > T(x_0),$$

where $\psi(t) = (\psi_2^{1-\gamma}||x_0||^{2(1-\gamma)} - \psi_3(1-\gamma)t)^{\frac{1}{2(1-\gamma)}}$ *and*

$$T(x_0) \leq \begin{cases} \max\left(0, \frac{\psi_2^{1-\gamma}||x_0||^{2(1-\gamma)} - \psi_1^{1-\gamma}\mu^{2(1-\gamma)}}{\psi_3(1-\gamma)}\right), \ \forall\gamma \in (0, 1), \\ \max\left(0, \frac{2[\ln(||x_0||) - \ln(\mu)]}{\psi_3}\right), \ \gamma = 1. \end{cases}$$

Let us consider the following interconnected nonlinear system

$$\dot{x}_1 = f_1(x_1, x_2, \omega), \tag{4}$$
$$\dot{x}_2 = f_2(x_1, x_2, \omega), \tag{5}$$

where $x_i \in \mathbb{R}^{n_i}, \omega \in \mathbb{R}^l$, and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^l \to \mathbb{R}^{n_i}$ ensures existence of the system solutions at least locally, for $i = 1, 2$. Suppose that there exist ISS Lyapunov functions V_1 and V_2, for (4) and (5), respectively; and some functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in \mathcal{K}_\infty$, $\gamma_i, \chi_i \in \mathcal{K}$, such that, for all $x_i \in \mathbb{R}^{n_i}$ and any $w \in \mathcal{L}_\infty$, the following hold:

$$\psi_{i1}(||x_i||) \leq V_i(x_i) \leq \psi_{i2}(||x_i||), i = 1, 2, \tag{6}$$
$$V_1(x_1) \geq \max[\chi_1(V_2(x_2)), \gamma_1(||w||)]$$
$$\Rightarrow \nabla V_1(x_1)f_1(x_1, x_2, \omega) \leq -\psi_{13}(V_1(x_1)), \tag{7}$$
$$V_2(x_2) \geq \max[\chi_2(V_1(x_1)), \gamma_2(||w||)]$$
$$\Rightarrow \nabla V_2(x_2)f_2(x_1, x_2, \omega) \leq -\psi_{23}(V_2(x_2)). \tag{8}$$

Thus, the following nonlinear small-gain result is introduced, in terms of ISS Lyapunov properties, for the interconnected system (4)-(5).

Theorem 1. [18]. *Suppose that the interconnected system* (4)-(5) *has ISS Lyapunov functions* V_1 *and* V_2 *satisfying the conditions* (6)-(8)*. Then, the system* (4)-(5) *is ISS if*

$$\chi_1 \circ \chi_2(r) < r, \ \forall r > 0. \tag{9}$$

IV. Adaptive Sliding-Mode Observer

Let us introduce the following adaptive observer

$$\dot{\Omega} = (A - LC)\Omega + G(t, \hat{x}, u), \tag{10}$$
$$\dot{\hat{\theta}} = \Gamma\Omega^T C^T \lceil y - C\hat{x}\rfloor^\alpha, \tag{11}$$
$$\dot{\hat{x}} = A\hat{x} + \phi(\hat{x}, u) + G(t, \hat{x}, u)\hat{\theta} + L(y - C\hat{x})$$
$$+ kD\text{sign}[F(y - C\hat{x})] + \Omega\dot{\hat{\theta}}, \tag{12}$$

Define the function $\lceil a\rfloor^\alpha := |a|^\alpha \text{sign}(a)$, for any $\alpha \in [0, 1)$ and any $a \in \mathbb{R}$; and the function $\text{sign}[q] := \frac{q}{||q||}$, for any $q \in \mathbb{R}^m$, also $\hat{\theta} \in \mathbb{R}^q$ and $\hat{x} \in \mathbb{R}^n$ are the estimations for θ and x, respectively; and $\Omega \in \mathbb{R}^{n \times q}$ is an auxiliary variable. If the signal G is persistently exciting (PE), then, due to the filtering property of the variable Ω, the variable $C\Omega$ is also PE. Note that the function $\lceil y - C\hat{x}\rfloor^\alpha$ in (11) is understood in the component-wise sense. The observer matrix gain $L \in \mathbb{R}^{n \times p}$ has to be selected such that $(A - LC)$ is Hurwitz, $\Gamma = \Gamma^T > 0 \in \mathbb{R}^{q \times q}$, while $F \in \mathbb{R}^{l \times p}$, $k \geq 0$ are designed later.

Let us consider that the following assumption holds.

Assumption 3. *Let* $0 < \varrho_{\min} \leq \sigma_{\min}(C\Omega(t))$ *for all* $t \geq 0$ *and* $||C\Omega||_\infty \leq \varrho_{\max} < +\infty$.

The existence of $\varrho_{\max}$ is guaranteed by Assumption 1 and the fact that matrix $(A - LC)$ is Hurwitz; while if the matrix $C\Omega$ has a full column rank then the existence of $\varrho_{\min}$ is also assured. The values of $\varrho_{\max}$ and $\varrho_{\min}$ can be evaluated numerically during experiments.

A. Nonlinear Parameter Identification Algorithm

Define $\tilde{\theta} := \hat{\theta} - \theta$ and $\delta := \tilde{x} + \Omega\tilde{\theta}$ with $\tilde{x} := x - \hat{x}$. Then, the error dynamics are given by:

$$\dot{\tilde{\theta}} = -\Gamma\Omega^T C^T \lceil C\Omega\tilde{\theta} - C\delta\rfloor^\alpha, \tag{13}$$
$$\dot{\delta} = (A - LC)\delta + D(\omega - k\text{sign}[F(y - C\hat{x})])$$
$$+ \Delta_\phi(x, \hat{x}, u) + \Delta_g(t, x, \hat{x}, u)\theta, \tag{14}$$

where $\Delta_\phi(x, \hat{x}, u) := \phi(x, u) - \phi(\hat{x}, u)$ and $\Delta_g(t, x, \hat{x}, u) := G(t, x, u) - G(t, \hat{x}, u)$. The ISS convergence properties of the error dynamics (13) with respect to the input δ for $\alpha \in [0, 1)$ are given by the following result.

Lemma 2. [16]. *Let Assumption 3 be satisfied. Then, the error dynamics* (13)*, with* $\alpha \in [0, 1)$ *and* $\Gamma = \Gamma^T > 0$*, is ISS with respect to* δ *and the trajectories satisfy:*

$$||\tilde{\theta}(t)|| \leq \sqrt{r_1}\left(r_2^{\frac{\alpha-1}{2}}||\tilde{\theta}_0||^{1-\alpha} + \psi_{\tilde{\theta}}t\right)^{\frac{1}{1-\alpha}}, \ \forall t \leq T_{\tilde{\theta}_0}, \tag{15}$$
$$||\tilde{\theta}(t)|| \leq \sqrt{\frac{r_1}{r_2}}\mu_{\tilde{\theta}}, \ \forall t > T_{\tilde{\theta}_0}, \tag{16}$$

where $\tilde{\theta}_0 := \tilde{\theta}(0)$, $\psi_{\tilde{\theta}} := 2^{\alpha-1}\alpha^2\varrho_{\min}^{\alpha+1}(2r_2)^{\frac{\alpha+1}{2}}/2(1+\alpha)$, $r_1 := 2\lambda_{\max}(\Gamma)$, $r_2 := 2\lambda_{\min}(\Gamma)$ *and*

$$\mu_{\tilde{\theta}} = \frac{p^{\frac{1}{\alpha+1}}[(1+\alpha)^\alpha + 2^{2-\alpha}\alpha^{\alpha-1}]^{\frac{1}{\alpha+1}}}{\varrho_{\min}(1+\alpha)^{\frac{\alpha}{\alpha+1}}}||C||||\delta||_\infty,$$

$$T_{\tilde{\theta}_0} \leq \max\left[0, \frac{2^{1-\alpha}(1+\alpha)\left(||\tilde{\theta}_0||^{1-\alpha} - \mu_{\tilde{\theta}}^{1-\alpha}\right)}{\alpha^2\varrho_{\min}^{\alpha+1}r_2}\right],$$

for any $\tilde{\theta}_0 \in \mathbb{R}^q$, *and* p *the dimension of the output.*

The results of Lemma 2 establish that the trajectories of the error dynamics (13), *i.e.* $\tilde{\theta}(t)$, enter into the bound $\sqrt{r_1/r_2}\mu_{\tilde{\theta}}$ in a finite time for any $\alpha \in [0, 1)$. It is clear that the size of such a bound depends on α. In the same vein, the following lemma illustrates the ISS convergence properties of the error dynamics (14) with respect to the inputs $\tilde{\theta}$ and ω.

Lemma 3. *Let Assumption 1 and 2 hold. Assume that* $k = \|\omega\|_\infty$ *and the following matrix inequalities*

$$PD \leq C^T F^T, \tag{17}$$

$$\begin{bmatrix} \zeta & \frac{\beta}{2}I_n & P & P \\ \star & -\Lambda^{-1} & 0 & 0 \\ \star & \star & -\Lambda & 0 \\ \star & \star & \star & -\frac{\varepsilon}{\eta_2^2}I_n \end{bmatrix} \leq 0, \tag{18}$$

$$\zeta := (A - LC)^T P + P(A - LC) + \eta_1 P + \sigma C^T C,$$

are feasible for $0 < P^T = P \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{l\times p}$, $L \in \mathbb{R}^{n\times p}$, $0 < \Lambda^T = \Lambda \in \mathbb{R}^{n\times n}$, $\eta_1 := 2(L_\phi + L_g\theta^+)$, $\eta_2 := \eta_1||\Omega||_\infty$, *and some positive constants* $\beta, \varepsilon, \sigma > 0$. *Thus, system* (14) *is ISS with respect to the inputs* $\tilde{\theta}$ *and* ω. *Furthermore, the trajectories satisfy:*

$$||\delta(t)|| \leq e^{-\frac{\psi_\delta}{2}t}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||\delta_0||, \ \forall t \leq T_{\delta_0}, \tag{19}$$

$$||\delta(t)|| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\mu_\delta, \ \forall t > T_{\delta_0}, \tag{20}$$

where $\tilde{\delta}_0 := \delta(0)$, $\psi_\delta := \alpha_\delta(1-\rho)/\lambda_{\max}(P)$, $\alpha_\delta := \beta\lambda_{\min}(P) + \sigma\|C\|^2/2$, $\rho \in (0, 1)$ *and*

$$\mu_\delta = \sqrt{\frac{2\varepsilon + \sigma\varrho_{\max}^2}{2\rho\alpha_\delta}}||\tilde{\theta}||_\infty + \sqrt{\frac{8||F||^2}{\rho\alpha_\delta\sigma}}||w||_\infty,$$

$$T_{\delta_0} \leq \max\left(0, \frac{2[\ln(||\delta_0||) - \ln(\mu_\delta)]}{\psi_\delta}\right),$$

for any $\delta_0 \in \mathbb{R}^n$.

All the proofs of the proposed results are omitted due to the lack of space. It is clear that the size of μ_δ could be minimized in order to attenuate the effects of w. On the other hand, note that condition (17) introduces some structural restrictions. Particularly, one has that the triple (A, D, C) must not have invariant zeros, and the relative degree of the output y with respect to the input w must be equal to one. The feasibility of (18) can be ensured, for small enough η_1 and η_2, due to the fact that the pair (A, C) is detectable. In this sense, it is worth highlighting that the matrix inequality (20) can be easily rewritten to obtain an LMI with respect to the matrices P and $Y := PL$ fixing the values for β, σ, ε and Λ.

B. Convergence of the Adaptive Observer

In order to show the convergence properties of the adaptive observer (10)-(12), the statements provided by Lemmas 2, 3, and Theorem 1 are applied. The following result shows that the interconnected error system (13)-(14) is ISS with respect to the external disturbances ω, for any $\alpha \in [0, 1)$.

Theorem 2. *Let Assumptions 1, 2 and 3 hold. Let the matrix inequalities* (17), (18) *and*

$$\left(\frac{(p\kappa_\delta)^{\frac{2}{\alpha+1}}\|C\|^2(2\varepsilon + \sigma\varrho_{\max}^2)\lambda_{\min}(P)}{\varrho_{\min}^2(1+\alpha)^{\frac{2\alpha}{\alpha+1}}\lambda_{\max}(P)\rho\alpha_\delta}\right) < 1, \tag{21}$$

be feasible for $0 < P^T = P \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{l\times p}$, $L \in \mathbb{R}^{n\times p}$, $0 < \Lambda^T = \Lambda \in \mathbb{R}^{n\times n}$, $0 < \Gamma^T = \Gamma \in \mathbb{R}^{q\times q}$, *constants* $\alpha_\delta = \beta\lambda_{\min}(P) + \sigma\|C\|^2/2$, $\eta_1 = 2(L_\phi + L_g\theta^+)$, $\eta_2 = \eta_1||\Omega||_\infty$, $\kappa_\delta := (1+\alpha)^\alpha + 2^{2-\alpha}\alpha^{\alpha-1}$, $\beta, \varepsilon, \sigma > 0$, $\alpha \in [0, 1)$, $r_1 = 2\lambda_{\max}(\Gamma)$, $\rho \in (0, 1)$ *and* p *the dimension of the output. Then, the interconnected error system* (13)-(14) *is ISS with respect to the input* w.

Theorem 2 implies that the estimation error $e := x - \hat{x} = \delta + \Omega\tilde{\theta}$ is also ISS with respect to ω, for any $\alpha \in [0, 1)$, since

$$||e(t)|| \leq (1 + ||\Omega||)\left\|\begin{bmatrix}\tilde{\theta}(t)\\ \delta(t)\end{bmatrix}\right\|, \ \forall t \geq 0.$$

In the ideal case, *i.e.* $w \equiv 0$ and $k = 0$, the estimations $\hat{\theta}$ and $\hat{x}$ converge to their real values and the rate of convergence for $\tilde{\theta}$ is faster than exponential for any $\alpha \in [0, 1)$. For the perturbed case, *i.e.* $w \not\equiv 0$ with $k = ||w||_\infty$, the effect of the external disturbance w is completely attenuated.

V. Simulation Results

Let us consider a robotic system depicted by Fig. 1.

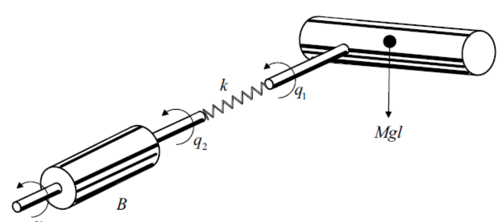

Figure 1. Model of a single-link flexible-joint robot manipulator.

The dynamics of such a system is given as follows

$$J_1\ddot{q}_1 + Mgl\sin(q_1) + k(q_1 - q_2) = w(t),$$
$$J_m\ddot{q}_2 + c_f\dot{q}_2 - k(q_1 - q_2) = u(t),$$

where q_1 and q_2 are the angular position of the link and motor, respectively; J_1 and J_m denote the inertia of the link and the motor, M is the mass of the link, g is the gravitational acceleration, l represents the length of the link and $c_f = 1$ is the friction coefficient of the motor which is the unknown parameter to estimate. The numerical values of the parameters of the system are shown in Table I.

Table I
PARAMETERS OF THE ROBOT MANIPULATOR.

Parameters	Value
J_1	0.8$[Kgm^2]$
J_m	1$[Kgm^2]$
g	9.8 $[m/s^2]$
l	0.6 $[m]$
k	20 $[Nm/\text{rad}]$

Define $\zeta_1 = \dot{q}_1$, $\zeta_2 = \dot{q}_2$, $x = [x_1, x_2, x_3, x_4]^T := [q_1, \zeta_1, q_2, \zeta_2]^T$ and $\theta = c_f$. Thus, the system in state-space form can be written as follows

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_1} & 0 & \frac{k}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 \\ -\frac{Mgl}{J_1}\sin(x_1) \\ 0 \\ \frac{1}{J_m}u \end{bmatrix}}_{\phi(x,u)}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{J_m}x_4 \end{bmatrix}}_{G(t,x,u)} \theta + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{D} w(t),$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x.$$

It can be shown that such a system satisfies Assumptions 1 and 2. Let us consider that the initial conditions for the system are $x(0) = [-2, -2, -1.5, 2]^T$ while the initial conditions for the adaptive observer (10)-(12) are given as $\Omega(0) = 0$, $\hat{\theta}(0) = 0$ and $\hat{x}(0) = 0$. For simulation purposes, the input is given as $u(t) = \sin(0.2t) + 0.5\sin(3t) + 2\cos(50t)$ while the disturbance term is taken as $w(t) = 0.3\cos(2t) + 0.1$.

Fixing $\Gamma = 630$, $k = 1$ and $\Lambda = I_4$, one can use SeDuMi solver among Yalmip in Matlab to find a solution for the matrix inequalities (17) and (18), and thus, verify (21). The following feasible solution is found with $L_{\phi_1} = 7$, $L_{g_1} = 2$, $\|\Omega\|_\infty = 0.492$,

$$P = \begin{pmatrix} 14.3132 & -0.2447 & 0.1127 & 0.0013 \\ -0.2447 & 21.0065 & -8.7811 & -0.1673 \\ 0.1127 & -8.7811 & 12.1135 & -0.3947 \\ 0.0013 & -0.1673 & -0.3947 & 0.0395 \end{pmatrix},$$

$$L = \begin{pmatrix} 30.2427 & 0.4684 \\ -25.0825 & 93.6232 \\ -0.3414 & 127.9623 \\ 18.4329 & 1412.0191 \end{pmatrix},$$

$$F = \begin{pmatrix} 293.5014 & 322.8523 \end{pmatrix}, \ \beta = 0.6324, \ \varepsilon = 1023.4, \ \sigma = 281.3023.$$

The corresponding simulations have been done in Matlab with Euler explicit discretization method and sampling time equal to 0.001. Based on some simulations, one can show that Assumption 3 holds with $\varrho_{max} = 0.008$, $\varrho_{min} = 0.001$ and $\rho = 0.9$. Taken all the corresponding values, one can show that

$$\left. \left(\frac{(p\kappa_\delta)^{\frac{2}{\alpha+1}} \|C\|^2 (2\varepsilon + \sigma\varrho_{\max}^2)\lambda_{\min}(P) r_1}{\varrho_{\min}^2 (1+\alpha)^{\frac{2\alpha}{\alpha+1}} \lambda_{\max}(P)\rho\alpha_\delta\sigma} \right) \right|_{\alpha=0} = 0.619,$$

$$\left. \left(\frac{(p\kappa_\delta)^{\frac{2}{\alpha+1}} \|C\|^2 (2\varepsilon + \sigma\varrho_{\max}^2)\lambda_{\min}(P) r_1}{\varrho_{\min}^2 (1+\alpha)^{\frac{2\alpha}{\alpha+1}} \lambda_{\max}(P)\rho\alpha_\delta\sigma} \right) \right|_{\alpha=0.5} = 0.169,$$

$$\left. \left(\frac{(p\kappa_\delta)^{\frac{2}{\alpha+1}} \|C\|^2 (2\varepsilon + \sigma\varrho_{\max}^2)\lambda_{\min}(P) r_1}{\varrho_{\min}^2 (1+\alpha)^{\frac{2\alpha}{\alpha+1}} \lambda_{\max}(P)\rho\alpha_\delta\sigma} \right) \right|_{\alpha=1} = 0.038,$$

and thus, the conditions of Theorem 2 are satisfied. The simulation results for different values of $\alpha \in [0, 1)$ are depicted by Figs. 2, 3 and 4. It is clear that due to the disturbances, the corresponding estimations converge to a neighborhood of the real value but faster than the linear algorithm, *i.e.* when $\alpha = 1$.

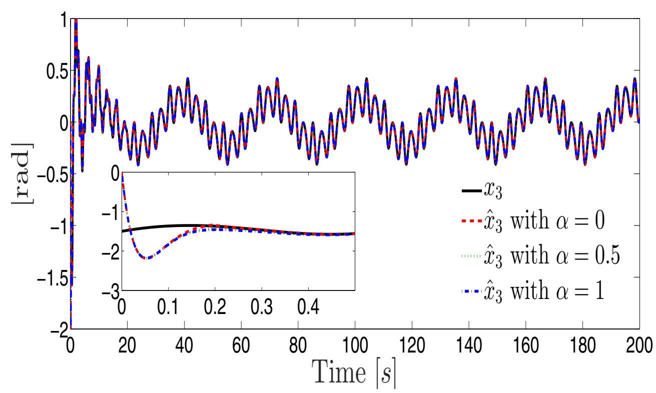

Figure 2. Estimation of x_3.

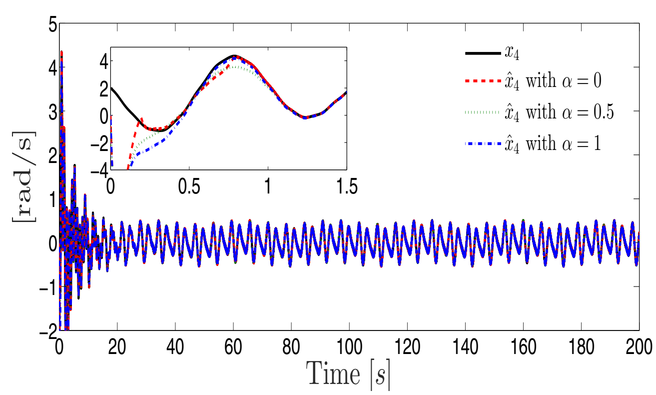

Figure 3. Estimation of x_4.

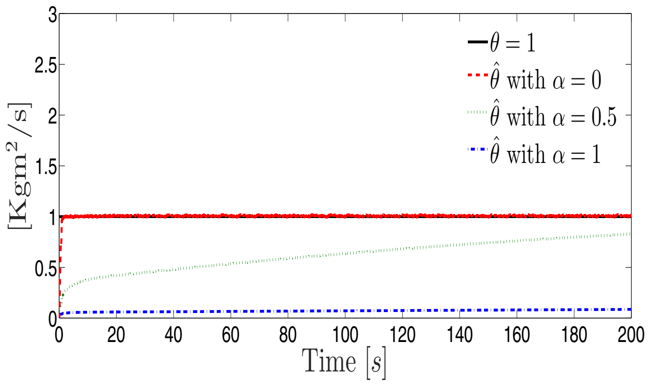

Figure 4. Estimation of θ.

In order to better illustrate that the size of the convergence neighborhood can be improved with respect to the selection of α, the following error index is proposed:

$$e_{RMS}(t) = \left(\frac{1}{\Delta T} \int_{t-\Delta t}^{t} ||\tilde{\theta}(\tau), e(\tau)||^2 d\tau \right)^{\frac{1}{2}},$$

where $\Delta T = 2$ is a time window width in which the corresponding signal is evaluated. The results are depicted by Fig. 5. In this case, it is clear that for $\alpha = 0.5$ provides the better result.

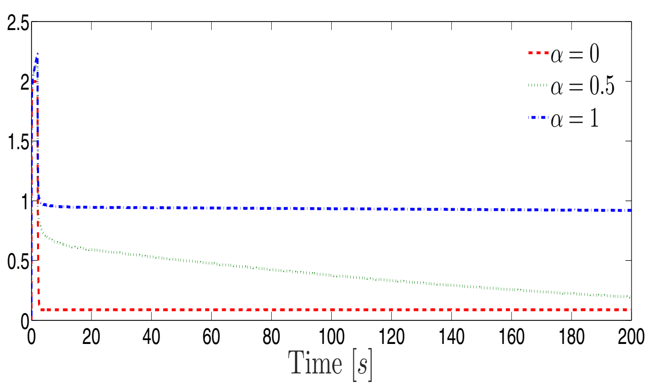

Figure 5. Estimation Error Index $e_{RMS}(t)$.

VI. Conclusions

This paper contributes to a nonlinear adaptive sliding-mode observer based on a nonlinear parameter identification algorithm for uncertain nonlinear systems. The proposed nonlinear adaptive observer is an extension of the one presented by [16]. This extends the class of nonlinear systems for which the adaptive observer can be applied allowing to have a parameter distribution matrix and nonlinear terms depending on the full state and the input. In [16], such functions were considered just depending on the measurable output and input. The proposed parameter identification algorithm provides an ultimate bound for the state and parameter estimation error and attenuating the effects of the external disturbances. The synthesis of the observer is given in terms of LMIs. The proofs are developed based on the Lyapunov function approach and the input-to-state stability theory. Some simulation results illustrate the efficiency of the proposed nonlinear adaptive sliding-mode observer.

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