Algorithms for Computing Cubatures Based on Moment Theory

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Abstract Quadrature is an approximation of the definite integral of a function by a weighted sum of function values at specified points, or nodes, within the domain of integration. Gaussian quadratures are constructed to yield exact results for any polynomials of degree $2r-1$ or less by a suitable choice of r nodes and weights. Cubature is a generalization of quadrature in higher dimension.

In this article we elaborate algorithms to compute all minimal cubatures for a given domain and a given degree. We propose first an algorithm in symbolic computation to characterize all cubatures of a given degree with a fixed number of nodes. The determination of the nodes and weights is then left to the computation of the eigenvectors of the matrix identified at the characterization stage and can be performed numerically.

The characterisation of cubatures on which our algorithms are based stems from moment theory. We formulate the results there in a basis independent way : Rather than considering the moment matrix, the central object in moment problems, we introduce the underlying linear map from the polynomial ring to its dual, the Hankel operator. This makes natural the use of bases of polynomials other than the monomial basis, and proves to be computationally relevant, either for numerical properties or to exploit symmetry.

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1 Introduction

Quadrature and cubature generally refer to the approximation of integrals. Among the possible interpretations, this article looks at the approximation of the integral of a function f over a domain $\mathcal{D}$ in $\mathbb{R}^n$ by a weighted sum of evaluations of f at well chosen points, known as *nodes*, in $\mathcal{D}$. The nodes shall be independent of the function f and the weights are preferably positive. The criteria best suited to choose the nodes and weights can be open for debate and might depend on the intention or application. In this paper we adopt the degree criterion that presides over the well known Gaussian quadratures in the univariate case : the formula shall be exact for all polynomials up to a certain degree d. The algebraic techniques developed in this manuscript can nonetheless be adapted for the formula to be exact on alternative subspaces of polynomials, or even function spaces. Minimizing the number of nodes used for the criterion to be satisfied is the spirit in which the techniques are developed.

In dimension $n = 1$, Gaussian quadratures optimally achieve an exact formula for degree $2r - 1$ with r nodes. These nodes are the roots of the orthogonal polynomial of degree r associated with the considered integral operator. In higher dimension, the analogous problem does not enjoy such a clear-cut answer. Relevant integration schemes are still being investigated for standard domains like the simplex and the hypercube. Those are necessary for the discretization of partial differential equations, whether in high degree for isogeometric analysis [36, 52] or in high dimensions [11, 46]. A simple product of Gaussian quadratures is "cursed by dimensionality". Our original motivation resides in the use of efficient integration schemes in geometric modeling [24, 34, 35].

Cubatures are the multidimensional analogues of quadratures. We mostly consider the linear forms

$$\begin{array}{rccc} \Omega: & \mathbb{R}[\mathrm{x}] & \to & \mathbb{R} \\ & p & \mapsto & \int_{\mathbb{R}^n} p(x) d\mu(x) \end{array}$$

where μ is a positive Borel measure on $\mathbb{R}^n$, the closed support $\operatorname{supp}\mu$ of which is compact, and $\mathbb{R}[\mathrm{x}] = \mathbb{R}[x_1, \dots, x_n]$ denotes the $\mathbb{R}$-linear space of polynomials with n variables. A linear form

$$\begin{array}{rccc} \Lambda: & \mathbb{R}[\mathrm{x}] & \to & \mathbb{R} \\ & p & \mapsto & \displaystyle\sum_{j=1}^{r} a_j\, p(\xi_j) \end{array}$$

with $r \in \mathbb{N}$, $a_j > 0$ and $\xi_j \in \mathbb{R}^n$ pairwise distinct, is called a *cubature of degree* d for Ω if it satisfies $\Lambda(p) = \Omega(p)$ for all $p \in \mathbb{R}[\mathrm{x}]_{\leq d}$, where $\mathbb{R}[\mathrm{x}]_{\leq d}$ is the space of polynomials of degree d or less. The points ξ_j are the *nodes* and the numbers a_j are the *weights* of the cubature. Such a cubature is called *inside* if the nodes ξ_j lie on $\operatorname{supp}\mu$ and *minimal* if the number of nodes r is minimal.

When Ω is definite, i.e. when $\Omega(p^2) = 0$ implies that $p = 0$, at least for $p \in \mathbb{R}[\mathrm{x}]_{\leq d}$, it will appear clearly in our constructive approach that a cubature of degree d cannot have less than $\binom{\lfloor d/2 \rfloor + n}{n}$ nodes. This is known as Mysovskikh's theorem [21, Theorem 3.7.4]. A cubature that attains the lower bound is called a *Gaussian cubature* as it is then analogous to the univariate case. Gaussian cubatures are rather rare and the measures that admit Gaussian cubatures for any degree are intriguing [47, 67]. A review of sharper lower bounds can be found in [54], [15, Section 7].

The existence of inside cubatures of any degree is guaranteed by Tchakaloff's theorem [73] [70, Section 3.3]. It also provides a bound on the number of nodes of minimal cubatures, namely $\binom{d+n}{n}$, where d is the degree of the cubature. Analogous results [6, 58] were later proved with less constraint on the measure μ and recently this upper bound was revisited for cubatures in the plane [62]. However, the proof of Tchakaloff's theorem is not constructive.

Radon's work [59] marked a starting point in the construction of cubatures thanks to orthogonal polynomials. As presented in reviews on the topic [14, 70], cubatures have been constructed using several techniques based on zeros of multivariate orthogonal polynomials, on product of Gaussian quadratures, or on solutions of multivariate nonlinear systems. As a result, cubature formulae are available for a large variety of regions and weight functions [16, 70].

More recently, a characterization of cubatures based on *moment matrices* was initiated in [23]. Additional contributions in this direction were made regarding Gaussian cubatures [42] or on a numerical method to compute some of them [1]. They are based on algebraic properties of moment matrices that emerged in the realm of global optimization [41, 45].

The moment matrix relating to the linear form Λ can be interpreted as the matrix of a linear map from the polynomial ring to its dual in the monomial basis. In the univariate case, a moment matrix is Hankel. Alternative bases are relevant, for instance when symmetry is to be accounted for [13] or when better numerical stability is sought [7], [27, Section 5.3]. Rather than focus on the moment matrix, we introduce the Hankel operator as the central object. An algebraic characterisation of cubatures can then be expressed in terms of the rank and the positivity of the Hankel operator. This characterisation translates naturally into linear algebra for any choice of a basis for the polynomial ring.

We present an algorithm in symbolic computation that provides necessary and sufficient conditions for the existence of a cubature of degree d with r nodes. The algorithm improves on Gauss-Bareiss elimination by taking advantage of the positivity constraint. Beside being fraction free, the diagonalisation it performs avoids the complicated pivoting thanks to a pertinent observation about positive semidefinite matrices. The output of the algorithm consists of sets equalities and inequalities bearing on unknown generalized moments, up to a determined finite order. Each solution of these correspond in a unique way to a cubature. The nodes and the weights of the characterized cubature can then be computed as eigenvectors of a matrix identified at the first stage. This second step can safely be performed numerically.

The paper is organized as follows. Section 3 introduces the Hankel operator associated to a linear form and shows how to characterize cubatures with linear algebra. Section 4 describes the fraction and pivoting free algorithm to diagonalize parametric symmetric matrices over their locus of positivity. Section 5 makes precise how the tools exposed so far apply to algorithmically characterize the cubatures for a given integration form and compute its nodes and weights. Section 6 applies the algorithm to find all the minimal cubatures of degree 5 for the hexagon. We show that there is a one parameter family of such cubatures, when only two were known. We also demonstrate on this example how symmetry can be exploited to alleviate the computational cost.

In Section 2 we treat anew the well known univariate case of Gaussian quadratures with a view to its present generalization to cubatures. We then exhibit a number of related univariate problems that can be generalized with the multivariate techniques presented in this mansucript.

2 The univariate case : Gaussian quadratures

In this section, the classical quadrature problem is introduced (see *e.g.* [69, Chapter 3] or [70, Chapter 1.3] and references herein) and methods to determine Gaussian quadratures are presented. Though the subject is tied in with orthogonal polynomials (see *e.g.* [21, Chapter 1]), we favor here a presentation that extends to the multidimensional case that we shall deal with. Here, $\mathbb{R}[x]$ denotes the $\mathbb{R}-$vector space of polynomials in one variable and coefficients in $\mathbb{R}$. Then $\mathbb{R}[x]_{\leq d}$ is the space of polynomials of degree d and less.

Consider the linear form Ω on $\mathbb{R}[x]$ defined by

$$\Omega : \mathbb{R}[x] \to \mathbb{R}, p \mapsto \int_a^b p(x)\omega(x)dx,$$

where $[a, b] \subset \mathbb{R}$ is an interval (finite or infinite) and ω is a nonnegative *weight function* defined on this interval. It is typically assumed that this function is measurable and and non negative on $[a, b]$ and even

that
$$\int_a^b p^2(x)\omega(x)dx = 0 \quad \Rightarrow \quad p(x) = 0.$$
Furthermore, the *moments* are given by
$$m_k = \int_a^b x^k \omega(x)dx, \text{ for } k \in \mathbb{N}.$$

A linear form Λ on $\mathbb{R}[x]$ defined by
$$\Lambda : \mathbb{R}[x] \to \mathbb{R}, p \mapsto \sum_{j=1}^{r} a_j p(\xi_j),$$
with $r \in \mathbb{N}$, $r \geq 1$, $a_j \in \mathbb{R}$, $a_j > 0$, and $\xi_j \in \mathbb{R}$, pairwise distinct, is called a *quadrature of degree* d for Ω if it satisfies
$$\Omega(p) = \Lambda(p), \text{ i.e. } \int_a^b p(x)\omega(x)dx = \sum_{j=1}^{r} a_j p(\xi_j), \quad \forall p \in \mathbb{R}[x]_{\leq d}$$
and if this equality does not hold for at least one polynomial p of degree $d+1$. The points ξ_j are the *nodes* and the coefficients a_j are the *weights* of the quadrature. Such a quadrature is called *inside* if the nodes ξ_j lie on the interval $[a,b]$ and *minimal* if the number of nodes r is minimal for a fixed degree d.

In this univariate case, minimal inside quadratures with positive weights are known and called *Gaussian quadratures* (see *e.g.* [69, Chapter 3.6] for more details). A Gaussian quadrature with r nodes is of degree $2r-1$.

In next paragraph we show two related ways of computing the nodes. This is done thanks to the properties of the moment matrices:
$$\left(\Omega(x^{i+j-2})\right)_{1\leq i,j\leq r+1} = \begin{pmatrix} \Omega(1) & \Omega(x) & \dots & \Omega(x^r) \\ \Omega(x) & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ \Omega(x^r) & \dots & \dots & \Omega(x^{2r}) \end{pmatrix} = \begin{pmatrix} m_0 & m_1 & \dots & m_r \\ m_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ m_r & \dots & \dots & m_{2r} \end{pmatrix} = \left(m_{i+j-2}\right)_{1\leq i,j\leq r+1}.$$
These are Hankel matrices. But rather than their displacement structure, let us observe that these are the matrices of the scalar product
$$\Phi : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}, (p,q) \mapsto \Omega(pq)$$
restricted to $\mathbb{R}$-vector space $\mathbb{R}[x]_{\leq r}$ endowed with the monomial basis $\{1, x, \dots, x^r\}$. The bilinear map defines a duality pairing to which is associated the injective map
$$\begin{array}{rccc} \widehat{\mathcal{H}} : & \mathbb{R}[x] & \to & \mathbb{R}[x]^* \\ & p & \mapsto & \Lambda_p, \end{array} \quad \text{where} \quad \begin{array}{rccc} \Lambda_p : & \mathbb{R}[x] & \to & \mathbb{R} \\ & q & \mapsto & \Lambda(pq). \end{array}$$

In the multivariate case we shall actually focus on this *Hankel operator* $\widehat{\mathcal{H}}$ rather than the bilinear map Φ.

Once the nodes of the cubature are found, the weights of a quadrature are generally obtained afterwards by solving the Vandermonde linear system
$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \xi_1 & \xi_2 & \dots & \xi_r \\ \vdots & \vdots & & \vdots \\ \xi_1^{r-1} & \xi_2^{r-1} & \dots & \xi_r^{r-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{r-1} \end{pmatrix} \tag{1}$$
obtained from the equations
$$\Lambda(x^k) = \sum_{j=1}^{r} a_j \xi_j^k = m_k = \Omega(x^k) \quad \forall k = 0, \dots, r-1. \tag{2}$$

2.1 Nodes as roots of orthogonal polynomials

Assume that a Gaussian quadrature of degree $2r-1$ exists for Ω, i.e. there exists a linear form

$$\Lambda : \mathbb{R}[x] \to \mathbb{R}, p \mapsto \sum_{j=1}^{r} a_j p(\xi_j)$$

with $a_j > 0$ and $\xi_j \in \mathbb{R}$ pairwise distinct such that $\Omega(x^k) = m_k = \Lambda(x^k)$ for all $k \le 2r-1$. Consider then the monic polynomial π_r of degree r whose roots are the nodes $\xi_1, \dots, \xi_r$

$$\pi_r(x) = \prod_{j=1}^{r} (x - \xi_j) = x^r - \tau_{r-1} x^{r-1} - \dots - \tau_0.$$

Since $\pi_r(\xi_j) = 0$ for all $j = 1, \dots, r$, we have $\Lambda(\pi_r) = 0$ and even $\Lambda(p\pi_r) = 0 \quad \forall p \in \mathbb{R}[x]$. Taking $p(x) = x^k$ for $k = 0, \dots, r$, we deduce by linearity of Λ that

$$\sum_{i=0}^{r-1} \tau_i \Lambda(x^{k+i}) - \Lambda(x^{k+r}) = 0 \quad \forall k = 0, \dots, r.$$

Thus, the vector $\begin{pmatrix} \tau_0 & \dots & \tau_{r-1} & -1 \end{pmatrix}^t$ is in the kernel of the Hankel matrix

$$H_1^{(r)} = \left(\Lambda(x^{i+j-2})\right)_{1 \le i,j \le r+1} = \begin{pmatrix} \Lambda(1) & \Lambda(x) & \dots & \Lambda(x^r) \\ \Lambda(x) & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ \Lambda(x^r) & \dots & \dots & \Lambda(x^{2r}) \end{pmatrix}.$$

By hypothesis $\Lambda(x^k) = m_k = \Omega(x^k)$ for $0 \le k \le 2r-1$. To obtain the nodes of Λ one can thus:

- Determine the polynomial π_r, through its coefficients $\tau_0, \dots, \tau_{r-1}$ in the monomial basis, by solving the Hankel linear system

$$\begin{pmatrix} m_0 & m_1 & \dots & m_{r-1} \\ m_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ m_{r-1} & \dots & \dots & m_{2r-2} \end{pmatrix} \begin{pmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \vdots \\ \tau_{r-1} \end{pmatrix} = \begin{pmatrix} m_r \\ m_{r+1} \\ \vdots \\ \vdots \\ m_{2r-1} \end{pmatrix}$$

- Determine the roots of π_r.

To make the link with a classical approach to Gaussian quadratures, we remark here that the polynomial π_r constructed above is the monic orthogonal polynomial of degree r for the scalar product Φ defined by Ω. Indeed, since Λ is a cubature of degree $2r-1$, we have

$$\Phi(p, \pi_r) = \Omega(p\pi_r) = \Lambda(p\pi_r) = 0, \quad \forall p \in \mathbb{R}[x]_{\le r-1}. \tag{3}$$

Orthogonal polynomials have real roots, within the support $[a, b]$ of the measure [72], as is expected for a quadrature.

2.2 Nodes as generalized eigenvalues

Adding some columns to the Hankel linear system above, which results in shifting left the columns of $H_1^{(r-1)}$, one observes

$$\underbrace{\begin{pmatrix} m_0 & m_1 & \dots & m_{r-1} \\ m_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ m_{r-1} & \dots & \dots & m_{2r-2} \end{pmatrix}}_{H_1^{(r-1)}} \underbrace{\begin{pmatrix} 0 & \dots & \dots & 0 & \tau_0 \\ 1 & \ddots & & \vdots & \tau_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & \tau_{r-1} \end{pmatrix}}_{M_x^{(r-1)}} = \underbrace{\begin{pmatrix} m_1 & m_2 & \dots & m_r \\ m_2 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ m_r & \dots & \dots & m_{2r-1} \end{pmatrix}}_{H_x^{(r-1)}},$$

that is

$$H_1^{(r-1)} M_x^{(r-1)} = H_x^{(r-1)}, \text{ where } H_q^{(r-1)} = \left(\Lambda(q\, x^{i+j-2})\right)_{1 \le i,j \le r} \text{ for } q \in \mathbb{R}[x]. \tag{4}$$

$M_x^{(r-1)}$ is the companion matrix of the polynomial π_r. Its eigenvalues are thus the roots of the polynomial π_r. Hence the generalized eigenvalues of the pair of Hankel matrices $(H_x^{(r-1)}, H_1^{(r-1)})$ are the sought nodes $\xi_1, \dots, \xi_r$.

Both $H_1^{(r-1)}$ and $H_x^{(r-1)}$ are real and symmetric matrices. As the matrix of a scalar product, $H_1^{(r-1)}$ is furthermore positive definite. Let v be a generalized eigenvector of $(H_x^{(r-1)}, H_1^{(r-1)})$ associated to the eigenvalue v, i.e. $H_x^{(r-1)} v = \xi H_1^{(r-1)} v$. Then $\xi = \frac{\bar{v}^T H_x v}{\bar{v}^T H_1 v} = \frac{v^T H_x \bar{v}}{v^T H_1 \bar{v}} = \bar{\xi}$. This proves that the generalized eigenvalues of $(H_x^{(r-1)}, H_1^{(r-1)})$ are real.

In the multivariate setting, the matrices of the Hankel operator will take the role of the above Hankel matrices. The way to generalize the companion matrix $M_x^{(r-1)}$ is to see it as the matrix of the multiplication by x, modulo π_r, expressed in the monomial basis $\{1, x, \dots, x^r\}$ of quotient space $\mathbb{R}[x]/(\pi_r)$.

2.3 Jacobi matrix and modified moment matrices

In the above, the matrices that come in the determination of a quadrature were interpreted as matrices of symmetric bilinear forms or linear maps in the monomial basis. Alternative bases are of interest from different points of view.

On one hand we considered the bilinear forms

$$\begin{aligned} \Phi : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}, (p,q) &\mapsto \int_a^b p(x) q(x) \omega(x) dx \\ \text{and} \quad \Phi_x : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}, (p,q) &\mapsto \int_a^b x p(x) q(x) \omega(x) dx \end{aligned}.$$

The matrices, in the monomial basis, of their restrictions to $\mathbb{R}[x]_{\le r-1}$ are $H_1^{(r-1)}$ and $H_x^{(r-1)}$. But other bases are relevant.

Consider first the basis $\widetilde{\Pi}$ defined by the orthonormal polynomials $\widetilde{\pi}_r$, obtained by normalizing the monic orthogonal polynomials π_r. Then

- The matrix of Φ in the basis $\widetilde{\Pi}$ is the identity.

- The matrix of Φ_x in the basis $\widetilde{\Pi}$ is the tridiagonal *Jacobi matrix*

$$\begin{pmatrix} a_0 & b_0 & 0 & \dots \\ b_0 & a_1 & b_1 & \ddots \\ 0 & b_1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $(a_r)_{r\in\mathbb{N}}$ and $(b_r)_{r\in\mathbb{N}}$ are sequences of real numbers that reflect the recurrence equation of order 2 satisfied by the orthonormal polynomials $\widetilde{\pi}_r$ (see *e.g.* [21, Corollary 1.3.10],[2, Chapter 1]):

$$\begin{cases} x\widetilde{\pi}_r(x) = b_r\widetilde{\pi}_{r+1}(x) + a_r\widetilde{\pi}_r(x) + b_{r-1}\widetilde{\pi}_{r-1}(x) & \forall r \geq 1 \\ x\widetilde{\pi}_0(x) = b_0\widetilde{\pi}_1(x) + a_0\widetilde{\pi}_0(x) \end{cases}$$

The matrix of the multiplication by x in $\mathbb{R}[x]/(\pi_r)$ is thus the truncated $r \times r$ Jacobi matrix. Its eigenvalues are the roots of the orthonormal polynomial $\widetilde{\pi}_r$, or equivalently the nodes $\xi_1, \dots, \xi_r$. See [69, Theorem 3.6.20] for the classical link between eigenvalues of tridiagonal matrices and roots of orthogonal polynomials.

One may also consider the matrices of Φ and Φ_x in a basis of orthogonal polynomials for another measure. This correspond to considering *modified* moments and provide some computational advantage in terms of numerical stability [7], [65].

Another interesting choice of alternative basis arises in the presence of symmetry. Assume that the weight function $\omega : [-a, a] \to \mathbb{R}$ is even. Then the odd moments are zero : $\Omega(x^{2k+1}) = 0$. If we simply reorder the monomials in a basis by listing first the even degree monomials and then the odd degree monomials, we observe that the matrix $\tilde{H}_1$ of Φ in this basis becomes block diagonal. The same is true of the matrix $\tilde{H}_{x^2}$ of Φ_{x^2} and the generalized eigenvalues of the pair of truncated matrices $(\tilde{H}_{x^2}, \tilde{H}_1)$ are $\xi_1^1, \dots, \xi_r^2$. This all generalizes to the multivariate case that will be developed in [13].

2.4 Related problems

The techniques presented in this paper have other applications than cubatures. Indeed, a number of classical problems can be formulated as we did for quadratures, in the univariate case, and cubatures, in the multivariate case.

In the univariate case, one seeks to retrieve pairs $\{(a_1, \xi_1), \dots, (a_r, \xi_r)\}$ from the *moments* $(\mu_k)_{k\in\mathbb{N}}$ or $(\mu_k)_{0\leq k\leq R}$ with R an upper bound of r under the assumption that

$$\mu_k = \sum_{j=1}^{r} a_j\, \xi_j^k.$$

In the case of quadratures, $\mu_k \in \mathbb{R}$ and we expect a_j and ξ_j to be in $\mathbb{R}$. In general though μ_k, a_j and ξ_j are in $\mathbb{C}$. Furthermore, the number of terms r might be an additional unknown of the problem. As demonstrated above the problem can be solved (uniquely) when $R = 2r - 1$ moments are available.

In the multivariate case, the input is indexed by $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ and each ξ_j belongs to $\mathbb{C}^n$. The relationships are given by

$$\mu_k = \sum_{j=1}^{r} a_j\, \xi_{j,1}^{k_1} \dots \xi_{j,n}^{k_n}.$$

An additional problem in the multivariate case, even when r is known, is to find an appropriate $\mathbb{K}$-vector space basis of the quotient algebra $\mathbb{K}[x_1, \dots, x_n]/I_\xi$, where I_ξ is the ideal of all polynomials vanishing on $\{\xi_1, \dots, \xi_r\} \subset \mathbb{C}^n$.

Prony's method for exponential interpolation [60]. One wants to retrieve $f : \mathbb{C} \to \mathbb{C}$ as

$$f(x) = \sum_{j=1}^{r} a_j e^{\lambda_j x}$$

from the knowledge of $\mu_k = f(k)$ with $k \in \mathbb{N}$. One retrieves the above problem by considering $\xi_j = e^{\lambda_j}$.

In its multivariate version $f : \mathbb{C}^n \to \mathbb{C}$ and

$$f(x) = \sum_{j=1}^{r} a_j e^{\langle \lambda_j , x \rangle}$$

where $x = (x_1, \dots, x_n)$, $\lambda_j \in \mathbb{C}^n$ and $\langle y, x \rangle = y_1 x_1 + \dots + y_n x_n$. With $\xi_{j1} = e^{\lambda_{j1}}$, ..., $\xi_{jn} = e^{\lambda_{jn}}$ the problem is then recast into

$$f(x) = \sum_{j=1}^{r} a_j \xi_j^x.$$

Under some natural assumptions, solutions have been proposed for the univariate case [56], for the multivariate case with a univariate resolution (projection method) [57] and with a multivariate approach [39, 66, 55].

Sparse interpolation [8]. Assuming a (multivariate) polynomial f has a small support

$$f(x) = \sum_{j=1}^{r} a_j x^{\alpha_j},$$

one wishes to retrieve the exponents $\alpha_j \in \mathbb{N}^n$ from evaluations of the polynomial at chosen points. For $k \in \mathbb{N}$, one chooses $\mu_k = f(p_1^k, \dots, p_n^k)$ where p_j are distinct prime numbers. From $\mu_0, \dots, \mu_{2r-1}$ one can retrieve $\xi_j = p^{\alpha_j} = p_1^{\alpha_{j1}} \dots p_n^{\alpha_{jn}}$ so that the exponent can be found by factorization [8, 37, 38]. Replacing the prime numbers by roots of unity, the authors in [26] proposed a symbolic-numeric solution.

For latest development in Prony's method and sparse interpolation, one can consult the material attached to the 2015 Dagstuhl seminar *Sparse modeling and multi-exponential analysis* (`http://www.dagstuhl.de/15251`).

Pole estimation, Padé approximant [31]. The input data are the coefficients of the Taylor expansion at $z = 0$ of a function $f : \mathbb{C} \to \mathbb{C}$ and one wishes to find the poles of the function.

$$f(x) = \sum_{j=1}^{r} \frac{a_j}{1 - \xi_j \, x} = \sum_{k \in \mathbb{N}} \mu_k \, x^k.$$

The multivariate version that can be approached is

$$f(x) = \sum_{j=1}^{r} \frac{a_j}{\langle \xi_j , x \rangle} = \sum_{k \in \mathbb{N}^n} \mu_k \, x^k.$$

In the univariate case all polynomials can be factored over $\mathbb{C}$ into degree 1 polynomials. Therefore the problem covers all rational functions with distinct poles. The restriction to denominators that can be factored into linear form is more restrictive in the multivariate case.

Shape-from-moment problem [29, 53]. The problem consists in recovering the vertices of a convex $n-$dimensional polytope $\mathcal{V}$ from its moments. *Projection methods* have been developed in [12, 29] based on moments in a direction $\delta \in \mathbb{R}^n$ and Brion's identities

$$\begin{cases} \frac{(k+n)!}{k!} \int_{\mathcal{V}} < x, \delta >^k \, dx = & \sum_{j=1}^{r} a_j < v_j, \delta >^{n+k} \qquad k \geq 0 \\ \qquad\qquad\qquad\qquad 0 = & \sum_{j=1}^{r} a_j < v_j, \delta >^{n-k} \quad 1 \leq k \leq n. \end{cases}$$

The coefficients a_j are nonzero real numbers and $< v_j, \delta >$ are the projections of the vertices v_j on the direction δ. Taking $\xi_j =< v_j, \delta >$, the formula can be recast into

$$\mu_k = \sum_{j=1}^{r} a_j \xi_j^k \quad k \in \mathbb{N},$$

where μ_k is related to the left hand side of the system of equations above. The set of projected vertices can thus be recovered. Different projections are then required and matching processes are presented in [12, 29]. The case $n = 2$ treated with complex moments in [22, 28, 53] is linked to this general case in [12].

A multidimensional treatment of the shape-from-moment problem is however impossible due to the lack, up to our knowledge, of an equivalent formula for the moments $\int_{\mathcal{V}} x^k dx$ with $k \in \mathbb{N}^n$.

Symmetric tensor decomposition [10]. The general (multivariate) problem is to find, for a homogeneous polynomial

$$f(z) = \sum_{|k|=d} \binom{d}{k} \mu_k \, z^k,$$

the minimal *rank* r such that there exist $(a_1, \xi_1), \ldots, (a_r, \xi_r) \in \mathbb{C} \times \mathbb{C}^n$ such that f can be written as a linear combination of linear forms to the d^{th} power:

$$f(z) = \sum_{i=1}^{r} a_i \, \langle \xi_i, z \rangle^d.$$

De-homogenizing the binary case ($n = 2$), we obtain an equivalent univariate problem

$$f(z) = \mu_d \, z^d + \ldots + \binom{d}{k} \mu_k \, z^k + \ldots + \mu_0 = \sum_{i=1}^{r} a_i \, (1 - \xi_i \, z)^d$$

initially solved by Sylvester [71].

3 Characterization of cubatures through Hankel operators

In the following, $\mathbb{K}$ denotes a subfield of $\mathbb{C}$. At some points we shall need to make clear whether $\mathbb{K}$ is $\mathbb{C}$, $\mathbb{R}$ or a *computable* extension of $\mathbb{Q}$, in the sense that exact arithmetic can be performed in $\mathbb{K}$. $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$ denotes the ring of polynomials in the variables $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{K}$ and $\mathbb{K}[\mathrm{x}]_{\leq \delta}$ the $\mathbb{K}-$vector space of polynomials of degree at most δ.

The *dual* of $\mathbb{K}[\mathrm{x}]$, the set of $\mathbb{K}-$linear forms on $\mathbb{K}[\mathrm{x}]$, is denoted by $\mathbb{K}[\mathrm{x}]^*$. A typical example of a linear form on $\mathbb{K}[\mathrm{x}]$ is the evaluation $\mathbb{e}_\xi$ at a point ξ of $\mathbb{K}^n$. It is defined by

$$\mathbb{e}_\xi : \mathbb{K}[\mathrm{x}] \to \mathbb{K}, p \mapsto p(\xi).$$

Other examples of linear forms on $\mathbb{K}[\mathrm{x}]$ are given by linear combinations of evaluations

$$\Lambda : \mathbb{K}[\mathrm{x}] \to \mathbb{K}, p \mapsto \sum_{j=1}^{r} a_j \mathbb{e}_{\xi_j}(p),$$

with $a_j \in \mathbb{K} \setminus \{0\}$ and $\xi_j \in \mathbb{K}^n$, or by the integration over a domain $\mathcal{D} \subset \mathbb{R}^n$

$$\Omega : \mathbb{R}[x] \to \mathbb{R}, p \mapsto \int_{\mathcal{D}} p(x)dx.$$

We shall say that $\Lambda : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ is a cubature if $\Lambda = \sum_{j=1}^{r} a_j \mathbb{e}_{\xi_j}$, with the *nodes* $\xi_j \in \mathbb{R}^n$, and the *weights* $a_j > 0$. This section is first aimed at characterizing such linear forms Λ through their *associated Hankel operators* : cubatures correspond to positive semi-definite Hankel operators of finite rank. Second, we present a criterion of existence for such a form Λ to exist from the knowledge of the values of Λ on $\mathbb{K}[\mathrm{x}]_{\le\delta}$, for some $\delta < \infty$: Hankel operators of a given finite rank can be uniquely determined from their restriction to $\mathbb{K}[\mathrm{x}]_{\le\delta}$ thanks to the flat extension criterion. Last, we shall show how to compute the nodes $\xi_1, \dots, \xi_r$ and the weights $a_1, \dots, a_r$ from the values of Λ on $\mathbb{K}[\mathrm{x}]_{\le\delta}$: The nodes of the cubature are the roots of the ideal formed by the kernel of the Hankel operator. They can be computed as generalized eigenvalues and eigenvectors.

When reviewing the above results as needed for cubatures, we strive to remain simple, providing the arguments and proofs that basically rely on linear algebra and matrix computations. A noteworthy feature of our review is that we free the results from the monomial basis by making the Hankel operator the central object rather than the moment matrix. Alternative bases are indeed useful, for instance when symmetry is to be accounted for [13] or when better numerical stability is sought [7], [27, Section 5.3]. Our formulations are thus readily amenable for application in a more varied context.

The concept of flat extension was introduced for the classical truncated moment problem [18, 19]. It was incorporated in a successful line of developments in global optimisation and algebraic geometry [40, 41]. Our reference text is the book chapter [44]. The link between cubatures and flat extensions was first expanded on in [23]; [42] then provided a criterion for the existence of a Gaussian cubature while [1] showed how to compute some cubatures as a global optimization problem. The connection between Hankel operators and multiplication maps, that allows one to compute the cubature nodes as an eigenvalue problem, was for instance used in [10, 55]. in the context of symmetric tensor decomposition and multivariate Prony's method.

3.1 Hankel operator associated to a linear form

For a linear form $\Lambda \in \mathbb{K}[\mathrm{x}]^*$, the associated *Hankel operator* $\widehat{\mathcal{H}}$ is the $\mathbb{K}$-linear map

$$\begin{array}{rccc} \widehat{\mathcal{H}}: & \mathbb{K}[\mathrm{x}] & \to & \mathbb{K}[\mathrm{x}]^* \\ & p & \mapsto & \Lambda_p, \end{array} \quad \text{where} \quad \begin{array}{rccc} \Lambda_p: & \mathbb{K}[\mathrm{x}] & \to & \mathbb{K} \\ & q & \mapsto & \Lambda(p\,q). \end{array}$$

The kernel

$$I_\Lambda := \{p \in \mathbb{K}[\mathrm{x}] \mid \Lambda_p = 0\} = \{p \in \mathbb{K}[\mathrm{x}] \mid \Lambda(p\,q) = 0,\ \forall q \in \mathbb{K}[\mathrm{x}]\}$$

of $\widehat{\mathcal{H}}$ is an ideal of $\mathbb{K}[\mathrm{x}]$. Let us observe that if $\Lambda = \sum_{j=1}^{r} a_j \mathbb{e}_{\xi_j}$, with $a_j \in \mathbb{K}$ and $\xi_j \in \mathbb{K}^n$, then the ideal

$$I(\xi) := \{p \in \mathbb{K}[\mathrm{x}] \,|\, p(\xi_i) = 0,\, 1 \le i \le r\}$$

is included in I_Λ. We shall see that the reciprocal is true as well.

The Hankel operator can be represented by an infinite matrix once we fix a basis for $\mathbb{K}[\mathrm{x}]$ and $\mathbb{K}[\mathrm{x}]^*$, but we shall examine the truncated operators to remain in the realm of finite dimension when it comes to matrices. If V is a $\mathbb{K}$-linear subspace of $\mathbb{K}[\mathrm{x}]$ we define $\widehat{\mathcal{H}}_{|V}$ thanks to the restrictions $\Lambda_{p|V}$ of Λ_p to V:

$$\begin{array}{rccc} \widehat{\mathcal{H}}_{|V}: & V & \to & V^* \\ & p & \mapsto & \Lambda_{p|V}, \end{array}$$

Assume V is the $\mathbb{K}$-linear span $\langle b_1, \ldots, b_s \rangle$ of a linearly independent set $B = \{b_1, \ldots, b_s\}$ in $\mathbb{K}[\mathrm{x}]$. Then the matrix of $\widehat{\mathcal{H}}_{|V}$ in the basis B and its dual basis is

$$H_1^B = \left(\Lambda(b_i b_j)\right)_{1 \le i,j \le s}.$$

Note that if $\Lambda = \sum_{i=1}^r a_i \mathfrak{e}_{\xi_j}$ we have the following matrix equality that generalizes the factorization of Hankel matrices in terms of Vandermonde matrices:

$$H_1^B = \left(W_\xi^B\right)^T A\, W_\xi^B, \text{ where } A = \operatorname{diag}(a_1, \ldots, a_r) \text{ and } W_\xi^B = \left(b_j(\xi_i)\right)_{1 \le i \le r, 1 \le j \le s}.$$

Hence $\widehat{\mathcal{H}}$ has rank at most r when $\Lambda = \sum_{j=1}^r a_j \mathfrak{e}_{\xi_j}$. In all cases, the rank of $\widehat{\mathcal{H}}$ is the dimension of $\mathbb{K}[\mathrm{x}]/I_\Lambda$ as a $\mathbb{K}$-vector space.

Before further investigation of Hankel operators, we shall review some results about ideals such as $I(\xi)$ and I_Λ, when $\widehat{\mathcal{H}}$ is of finite rank.

3.2 Zero-dimensional ideals

In the univariate case, the nodes of a quadrature are characterized as the roots of a univariate polynomial p. If $p \in \mathbb{K}[x]$ is of degree r then $\mathbb{K}[x] = \langle 1, x, \ldots, x^{r-1} \rangle \oplus (p)$, where (p) is the ideal generated by p. In other words the classes of $1, x, \ldots, x^{r-1}$ in the quotient $\mathbb{K}[x]/(p)$ form a basis of this quotient when considered as a $\mathbb{K}$-linear space. The number of roots of p, counted with multiplicities, is equal to the dimension of $\mathbb{K}[x]/(p)$. Observe furthermore that the companion matrix associated to p corresponds to the matrix of the multiplication by x in $\mathbb{K}[\mathrm{x}]/(p)$; Its eigenvalues are the roots of p. In this section, we generalize these notions to the multivariate case. The set of nodes of a cubature shall be characterized as the roots, or *variety*, of a *zero dimensional* ideal in the multivariate polynomial ring $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$.

Given an ideal I in $\mathbb{K}[\mathrm{x}]$, its variety is understood in $\mathbb{C}^n$ and is denoted by

$$V_{\mathbb{C}}(I) = \{\xi \in \mathbb{C}^n \,|\, p(\xi) = 0, \ \forall p \in I\}.$$

Conversely, for $V \subset \mathbb{C}^n$, $I(V) = \{p \in \mathbb{K}[\mathrm{x}] \,|\, p(\xi) = 0, \forall \xi \in V\}$ is a *radical* ideal. In the univariate case, radical ideals are generated by square-free polynomials. In the multivariate case, an ideal I is radical if $p^k \in I \Rightarrow p \in I$, for any $k > 1$.

Theorem 3.1 *[17, Chapter 2.2] Let $I \subset \mathbb{K}[\mathrm{x}]$ be an ideal. The variety $V_{\mathbb{C}}(I) \subset \mathbb{C}^n$ is a finite set if and only if the algebra $\mathbb{K}[\mathrm{x}]/I$ is finite-dimensional as a $\mathbb{K}$−vector space. Then the dimension of $\mathbb{K}[\mathrm{x}]/I$ as a $\mathbb{K}$-vector space is greater than or equal to the number of points in $V_{\mathbb{C}}(I)$. Equality occurs if and only if I is a radical ideal.*

When I is not a radical ideal, an appropriate notion of multiplicity for the roots of I allows to obtain a more general version of the above result [17, Chapter 4]. In either case, an ideal I is said to be *zero-dimensional* if $\mathbb{K}[\mathrm{x}]/I$ is finite-dimensional as a $\mathbb{K}$−vector space. Contrary to the univariate case, there is no universal bases for the quotient spaces $\mathbb{K}[\mathrm{x}]/I$ of a given dimension.

The class of any polynomial $f \in \mathbb{K}[\mathrm{x}]$ modulo I is denoted by $[f]$. To shorten some statements we shall say that $B = \{b_1, \ldots, b_r\} \subset \mathbb{K}[\mathrm{x}]$ is a basis of I_Λ when $[B] = \{[b_1], \ldots, [b_r]\}$ is.

Given a polynomial $p \in \mathbb{K}[\mathrm{x}]$, the map

$$\begin{array}{rccc} \mathcal{M}_p : & \mathbb{K}[\mathrm{x}]/I & \to & \mathbb{K}[\mathrm{x}]/I \\ & [f] & \mapsto & [pf] \end{array}$$

is a well defined linear map, simply called the *multiplication by p*. Its eigenvalues consist of the values of the function p on $V_{\mathbb{C}}(I)$ [17, Theorem 4.5]. We shall give a proof of this result by exhibiting the left eigenvectors of the matrix of $\mathcal{M}_p$. This is a hint that duality can shed further light on the structure of $\mathbb{K}[\mathrm{x}]/I$ in general [55].

Theorem 3.2 *Let I be a zero-dimensional ideal in $\mathbb{K}[\mathrm{x}]$ and assume $B = \{b_1, \dots, b_r\}$ is a basis of $\mathbb{K}[\mathrm{x}]/I$. For p a polynomial in $\mathbb{K}[\mathrm{x}]$ consider M_p^B the matrix of the multiplication operator $\mathcal{M}_p$ in the basis B. Then for any $\xi \in V_{\mathbb{C}}(I)$ the row vector $\begin{pmatrix} b_1(\xi) & \dots & b_r(\xi) \end{pmatrix}$ is a left eigenvector of M_p^B associated to the eigenvalue $p(\xi)$.*

PROOF: If $\mathcal{M}_p^B = \left(m_{ij}\right)_{1 \le i,j \le r}$ is the matrix of the multiplication operator $\mathcal{M}_p$, we have $pb_j \equiv \sum_{i=1}^r m_{ij} b_i \mod I$, $\forall j = 1, \dots, r$. Evaluating this equality at a root ξ of I brings

$$p(\xi) b_j(\xi) = \sum_{k=1}^{r} m_{kj} b_k(\xi) \quad \forall j = 1, \dots, r,$$

which can be rewritten as $p(\xi_i) \begin{pmatrix} b_1(\xi_i) & \dots & b_r(\xi_i) \end{pmatrix} = \begin{pmatrix} b_1(\xi_i) & \dots & b_r(\xi_i) \end{pmatrix} M_p^B$. ∎

Assume $V_{\mathbb{C}}(I) = \{\xi_1, \dots, \xi_s\}$. Introducing the *(generalized) Vandermonde matrix*

$$W_\xi^B = \left(b_j(\xi_i)\right)_{1 \le i \le s,\, 1 \le j \le r} \tag{5}$$

the above result can be written as a matricial identity $W_\xi^B M_p^B = D W_\xi^B$, where D is the diagonal matrix whose diagonal elements are $p(\xi_1), \dots, p(\xi_s)$.

As in the univariate case, the Vandermonde matrix appears naturally in Lagrange interpolation. We shall call a set $\{\ell_1, \dots, \ell_r\} \subset \mathbb{K}[\mathrm{x}]$ a *Lagrange basis* associated to $\xi = \{\xi_1, \dots, \xi_r\}$ if it satisfies $\ell_i(\xi_j) = \delta_{ij}$. The above theorem shows their existence. It is unique only modulo the ideal $I(\xi)$ of all the polynomials vanishing at $\xi_1, \dots, \xi_r$.

Theorem 3.3 *Let I be a zero-dimensional radical ideal in $\mathbb{K}[\mathrm{x}]$ with $V_{\mathbb{C}}(I) = \{\xi_1, \dots, \xi_r\} \subset \mathbb{K}^n$. Then $B = \{b_1, \dots, b_r\}$ is a basis of $\mathbb{K}[\mathrm{x}]/I$ if and only if the matrix $W_\xi^B = \left(b_j(\xi_i)\right)_{1 \le i,j \le r}$ is invertible. In this case the r polynomials $\ell_1, \dots, \ell_r$ in $\mathbb{K}[\mathrm{x}]$ defined by*

$$\begin{pmatrix} \ell_1 & \dots & \ell_r \end{pmatrix} = \begin{pmatrix} b_1 & \dots & b_r \end{pmatrix} \left(W_\xi^B\right)^{-1} \tag{6}$$

form a basis of $\mathbb{K}[\mathrm{x}]/I$ that satisfies $\ell_i(\xi_j) = \delta_{ij}$. For any $p \in \mathbb{K}[\mathrm{x}]$, $[\ell_i]$ is an eigenvector of $\mathcal{M}_p$ associated to the eigenvalue $p(\xi_i)$.

PROOF: Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{K}^r$ such that $\alpha_1 b_1 + \dots + \alpha_r b_r \equiv 0 \mod I$. Since $V_{\mathbb{C}}(I) = \{\xi_1, \dots, \xi_r\}$, we have

$$\alpha_1 b_1(\xi_i) + \dots + \alpha_r b_r(\xi_i) = 0 \quad \forall i = 1, \dots, r.$$

With the help of the matrix $W_\xi^B = \left(b_j(\xi_i)\right)_{1 \le i,j \le r}$, this equality can be rewritten as $W_\xi^B \alpha = 0$. Thus, the matrix W_ξ^B is invertible if and only if $(\alpha_1, \dots, \alpha_r) = (0, \dots, 0)$. This means that W_ξ^B is invertible if and only if $b_1, \dots, b_r$ are linearly independent modulo I. Since $\operatorname{Card} B = \dim \mathbb{K}[\mathrm{x}]/I$, we have that W_ξ^B is invertible if and only if B is a basis of $\mathbb{K}[\mathrm{x}]/I$.

Evaluating the defining equality for ℓ at ξ_i we obtain that $v = \begin{pmatrix} \ell_1(\xi_i) & \dots & \ell_r(\xi_r) \end{pmatrix}$ is the solution to the linear system $v W_\xi^B = \begin{pmatrix} b_1(\xi_i) & \dots & b_r(\xi_i) \end{pmatrix}$. The unique solution to this system is the row vector that has zero entries except for a 1 at the i-th position.

The last point comes from the matricial equality $W_\xi^B M_p^B = D W_\xi^B$ following Theorem 3.2, mutipliying it on the right and left by $\left(W_\xi^B\right)^{-1}$. ∎

3.3 Hankel operators of finite rank

In this section we examine the Hankel operators associated to linear forms $\sum_{i=1}^{r} a_i \mathbb{e}_{\xi_j}$. Several results in this section require additionally that I_Λ is radical. This property will appear in Section 3.4 as a consequence of the positivity of the Hankel operator associated to cubature. The goal here is to show how the weights $a_1, \ldots, a_r$ and the nodes $\xi_1, \ldots, \xi_r$ can be retrieved as generalized eigenvectors.

Theorem 3.4 *A linear form Λ is equal to $\sum_{i=1}^{r} a_i \mathbb{e}_{\xi_j}$ for some $\xi_i \in \mathbb{K}^n$ and $a_i \in \mathbb{K} \setminus \{0\}$, if and only if its associated Hankel operator $\widehat{\mathcal{H}}$ is of finite rank r and I_Λ radical. In this case, $V_{\mathbb{C}}(I_\Lambda) = \{\xi_1, \ldots, \xi_r\}$ and $a_i = \Lambda(\ell_i)$, where $\{\ell_1, \ldots, \ell_r\}$ is a Lagrange basis associated to $\{\xi_1, \ldots, \xi_r\}$.*

PROOF: The rank of $\widehat{\mathcal{H}}$ is the dimension of $\mathbb{K}[\mathrm{x}]/I_\Lambda$. Assume that $\mathbb{K}[\mathrm{x}]/I_\Lambda$ is of dimension r and that I_Λ is radical. There are thus distinct $\xi_1, \ldots, \xi_r \in \mathbb{C}^n$ that form the variety of I_Λ. We can call on Theorem 3.3 to exhibit a Lagrange basis for $\mathbb{K}[\mathrm{x}]/I_\Lambda$. Since $\mathbb{K}[\mathrm{x}] = \langle \ell_1, \ldots, \ell_r \rangle \oplus I_\Lambda$, Λ is entirely determined by its values on $\ell_1, \ldots, \ell_r$. Hence $\Lambda = \sum_{i=1}^{r} a_i \mathbb{e}_{\xi_j}$.

Assume now that $\Lambda = \sum_{j=1}^{r} a_j \mathbb{e}_{\xi_j}$ with $a_j \in \mathbb{C} \setminus \{0\}$ and $\xi_j \in \mathbb{C}^n$ pairwise distinct. Let $I(\xi)$ be the ideal of polynomials vanishing at the points $\xi_1, \ldots, \xi_r$. By the *Strong Nullstellensatz* [17, Chapter 1.4], the ideal $I(\xi)$ is radical. Furthermore $\mathbb{K}[\mathrm{x}]/I(\xi)$ is of dimension r. We prove next that $I_\Lambda = I(\xi)$.

Let $f \in I(\xi)$, then, for all $q \in \mathbb{K}[\mathrm{x}]$, $\Lambda(qf) = \sum_{j=1}^{r} a_j q(\xi_j) f(\xi_j) = 0$. Thus, $I(\xi) \subset I_\Lambda$. Conversely, consider $q \in I_\Lambda$ and take $\ell_1, \ldots, \ell_r$ to be a Lagrange basis associated to $\xi_1, \ldots, \xi_r$. Then $a_i q(\xi_i) = \Lambda(q \ell_i) = 0$ for all $1 \le i \le r$. Since $a_i \neq 0$, we conclude that $q \in I(\xi)$ so that $I_\Lambda \subset I(\xi)$. ■

One can formulate a similar result without assuming that I_Λ is radical [55, Theorem 3.1], with some derivation operators coming into play in Λ. One then touches on Hermite interpolation instead of Lagrange interpolation.

In the spirit of Section 3.2, our next endeavor is to characterize the bases $\mathbb{K}[\mathrm{x}]/I_\Lambda$ and determine the matrices of the multiplication maps therein. This shall be done in terms of the matrices of the Hankel operators and their inverses.

Bases of $\mathbb{K}[\mathrm{x}]/I_\Lambda$

A basis of $\mathbb{K}[\mathrm{x}]/I_\Lambda$ as a $\mathbb{K}$-vector space is the image by the natural projection $\mathbb{K}[\mathrm{x}] \to \mathbb{K}[\mathrm{x}]/I_\Lambda$ of a linearly independent set $B \subset \mathbb{K}[\mathrm{x}]$ s.t. $\mathbb{K}[\mathrm{x}] = \langle B \rangle \oplus I_\Lambda$. Recall from linear algebra, see for instance [30, Proposition V, Section 2.30], that this latter equality implies:

$$\mathbb{K}[\mathrm{x}]^* = (I_\Lambda)^\perp \oplus B^\perp \quad \text{and} \quad \begin{array}{ccc} (I_\Lambda)^\perp & \to & \langle B \rangle^* \\ \Lambda & \mapsto & \Lambda_{|\langle B \rangle} \end{array} \text{ is an isomorphism,}$$

where, for any $V \subset \mathbb{K}[\mathrm{x}]$, $V^\perp = \{\Gamma \in \mathbb{K}[\mathrm{x}]^* \mid \Gamma(v) = 0, \forall v \in V\}$ is a linear subspace of $\mathbb{K}[\mathrm{x}]^*$.

Note that the image of $\widehat{\mathcal{H}}$ lies in $(I_\Lambda)^\perp$. With the natural identification of $\langle B \rangle$ with $\mathbb{K}[\mathrm{x}]/I_\Lambda$, and $\langle B \rangle^*$ with $(\mathbb{K}[\mathrm{x}]/I_\Lambda)^*$, the factorization of $\widehat{\mathcal{H}}$ by the natural projection $\mathbb{K}[\mathrm{x}] \to \mathbb{K}[\mathrm{x}]/I_\Lambda$ defines the injective morphism $\mathcal{H} : \mathbb{K}[\mathrm{x}]/I_\Lambda \to (\mathbb{K}[\mathrm{x}]/I_\Lambda)^*$ through the commuting diagram:

$$\begin{tikzcd} \mathbb{K}[\mathrm{x}] \arrow[r, "\widehat{\mathcal{H}}"] \arrow[d] & (I_\Lambda)^\perp \arrow[d, leftrightarrow, "\cong"] \\ \mathbb{K}[\mathrm{x}]/I_\Lambda \arrow[r, "\mathcal{H}"'] & (\mathbb{K}[\mathrm{x}]/I_\Lambda)^* \end{tikzcd}$$

If the rank of $\widehat{\mathcal{H}}$ is finite and equal to r, then the dimension of $\mathbb{K}[\mathrm{x}]/I_\Lambda$ and $(\mathbb{K}[\mathrm{x}]/I_\Lambda)^*$, as $\mathbb{K}$-vector spaces, is r and the injective linear operator $\mathcal{H}$ is then an isomorphism. When $B \subset \mathbb{K}[\mathrm{x}]$ is a basis for $\mathbb{K}[\mathrm{x}]/I_\Lambda$, we can identify $\mathbb{K}[\mathrm{x}]/I_\Lambda$ with $\langle B\rangle$ and thus H_1^B is the matrix of $\mathcal{H}$ in this basis and its dual basis. It follows that H_1^B is invertible. The converse assertion is also true. It gives us a criterion for $B \subset \mathbb{K}[\mathrm{x}]$ to define a basis of $\mathbb{K}[\mathrm{x}]/I_\Lambda$.

Theorem 3.5 *Assume that* $\operatorname{rank}\widehat{\mathcal{H}} = r < \infty$ *and consider* $B = \{b_1, \dots, b_r\} \subset \mathbb{K}[\mathrm{x}]$*. Then B is a basis of* $\mathbb{K}[\mathrm{x}]/I_\Lambda$ *if and only if the matrix* H_1^B *is invertible.*

PROOF: If H_1^B is invertible then B is linearly independent modulo I_Λ and thus B is a basis of the r-dimensional vector space $\mathbb{K}[\mathrm{x}]/I_\Lambda$. Indeed, assume $c = (c_1, \dots, c_r) \in \mathbb{K}^r$ is such that $c_1 b_1 + \cdots + c_r b_r \equiv 0 \mod I_\Lambda$. Using the definition of I_Λ, we get $c_1\Lambda(b_1 b_i) + \cdots + c_r\Lambda(b_r b_i) = 0$, $\forall i = 1, \dots, r$. These equalities amount to $H_1^B c = 0$.

When $\mathbb{K}[\mathrm{x}] = \langle B\rangle \oplus I_\Lambda$, i.e. when B is a basis of $\mathbb{K}[\mathrm{x}]/I_\Lambda$, then, with the natural identification of $\langle B\rangle$ with $\mathbb{K}[\mathrm{x}]/I_\Lambda$, H_1^B is the matrix of $\mathcal{H}$. Since $\mathcal{H}$ is an isomorphism, H_1^B must be invertible. ■

Multiplication maps in $\mathbb{K}[\mathrm{x}]/I_\Lambda$

We now assume that the Hankel operator $\widehat{\mathcal{H}}$ associated to Λ has finite rank r. Then $\mathbb{K}[\mathrm{x}]/I_\Lambda$ is of dimension r when considered as a linear space over $\mathbb{K}$. In this paragraph we make explicit how to obtain the matrices of the multiplication maps from the matrices of associated Hankel operators.

For $p \in \mathbb{K}[\mathrm{x}]$, consider the multiplication map

$$\begin{array}{cccc} \widehat{\mathcal{M}}_p: & \mathbb{K}[\mathrm{x}] & \to & \mathbb{K}[\mathrm{x}] \\ & q & \mapsto & p\,q \end{array}$$

and

$$\begin{array}{cccc} \mathcal{M}_p: & \mathbb{K}[\mathrm{x}]/I_\Lambda & \to & \mathbb{K}[\mathrm{x}]/I_\Lambda \\ & [q] & \mapsto & [p\,q]. \end{array}$$

$\mathcal{M}_p$ is a well defined linear map respecting the following commuting diagram

$$\begin{tikzcd} \mathbb{K}[\mathrm{x}] \arrow[r, "\widehat{\mathcal{M}}_p"] \arrow[d] & \mathbb{K}[\mathrm{x}] \arrow[d] \\ \mathbb{K}[\mathrm{x}]/I_\Lambda \arrow[r, "\mathcal{M}_p"'] & \mathbb{K}[\mathrm{x}]/I_\Lambda \end{tikzcd}$$

Let us temporarily introduce the Hankel operator $\widehat{\mathcal{H}}_p$ associated to Λ_p. Note that $\widehat{\mathcal{H}}_p = \widehat{\mathcal{H}} \circ \widehat{\mathcal{M}}_p$. Therefore the image of $\widehat{\mathcal{H}}_p$ is included in the image of $\widehat{\mathcal{H}}$ and $\ker\widehat{\mathcal{H}} \subset \ker\widehat{\mathcal{H}}_p$. We can thus construct the map $\mathcal{H}_p : \mathbb{K}[\mathrm{x}]/I_\Lambda \to (\mathbb{K}[\mathrm{x}]/I_\Lambda)^*$ that satisfies $\mathcal{H}_p = \mathcal{H} \circ \mathcal{M}_p$ and the following commuting diagram.

$$\begin{tikzcd}
\mathbb{K}[\mathrm{x}] \arrow[rr, "\widehat{\mathcal{H}}_p"] \arrow[dr, "\widehat{\mathcal{M}}_p"'] \arrow[ddd] & & (I_\Lambda)^\perp \arrow[ddd, leftrightarrow, "\cong"] \\
& \mathbb{K}[\mathrm{x}] \arrow[ur, "\widehat{\mathcal{H}}"'] \arrow[d] & \\
& \mathbb{K}[\mathrm{x}]/I_\Lambda \arrow[dr, "\mathcal{H}"] & \\
\mathbb{K}[\mathrm{x}]/I_\Lambda \arrow[ur, "\mathcal{M}_p"] \arrow[rr, "\mathcal{H}_p"'] & & (\mathbb{K}[\mathrm{x}]/I_\Lambda)^*
\end{tikzcd}$$

Proposition 3.6 *Assume the Hankel operator $\widehat{\mathcal{H}}$ associated to the linear form Λ has finite rank r. Let $B = \{b_1, \ldots, b_r\}$ be a basis of $\mathbb{K}[\mathrm{x}]/I_\Lambda$. Then the matrix M_p^B of the multiplication by p in $\mathbb{K}[\mathrm{x}]/I_\Lambda$ satisfies*

$$H_p^B = H_1^B\, M_p^B \quad \text{where} \quad H_1^B = \left(\Lambda(b_i b_j)\right)_{1\le i,j\le r} \;\text{ and }\; H_p^B = \left(\Lambda(p\, b_i b_j)\right)_{1\le i,j\le r}$$

PROOF: The matrix of $\mathcal{H}_p$ in B and its dual basis is H_p^B while $\mathcal{H}_p = \mathcal{H} \circ \mathcal{M}_p$. ∎

Combining the above with Theorem 3.2 and Theorem 3.3 leads then to the following result.

Corollary 3.7 *Let $\mathcal{H} : \mathbb{K}[\mathrm{x}]/I_\Lambda \to \mathbb{K}[\mathrm{x}]/I_\Lambda$ be the Hankel operator associated to $\Lambda = \sum_{i=1}^r a_i \mathbb{e}_{\xi_i}$. Consider $\{\ell_1, \ldots, \ell_r\} \subset \mathbb{K}[\mathrm{x}]$ a Lagrange basis associated to $\xi_1, \ldots, \xi_r$. For any $p \in \mathbb{K}[\mathrm{x}]$, ℓ_i is a generalized eigenvector of $(\mathcal{H}_p, \mathcal{H})$ associated to $p(\xi_i)$.*

3.4 Positivity

Associating a Hankel operator $\widehat{\mathcal{H}} : \mathbb{K}[\mathrm{x}] \to \mathbb{K}[\mathrm{x}]^*$ to a linear form $\Lambda : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ is somewhat equivalent to associating to Λ the bilinear form $(p, q) \mapsto \Lambda(pq) = \widehat{\mathcal{H}}(p)(q)$. When this associated bilinear form is *positive semi-definite*, that is $\Lambda(p^2) = \widehat{\mathcal{H}}(p)(p) \ge 0$ for all $p \in \mathbb{R}[\mathrm{x}]$, then Λ and $\widehat{\mathcal{H}}$ are also said to be *positive semi-definite*. It is easily seen to be the case when Λ is the Hankel operator associated the linear form $\Omega : p \mapsto \int_{\mathcal{D}} p(x)dx$ or $\Lambda = \sum_{i=1}^r a_i \mathbb{e}_{\xi_j}$, with $\xi_i \in \mathbb{R}^n$ and $a_i > 0$. Note that the matrices H_1^B introduced in Section 3.1 are positive semi-definite when $\widehat{\mathcal{H}}$ has this property.

Theorem 3.8 *If $\Lambda : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ is a positive semi-definite linear form then $I_\Lambda = \{p \mid \Lambda(pq) = 0, \forall q \in \mathbb{K}[\mathrm{x}]\}$ is a real radical ideal.*

What *real radical* means in general is to be found in [9]. Of relevance in this article is the fact that a zero dimensional real radical ideal of $\mathbb{R}[x]$ has all its roots in $\mathbb{R}^n$, as opposed to $\mathbb{C}^n$, and they are simple roots. Based on fundamental results in real algebraic geometry, the above assertion is proved by an elegant and short argument in [45, Lemma 5.2].

A *cubature* is thus well characterized by its Hankel operator by combining Theorem 3.8 and Theorem 3.4.

Theorem 3.9 *Let $\widehat{\mathcal{H}}$ be the Hankel operator associated to the linear form $\Lambda : \mathbb{R}[x] \to \mathbb{R}[x]$. Then $\Lambda = \sum_{i=1}^r a_i \mathbb{e}_{\xi_i}$ for some $a_i > 0$ and $\xi_i \in \mathbb{R}^n$ if and only if $\widehat{\mathcal{H}}$ is positive semi-definite of finite rank r. When this is the case, $\xi_1, \ldots, \xi_r$ are the roots of the kernel I_Λ of $\widehat{\mathcal{H}}$.*

The algorithm we shall develop in Section 5 establishes the existence of a cubature thanks to the above criterion before it goes on to compute the nodes. The following property specific to positive semi-definite forms allows to restrict the search space of the algorithm.

Proposition 3.10 *Assume that the linear form $\Lambda : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ is positive semi-definite and let $\widehat{\mathcal{H}}$ be its associated Hankel operator. For each $\delta \in \mathbb{N}$ consider furthermore*

$$\begin{array}{rccc} \widehat{\mathcal{H}}^{(\delta)}: & \mathbb{R}[\mathrm{x}]_{\leq\delta} & \to & \mathbb{R}[\mathrm{x}]^*_{\leq\delta} \\ & p & \mapsto & \Lambda_p^{(\delta)} \end{array} \quad \textit{where} \quad \begin{array}{rccc} \Lambda_p^{(\delta)}: & \mathbb{R}[\mathrm{x}]_{\leq\delta} & \to & \mathbb{R} \\ & q & \mapsto & \Lambda(pq) \end{array}.$$

Then

$$\ker \widehat{\mathcal{H}}^{(\delta)} = \ker \widehat{\mathcal{H}} \cap \mathbb{R}[\mathrm{x}]_{\leq\delta}.$$

Furthermore $\operatorname{rank} \widehat{\mathcal{H}}^{(\delta)} = \operatorname{rank} \widehat{\mathcal{H}}^{(\delta+1)}$ *implies that $\widehat{\mathcal{H}}$ is of finite rank $r = \operatorname{rank} \widehat{\mathcal{H}}^{(\delta)}$.*

PROOF: The inclusion $\ker \widehat{\mathcal{H}} \cap \mathbb{R}[\mathrm{x}]_{\leq\delta} \subset \ker \widehat{\mathcal{H}}^{(\delta)}$ is easy. The converse inclusion follows from the semi-definiteness of the bilinear form $(p,q) \mapsto \Lambda(pq)$. If $p \in \ker \widehat{\mathcal{H}}^{(\delta)}$ then in particular $\Lambda(p^2) = 0$. The Cauchy-Schwarz inequality $(\Lambda(pq)^2 \leq \Lambda(p^2)\Lambda(q^2))$ implies then that $\Lambda(pq) = 0$ for all $q \in \mathbb{R}[\mathrm{x}]$. Hence $p \in \ker \widehat{\mathcal{H}}$.

It follows that $\ker \widehat{\mathcal{H}}^{(\delta)} = \ker \widehat{\mathcal{H}}^{(\delta+1)} \cap \mathbb{R}[\mathrm{x}]_{\leq\delta}$. Thus if $\operatorname{rank} \widehat{\mathcal{H}}^{(\delta)} = \operatorname{rank} \widehat{\mathcal{H}}^{(\delta+1)}$, all the element of $\mathbb{R}[\mathrm{x}]_{\leq\delta+1} \setminus \mathbb{R}[\mathrm{x}]_{\leq\delta}$ are linearly dependent on the elements of $\mathbb{R}[\mathrm{x}]_{\leq\delta}$ modulo $\ker \widehat{\mathcal{H}}^{(\delta+1)}$. In particular for each x^α with $|\alpha| = \delta + 1$ there exists $p_\alpha \in \mathbb{K}[\mathrm{x}]_{\leq\delta}$ such that $x^\alpha + p_\alpha \in \ker \widehat{\mathcal{H}}^{(\delta+1)} \subset \ker \widehat{\mathcal{H}}$. The latter is an ideal hence, for all $1 \leq i \leq n$, the polynomial $x_i x^\alpha + x_i p_\alpha$ of degree $\delta+2$ belongs to $\ker \widehat{\mathcal{H}}$, and thus to $\ker \widehat{\mathcal{H}}^{(\delta+2)} = \ker \widehat{\mathcal{H}} \cap \mathbb{R}[\mathrm{x}]_{\leq\delta+2}$. It follows that $\operatorname{rank} \widehat{\mathcal{H}}^{(\delta+2)} = \operatorname{rank} \widehat{\mathcal{H}}^{(\delta+1)} = \operatorname{rank} \widehat{\mathcal{H}}^{(\delta)}$. The claim follows by induction. ∎

Observe that in general $\operatorname{rank} \widehat{\mathcal{H}}^{(0)} \leq \operatorname{rank} \widehat{\mathcal{H}}^{(1)} \leq \ldots \leq \operatorname{rank} \widehat{\mathcal{H}}^{(\delta)} \leq \operatorname{rank} \widehat{\mathcal{H}}^{(\delta+1)} \leq \ldots$ and this sequence of ranks becomes stationary when $\widehat{\mathcal{H}}$ is of finite rank. Yet you may have $\widehat{\mathcal{H}}^{(\delta)} = \operatorname{rank} \widehat{\mathcal{H}}^{(\delta+1)} < \operatorname{rank} \widehat{\mathcal{H}}^{(\delta+1)}$ when $\widehat{\mathcal{H}}$ is not positive semi-definite.

The proof above actually points to the fact that a basis for $\ker \widehat{\mathcal{H}}^{(\delta+1)}$ provides a set of generators for I_Λ (cf. also [44, Theorem 5.19]). We do not dwell on this as we wish to emphasize that the nodes of the cubature are more straightforwardly obtained as generalized eigenvalues, a problem for which that are well studied numerical algorithms.

Before computing the nodes, we shall first determine if the characterized cubature is *inside*. Assuming that the support K of the cubature is given as a semi algebraic set, i.e. $K = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0\}$, for some polynomials $g_1, \ldots, g_s \in \mathbb{R}[\mathrm{x}]$, the following criterion allows one to test if the cubature is inside.

Corollary 3.11 *Consider $\Lambda = \sum_{i=1}^r a_i \mathbb{e}_{\xi_i}$, with $\xi_i \in \mathbb{R}^n$ and $a_i > 0$, and $q \in \mathbb{R}[\mathrm{x}]$. Then $q(\xi_i) \geq 0$, for $1 \leq i \leq r$, if and only if the Hankel operator $\widehat{\mathcal{H}}_q$ associated to Λ_q is positive semi-definite.*

PROOF: As $\Lambda_q = \sum_{i=1}^r a_i q(\xi_i) \mathbb{e}_{\xi_i}$, by Theorem 3.9, $\widehat{\mathcal{H}}_q$ is positive semi-definite if and only if $a_i q(\xi_i) \geq 0$ for all $1 \leq i \leq r$. ∎

3.5 Flat extensions

In determining a cubature of degree $2d$ for a linear form $\Omega : p \mapsto \int_{\mathcal{D}} p(x)dx$ one shall seek to extend the linear form $\Lambda^{(d)} = \Omega_{|\mathbb{R}[\mathrm{x}]_{\leq 2d}}$ to a linear form Λ on $\mathbb{R}[\mathrm{x}]$ such that the associated Hankel operator is positive semi-definite of finite rank. The existence of such a form needs to be verified in some finite way though.

A linear form $\Lambda^{(\delta+\kappa)}$ on $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$ is an *extension* of a given linear form $\Lambda^{(\delta)}$ on $\mathbb{R}[x]_{\leq 2\delta}$ if its restriction to $\mathbb{R}[x]_{\leq 2\delta}$ is $\Lambda^{(\delta)}$, that is if

$$\Lambda^{(\delta+\kappa)}(p) = \Lambda^{(\delta)}(p) \quad \forall p \in \mathbb{R}[x]_{\leq 2\delta}.$$

Let $\Lambda^{(\delta)}$ be a linear form on $\mathbb{K}[\mathrm{x}]_{\leq 2\delta}$. Similarly to Section 3.1, we associate with $\Lambda^{(\delta)}$ the Hankel operator

$$\begin{array}{rccc} \mathcal{H}^{(\delta)}: & \mathbb{K}[\mathrm{x}]_{\leq\delta} & \to & \mathbb{K}[\mathrm{x}]^*_{\leq\delta} \\ & p & \mapsto & \Lambda_p^{(\delta)} \end{array}, \quad \text{where} \quad \begin{array}{rccc} \Lambda_p^{(\delta)}: & \mathbb{K}[\mathrm{x}]_{\leq\delta} & \to & \mathbb{K} \\ & q & \mapsto & \Lambda(pq). \end{array}$$

A linear form $\Lambda^{(\delta+\kappa)}$ on $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$ is a *flat extension* of a given linear form $\Lambda^{(\delta)}$ if furthermore the rank of its associated Hankel operator $\mathcal{H}^{(\delta+\kappa)}$ is the rank of the Hankel operator $\mathcal{H}^{(\delta)}$ associated with the linear form $\Lambda^{(\delta)}$. In this case, $\mathcal{H}^{(\delta+\kappa)}$ is positive semi-definite if and only if $\mathcal{H}^{(\delta)}$ is positive semi-definite.

The flat extension theorem below was proved in terms of moment matrices, i.e. matrices of $\widehat{\mathcal{H}}^{(\delta)}$ in the monomial basis of $\mathbb{K}[\mathrm{x}]_{\leq\delta}$ [18, 19, 43, 44, 45]. It can be interpreted as an involutivity or regularity criterion [50, 51], and was generalized as such in [49].

Theorem 3.12 (Flat extension theorem) *Let $\Lambda^{(\delta)}$ be a linear form on $\mathbb{R}[x]_{\leq 2\delta}$. Assume that $\Lambda^{(\delta)}$ is a flat extension of its restriction to $\mathbb{R}[x]_{\leq 2\delta-2}$. Then there exists a unique flat extension of $\Lambda^{(\delta)}$ to $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$ for all $\kappa \geq 1$.*

4 Fraction-free diagonalization of positive semidefinite matrices

In this section we consider a symmetric matrix whose entries are polynomials in some parameters $h_1, \ldots, h_t$ over a computable field extension $\mathbb{K}$ of $\mathbb{Q}$ is. We present an algorithm in symbolic computation that computes a partition of the locus of positive semi-definiteness of the matrix in $\mathbb{R}^t$. Each region in this partition is given by a set of equalities and inequalities together with a parametrization of the diagonalization valid on this region of the parameter space. Additional specifications for this algorithm provides the central computational ingredient for determining the existence of a cubature with given degree and number of nodes.

While the problem can be approached with Gaussian elimination it is preferable to provide a fraction free approach that limits the expression growth, a key difficulty in symbolic computation. Bareiss' algorithm for the triangularization of a given matrix [3, 4, 25] is the model for such an approach, one that does not require the computation of greatest common divisors. It was revisited for the diagonalization of a given symmetric matrix [33, 64]. An algorithmic difficulty in the description and implementation of the algorithm is the necessity of *pivoting* the rows and columns when the next diagonal element is tested to be zero. We observe, and this appears to be new, that the pivoting is unnecessary when we restrict to positive semi-definite symmetric matrices.

4.1 Positive semi-definite matrices

A symmetric matrix $A = (a_{ij})_{1\leq i,j\leq c}$, with entries in $\mathbb{R}$, is called *positive semi-definite* if $v^tAv \geq 0$, $\forall v \in \mathbb{R}^c$. It is called *positive definite* if it satisfies furthermore $v^tAv = 0 \Leftrightarrow v = 0$. Given any matrix $A = (a_{ij})_{1\leq i,j\leq c}$, the determinant of its submatrix $A_J = (a_{ij})_{(i,j)\in J^2}$ with $J \subset \{1, \ldots, c\}$ is called a *principal minor*. These minors provide a characterization of positivity for symmetric matrices [74, Theorem 7.2].

Theorem 4.1 *Let A be a symmetric matrix with entries in $\mathbb{R}$. Then*

- *A is positive semi-definite if and only if every principal minor of A is nonnegative.*
- *A is positive definite if and only if every leading principal minor of A is positive.*

Instead of requiring all minors to be nonnegative, we can consider c specific linear combinations of the minors, namely the coefficients of the characteristic polynomial [5, Chapter 8.2.4].

When the matrix A depends polynomially on some parameters the locus of positivity is thus a semi algebraic set. Describing semi-algebraic sets is a central question in computational real algebraic geometry [5]. When one is interested in finding a (single) point in this semi-algebraic set, SemiDefinite Program solvers can provide a numerical solution. Exact algorithms were also developed to decide whether this particular semi-algebraic set is empty or not and, in the negative case, exhibits an algebraic representation of a point in this semi-algebraic set. We refer to [32] and the reference therein. We shall develop an approach that provides the basis in which the associated bilinear form is diagonal in addition to a description of the semi-algebraic description of the locus of positivity. As a side note, the Maple implementation of our approach actually succeeded with Section 6 when the Maple implementation of [32] did not.

We think it worth sharing the original observation on Gauss algorithm that prompted our approach. Gaussian elimination is a staple algorithm to compute, among other things, the rank of a matrix. Proposition 4.2 below basically offers a simplification of this algorithm that decides if a symmetric matrix is positive semi-definite. It is based on an observation that shows that there is no need to permute the columns when a pivot is zero.

Proposition 4.2 *Let $A = (a_{ij})_{1\le i,j\le c}$ be a symmetric matrix with entries in $\mathbb{R}$.*

1. *If A is a 1×1 matrix, then A is positive semi-definite if and only if $a_{11} \ge 0$.*
2. *Otherwise:*
   (a) *If $a_{11} < 0$, then A is not positive semi-definite.*
   (b) *If $a_{11} = 0$, then A is positive semi-definite if and only if $a_{1j} = 0$ for every $j = 1,\dots,c$ and the submatrix obtained by deleting the first row and the first column is positive semi-definite.*
   (c) *If $a_{11} > 0$, then for each $i > 1$ subtract $\frac{a_{i1}}{a_{11}}$ times row 1 from row i and delete the first row and the first column. Then A is positive semi-definite if and only if the resulting matrix is positive semi-definite.*

PROOF: Since the coefficient a_{11} is a principal minor, $a_{11} \ge 0$ is a necessary condition for the positive semi-definiteness of the matrix A. In the particular case of a 1×1 matrix A, this is also a sufficient condition by Theorem 4.1. It remains now to study the cases 2.(b) and 2.(c).

Assuming that $a_{11} = 0$, then we distinguish two cases:

- If $a_{1j} = 0$ for all $j = 1,\dots,c$, then every principal minor $\det A_J$ with $1 \in J$ is zero since the coefficients of a row of the matrix A_J are zero. As a consequence, A is positive semi-definite if and only if the matrix $(a_{ij})_{2\le i,j\le c}$ is positive semi-definite.
- If there is a nonzero coefficient in the first row of A, that is if there exists j such that $a_{1j} \ne 0$, then the principal minor $\det\begin{pmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj}\end{pmatrix}$ is negative since
$$\det\begin{pmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj}\end{pmatrix} = a_{11}a_{jj} - a_{1j}a_{j1} = -a_{1j}^2 < 0.$$
  Then A is not positive semi-definite.

Assuming that $a_{11} > 0$, the row operations describe the matrix equality $A = LU$, that is

$$\begin{pmatrix} a_{11} & \dots & a_{1c} \\ \vdots & & \vdots \\ a_{c1} & \dots & a_{cc}\end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{a_{c1}}{a_{11}} & 0 & \dots & 0 & 1\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1c} \\ 0 & \widetilde{a}_{22} & \dots & \dots & \widetilde{a}_{2c} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & \widetilde{a}_{c2} & \dots & \dots & \widetilde{a}_{cc}\end{pmatrix}$$

with $\widetilde{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j}$ for all $(i,j) \in \mathbb{N}^2$ such that $2 \le i,j \le c$. Since we have

$$\widetilde{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j} = a_{ji} - \frac{a_{1i}}{a_{11}}a_{j1} = \widetilde{a}_{ji} \quad \forall 2 \le i,j \le c,$$

the matrix $\widetilde{A} = (\widetilde{a}_{ij})_{2\le i,j\le c}$ is a symmetric matrix.

For every $J \subset \{1,\dots,c\}$ with $1 \in J$, we also have $\det A_J = \det L_J\ \det U_J$ and $\det L_J = 1$ so that the principal minor $\det A_J$ of A is the principal minor $\det U_J$ of U. We have furthermore that $\det U_J = a_{11}\det U_{J\smallsetminus\{1\}}$. Since a_{11} is positive, the principal minors of U are nonnegative if and only if the ones of the matrix U without the first row and the first column are nonnegative. ■

When the matrix A depends polynomially on some parameters, one can add a case distinction at each step to provide the conditions on the parameters for the matrix to be positive semidefinite and compute the rank. However the algorithm suggested by Proposition 4.2 runs then with rational functions whose numerators and denominators grow fast in degree. We shall thus introduce a fraction-free variant to this algorithm, elaborating on Bareiss elimination algorithm.

4.2 Fraction-free triangularization and diagonalization

We seek to perform elimination in matrices the entries of which lie in a polynomial ring $\mathrm{R} = \mathbb{K}[h_1,\dots,h_t]$ where $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$ is a field that allows to perform exact arithmetic operations. Typically $\mathbb{K}$ is an algebraic extension of $\mathbb{Q}$ as for instance $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$.

The fraction-free approach of Bareiss [3, 4, 25] to Gaussian elimination limits the growth of the entries by dividing a known extraneous factor at each step. The intermediate results can furthermore be expressed in terms of minors of the original matrix. This gives a clear idea on the growth and the specialization property of the elimination obtained. The scheme was revisited in [33, 64] for the diagonalization of symmetric matrices. The essence of these algorithms lie in the following results. The first statement is the heart of Bareiss's algorithm [3, 4, 25]. The second observation was presented in [33, 64].

Theorem 4.3 *Consider $A = (a_{ij})_{1\le i,j\le c}$ a $c\times c$ matrix with entries in an integral domain* R. *Let $A^{(0)} = A$ and $a_{0,0}^{(-1)} = 1$ and for an integer ℓ such that $1 \le \ell \le c-1$, let*

$$A^{(\ell)} = \begin{pmatrix} a_{1,1}^{(0)} & a_{1,2}^{(0)} & \cdots & \cdots & \cdots & \cdots & a_{1,c}^{(0)} \\ 0 & a_{2,2}^{(1)} & & & & & a_{2,c}^{(1)} \\ \vdots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & a_{\ell,\ell}^{(\ell-1)} & \cdots & \cdots & a_{\ell,c}^{(\ell-1)} \\ \vdots & & & 0 & a_{\ell+1,\ell+1}^{(\ell)} & \cdots & a_{\ell+1,c}^{(\ell)} \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & a_{c,\ell+1}^{(\ell)} & \cdots & a_{c,c}^{(\ell)} \end{pmatrix} \text{ with } a_{i,j}^{(\ell)} = \det\begin{pmatrix} a_{1,1} & \cdots & a_{1,\ell} & a_{1,j} \\ \vdots & & \vdots & \vdots \\ a_{\ell,1} & \cdots & a_{\ell,\ell} & a_{\ell,j} \\ a_{i,1} & \cdots & a_{i,\ell} & a_{i,j} \end{pmatrix} \text{ for } \ell < i,j \le c.$$

Let ℓ be an integer such that $1 \le \ell \le c-1$.

1. *If $a_{k-1,k-1}^{(k-2)} \neq 0$ for all $k = 1,\dots,\ell$, then $A^{(\ell)} = L_\ell \dots L_1 A$ with*

$$L_k = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & 1 & \ddots & & & \vdots \\ \vdots & & \vdots & -l_{k+1} & l_k & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \vdots & & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -l_c & 0 & \dots & 0 & l_k \end{pmatrix} \text{ with } l_k = \frac{a_{k,k}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}} \text{ and } l_j = \frac{a_{j,k}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}} \text{ for } k+1 \le j \le c.$$

2. *If A is furthermore symmetric, then the matrix $\widetilde{A}^{(\ell)} = L^{(\ell)} A L^{(\ell)^t}$, with $L^{(\ell)} = L_\ell \dots L_1$, is also symmetric. This matrix $\widetilde{A}^{(\ell)}$ is given by*

$$\widetilde{A}^{(\ell)} = \begin{pmatrix} \tilde{a}_{11}^{(0)} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \tilde{a}_{\ell,\ell}^{(\ell-1)} & 0 & \cdots & 0 \\ & & 0 & \tilde{a}_{\ell+1,\ell+1}^{(\ell)} & \cdots & \tilde{a}_{\ell+1,c}^{(\ell)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 0 & \tilde{a}_{c,\ell+1}^{(\ell)} & \cdots & \tilde{a}_{c,c}^{(\ell)} \end{pmatrix}, \text{ with } \tilde{a}_{i,j}^{(k)} = a_{i,j}^{(k)} a_{k,k}^{(k-1)}. \tag{7}$$

The multiplication on the left by the matrix L_ℓ correspond to an elimination step : a multiple of the ℓ-th row is subtracted to a non zero multiple of each of the row below so as to have zeros appear under the diagonal in the ℓ-th column. The simplest choice of multiples would be $a_{i,\ell}^{(\ell-1)}$ and $a_{\ell,\ell}^{(\ell-1)}$. But one proves [3, 4, 25] that the so obtained entries are all divisible by the previous *pivot* $a_{\ell-1,\ell-1}^{(\ell-2)}$. It can thus be divided out.

If we consider a symmetric $c \times c$ matrix A with entries in $\mathbb{R}$ that is positive definite then, according to Theorem 4.1, $a_{\ell,\ell}^{(\ell-1)} > 0$ for all $1 \le \ell \le c$. We can thus proceed with elimination steps without any pivoting and obtain a diagonal matrix $\widetilde{A}^{(c-1)}$. In general though one might encounter a pivot $a_{k,k}^{(k-1)}$ that is zero. To go further in the elimination one then has to introduce a pivoting strategy, which differs for diagonalization [33] compared to triagularization [4, 25]. We are here concerned with positive semi-definite matrices. A property of the matrices $A^{(l)}$ similar to the one disclosed in Corollary 4.2 allows us to forgo the pivoting.

4.3 Locus of semi-positivity by diagonalization

Consider now the integral domain R to be the polynomial ring $\mathbb{K}[\mathrm{h}] = \mathbb{K}[h_1, \dots, h_t]$, where $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$. Our goal is to determine the locus, in $\mathbb{R}^t$, such that the symmetric matrix A with entries in $\mathbb{K}[\mathrm{h}]$ becomes semi-definite under substitution. This locus is a semi- algebraic set : it is given as a union of sets, each of which is described by a finite set of polynomial equalities and inequalities.

To formalize our statements let us introduce a *specialization* $\phi : \mathbb{K}[\mathrm{h}] \to \mathbb{R}$ to be a $\mathbb{K}$-morphism. A typical example is given by

$$\phi_\hbar : \mathbb{K}[\mathrm{h}] \to \mathbb{R}, p \mapsto p(\hbar)$$

with $\hbar = (\hbar_1, \dots, \hbar_t)$ a point in $\mathbb{R}^t$. Given a matrix $A = (a_{ij})_{1 \le i,j \le c}$ with $a_{ij} \in \mathbb{K}[\mathrm{h}]$, we denote by $\phi(A)$ the matrix obtained by applying ϕ to the coefficients of A, that is

$$\phi(A) = (\phi(a_{ij}))_{1 \le i,j \le c} = (\bar{a}_{ij})_{1 \le i,j \le c}.$$

Likewise

$$\bar{a}_{ij}^{(\ell)} = \det \begin{pmatrix} \bar{a}_{1,1} & \cdots & \bar{a}_{1,\ell} & \bar{a}_{1,j} \\ \vdots & & \vdots & \vdots \\ \bar{a}_{\ell,1} & \cdots & \bar{a}_{\ell,\ell} & \bar{a}_{\ell,j} \\ \bar{a}_{i,1} & \cdots & \bar{a}_{i,\ell} & \bar{a}_{i,j} \end{pmatrix} = \phi\left(a_{ij}^{(\ell)}\right) \quad \forall 1 \le \ell \le c-1, \forall \ell < i, j \le c.$$

Theorem 4.4 *With the hypotheses and notations of Theorem 4.3 with A symmetric and $\mathrm{R} = \mathbb{K}[\mathrm{h}]$, $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$, let $\phi : \mathrm{R} \to \mathbb{R}$ be a specialisation. Assume that $\phi\left(a_{k,k}^{(k-1)}\right) > 0$ for $1 \le k < \ell$.*

For $\phi(A)$ to be semi-definite positive it is necessary that $\phi\left(a_{\ell,\ell}^{(\ell-1)}\right) \ge 0$. If $\phi\left(a_{\ell,\ell}^{(\ell-1)}\right) = 0$ then $\phi(A)$ is positive semidefinite of rank $\ell + r$ if and only if $\phi\left(a_{\ell,\ell+j}^{(\ell-1)}\right) = 0$ for all $1 \le j \le c - \ell$ and the principal submatrix $\phi\left(A_{\hat{\ell}}\right)$, where $A_{\hat{\ell}}$ is obtained from A by removing the ℓ-th row and ℓ-th column, is positive semi-definite of rank r.

PROOF: Let A_J be the $(\ell+2)\times(\ell+2)$ leading principal submatrix of A. If we apply the adequately truncated $L^{(\ell)}$ to A_J, we obtain the $(\ell+2)\times(\ell+2)$ leading principal submatrix of $A^{(\ell)}$. We have then

$$\left(a_{\ell,\ell}^{(\ell-1)}\right)^2\left(\prod_{k=1}^{\ell-1} a_{k,k}^{(k-1)}\right)\det(A_J) = \left(\prod_{k=1}^{\ell} a_{k,k}^{(k-1)}\right)\left(a_{\ell+1,\ell+1}^{(\ell)}a_{\ell+2,\ell+2}^{(\ell)} - \left(a_{\ell+1,\ell+2}^{(\ell)}\right)^2\right).$$

If $a_{kk}^{(k-1)} > 0$ for all $1\le k\le \ell$ and $a_{\ell+1,\ell+1}^{(\ell)} = 0$ while $a_{\ell+1,\ell+2}^{(\ell-1)} \neq 0$ then $\det A_J < 0$. According to Theorem 4.1 this contradicts A being positive semi-definite.

Notice that we used the fact that

$$\det L_k = \left(\frac{a_{k,k}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}}\right)^{c-k} \quad \text{so that} \quad \det L^{(\ell)} = \left(a_{\ell,\ell}^{(\ell-1)}\right)^{c-\ell}\prod_{k=1}^{\ell-1} a_{k,k}^{(k-1)}$$

and replaced c by $\ell+2$ to reflect the truncation.

The same argument, changing A_J to be another principal matrix, shows that for A to be positive semi-definite when $a_{\ell+1,\ell+1}^{(\ell)} = 0$, it is required that $a_{\ell+1,\ell+j}^{(\ell)} = 0$ for all $2\le j\le c-\ell$. ■

Notice that for $k\le\ell$ we have $\left(A_{\hat\ell}\right)^{(k)} = \left(A^{(k)}\right)_{\hat\ell}$. So when we encounter a pivot that is zero, we ignore that row and column and continue on. The entries of the successive matrices we encounter are thus determinants of a principal submatrix of A, whose size is the rank of A.

We give a recursive description of our algorithm. We will use the following notation. For a matrix $A = (a_{ij})_{1\le i,j\le c}$ with entries in R and $p\in R$, we define the first elimination matrix with the last non-zero pivot p as

$$L(A,p) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\dfrac{a_{2,1}}{p} & \dfrac{a_{1,1}}{p} & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\dfrac{a_{c,1}}{p} & 0 & \dots & 0 & \dfrac{a_{1,1}}{p} \end{pmatrix}$$

Algorithm 4.5 ***Diagonalization*** & ***Positivity***

Input : ▷ $A = (a_{ij})_{1\le i,j\le c}$ *a* $c\times c$ *symmetric matrix with entries in* $\mathbb{K}[\mathrm{h}]$.
▷ P *a list of polynomials in* $\mathbb{K}[\mathrm{h}]$ *-* P *stands for* pivots *or* positive. *The default value is* $[1]$.
▷ Z *a set of polynomials in* $\mathbb{K}[\mathrm{h}]$ *-* Z *stands for* zero. *The default value is* $\varnothing$.
▷ E *a list of elimination matrices. The default value is the empty list* $[]$.

Output: ▷ *A set* $\mathcal{S}$ *of triplets [P, Z, E], where*
- P *is a list of polynomials in* $\mathbb{K}[\mathrm{h}]$,
- Z *is a set of polynomials in* $\mathbb{K}[\mathrm{h}]$,
- E *is a list of elimination matrices in* $\mathbb{K}(h)$ *whose denominators are power products of elements in* P.

The set $\mathcal{S}$ *satisfies the following property: For a specialization* $\phi:\mathbb{K}[\mathrm{h}]\to\mathbb{R}$, $\phi(A)$ *is positive semi-definite if and only if there is a triplet [P, Z, E] in* $\mathcal{S}$ *such that*

$$\phi(p) > 0 \ \forall p\in P \text{ and } \phi(q) = 0 \ \forall q\in Z.$$

In this case $\mathrm{Card}(P)-1$ *is the rank of* $\phi(A)$ *and, letting* L *be the product of the elements of* E, $\phi(LAL^t)$ *is a diagonal matrix whose non zero entries are the product of two consecutive elements in* $\phi(P) = \{\phi(p), p\in P\}$.

If $c = 0$ then ***return*** *[P, Z, E];* Otherwise*:*

If $a_{11} \notin \mathbb{K}$ *or* $a_{11} > 0$ then

- *Let p be the last element of P.*
- *Let A_1 be the submatrix of $L(A,p)\,A$ obtained by removing the first row and first column.*
- *Append a_{11} to P to give P_1.*
- *Append $L(A,p)$ to E to give E_1.*
- *$\mathcal{S}_1$:= Diagonalization & Positivity (A_1, P_1, Z, E_1, T_1).*

If $a_{11} \notin \mathbb{K}$ *or* $a_{11} = 0$ then

- *$Z_2 := \{a_{1,j} \,|\, 1 \le j \le n\} \cup Z$.*
- *A_2 is obtained from A by removing the first row and first column.*
- *$\mathcal{S}_2$:= Diagonalization & Positivity (A_2, P, Z_2, E, T_2).*

return $\mathcal{S} \coloneqq \mathcal{S}_1 \cup \mathcal{S}_2$.

The elements in P and Z are principal minors of A of size up to order $\mathrm{Card}(P) - 1$. This determines their degrees in terms of the degrees of the entries of A. The output set $\mathcal{S}$ can have as many as 2^c triplets. One can lower this number if we can dismiss certain branches by checking if the semi-algebraic set defined by a pair (P, Z) is consistent. Though algorithmic [5], this is no light task. In its application to cubature, the algorithm will be made more specific, which shall result in a restricted output.

4.4 Diagonalization for determining the existence of cubatures

Taking into account the specificity of the cubature problem, we revisit the procedure in the previous section. There are less cases to be distinguished so that the size of the output is smaller.

The cubature problem comes with not only a $c \times c$ symmetric matrix A whose entries belong to $\mathbb{K}[\mathrm{h}] = \mathbb{K}[h_1, \dots, h_t]$ but also three input indices c', c'' and r, which satisfy $c' \le r \le c'' \le c$. With these indices are considered two leading principal submatrices of A, A' and A'', respectively of size c' and c''. The entries of A are of degree 1 or less in general, but the entries of A' more specifically belong to $\mathbb{K}$, and A' is positive definite. We are now looking for the locus in $\mathbb{R}^t$ where the respective specializations $\bar{A}''$ and $\bar{A}$ of A'' and A are semi-definite positive of rank r - extending the terminology introduced in Section 3.5, $\bar{A}$ is a *flat extension* of $\bar{A}''$. Algorithm 4.5 can then be appropriately modified to determine this locus. The output size is then reduced to $\binom{c''-c'}{r-c'}$ triplets :

- Since A' is positive definite, the algorithm runs without splitting as long as the matrix is of size bigger than $c - c'$. The first c' pivots are added to P.
- Since A'' is required to have the same rank as A, the algorithm is stopped when the matrix is of size $c - c''$. All its entries are added to Z.
- Only the branches that can terminate with $\mathrm{Card}\, P = r + 1$ (as we included 1 as first pivot in P) need to be continued.

The changes are straightforward. We simply provide the formal description of the input and output so as to serve as reference when this algorithm is called upon.

Algorithm 4.6 ***Diagonalization*** **&** ***Positivity with Rank Constraints***

Input : ▷ *Integers* $c' \le r \le c'' \le c$.
▷ *A* $c \times c$ *symmetric matrix* A *with entries in* $\mathbb{K}[\mathrm{h}]_{\le 1}$,
whose $c' \times c'$ *leading principal submatrix is positive definite with entries in* $\mathbb{K}$.
▷ P *a list of polynomials in* $\mathbb{K}[\mathrm{h}]$.
▷ Z *a set of polynomials in* $\mathbb{K}[\mathrm{h}]$.
▷ E *a list of elimination matrices.*

Output: ▷ *A set* $\mathcal{S}$ *of* $\binom{c''-c'}{r-c'}$ *triplets* $[P, Z, E]$, *where*
- P *is a list of polynomials in* $\mathbb{K}[\mathrm{h}]$ *with* $\operatorname{Card} P = r+1$,
- Z *is a set of polynomials in* $\mathbb{K}[\mathrm{h}]$,
- E *is a list of elimination matrices in* $\mathbb{K}(h)$ *whose denominators are power products of elements in* P.

The set $\mathcal{S}$ *satisfies the following property: For a specialization* $\phi : \mathbb{K}[\mathrm{h}] \to \mathbb{R}$, $\phi(A)$ *and* $\phi(A'')$, *where* A'' *is the leading principal submatrix of size* c'', *are positive semi-definite of rank* r *if and only if there is a triplet* $[P, Z, E]$ *such that*

$$\phi(p) > 0 \quad \forall p \in P \text{ and } \phi(q) = 0 \quad \forall q \in Z.$$

In this case, letting L *be the product of the elements of* E, $\phi(LAL^t)$ *is a diagonal matrix whose non zero entries are the product of two consecutive elements in* $\phi(P) = \{\phi(p), p \in P\}$.

5 Algorithms to determine cubatures

This section offer the synthesis of our approach to determine cubatures of degree d for an integral operator. Our approach is two fold. We first parametrize the cubatures with a given number of nodes, hence establishing their existence, or inexistence. We then provide the means to compute their nodes and weights. A salient feature in this approach is that the cubatures of a given degree d and with a given number of nodes are described exhaustively.

All along this section, the polynomial ring $\mathbb{K}[\mathrm{x}]$ is endowed with a basis $\mathfrak{M}$ such that, for any $\delta \in \mathbb{N}$, $\mathfrak{M}^{(\delta)} = \mathfrak{M} \cap \mathbb{K}[\mathrm{x}]_{\le \delta}$ is a basis of $\mathbb{K}[\mathrm{x}]_{\le \delta}$. Any basis consisting of homogeneous polynomials fits this description, as for instance the monomial basis. But so does a set of orthogonal polynomials for a given measure. We assume that $\mathfrak{M} = \{m_1, m_2, \ldots\}$ with $\mathfrak{M}^{(\delta)} = \{m_1, m_2, \ldots, m_{r_\delta}\}$, where $r_\delta = \dim \mathbb{K}[\mathrm{x}]_{\le \delta}$.

We seek to determine the cubatures of degree d for a positive linear form

$$\Omega : \quad \mathbb{R}[x] \quad \to \mathbb{R}$$

i.e. we look for the linear forms

$$\begin{array}{rccc} \Lambda : & \mathbb{R}[x] & \to & \mathbb{R} \\ & p & \mapsto & \sum_{i=1}^{r} a_i p(\xi_i) \end{array},$$

for some $r \in \mathbb{N}$, $a_i > 0$ and $\xi_i \in \mathbb{R}^n$, such that

$$\Lambda(m_i) = \Omega(m_i), \text{ for all } 1 \le i \le \dim \mathbb{R}[\mathrm{x}]_{\le d}.$$

One is either given r or one might be seeking the minimal possible r. Our emphasis is to describe the algorithmic aspects of determining cubatures and will not dwell on the finest hypotheses under which the existence is secured, or on the sharper bound known on the number of nodes. We are mostly considering the forms Ω defined as $\Omega(p) = \int_{\mathbb{R}^n} p d\mu$ where μ is a positive Borel measure with compact support. It is also reasonable to assume that Ω is definite, i.e. $\Omega(p^2) > 0 \Rightarrow p = 0$, at least when $\deg p \le d$.

In Section 3 we reviewed the conditions for a linear form to be a cubature. These conditions translate as rank and positivity constraints on the associated Hankel operator. For the specific problem of determining an interpolatory cubature of degree d for a integral operator Ω, these conditions can be checked in a finite dimension associated with an integer $\delta \le d+1$. This is explained in Section 5.1. In Section 5.2 we thus parametrize the space of linear forms on $\mathbb{R}[\mathrm{x}]_{\le 2\delta}$ whose restrictions to $\mathbb{R}[\mathrm{x}]_{\le d}$ coincide with Ω and apply the algorithm established in Section 4 to determine the locus of cubatures with r nodes among them. Following additional results in Section 3 we then show how, for each characterized cubature, we can retrieve its nodes as eigenvectors in Section 5.4. The weights are then obtained as the solution of a nonsingular linear system. The whole process is demonstrated for the minimal cubatures of degree 5 of the hexagon in Section 6.

5.1 Upper bound on the minimal number of nodes

The question of existence of a cubature of any given degree and a bound for the minimal number of nodes was provided by Tchakaloff [73] [70, Section 3.3].

Theorem 5.1 *Let μ be a positive measure with compact support K in $\mathbb{R}^n$ and d a fixed integer. For some $r \le \dim \mathbb{R}[\mathrm{x}]_{\le d}$, there exist r points $\xi_1, \dots, \xi_r \in K$, and r positive real numbers $a_1, \dots, a_r$ such that*

$$\Omega(p) = \sum_{i=1}^{r} a_i p(\xi_i), \ \forall p \in \mathbb{R}[\mathrm{x}]_{\le d}.$$

The above theorem was generalized in [58, 6] by reducing the hypothesis on the measure μ. The theorem remains true for any finite dimensional spaces of polynomials. It actually generalizes to spaces of functions other than polynomials [61]. See [68, Theorem 1.2.4]. These non contructive proofs call on Carathéodory's theorem in convex geometry.

Cubatures whose weights are obtained uniquely from the nodes are called *interpolatory*. Since the weights are solution of the linear system

$$W_\xi^{(d)} \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \Omega(m_1) \\ \vdots \\ \vdots \\ \Omega(m_{r_d}) \end{pmatrix}, \text{ where } W_\xi^{(d)} = \left(m_j(\xi_i)\right)_{1\le i\le r 1\le j\le r_\delta}$$

a cubature is interpolatory if the rank $w_\xi^{(d)}$ of $W_\xi^{(d)}$ is equal to the number r of nodes. The following result shows that a subset of the nodes of a non-interpolatory cubature can be used as the set of nodes of an interpolatory one [14, Section 6.1]. In the litterature on computing cubatures, this was observed in [20], [70, Theorem 3.3-3]. It is also a consequence of Carathéodory's theorem [63, Corollary 17.1.2].

Lemma 5.2 *Assume $\Lambda = \sum_{i=1}^{s} a_i \mathbb{e}_{\xi_i}$ is a cubature of degree d for $\Omega : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$. Then there is a cubature with $w_\xi^{(d)}$ nodes in $\{\xi_1, \dots, \xi_s\}$.*

A corollary to Theorem 5.1 is thus the existence of an inside interpolatory cubature of degree d with at most $\dim \mathbb{R}[\mathrm{x}]_{\le d}$ nodes. Furthermore any minimal cubature needs to be interpolatory.

5.2 Algorithm to determine the existence of a cubature with r nodes

According to Theorem 3.9 and Theorem 3.12, a cubature for Ω of degree d with r nodes exists if we can find $\delta \in \mathbb{N}$ and a linear form $\Lambda^{(\delta)} : \mathbb{R}[\mathrm{x}]_{\le 2\delta} \to \mathbb{R}$ such that

(1) $\Lambda^{(\delta)}(p) = \Omega(p)$, for all $p \in \mathbb{R}[\mathrm{x}]_{\le d}$;

(2) $\Lambda^{(\delta)}$ is positive semi-definite and its associated Hankel operator $\widehat{\mathcal{H}}^{(\delta)} : \mathbb{R}[\mathrm{x}]_{\le 2\delta} \to \mathbb{R}[\mathrm{x}]^*_{\le 2\delta}$ is of rank r;

(3) $\Lambda^{(\delta)}$ is a flat extension of its resriction $\Lambda^{(\delta-1)}$ to $\mathbb{R}[\mathrm{x}]_{\leq 2\delta-2}$. In other words $\widehat{\mathcal{H}}^{(\delta)}$ has the same rank as the Hankel operator $\widehat{\mathcal{H}}^{(\delta-1)} : \mathbb{R}[\mathrm{x}]_{\leq \delta-1} \to (\mathbb{R}[\mathrm{x}]_{\leq \delta-1})^*$ associated to $\Lambda^{(\delta-1)}$.

The cubature is then the unique flat extension $\Lambda : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ of $\Lambda^{(\delta)}$ predicted by Theorem 3.12. Theorem 3.9 ensures then that $\Lambda = \sum_{i=1}^{r} a_i \mathbb{e}_{\xi_i}$ for

- $\xi_1, \dots, \xi_r \in \mathbb{R}^n$, the roots of the ideal $I_\Lambda = \{p \,|\, \Lambda(pq) = 0, \forall q \in \mathbb{K}[\mathrm{x}]\}$;
- $a_i = \Lambda(\ell_i) > 0$, where $\{\ell_1, \dots, \ell_r\}$ is a Lagrange basis for $\{\xi_1, \dots, \xi_r\}$.

Some inequalities need to be satisfied between d, δ and r for the above to make sense. First $\lfloor \frac{d}{2} \rfloor < \delta$ and $r \leq \dim \mathbb{R}[\mathrm{x}]_{\leq \delta-1}$. Note also that the restriction to $\mathbb{R}[\mathrm{x}]_{\leq \lfloor \frac{d}{2} \rfloor}$ of the Hankel operator associated to Ω agrees with $\widehat{\mathcal{H}}^{(\lfloor \frac{d}{2} \rfloor)}$. If Ω is definite positive then we must have $\dim \mathbb{R}[\mathrm{x}]_{\leq \lfloor \frac{d}{2} \rfloor} \leq r$.

Given $\Lambda^{(\delta)} : \mathbb{R}[\mathrm{x}]_{\leq 2\delta} \to \mathbb{R}$ such that Property (1) stands, Properties (2) and (3) can be checked computationally on the matrix

$$H_1^{(\delta)} = (\Lambda(m_i m_j))_{1 \leq i,j \leq r_\delta}$$

of $\widehat{\mathcal{H}}^{(\delta)}$ in the basis $\mathfrak{M}^{(\delta)}$ of $\mathbb{R}[\mathrm{x}]_{\leq \delta}$, and its dual. Namely: $H_1^{(\delta)}$ is positive semi-definite of rank r and its leading principal submatrix of size $r_{\delta-1}$ is already of rank r. One can then see how Algorithm 4.5 applies in determining the existence of cubatures.

Let us introduce $\tilde{\Lambda}^{(\delta)} : \mathbb{K}[\mathrm{x}]_{\leq 2\delta} \to \mathbb{K}[h_{r_d+1}, \dots, h_{r_{2\delta}}]$ the linear map such that

$$\tilde{\Lambda}(m_i) = \begin{cases} \Omega(m_i) & \text{if } i \leq r_d, \text{ i.e. } \deg(m_i) \leq d, \\ h_i & \text{if } i > r_d, \text{ i.e. } \deg(m_i) > d. \end{cases}$$

and build the matrix

$$\tilde{H}_1^{(\delta)} = \left(\tilde{\Lambda}(m_i m_j)\right)_{1 \leq i,j \leq r_\delta}$$

whose entries are polynomials in $\mathbb{K}[\mathrm{h}] = \mathbb{K}[h_{r_d+1}, \dots, h_{r_{2\delta}}]$ of degree at most one. The existence of the desired cubature boils down to determining values $\hbar = (\hbar_{r_d+1}, \dots, \hbar_{r_{2\delta}}) \in \mathbb{R}^{r_{2\delta}-r_d}$ such that under the specialization

$$\begin{array}{cccc} \phi: & \mathbb{K}[\mathrm{h}] & \to & \mathbb{R} \\ & f & \mapsto & f(\hbar) \end{array}$$

the matrix $\phi\left(\tilde{H}_1^{(\delta)}\right)$ is positive semi-definite of rank r and has its principal leading submatrix of size $r_{\delta-1}$ already of rank r. The polynomial equalities and inequalities on the unknowns h for these conditions to be fulfilled are given by the output of Algorithm 4.6 applied to $A := \tilde{H}_1^{(\delta)}$, $c' := \dim \mathbb{R}[\mathrm{x}]_{\lfloor \frac{d}{2} \rfloor}$, $c'' := \dim \mathbb{R}[\mathrm{x}]_{\delta-1}$, and $c := \dim \mathbb{R}[\mathrm{x}]_\delta$. Note though that some branches of the algorithm can be dismissed at an early stage by taking into account Proposition 3.10.

The remaining issue is how to choose δ to be certain that any cubature Λ with r nodes is such that $\widehat{\mathcal{H}}^{(\delta)}$ is of rank r. Were we to look for all possible cubatures with r nodes, we would need to take $\delta = r$. If we restrict to searching for interpolatory cubatures we can limit ourselves to $\delta = d + 1$ thanks to Lemma 5.2 : With Theorem 3.3 and Theorem 3.9, the latter indeed implies that for an interpolatory cubature $\Lambda = \sum_{i=1}^{r} a_i \mathbb{e}_{\xi_i}$ a subset of $\mathfrak{M}^{(d)}$ needs to be a basis for $\mathbb{R}[\mathrm{x}]/I_\Lambda$, where I_Λ is the ideal of $\{\xi_1, \dots, \xi_r\}$. It follows that $\widehat{\mathcal{H}}^{(d)}$ is of rank r from Theorem 3.5. One can furthermore take into account Proposition 3.10: when looking for all interpolatory cubatures with r nodes, it is sufficient to take

$$\delta = 1 + \min\left\{ \dim \mathbb{K}[\mathrm{x}]_{\leq d}, \left\lfloor \frac{d}{2} \right\rfloor + r - \dim \mathbb{K}[\mathrm{x}]_{\lfloor \frac{d}{2} \rfloor} \right\}.$$

Algorithm 5.3 ***Existence-of-a-cubature***

Input : $\triangleright$ *The degree d of the sought cubature.*
$\triangleright$ *A number r of nodes, $\dim \mathbb{R}[\mathrm{x}]_{\le \lfloor \frac{d}{2} \rfloor} \le r \le \dim \mathbb{R}[\mathrm{x}]_{\le d}$.*
$\triangleright$ *The values of the linear form Ω on the basis $\{m_1, \dots, m_{r_d}\}$ of $\mathbb{R}[\mathrm{x}]_{\le d}$.*

Output : A set $\mathcal{S}$ of triplets $[P, Z, J]$ where
$\triangleright$ *P is a list of r polynomials in $\mathbb{K}[h_{r_d+1}, \dots, h_{r_{2\delta}}]$;*
$\triangleright$ *Z is a set of polynomials in $\mathbb{K}[h_{r_d+1}, \dots, h_{r_{2\delta}}]$;*
$\triangleright$ *J is an increasing sequence of r integers between 1 and $\dim \mathbb{R}[\mathrm{x}]_{\delta-1}$.*

The set $\mathcal{S}$ satisfies the following two converse properties:

- *For any $\hbar = (\hbar_{r_d+1}, \dots, \hbar_{r_{2\delta}}) \in \mathbb{R}^{r_{2\delta}-r_d}$ such that there is a triplet $[P, Z, J]$ in $\mathcal{S}$ with*

$$p(\hbar) > 0 \ \forall p \in P \ \text{ and } \ q(\hbar) = 0 \ \forall q \in Z,$$

there exists a unique cubature Λ of degree d for Ω with r nodes satisfying

$$\Lambda(m_j) = \begin{cases} \Omega(m_j) & \text{if } j \le r_d \\ \hbar_j & \text{if } r_d < j \le r_{2\delta} \end{cases}$$

This cubature is interpolatory and $\left(\Lambda(m_i m_j)\right)_{i,j \in J}$ is non singular.

- *For a linear form $\Lambda : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ to be an interpolatory cubature of degree d for Ω with r nodes there must be a triplet $[P, Z, J]$ in $\mathcal{S}$ such that the specialization*

$$\begin{array}{rccc} \phi: & \mathbb{K}[h_{r_d+1}, \dots, h_{r_{2\delta}}] & \to & \mathbb{R} \\ & h_j & \mapsto & \Lambda(m_j) \end{array}$$

satisfies

$$\phi(p) > 0 \ \forall p \in P \ \text{ and } \ \phi(q) = 0 \ \forall q \in Z.$$

1. *Let $\delta := 1 + \min\left\{ \dim \mathbb{K}[\mathrm{x}]_d,\ \left\lfloor \frac{d}{2} \right\rfloor + r - \dim \mathbb{K}[\mathrm{x}]_{\lfloor \frac{d}{2} \rfloor} \right\}$.*

2. *Construct the matrix $\tilde{H}_1^{(\delta)} = \left(\tilde{\Lambda}(m_i m_j)\right)_{1 \le i,j \le c}$ where $\tilde{\Lambda} : \mathbb{K}[\mathrm{x}]_{\le 2\delta} \to \mathbb{K}[h_{r_d+1}, \dots, h_{r_{2\delta}}]$ is the linear map such that*

$$\tilde{\Lambda}(m_i) = \begin{cases} \Omega(m_i) & \text{if } i \le r_d, \text{ i.e. } \deg(m_i) \le d, \\ h_i & \text{if } i > r_d, \text{ i.e. } \deg(m_i) > d. \end{cases}$$

3. *Call Algorithm 4.6 [Diagonalization & Positivity with Rank Constraints] with $A := \tilde{H}_1^{(\delta)}$, $c := \dim \mathbb{R}[\mathrm{x}]_{\le \delta}$, $c' := \dim \mathbb{K}[\mathrm{x}]_{\lfloor \frac{d}{2} \rfloor}$, and $c'' := \dim \mathbb{R}[\mathrm{x}]_{\le \delta - 1}$;*

4. *For each triplets $[P, Z, E]$ in the output of the above, consider J to be sequence of the indices of the nonzero diagonal elements appearing in P.*

This algorithm gives us a modus operandi to determine the minimal number of nodes in the cubatures of degree d for Ω. As seen in Lemma 5.2 these need to be interpolatory cubatures. The above algorithm is run successively on an increasing r, starting with a lower bound. When Ω is positive definite, the most general lower bound is $\dim \mathbb{R}[\mathrm{x}]_{\le \lfloor \frac{d}{2} \rfloor}$. Yet, under certain symmetry properties of Ω, or dimension condition, sharper lower bounds are known [14, Section 8.3]

5.3 Extension and inside-ness

When looking at the cubatures for $\Omega : \mathbb{R}[x] \to \mathbb{R}$ given as the integration according to a measure μ, i.e. $\Omega(p) = \int_{\mathbb{R}^n} p d\mu$, it might be necessary to ensure that the nodes lie on the support K of this measure, i.e. that the cubature is *inside*. When K is a semi-algebraic set it is possible to test this property of a cubature before actually computing its nodes. The key property is Corollary 3.11. Assume

$$K = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0\}$$

and consider $\kappa \in \mathbb{N}$ be such that $2\kappa \geq \deg g_k$ for all $k = 1, \ldots, s$.

In previous section we have characterized a cubature Λ of degree d for Ω with r nodes thanks to its restriction $\Lambda^{(\delta)}$ to $\mathbb{R}[x]_{\leq 2\delta}$, for some $\delta \in \mathbb{N}$. $\Lambda^{(\delta)}$ being a flat extension of $\Lambda^{(\delta-1)}$, it determines uniquely $\Lambda : \mathbb{R}[x] \to \mathbb{R}$. There are several reasons for wanting to know an intermediate flat extension $\Lambda^{(\delta+\kappa)}$ of $\Lambda^{(\delta)}$, as for instance testing if the cubature is inside.

To directly obtain this extension Algorithm 5.3 can be slightly modified as follow: In Step 2. we introduce $H_1^{(\delta+\kappa)} = \left(\tilde{\Lambda}(m_i m_j)\right)_{1 \leq i,j \leq c}$ where $\tilde{\Lambda} : \mathbb{K}[x]_{\leq 2\delta} \to \mathbb{K}[h_{r_d+1}, \ldots, h_{r_{2\delta+2\kappa}}]$ is the linear map such that

$$\tilde{\Lambda}(m_i) = \begin{cases} \Omega(m_i) & \text{if } i \leq r_d, \text{ i.e. } \deg(m_i) \leq d, \\ h_i & \text{if } i > r_d, \text{ i.e. } \deg(m_i) > d. \end{cases}$$

In Step 3. we then call Algorithm 4.6 with $A := H_1^{(\delta+\kappa)}$, $c := \dim \mathbb{R}[x]_{\leq \delta+\kappa}$, $c' := \dim \mathbb{K}[x]_{\lfloor \frac{d}{2} \rfloor}$, and $c'' := \dim \mathbb{R}[x]_{\leq \delta-1}$.

The triplets produced by this modification are in one to one correspondence with the triplets produced by the original algorithm, only their component Z has more elements. As guaranteed by Theorem 3.12 one can combine the elements of Z so as to express $h_{r_{2\delta}+1}, \ldots, h_{r_{2\delta+\kappa}}$ explicitly in terms of $h_{r_d+1}, \ldots, h_{r_{2\delta}}$.

5.4 Computation of the weights and the coordinates of the nodes

Assume that a cubature Λ of degree d for Ω with r nodes has been identified by a solution $(\hbar_{r_d+1}, \ldots, \hbar_{r_{2\delta}})$ of one triplet $[P, Z, J]$ output by Algorithm 5.3. This means that Λ is the unique flat extension of $\Lambda^{(\delta)} : \mathbb{R}[x]_{\leq 2\delta} \to \mathbb{R}$ defined by

$$\Lambda^{(\delta)}(m_j) = \begin{cases} \Omega(m_i) & \text{if } i \leq r_d, \text{ i.e. } \deg(m_i) \leq d, \\ \hbar_i & \text{if } i > r_d, \text{ i.e. } \deg(m_i) > d. \end{cases}$$

and that $B = \{m_j \,|\, i \in J\} \subset \mathbb{R}[x]_{\leq \delta-1}$ is a basis for $\mathbb{K}[x]/I_\Lambda$, where $I_\Lambda = \{p \in \mathbb{K}[x] \,|\, \Lambda(pq) = 0, \forall q \in \mathbb{R}[x]\}$.

$\Lambda^{(\delta)}$ and hence Λ are positive semi-definite. Their Hankel operators are of rank r. By Theorem 3.9 $\Lambda = \sum_{i=1}^r a_i \mathfrak{e}_{\xi_i}$ for some $a_i > 0$ and $\xi_i \in \mathbb{R}^n$. The nodes $\{\xi_1, \ldots, \xi_r\}$ are the roots of the ideal I_Λ, which is generated by $\ker \widehat{\mathcal{H}}^{(\delta)}$.

Following Section 3 we can compute the nodes and the weights of the cubature Λ with linear algebra operations. We describe the process with the light assumption that $d \geq 2$ and that $\Omega(p^2) > 0$ for all $p \in \mathbb{R}[x]_{\leq 1} \setminus \{0\}$ as it avoids some cumbersome case distinctions. With this assumption, $B = \{b_1, \ldots, b_r\}$ contains $\mathfrak{M}^{(1)}$, the basis of $\mathbb{R}[x]_{\leq 1}$.

Computing the nodes as eigenvectors

First we form the symmetric matrices

$$H_1^B = \left(\Lambda(b_i b_j)\right)_{1 \leq i,j, \leq r} \text{ and } H_p^B = \left(\Lambda(p b_i b_j)\right)_{1 \leq i,j, \leq r}$$

for a polynomial $p \in \mathbb{R}[\mathrm{x}]_{\le 1} \setminus \mathbb{R}$. According to Theorem 3.5 and Theorem 3.6 H_1^B is invertible and $M_p^B = \left(H_1^B\right)^{-1} H_p^B$ is the matrix of the multiplication by p in $\mathbb{K}[\mathrm{x}]/I_\Lambda$ expressed in the basis B. By Theorem 3.2, the eigenvalues of M_p^B are the values of p at the nodes $\{\xi_1, \dots, \xi_r\}$ of Λ and

$$W_\xi^B M_p^B = D W_\xi^B \quad \text{where } D = \operatorname{diag}\left(p(\xi_1), \dots, p(\xi_r)\right).$$

Hence

$$H_p^B \left(H_1^B\right)^{-1} \left(W_\xi^B\right)^T = \left(W_\xi^B\right)^T D.$$

Most polynomials p of degree 1 separate the roots of I_Λ, i.e. are such that $p(\xi_1), \dots, p(\xi_r)$ are pairwise distinct. It is thus fair to assume that we chose p as such, by taking a random combination of the polynomials in $\mathfrak{M}^{(1)} = \{m_1, m_2, \dots, m_{n+1}\}$. The associated eigenvectors are thus unique, up to a scalar factor.

For any computed eigenvector v of $H_p^B \left(H_1^B\right)^{-1}$ there is thus a $\lambda \in \mathbb{R} \setminus \{0\}$ and a node ξ such that

$$\mathrm{v} = \lambda \begin{pmatrix} b_1(\xi) \\ \vdots \\ b_r(\xi) \end{pmatrix}.$$

Since $b_1 \in \mathbb{R}$, we can determine λ and hence the whole vector $\begin{pmatrix} b_1(\xi) & \dots & b_r(\xi) \end{pmatrix}^T$. By computing the r eigenvectors we can thus determine the Vandermonde matrix W_ξ^B.

With the hypotheses on d and Ω, we have $\mathfrak{M}^{(1)} = \{m_1, \dots, m_{n+1}\} \subset B$. With the previous computation we have thus determined the vector $\begin{pmatrix} m_2(\xi_i) & \dots & m_{n+1}(\xi_i) \end{pmatrix}^T$ for each $1 \le i \le r$. This latter vector provides us with the coordinates of each node ξ_i, for $1 \le i \le r$.

Computing the weights by solving a non singular linear system

According to Theorem 3.3, the matrix W_ξ^B that we determined at the previous stage is invertible. The unique solution $\mathrm{a} \in \mathbb{R}^r$ to the linear system

$$W_\xi^B \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \Omega(b_1) \\ \vdots \\ \Omega(b_r) \end{pmatrix} \tag{8}$$

thus provides us with the weights of the cubature Λ.

6 Minimal cubatures of degree 5 for the regular hexagon

A minimal cubature of degree 5 for the regular hexagon is described in [70] under the name $H_2 :\ 5-1$. Only two such cubatures were known [48, 70], and both are D_6−invariant, where D_6 is the group of symmetry of the hexagon; It consists of reflections and rotations. As a mean to illustrate the algorithms presented in this article we show that there is actually a one parameter family of minimal cubatures. They are C_6 invariant, where C_6 is the subgroup of D_6 consisting of rotations.

We first use the monomial basis and straightforwardly apply the algorithm presented in Section 5 to determine all the cubatures of degree 5 with 7 nodes for the hexagon. The C_6-invariance reveals itself ther experimentally. We then illustrate the use of a *symmetry adapted basis* to exploit and ascertain this invariance. Computing symmetric cubatures is the subject of a forthcoming paper. We give here a foretaste of the results on this example. We exploit the obtained block diagonal structure of the matrices of the Hankel operators to alleviate the computational cost the algorithms and deduce properties of the cubature from the size of the blocks.

Let $\mathcal{H}$ be the regular hexagon whose vertices are given by $(\pm 1, 0), (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ (see Figure 1(a)). Its moments of order less than or equal to 5 are

$$\int_{\mathcal{H}} 1dx = \frac{3\sqrt{3}}{2}, \qquad \int_{\mathcal{H}} x_1^2 dx = \int_{\mathcal{H}} x_2^2 dx = \frac{5\sqrt{3}}{16}$$
$$\int_{\mathcal{H}} x_1^4 dx = \int_{\mathcal{H}} x_2^4 dx = \frac{21\sqrt{3}}{160}, \qquad \int_{\mathcal{H}} x_1^2 x_2^2 dx = \frac{7\sqrt{3}}{160}$$

and zero otherwise.

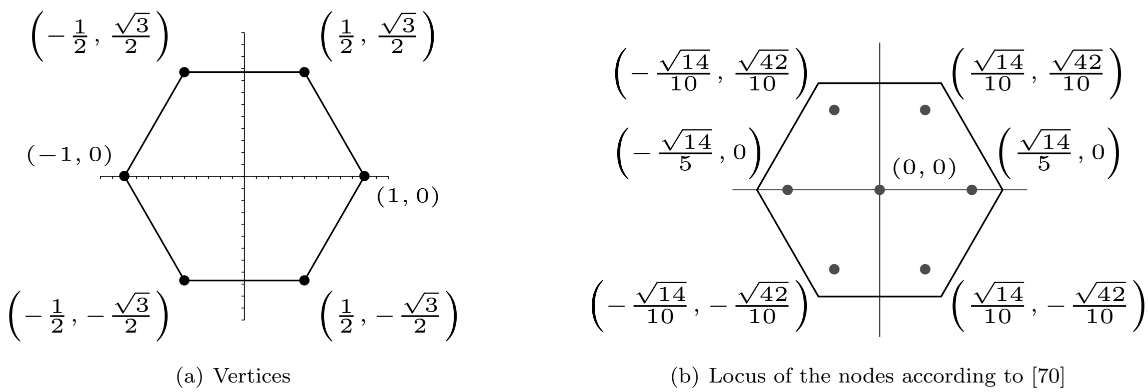

Figure 1: The regular hexagon $\mathcal{H}$ and the cubature $H_2 : 5-1$

Note that in virtue of [54] [14, Theorem 8.3] there cannot be cubature of degree 5 with $\dim \mathbb{R}[x]_{\le 2} = 6$ nodes, the hexagon being centrally symmetric. This could also be uncovered computationally following the process described in Section 5. We thus look for a cubature with 7 nodes.

6.1 Existence

We consider first the monomial basis $\mathfrak{M} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, \ldots\}$ ordered with the graded reverse lexicographic order. We proceed as described in Section 5.2 to determine the sets of equalities and inequalities that characterize the cubatures of degree 5 for $\mathcal{H}$ with 7 nodes.

1. $\delta := 1 + \min\left\{\dim \mathbb{K}[x,y]_{\le 5},\ \left\lfloor \frac{5}{2} \right\rfloor + 7 - \dim \mathbb{K}[x,y]_{\le \left\lfloor \frac{5}{2} \right\rfloor}\right\} = 4;$

   We thus consider the monomial basis $\mathfrak{M}^{(4)}$ of $\mathbb{R}[x]_{\le 4}$ ordered according to the graded reverse lexicographic order

   $$\mathfrak{M} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4\}.$$

2. The parameterized moment matrix, with $\tilde{\Lambda}(x_1^i x_2^j) = \frac{\sqrt{3}}{160} h_{ij}$ for $i + j > 5$, is

$$\tilde{H}_1^{\mathfrak{M}} = \frac{\sqrt{3}}{160}\left(\begin{array}{cccccc|cccc|ccccc}
240 & 0 & 0 & 50 & 0 & 50 & 0 & 0 & 0 & 0 & 21 & 0 & 7 & 0 & 21 \\
0 & 50 & 0 & 0 & 0 & 0 & 21 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 50 & 0 & 0 & 0 & 0 & 7 & 0 & 21 & 0 & 0 & 0 & 0 & 0 \\
50 & 0 & 0 & 21 & 0 & 7 & 0 & 0 & 0 & 0 & h_{60} & h_{51} & h_{42} & h_{33} & h_{24} \\
0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & h_{51} & h_{42} & h_{33} & h_{24} & h_{15} \\
50 & 0 & 0 & 7 & 0 & 21 & 0 & 0 & 0 & 0 & h_{42} & h_{33} & h_{24} & h_{15} & h_{06} \\
\hline
0 & 21 & 0 & 0 & 0 & 0 & h_{60} & h_{51} & h_{42} & h_{33} & h_{70} & h_{61} & h_{52} & h_{43} & h_{34} \\
0 & 0 & 7 & 0 & 0 & 0 & h_{51} & h_{42} & h_{33} & h_{24} & h_{61} & h_{52} & h_{43} & h_{34} & h_{25} \\
0 & 7 & 0 & 0 & 0 & 0 & h_{42} & h_{33} & h_{24} & h_{15} & h_{52} & h_{43} & h_{34} & h_{25} & h_{16} \\
0 & 0 & 21 & 0 & 0 & 0 & h_{33} & h_{24} & h_{15} & h_{06} & h_{43} & h_{34} & h_{25} & h_{16} & h_{07} \\
\hline
21 & 0 & 0 & h_{60} & h_{51} & h_{42} & h_{70} & h_{61} & h_{52} & h_{43} & h_{80} & h_{71} & h_{62} & h_{53} & h_{44} \\
0 & 0 & 0 & h_{51} & h_{42} & h_{33} & h_{61} & h_{52} & h_{43} & h_{34} & h_{71} & h_{62} & h_{53} & h_{44} & h_{35} \\
7 & 0 & 0 & h_{42} & h_{33} & h_{24} & h_{52} & h_{43} & h_{34} & h_{25} & h_{62} & h_{53} & h_{44} & h_{35} & h_{26} \\
0 & 0 & 0 & h_{33} & h_{24} & h_{15} & h_{43} & h_{34} & h_{25} & h_{16} & h_{53} & h_{44} & h_{35} & h_{26} & h_{17} \\
21 & 0 & 0 & h_{24} & h_{15} & h_{06} & h_{34} & h_{25} & h_{16} & h_{07} & h_{44} & h_{35} & h_{26} & h_{17} & h_{08}
\end{array}\right)$$

There are $\dim \mathbb{R}[x]_{\leq 8} - \dim \mathbb{R}[x]_{\leq 5} = 24$ parameters. The 6×6 leading principal submatrix has entries in $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$ and is positive definite.

3. We call Algorithm 4.6 with the matrix $A = \dfrac{160}{\sqrt{3}} \tilde{H}_1^{(4)}$ and $c' = 6$, $r = 7$, $c'' = 10$ and $c = 15$.

4. There are then $\binom{4}{1} = 4$ triplets $[P, Z, J]$, where $J = \{1, 2, 3, 4, 5, 6, k\}$ with $k \in \{7, 8, 9, 10\}$.

For the two triplets $[P, Z, J]$ where $J = \{1, 2, 3, 4, 5, 6, k\}$ with $k \in \{9, 10\}$, the sets of equalities Z are inconsistent. These triplets thus bring no cubature.

For the triplet $[P, Z, J]$ where $J = \{1, 2, 3, 4, 5, 6, 8\}$, the set of equalities Z has a single solution

$$h_{24} = \frac{49}{50}, h_{42} = \frac{147}{50}, h_{15} = 0, h_{60} = \frac{441}{50}, h_{33} = 0, h_{51} = 0, h_{06} = \frac{539}{50}$$

$$h_{07} = 0, h_{16} = 0, h_{25} = 0, h_{34} = 0, h_{43} = 0, h_{52} = 0, h_{61} = 0, h_{70} = 0,$$

$$h_{08} = \frac{14749}{2500}, h_{17} = 0, h_{26} = \frac{343}{2500}, h_{35} = 0, h_{44} = \frac{1029}{2500}, h_{53} = 0, h_{62} = \frac{3087}{2500}, h_{71} = 0, h_{80} = \frac{9261}{2500}$$

while the set P has a single non constant polynomial, which is a positive multiple of $50h_{42} - 49$. This triplet corresponds thus to a single cubature.

We consider now the triplet $[P, Z, J]$ with $J = \{1, 2, 3, 4, 5, 6, 7\}$. The only non constant polynomial in P is a positive multiple of $50h_{60} - 441$. By computing a Gröbner basis we determine that the solutions of the equalities in Z for which $50h_{60} - 441 \neq 0$ satisfy :

$$h_{24} = -\frac{196}{25} + h_{60}, h_{42} = \frac{294}{25} - h_{60}, h_{06} = \frac{98}{5} - h_{60}, \quad h_{33} = -h_{15}, h_{51} = h_{15},$$

$$h_{07} = 0, h_{16} = 0, h_{25} = 0, h_{34} = 0, h_{43} = 0, h_{52} = 0, h_{61} = 0, h_{70} = 0,$$

$$h_{44} = \frac{1029}{2500},$$

$$h_{08} = \frac{7889}{500} - \frac{28}{25} h_{60}, h_{26} = -\frac{2401}{500} + \frac{14}{25} h_{60}, h_{62} = \frac{3087}{500} - \frac{14}{25} h_{60}, h_{80} = -\frac{3087}{500} + \frac{28}{25} h_{60},$$

$$h_{35} = -\frac{7}{25} h_{15}, h_{53} = -\frac{7}{25} h_{15}, h_{71} = \frac{21}{25} h_{15}, h_{17} = \frac{21}{25} h_{15}.$$

All the unknowns can be expressed explicitly in terms of either h_{60} or h_{15} while

$$50^2 \left(h_{60} - \frac{49}{5}\right)^2 + 50^2 \, {h_{15}}^2 = 49^2.$$

All the real solutions of this equation are on a circle and thus can be parameterized with a single parameter θ as

$$h_{60} = \frac{49}{50}(10 + \cos\theta), \quad h_{15} = \frac{49}{50}\sin\theta.$$

Taking into account the positivity constraint $50h_{60} - 441 = 49(1 + \cos\theta) > 0$, all the solutions of the triplet are parametrized explicitly by θ, for $\theta \in]-\pi,\pi[$. Note though that the solution of Z for $\theta = \pi$ is the solution of the previous triplet.

6.2 Inside-ness

Proposition 3.11 gives us a method to show that the cubatures characterized above are inside for every value of $\theta \in]-\pi,\pi]$.

The regular hexagon $\mathcal{H}$ is defined as a semi-algebraic set by

$$\mathcal{H} = \{x \in \mathbb{R}^2 \mid g_1(x) \geq 0, \ldots, g_6(x) \geq 0\},$$

where $g_1(x) = x_2+\frac{\sqrt{3}}{2}$, $g_2(x) = -x_2+\frac{\sqrt{3}}{2}$, $g_3(x) = -x_2-\sqrt{3}x_1+\sqrt{3}$, $g_4(x) = x_2+\sqrt{3}x_1+\sqrt{3}$, $g_5(x) = x_2-\sqrt{3}x_1+\sqrt{3}$ and $g_6(x) = -x_2 + \sqrt{3}x_1 + \sqrt{3}$. Taking the basis $\mathfrak{M}^{(3)} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$ of $\mathbb{R}[x]_{\leq 3}$, we construct the localizing matrices $H_{g_k}^{(3)}$ for all $k = 1, \ldots, 6$. Their coefficients are in $\mathbb{Q}[\sqrt{3}, \cos\theta, \sin\theta]$. The positivity constraints obtained using Algorithm 4.6 are here satisfied for all values of the parameter θ.

6.3 Weights and nodes

After examining the output of Algorithm 5.3 we know that the cubatures of degree 5 with 7 nodes for the hexagon are in one to one correspondance with the real numbers θ in $]-\pi,\pi]$.

We now proceed to compute the nodes and weights of the associated cubatures for a selection of values of the parameter θ following Section 5.4. We first consider the two cases $\theta = 0$ and $\theta = \pi$ where the computations can be done in exact arithmetic. We shall retrieve the two D_6-invariant cubatures that already exist in the literature. We then compute numerically the nodes and weights for a set of of other values of θ.

The symmetric cubatures previously known: The case $\theta = 0$ belongs to the triplet with $J = \{1,2,3,4,5,6,7\}$. Hence $B = \left\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3\right\}$ is a basis of $\mathbb{R}[x]/I_\Lambda$. For $p = x_1 + 5x_2$ we have

$$H_1^B = \frac{\sqrt{3}}{160}\begin{pmatrix} 240 & 0 & 0 & 50 & 0 & 50 & 0 \\ 0 & 50 & 0 & 0 & 0 & 0 & 21 \\ 0 & 0 & 50 & 0 & 0 & 0 & 0 \\ 50 & 0 & 0 & 21 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 50 & 0 & 0 & 7 & 0 & 21 & 0 \\ 0 & 21 & 0 & 0 & 0 & 0 & \frac{539}{50} \end{pmatrix} \text{ and } H_p^B = \frac{\sqrt{3}}{160}\begin{pmatrix} 0 & 50 & 250 & 0 & 0 & 0 & 21 \\ 50 & 0 & 0 & 21 & 35 & 7 & 0 \\ 250 & 0 & 0 & 35 & 7 & 105 & 0 \\ 0 & 21 & 35 & 0 & 0 & 0 & \frac{539}{50} \\ 0 & 35 & 7 & 0 & 0 & 0 & \frac{49}{50} \\ 0 & 7 & 105 & 0 & 0 & 0 & \frac{49}{50} \\ 21 & 0 & 0 & \frac{539}{50} & \frac{49}{50} & \frac{49}{50} & 0 \end{pmatrix}.$$

A matrix of normalized eigenvectors for $(M_p^B)^T = H_p^B \left(H_1^B\right)^{-1}$ is then given by the transpose of

$$W_\xi^B = \left(\begin{array}{c|cc|cccc} 1 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\ 1 & \frac{\sqrt{\mathbf{14}}}{\mathbf{5}} & \mathbf{0} & \frac{14}{25} & 0 & 0 & \frac{14\sqrt{14}}{125} \\ 1 & -\frac{\sqrt{\mathbf{14}}}{\mathbf{5}} & \mathbf{0} & \frac{14}{25} & 0 & 0 & -\frac{14\sqrt{14}}{125} \\ 1 & -\frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & \frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & \frac{7}{50} & -\frac{\sqrt{47}}{50} & \frac{21}{50} & -\frac{7\sqrt{14}}{500} \\ 1 & \frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & -\frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & \frac{7}{50} & -\frac{\sqrt{47}}{50} & \frac{21}{50} & \frac{7\sqrt{14}}{500} \\ 1 & \frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & \frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & \frac{7}{50} & \frac{\sqrt{47}}{50} & \frac{21}{50} & \frac{7\sqrt{14}}{500} \\ 1 & -\frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & -\frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & \frac{7}{50} & \frac{\sqrt{47}}{50} & \frac{21}{50} & -\frac{7\sqrt{14}}{500} \end{array}\right).$$

Since x_1 and x_2 are the second and third polynomials in B, the second and third columns of W_ξ^B are the coordinates of the nodes. The nodes of the cubature are thus:

$$\xi_1 = (0,0), \quad \xi_2 = \left(\frac{\sqrt{14}}{5}, 0\right), \quad \xi_3 = \left(-\frac{\sqrt{14}}{5}, 0\right),$$

$$\xi_4 = \left(\frac{\sqrt{14}}{10}, \frac{\sqrt{42}}{10}\right), \xi_5 = \left(-\frac{\sqrt{14}}{10}, \frac{\sqrt{42}}{10}\right), \xi_6 = \left(-\frac{\sqrt{14}}{10}, -\frac{\sqrt{42}}{10}\right), \xi_7 = \left(\frac{\sqrt{14}}{10}, -\frac{\sqrt{42}}{10}\right).$$

With this exact computation, we also get the exact values of the weights

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \left(\frac{43\sqrt{3}}{112}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}\right).$$

We thus recognize the cubature given in [70]. It is represented in golden yellow in Figure 2.

The case $\theta = \pi$ belongs to the triplet with $J = \{1,2,3,4,5,6,8\}$ so that $C = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^2x_2\}$ is a basis of $\mathbb{R}[x]/I_\Lambda$. For $p = x_1 + 5x_2$ we have

$$H_1^C = \frac{\sqrt{3}}{160}\begin{pmatrix} 240 & 0 & 0 & 50 & 0 & 50 & 0 \\ 0 & 50 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 & 7 \\ 50 & 0 & 0 & 21 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 50 & 0 & 0 & 7 & 0 & 21 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & \frac{147}{50} \end{pmatrix} \text{ and } H_p^C = \frac{\sqrt{3}}{160}\begin{pmatrix} 0 & 50 & 250 & 0 & 0 & 0 & 35 \\ 50 & 0 & 0 & 21 & 35 & 7 & 0 \\ 250 & 0 & 0 & 35 & 7 & 105 & 0 \\ 0 & 21 & 35 & 0 & 0 & 0 & \frac{147}{10} \\ 0 & 35 & 7 & 0 & 0 & 0 & \frac{147}{50} \\ 0 & 7 & 105 & 0 & 0 & 0 & \frac{147}{10} \\ 35 & 0 & 0 & \frac{147}{10} & \frac{147}{50} & \frac{147}{10} & 0 \end{pmatrix}$$

A matrix of normalized eigenvectors for $(M_p^C)^T = H_p^C \left(H_1^C\right)^{-1}$ is given by the transpose of

$$W_\xi^C = \left(\begin{array}{c|cc|cccc} 1 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\ 1 & \mathbf{0} & \frac{\sqrt{\mathbf{14}}}{\mathbf{5}} & 0 & 0 & \frac{14}{25} & 0 \\ 1 & \mathbf{0} & -\frac{\sqrt{\mathbf{14}}}{\mathbf{5}} & 0 & 0 & \frac{14}{25} & 0 \\ 1 & -\frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & \frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & \frac{21}{50} & -\frac{7\sqrt{3}}{50} & \frac{7}{50} & \frac{21\sqrt{14}}{500} \\ 1 & \frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & -\frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & \frac{21}{50} & -\frac{7\sqrt{3}}{50} & \frac{7}{50} & -\frac{21\sqrt{14}}{500} \\ 1 & \frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & \frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & \frac{21}{50} & \frac{7\sqrt{3}}{50} & \frac{7}{50} & \frac{21\sqrt{14}}{500} \\ 1 & -\frac{\sqrt{\mathbf{42}}}{\mathbf{10}} & -\frac{\sqrt{\mathbf{14}}}{\mathbf{10}} & \frac{21}{50} & \frac{7\sqrt{3}}{50} & \frac{7}{50} & -\frac{21\sqrt{14}}{500} \end{array}\right).$$

Since x_1 and x_2 are the second and third polynomials in C, the second and third columns of W_ξ^C are the coordinates of the nodes

$$\xi_1 = (0,0), \quad \xi_2 = \left(0, \frac{\sqrt{14}}{5}\right), \quad \xi_3 = \left(0, -\frac{\sqrt{14}}{5}\right),$$

$$\xi_4 = \left(\frac{\sqrt{42}}{10}, \frac{\sqrt{14}}{10}\right), \xi_5 = \left(\frac{\sqrt{42}}{10}, -\frac{\sqrt{14}}{10}\right), \xi_6 = \left(-\frac{\sqrt{42}}{10}, -\frac{\sqrt{14}}{10}\right), \xi_7 = \left(-\frac{\sqrt{42}}{10}, \frac{\sqrt{14}}{10}\right).$$

With this exact computation, we get the exact values of the weights

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \left(\frac{43\sqrt{3}}{112}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}\right)$$

and thus a second D_6-invariant cubature [48]. It is represented in purple in Figure 2.

New cubatures: In the case $\theta \in]-\pi, \pi[$, $B = \left\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3\right\}$ is a basis of $\mathbb{R}[x]/I_\Lambda$. For $p = x_1 + 5x_2$ we have

$$H_1^B = \frac{\sqrt{3}}{160}\begin{pmatrix} 240 & 0 & 0 & 50 & 0 & 50 & 0 \\ 0 & 50 & 0 & 0 & 0 & 0 & 21 \\ 0 & 0 & 50 & 0 & 0 & 0 & 0 \\ 50 & 0 & 0 & 21 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 50 & 0 & 0 & 7 & 0 & 21 & 0 \\ 0 & 21 & 0 & 0 & 0 & 0 & h_{60} \end{pmatrix} \text{ and } H_p^B = \frac{\sqrt{3}}{160}\begin{pmatrix} 0 & 50 & 250 & 0 & 0 & 0 & 21 \\ 50 & 0 & 0 & 21 & 35 & 7 & 0 \\ 250 & 0 & 0 & 35 & 7 & 105 & 0 \\ 0 & 21 & 35 & 0 & 0 & 0 & h_{60}+5h_{51} \\ 0 & 35 & 7 & 0 & 0 & 0 & h_{51}+5h_{42} \\ 0 & 7 & 105 & 0 & 0 & 0 & h_{42}+5h_{33} \\ 21 & 0 & 0 & h_{60}+5h_{51} & h_{51}+5h_{42} & h_{42}+5h_{33} & 0 \end{pmatrix}$$

where

$$h_{60} = \frac{49}{50}(10+\cos\theta),\ h_{60}+5h_{51} = \frac{49}{50}(10+\cos\theta+5\sin\theta),\ h_{51}+5h_{42} = \frac{49}{50}(10-5\cos\theta+\sin\theta),\ h_{42}+5h_{33} = \frac{49}{50}(2-\cos\theta-5\sin\theta).$$

θ	Nodes	
0.3	(-0.3413079252984122,-0.6659646387973565)	(0.3413079252984122, 0.6659646387973565)
	(0.4060883325714323,-0.6285636532200647)	(-0.4060883325714323, 0.6285636532200647)
	(-0.7473962578698446,-0.0374009855772917)	(0.7473962578698446, 0.0374009855772917)
0.9	(-0.2731172820572105,-0.6967115258424265)	(0.2731172820572105, 0.6967115258424265)
	(0.4668112394603545,-0.5848822673953174)	(-0.4668112394603545, 0.5848822673953174)
	(-0.7399285215175650,-0.1118292584471091)	(0.7399285215175650, 0.1118292584471091)
1.5	(-0.2021977412142636,-0.7204971016234900)	(0.2021977412142636, 0.7204971016234900)
	(0.5228699227518689,-0.5353569312911290)	(-0.5228699227518689, 0.5353569312911290)
	(-0.7250676639661324,-0.1851401703323610)	(0.7250676639661324, 0.1851401703323610)
2.1	(0.1292579073787906, 0.7370837085298087)	(-0.1292579073787906,-0.7370837085298087)
	(0.7029621699918544, 0.2566012228348556)	(-0.7029621699918544,-0.2566012228348556)
	(-0.5737042626130638, 0.4804824856949531)	(0.5737042626130638,-0.4804824856949531)
3	(0.0176580675312746, 0.7481231132982465)	(-0.0176580675312746,-0.7481231132982465)
	(-0.6390645875089480, 0.3893538917129482)	(0.6390645875089480,-0.3893538917129482)
	(0.6567226550402226, 0.3587692215852983)	(-0.6567226550402226,-0.3587692215852983)

Table 1: Nodes of the cubatures (excepted $(0,0)$) for selected values of θ.

For each selected value of the parameter $\theta \in]-\pi, \pi[$, we compute (numerically) the eigenvectors of the matrix $\left(M_p^B\right)^T = H_p^B \left(H_1^B\right)^{-1}$ and deduce the coordinates of the nodes (see Table 1 and Figure 2). We note that $(0,0)$ is a node in all the so obtained cubatures. We can also observe that the other 6 nodes all lie on a circle and form a C_6 orbit. Furthermore solving the Vandermonde-like linear system (8), we get the same weights for all selected values of the parameter θ

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} 0.6649837921916226 \\ 0.3221820698602823 \\ 0.3221820698602823 \\ 0.3221820698602823 \\ 0.3221820698602823 \\ 0.3221820698602823 \\ 0.3221820698602823 \end{pmatrix},$$

where the first weight is associated with the node $(0,0)$. It thus looks like the computed cubatures are C_6-invariant. Two cubatures are represented in different shades of brown in Figure 2.

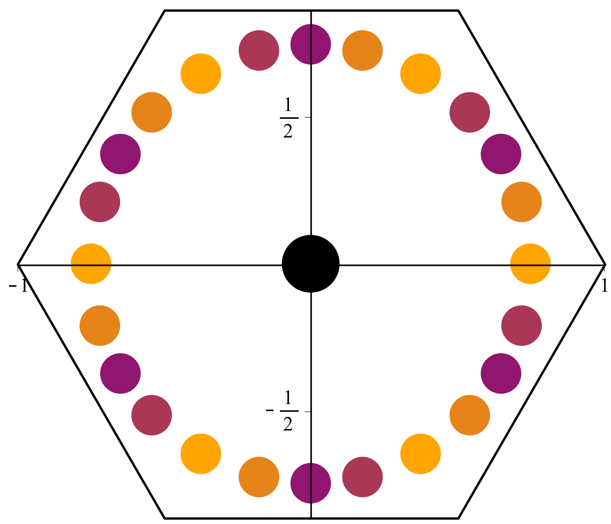

Figure 2: Four distinct minimal cubatures : the origin is a node for all the cubatures. The other nodes are represented with a distinct color for each distinct cubautre.

6.4 Alternative basis and C_6-invariant cubatures

The observed symmetry in the minimal cubatures computed above implies a certain structure. It manifested itself with the presence of many zeros in the matrices of the Hankel operators. We shall present here how to exploit the symmetry in the computation by working in an alternative basis. First it bring out a block diagonal structure in the matrices of the Hankel operators. Furthermore we can collect qualitative information at early stages and restrict our attention to a single sub-block in the multiplication and Vandermonde matrices. The results we apply results will actually be presented and proved in a forthcoming article [13]. This section is to be understood as a teaser.

We shall use a *symmetry adapted basis* of $\mathbb{K}[\mathrm{x}]$ according to the group D_6. It splits into 8 blocks. Their elements up to degree 8 are :

$$d_{11} = 1,\ d_{12} = x^2 + y^2,\ d_{13} = d_{12}^2,\ d_{14} = d_{12}^3,\ d_{15} = y^2\left(3\,x^2 - y^2\right)^2,\ d_{16} = d_{12}^4,\ ,d_{17} = d_{15}\,d_{12},$$

$$d_{21} = xy\left(x^2 - 3\,y^2\right)\left(3\,x^2 - y^2\right),\ d_{22} = d_{21}\,d_{12},$$

$$d_{31} = x\left(x^2 - 3\,y^2\right),\ d_{32} = d_{31}\,d_{12},\ d_{33} = d_{31}\,d_{12}^2$$

$$d_{41} = y\left(3\,x^2 - y^2\right),\ d_{42} = d_{41}\,d_{12},\ d_{43} = d_{41}\,d_{12}^2,$$

$$d_{51} = x,\ d_{52} = d_{51}\,d_{12},\ d_{53} = d_{51}\,d_{12}^2,\ d_{54} = 2\,xy^2\left(3\,x^2 - y^2\right),\ d_{55} = d_{51}\,d_{12}^3,\ d_{56} = d_{54}\,d_{12},\ d_{57} = d_{51}\,d_{15},$$

$$d_{61} = y,\ d_{62} = d_{61}\,d_{12},\ d_{63} = d_{61}\,d_{12}^2,\ d_{64} = y\left(x^2 - y^2\right)\left(3\,x^2 - y^2\right),\ d_{65} = d_{61}\,d_{12}^3,\ d_{66} = d_{64}\,d_{12},\ d_{67} = d_{61}\,d_{15},\ \ldots$$

$$d_{71} = x^2 - y^2,\ d_{72} = d_{71}\,d_{12},\ d_{73} = y^2\left(3\,x^2 - y^2\right),\ d_{74} = d_{71}\,d_{12}^2,\ d_{75} = d_{73}\,d_{12},\ d_{76} = d_{71}\,d_{12}^3,\ d_{77} = d_{73}\,d_{12}^2,\ d_{78} = d_{71}\,d_{15},$$

$d_{81} = 2\,xy,\ d_{82} = d_{81}\,d_{12},\ d_{83} = xy\left(3\,x^2 - y^2\right),\ d_{84} = d_{81}\,d_{12}^2,\ d_{85} = d_{83}\,d_{12},\ d_{86} = d_{81}\,d_{12}^3,\ d_{87} = d_{83}\,d_{12}^2,\ d_{88} = d_{83}\,d_{15}.$

Each block pertains to an *irreducible representation* of D_6. The first block correspond to the trivial representation and consists of invariants. The 5-th and 6-th blocks, as well as the 7-th and 8-th blocks, work in pairs as they relate to irreducible representations of degree 2. Note that the first and second blocks are C_6-invariants. Pairing all the blocks above actually provide a symmetry adapted basis according to C_6 that consists of 4 blocks. A crucial property is that if a linear form Λ is D_6*-invariant* then $\Lambda(d_{ij}) = 0$ when $i \geq 2$. If it is C_6*-invariant* then $\Lambda(d_{ij}) = 0$ when $i \geq 3$. Here we shall look for a cubature that is C_6 invariant and thus assume that it is the case.

A basis of $\mathbb{R}[x]_{\leq 4}$ is given by the union of the following four *blocks*, where we ordered the elements by degree:

$$\mathfrak{B} = \{d_{11}, d_{12}, d_{13}\} \cup \{d_{31}, d_{41}\} \cup \{d_{51}, d_{61}, d_{52}, d_{62}\} \cup \{d_{71}, d_{81}, d_{72}, d_{73}, d_{82}, d_{83}\}.$$

To ease following the argument we provide the degrees of these elements as follows:

$$\deg \mathfrak{B} = \{0,\ 2,\ 4\} \cup \{3,\ 3\} \cup \{1,\ 1,\ 3,\ 3\} \cup \{2,\ 2,\ 4,\ 4,\ 4,\ 4\}.$$

The computation gain appears as follow : with $\tilde{\Lambda}(d_{ij}) = h_{ij}$ for $i = 1,2$ and $\deg d_{ij} > 5$ and $\tilde{\Lambda}(d_{ij}) = 0$ for $3 \leq i \leq 8$, the parameterized matrix of the Hankel operator is block diagonal :

$$\tilde{H}_1 = \frac{\sqrt(3)}{2}\ \mathrm{diag}\left(\tilde{H}_1^{(1)},\ \tilde{H}_1^{(2)},\ \tilde{H}_1^{(3)},\ \tilde{H}_1^{(4)}\right)$$

where

$$\tilde{H}^{(1)} = \begin{bmatrix} 3 & \frac{5}{4} & \frac{7}{10} \\ \frac{5}{4} & \frac{7}{10} & 2\,h_{14} \\ \frac{7}{10} & 2\,h_{14} & 2\,h_{16} \end{bmatrix}, \quad \tilde{H}^{(2)} = \begin{bmatrix} 2\,h_{14} - 2\,h_{15} & 2\,h_{21} \\ 2\,h_{21} & 2\,h_{15} \end{bmatrix},$$

$$\tilde{H}^{(3)} = \begin{bmatrix} \frac{5}{8} & 0 & \frac{7}{20} & 0 \\ 0 & \frac{5}{8} & 0 & \frac{7}{20} \\ \frac{7}{20} & 0 & h_{14} & 0 \\ 0 & \frac{7}{20} & 0 & h_{14} \end{bmatrix}, \quad \tilde{H}^{(4)} = \begin{bmatrix} \frac{7}{20} & 0 & h_{14} & h_{15} & 0 & h_{21} \\ 0 & \frac{7}{20} & 0 & -h_{21} & h_{14} & h_{15} \\ h_{14} & 0 & h_{16} & h_{17} & 0 & h_{22} \\ h_{15} & -h_{21} & h_{17} & h_{17} & -h_{22} & 0 \\ 0 & h_{14} & 0 & -h_{22} & h_{16} & h_{17} \\ h_{21} & h_{15} & h_{22} & 0 & h_{17} & h_{17} \end{bmatrix}$$

To determine the existence conditions of a cubature we can thus apply Algorithm 4.6 separately to $\tilde{H}^{(1)}$, $\tilde{H}^{(2)}$, $\tilde{H}^{(3)}$ and $\tilde{H}^{(4)}$. We thus deal with matrices with sizes 3×3, 2×2, 4×4, 6×6 instead of a single matrix of size 15×15.

The triplets $[P, Z, J]$ now combine four blocks $[P^{(i)}, Z^{(i)}, J^{(i)}]$ for $1 \leq i \leq 4$. Two index sets $J^{(i)}$ are predetermined owing to the degrees appearing in the $\mathfrak{B}^{(i)}$: $J^{(1)} = \{1,2\}$ and $J^{(4)} = \{1,1\}$. There is a total of four possibilities for the triplets J. They are given by the pairs $\left(J^{(2)},\ J^{(3)}\right)$ taken to be either $(\{1\},\{1,2\})$, $(\{2\},\{1,2\})$, $(\{\},\{1,2,3\})$, or $(\{\},\{1,2,4\})$. The last two possibilities bring inconsistent systems of polynomial equations Z.

The case $\left(J^{(2)},\ J^{(3)}\right) = (\{2\},\{1,2\})$

The set of equations and inequations admit a single solution :

$$h_{14} = \frac{49}{250},\ h_{15} = \frac{49}{250},\ h_{16} = \frac{343}{3125},\ h_{17} = \frac{343}{3125},\ h_{21} = 0,\ h_{22} = 0.$$

Since $h_{21} = 0$, $h_{22} = 0$, we know that it determines a D_6-invariant cubature.

A basis of $\mathbb{R}[x]/I_\Lambda$ is given by $B = \{d_{11}, d_{12};\, d_{41};\, d_{51}, d_{61};\, d_{71}, d_{81}\}$ The distribution of the elements of B among the blocks of the symmetry adapted basis already bears some qualititative information. First, as there are two polynomials in the first block we know that there are two orbits of nodes. As there is a total of 7 nodes, we deduce that it can only be an orbit of 6 nodes in addition to the origin. We can furthermore deduce from the number of basis elements in each block the *types* of the orbits of nodes. In this case the orbit with 6 nodes has 2 nodes on the x-axis. The nodes are thus completely determined if we know the radius of the circle on which they lie.

H_p^B is also block diagonal for any invariant polynomial p. We have

$$H_1^B := \sqrt(3)\,\mathrm{diag}\left(\begin{bmatrix} \frac{3}{2} & \frac{5}{8} \\ \frac{5}{8} & \frac{7}{20} \end{bmatrix}, \begin{bmatrix} \frac{49}{250} \end{bmatrix}, \begin{bmatrix} \frac{5}{16} & 0 \\ 0 & \frac{5}{16} \end{bmatrix}, \begin{bmatrix} \frac{7}{40} & 0 \\ 0 & \frac{7}{40} \end{bmatrix}\right),$$

and for $p = x^2 + y^2$

$$H_p^B = \sqrt{3}\,\mathrm{diag}\left(\begin{bmatrix} \frac{5}{8} & \frac{7}{20} \\ \frac{7}{20} & \frac{49}{250} \end{bmatrix}, \begin{bmatrix} \frac{343}{6250} \end{bmatrix}, \frac{7}{40}\,\mathrm{I}_2, \frac{49}{500}\,\mathrm{I}_2\right),$$

so that

$$M_p^B = \left(H_1^B\right)^{-1} H_p^B = \mathrm{diag}\left(\begin{bmatrix} 0 & 0 \\ 1 & \frac{14}{25} \end{bmatrix}, \frac{14}{25}\,\mathrm{I}_1, \frac{14}{25}\,\mathrm{I}_2, \frac{14}{25}\,\mathrm{I}_2\right).$$

The eigenvalues of $\left(M_p^B\right)^T$ are the evaluations of p at the nodes. We hence see that the origin is one of the nodes, while the other six nodes lie on a circle of radius $\frac{\sqrt{14}}{5}$. We can thus conclude that the nodes are : $\{(0,0)\} \cup \left\{\left(\frac{\sqrt{14}}{5}\cos\left(k\frac{\pi}{3}\right), \frac{\sqrt{14}}{5}\sin\left(k\frac{\pi}{3}\right)\right) \mid 0 \le k \le 5\right\}$.

The case $\left(J^{(2)}, J^{(3)}\right) = (\{1\}, \{1,2\})$

The set of equations and inequations admit a one dimensional family of solutions. They can be parametrized by $\theta \in]0, 2\pi[$ and are then given by

$$h_{14} = \tfrac{49}{250},\ h_{15} = \tfrac{49}{500}(\cos\theta + 1),\ h_{16} = \tfrac{343}{3125},\ h_{17} = \tfrac{343}{6250}(\cos\theta + 1),\quad h_{21} = \tfrac{49}{500}\sin\theta,\ h_{22} = \tfrac{343}{6250}\sin\theta.$$

We can infer that there are two orbits, one being the origin and and the other an orbit of 6 nodes.

For $\theta = \pi$ we have $h_{21} = 0$ and $h_{22} = 0$ so that it determines a D_6 invariant cubature. From the distribution of the block elements in the basis, we deduce the type of the 6 node orbit : it has two nodes on the y-axis.

For all $0 < \theta < 2\pi$, a basis of $\mathbb{R}[x]/I_\Lambda$ is given by $C = \{d_{11}, d_{12};\, d_{31};\, d_{51}, d_{61};\, d_{71}, d_{81}\}$

Computing the nodes of the cubature determined by the parameter θ. We shall first determine the radius of the orbits of nodes and then one node per orbit.

$$H_1^C = \sqrt{3}\,\mathrm{diag}\left(\begin{bmatrix} \frac{3}{2} & \frac{5}{8} \\ \frac{5}{8} & \frac{7}{20} \end{bmatrix}, \begin{bmatrix} \frac{49}{500}(1 - \cos\theta) \end{bmatrix}, \frac{5}{16}\,\mathrm{I}_2, \frac{7}{40}\,\mathrm{I}_2\right)$$

For $p = x^2 + y^2$ we have

$$H_p^C = \sqrt{3}\,\mathrm{diag}\left(\begin{bmatrix} \frac{5}{8} & \frac{7}{20} \\ \frac{7}{20} & \frac{49}{250} \end{bmatrix}, \begin{bmatrix} \frac{343}{6250}(1 - \cos(\theta)) \end{bmatrix}, \frac{7}{40}\,\mathrm{I}_2, \frac{49}{500}\,\mathrm{I}_2\right),$$

so that

$$M_p^C = \left(H_1^C\right)^{-1} H_p^C = \sqrt{3}\,\mathrm{diag}\left(\begin{bmatrix} 0 & 0 \\ 1 & \frac{14}{25} \end{bmatrix}, \frac{14}{25}\,\mathrm{I}_1, \frac{14}{25}\,\mathrm{I}_2, \frac{14}{25}\,\mathrm{I}_2\right).$$

The eigenvalues of M_p are thus 0 and $\frac{14}{5}$, the latter with a multiplicity of 6. This implies that the nodes other than the origin are all on a circle of radius $\frac{\sqrt{14}}{25}$, independently of the parameter θ. The set of nodes is thus

$$\{(0,0)\} \cup \left\{\left(\frac{\sqrt{14}}{5}\cos\left(\alpha + k\frac{\pi}{3}\right), \frac{\sqrt{14}}{5}\sin\left(\alpha + k\frac{\pi}{3}\right)\right) \mid 0 \le k \le 5\right\}$$

where α is to be determined.

As can be observed above, all the values of an invariant p at the different orbits of nodes already appear as eigenvalues of the first block $M_p^{(1)}$ of the multiplication matrix M_p^C.

Consider now the second generating invariant $q = d_{15} = y^2\left(3\,x^2 - y^2\right)^2$. Observe that

$$q\left(\frac{\sqrt{14}}{5}\cos\left(\alpha + k\frac{\pi}{3}\right), \frac{\sqrt{14}}{5}\sin\left(\alpha + k\frac{\pi}{3}\right)\right) = \left(\frac{14}{25}\right)^3 \left(1 - \cos(\alpha)^2\right)\left(4\,\cos(\alpha)^2 - 1\right)^2$$

To find the value of q at the nodes in terms of the parameter θ describing the cubature we consider

$$H_q^{(1)} = \begin{bmatrix} \frac{49}{250}\left(1 + \cos(\theta)\right) & \frac{343}{3125}\left(1 + \cos(\theta)\right) \\ \frac{343}{3125}\left(1 + \cos(\theta)\right) & \frac{4802}{78125}\left(1 + \cos(\theta)\right) \end{bmatrix}$$

so that

$$M_q^{(1)} = \left(H_1^{(1)}\right)^{-1} H_q^{(1)} = \begin{bmatrix} 0 & 0 \\ \frac{2\cdot 7^2}{5^4}\left(1 + \cos(\theta)\right) & \frac{4\cdot 7^3}{5^6}\left(1 + \cos(\theta)\right) \end{bmatrix}.$$

The value of q at the nodes is thus either 0 or $\frac{47^3}{5^6}\left(1 + \cos(\theta)\right)$.

For the cubature determined by the parameter θ, the nodes are thus

$$\{(0,0)\} \cup \left\{\left(\frac{\sqrt{14}}{5}\cos(\alpha + k\frac{\pi}{3}), \frac{\sqrt{14}}{5}\sin(\alpha + k\frac{\pi}{3})\right) \mid 0 \le k \le 5\right\}$$

where α is such that

$$\left(1 - \cos(\alpha)^2\right)\left(4\,\cos(\alpha)^2 - 1\right)^2 = \frac{1}{2}\left(1 + \cos(\theta)\right).$$

Determining the weights. Since we have a C_6-invariant cubature, the nodes on the same orbit have equal weight. Here we have two orbits of nodes and we show that we only need to solve a 2×2 linear system to retrieve the two distinct weights.

The vector of weights $a = \begin{bmatrix} a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 \end{bmatrix}^T$ is solution of $W_\xi^B\, a = \mathrm{m}$ where $\mathrm{m} = \begin{bmatrix} \frac{3}{2}\sqrt{3} & \frac{5}{8}\sqrt{3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ The two first rows of W_ξ^B are

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{14}{25} & \frac{14}{25} & \frac{14}{25} & \frac{14}{25} & \frac{14}{25} & \frac{14}{25} \end{bmatrix}$$

so that we actually have to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ 0 & \frac{14}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\sqrt{3} \\ \frac{5}{8}\sqrt{3} \end{bmatrix}.$$

the solution of which is

$$a_1 = \frac{43\sqrt{3}}{112} \quad \text{and} \quad a_2 = \frac{125\sqrt{3}}{672}.$$

After the existence of the invariant cubature was obtained, we have thus determined the nodes and weights by dealing only with $c \times c$ matrices, where c is the number of orbits of nodes.

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