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A Stochastic Game Framework for Analyzing Computational Investment Strategies in Distributed Computing with Application to Blockchain Mining

Swapnil Dhamal, Tijani Chahed, Walid Ben-Ameur, Eitan Altman, Albert Sunny, and Sudheer Poojary

Abstract—We study a stochastic game framework with varying number of players, for modeling and analyzing their computational investment strategies in distributed computing, for solving a problem such as in blockchain mining. In particular, we propose a continuous time Markov chain model, where players arrive and depart according to a stochastic process, and determine their investment strategies based on the state of the system. The players obtain a certain reward for being the first to solve the problem, while incur a certain cost based on the time and computational power invested in the attempt to solve it. We consider that the players are Markovian, that is, they determine their strategies which maximize their expected utilities, while ignoring past payoffs. We first study a scenario where the rate of problem getting solved is proportional to the total computational power invested by the players. We show that, in statewise Nash equilibrium, players with costs exceeding a certain threshold do not invest, while players with costs less than this threshold invest maximum power. Further, we show that Markov perfect equilibrium follows a similar threshold policy, and the players do not need to have knowledge of the system state. We then consider a scenario where the rate of problem getting solved is independent of the computational power invested by players. Here, we show that, in statewise Nash equilibrium, only the players with cost parameters in a relatively low range, invest. We also show that, in Markov perfect equilibrium, players invest proportionally to the reward-cost ratio. Using simulations, we quantify the effects of arrival and departure rates on players' expected utilities and provide insights.

1 Introduction

A distributed computing system comprises computers which communicate and coordinate their actions by passing messages to each other, in order to achieve a common goal which is usually solving a large problem. Distributed computing has gained more popularity than ever before in recent years, owing to the advent of blockchain technology. Blockchain has found applications in a wide variety of fields [1], some of them being cryptocurrencies such as Bitcoin, smart contracts, reputation systems, Internet of Things, security services, public services, etc. The security of a blockchain relies primarily on a proof-of-work procedure, where the providers of computational power (referred to as *miners*) collect block data consisting of a number of transactions, and repeatedly compute hashes on inputs (an input usually contains the block data appended by a random number) from a very large search space. A miner receives reward for mining a block, if it finds one of the rare inputs that generates a hash value satisfying certain constraints (such as having a very low value in case of bitcoin) before everyone else. Owing to the usage of cryptographic hash function, the best method for finding such an input is randomized search.

Since the proof-of-work procedure is computationally very intensive, successful mining of blocks usually requires the miners to invest significantly with regard to computational power. This, in turn, involves a certain amount of cost for investing a certain amount of computational power for certain amount of time. Once a block is mined, it is transmitted to everyone, and the process repeats for mining a new block. In this paper, we study the stochastic game during the mining of one particular block, where miners arrive and depart while a block is being mined. A miner's objective is to optimize its utility based on the offered reward for successfully mining a block before everyone else, by strategizing on the amount of computational power it should invest. There is a natural tradeoff, since a higher computational investment by a player would give it a higher chance of solving the problem before everyone else, while a lower investment is advantageous since it would reduce the cost incurred.

In this paper, we study two of the most common scenarios in distributed computing. The first scenario considers that the rate of problem getting solved is proportional to the total computational power invested by the providers of computation. This is a more classical scenario, where the problem to be solved is fixed, and the more total power is invested to solve it, the faster it gets solved. The second scenario addresses a different class of problems for which, the rate of problem getting solved is independent of the computational power invested by the miners. This could be owing to some peculiar nature of the problem, or the system changing the complexity of the problem depending on the total computational power received from the providers, or if only a portion (which could vary) of the received power is utilized by the system while the remaining power is used for idling. Bitcoin mining is an example of the system changing the problem complexity based on the total computational power, such that the time required for solving it with the received power, is 10 minutes on average. While the problem complexity does not change amidst a problem getting solved in bitcoin mining, our study of this scenario is motivated

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by the idea that the rate of problem getting solved could be independent of the computational powers invested by the miners.

1.1 Preliminaries

We now provide a non-technical introduction to the important concepts used in this paper. We will describe their technical details after we propose our model in the next section.

1.1.1 Stochastic Game

A stochastic game [2] is a dynamic game with probabilistic transitions across different states of the system. The players' payoffs and the probabilistic transitions between states depend on the current state and the players' strategies. The game continues until it reaches a terminal state (or continues for infinite time in absence of a terminal state). Stochastic games are thus a generalization of both Markov decision processes and repeated games.

1.1.2 Statewise Nash Equilibrium (SNE)

Nash equilibrium of a game is a profile of strategies of the involved players, from which no player can gain by a unilateral change of strategy, if the strategies of other players remain unchanged. We refer to statewise Nash equilibrium in stochastic game as a Nash equilibrium when players play myopically by only looking at the current state (without foreseeing the effects of their actions on the state transitions and resulting utilities).

1.1.3 Markov Perfect Equilibrium (MPE)

MPE [3] is an adaptation of subgame perfect Nash equilibrium to stochastic games. An MPE strategy of a player is a policy function describing its strategy for each state (while ignoring any history apart from the state information). Each player computes its best response strategy in each state by foreseeing the effects of its actions on the state transitions and resulting utilities, and the strategies of the other players. A player's MPE strategy is a best response to the other players' MPE strategies. In literature, for analytical tractability, MPE strategies are assumed to be symmetric (the decision policies of all players are identical in a given state). Moreover, Schelling [4] argues that symmetric equilibria can be perceived as stronger focal points than asymmetric ones.

1.2 Related Work

Stochastic games have been studied from a theoretical perspective [5], [6], [7], [8], [9] as well as in a number of applications such as computer networks [10], cognitive radio networks [11], wireless network virtualization [12], queuing systems [13], multiagent reinforcement learning [14], committee bargaining [15], and modeling complex living systems [16].

We enlist some of the important works on stochastic games. Altman and Shimkin [17] consider a processor-sharing service system, wherein an arriving customer observes the current load on the shared system and chooses whether to join it or use a constant-cost alternative such as a personal computer. The authors show that any Nash equilibrium consists of threshold decision rules, with a threshold on the queue length in the shared system. Nahir, Orda, and Raz [18] also consider a similar setup, with the difference that the customers consider using the shared system over a long time scale and for multiple jobs; hence the decision of each customer is a strategy on the portion of its jobs that would be served by the shared system, instead of a single one-time binary decision in [17]. Hassin and Haviv [19] propose a version of subgame perfect Nash equilibrium for games where the players are identical, and each player selects a strategy based on its private information regarding the state of the system. Wang and Zhang [20] investigate the Nash equilibrium in a queuing system, where reentering the system is a strategic decision of the customers. Hu and Wellman [21] use the framework of general-sum stochastic games in order to extend Q-learning to a noncooperative multiagent context, and find that agents are more likely to reach a joint optimal path with Nash Q-learning than with single-agent Q-learning.

There also exist game theoretic studies of distributed computing [22], [23]. Wei et al. [24] consider a resource allocation problem with constraints on quality of service, where customers request resources across a cloud-based network. The authors show existence of Nash equilibrium, if the resource allocation game has feasible solutions. Chun et al. [25] analyze the selfish caching problem, where selfish server nodes incur either cost for replicating resources or cost for access to a remote replica, and hence show the existence of pure strategy Nash equilibria. Grosu and Chronopoulos [26] propose a game theoretic framework for obtaining a user-optimal load balancing scheme in heterogeneous distributed systems.

Zheng and Xie [1] present a broad survey on blockchain technology. Sapirshtein, Sompolinsky, and Zohar [27] consider the case of selfish mining attacks, where a miner postpones the transmission of its mined blocks to other miners (since other miners would start mining the next block immediately after receiving the mined block, which may not be desired by the considered miner). Lewenberg et al. [28] study pooled mining, in which miners form coalitions where the obtained rewards are shared among their members, thus reducing the variance of the reward obtained by each player. Xiong et al. [29] consider a setting where miners can offload the mining process to an edge computing service provider. The authors models this as a Stackelberg game, where the service provider sets the price of edge computing services, and then the miners determine the amount of service to purchase based on the observed prices. Kiayias et al. [30] consider a stochastic game, where each state of the game corresponds to the mined blocks and the players who mined them. A player strategizes on which blocks to mine and when to transmit the blocks to other players.

In general, there exist game theoretic studies for distributed computing, as well as stochastic games for other applications including blockchain mining where a state signifies the state of the chain or tree of blocks. To the best of our knowledge, this is the first work to study a stochastic game framework for distributed computing considering the number of players to be dynamic.

2 Our Model

Consider a distributed computing system wherein agents provide their computational power to the system, and receive a certain reward r in return. In order to model the real-world scenario where the set of such agents is not static owing to their arrivals and departures, we consider a standard queueing setting where agents arrive according to a Poisson process with rate λ, and the departure times of the agents are independent and identically distributed as per exponential distribution with rate μ. The stochastic arrival of agents is natural, like in most applications. However, even their stochastic departure is well justified here. Agents would usually shut down their computers on a regular basis, or terminate computationally demanding mining task so as to run other tasks. Since mining is a high-speed continuous process, it is not practical for agents to consistently keep track of whether the mining of

the current block has completed. So agents may randomly depart before its completion.

We could hence view this setup as a continuous time multi-state process, where each state corresponds to a set of players in the system. Since in general, the agents involved at any given time would influence each others' payoffs, it results in a game. The stochastic arrival and departure of players makes it a stochastic game. As we shall see, there are other stochastic events in addition to arrivals and departures, and which depend on the players' strategies.

2.1 Model Elements

2.1.1 State Space

We analyze this system from the viewpoint of a typical player, which we label as player i. Hence, instead of considering a state as the set of players in the system, we could consider it as the set of players apart from player i. We assume that the players are homogeneous, that is, the parameters corresponding to all the players are the same. This is a standard assumption in the literature on stochastic games in order to have the size of the state space, linear rather than exponential in the number of players. Hence a system state, say n, corresponds to the number of other players in system.

From the viewpoint of player i, the process initiates from some initial state m at time $t = 0$, that is, there are m other players in system when i enters. Since a typical blockchain system requires a majority consensus to ensure validity of a mined block, we assume a minimum of 2 players in the system for blockchain's sustainability (for instance, the system could be configured so that a system-controlled miner is triggered whenever the number of miners is detected to be less than 2). Also, the system can typically accommodate a certain maximum number of players at any given time, say N (which could be very large). Hence the possible states in which player i can enter are $1, \ldots, N-1$. Let $[n]$ denote the set of players in state n apart from player i, and $[n \oplus i]$ denote the set of players in state n including player i.

2.1.2 Players' Strategies

Let $x_i^{(n,t)}$ denote the strategy of player i (that is, the computational power that it decides to invest) at time t when the system is in state n. Since players use a randomized search approach over a very large search space, the search follows a memoryless property, that is, the time required to find the solution in this search space is independent of the search space explored thus far. Also, note that a player does not have incentive to change its strategy amidst a state owing to this memoryless property and if no other player changes its strategy. Hence in our analysis, we consider that no player changes its strategy within a state. So we have $x_i^{(n,t)} = x_i^{(n,t')}$ for any t, t'; hence player i's strategy could be written as a function of the state, that is, $x_i^{(n)}$. Let $\mathbf{x}^{(n)}$ denote the strategy profile of the players in state n. Let $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N-1)})$ denote the policy profile.

We denote by a cost parameter c_i, the cost incurred by player i for investing unit amount of computational power for unit amount of time. We assume that players have no constraint on how much cost they can afford to incur. Instead, we consider that players aim to maximize their expected payoffs (the expected reward they would obtain minus the expected cost they would incur), while forgetting the cost they have incurred thus far. That is, players are Markovian.

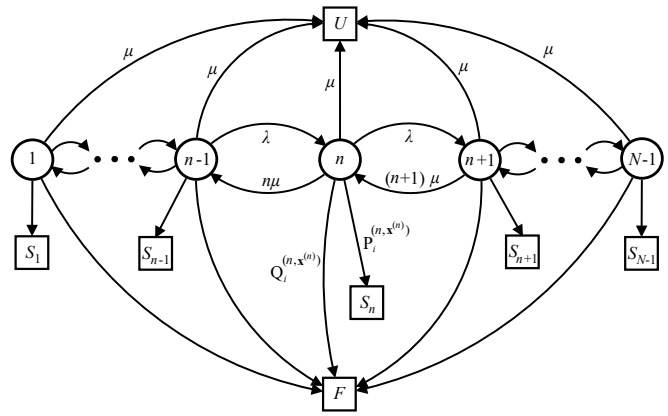


Fig. 1. The underlying continuous time Markov chain from the viewpoint of a typical player i

2.1.3 Rate of the Problem Getting Solved

Recall that the time required to find solution in the very large search space is independent of the search space explored thus far. We consider this time to be exponentially distributed owing to its memoryless property. Let $\Gamma^{(n,\mathbf{x}^{(n)})}$ be the corresponding rate of problem getting solved in state n when the players' strategy profile is $\mathbf{x}^{(n)}$. Since the time required is independent of the search space explored thus far, the probability that a player finds the solution before all other players at time t is proportional to its invested computational power at time t.

Note that the time required for the problem to get solved is the minimum of the times required by the players to solve the problem. Also, the minimum of exponentially distributed random variables, is another exponentially distributed random variable with rate which is the sum of the rates corresponding to the original random variables. And the probability of an original random variable being the minimum, is proportional to its rate. Let $\mathrm{P}_j^{(n,\mathbf{x}^{(n)})}$ be the rate (corresponding to an exponentially distributed random variable) of player j solving the problem in state n, when the strategy profile is $\mathbf{x}^{(n)}$. So we have $\Gamma^{(n,\mathbf{x}^{(n)})} = \sum_{j \in [n \oplus i]} \mathrm{P}_j^{(n,\mathbf{x}^{(n)})}$. Since the probability that player i solves the problem before the other players is proportional to its invested computational power at that time, we have that the rate of player i solving the problem is $\mathrm{P}_i^{(n,\mathbf{x}^{(n)})} = \frac{x_i^{(n)}}{\sum_{j \in [n \oplus i]} x_j^{(n)}} \Gamma^{(n,\mathbf{x}^{(n)})}$, and the rate of other players solving the problem is $\mathrm{Q}_i^{(n,\mathbf{x}^{(n)})} = \frac{\sum_{j \in [n]} x_j^{(n)}}{\sum_{j \in [n \oplus i]} x_j^{(n)}} \Gamma^{(n,\mathbf{x}^{(n)})}$.

2.1.4 The Continuous Time Markov Chain

Figure 1 depicts the CTMC from the viewpoint of player i (only the relevant transition rates are shown, for clarity). The circled states are the states in which player i can enter the system. With our assumption of Markovian players, coupled with the memoryless property of the computation done thus far, the system transiting from state n to state n' is equivalent to the player reentering the system in state n'. So the expression for expected utility could be written in a recursive form, which we now derive. Table 1 presents the notation. The possible events that can occur in a state $n \in \{2, \ldots, N-2\}$ are:

1) the problem gets solved by player i with rate $\mathrm{P}_i^{(n,\mathbf{x}^{(n)})} = \frac{x_i^{(n)}}{\sum_{j \in [n \oplus i]} x_j^{(n)}} \Gamma^{(n,\mathbf{x}^{(n)})}$, thus terminating the game in the absorbing state S_n where i gets a reward of r;

2) the problem gets solved by another player $j \in [n]$ with rate $\mathrm{Q}_i^{(n,\mathbf{x}^{(n)})} = \frac{\sum_{j\in[n]} x_j^{(n)}}{\sum_{j\in[n\oplus i]} x_j^{(n)}} \Gamma^{(n,\mathbf{x}^{(n)})}$, thus terminating the game in the absorbing state F where i gets no reward;
3) a new player arrives and the system transits to state $n+1$ with rate λ;
4) one of the other players $j \in [n]$ departs and the system transits to state $n-1$ with rate $n\mu$;
5) player i departs with rate μ, thus ending the game for player i with no reward in absorbing state U.

So, the sojourn time in state n is $(\Gamma^{(n,\mathbf{x}^{(n)})}+\lambda+(n+1)\mu)^{-1}$. The corresponding expected cost to i is $\frac{c_i x_i^{(n)}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}$.

2.1.5 Expression for Expected Utility

The probability of an event occurring before any other event is equivalent to the corresponding exponentially distributed random variable being the minimum, which in turn, is proportional to its rate. So the expected utility of player i as computed in state $n \in \{2,\ldots,N-2\}$ is

$$\begin{aligned} R_i^{(n,\mathbf{x})} &= \frac{\Gamma^{(n,\mathbf{x}^{(n)})}\frac{x_i^{(n)}}{\sum_{j\in[n\oplus i]} x_j^{(n)}}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}(r) + \frac{\Gamma^{(n,\mathbf{x}^{(n)})}\frac{\sum_{j\in[n]} x_j^{(n)}}{\sum_{j\in[n\oplus i]} x_j^{(n)}}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}(0) \\ &+ \frac{\lambda}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} R_i^{(n+1,\mathbf{x})} + \frac{n\mu}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} R_i^{(n-1,\mathbf{x})} \\ &+ \frac{\mu}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}(0) - \frac{c_i x_i^{(n)}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} \end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma^{(n,\mathbf{x}^{(n)})}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} r \frac{x_i^{(n)}}{\sum_{j\in[n\oplus i]} x_j^{(n)}} - \frac{c_i x_i^{(n)}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} \\ &+ \frac{\lambda}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} R_i^{(n+1,\mathbf{x})} + \frac{n\mu}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} R_i^{(n-1,\mathbf{x})} \end{aligned} \tag{1}$$

For $n=1$, the common denominator is $\lambda+\mu+\Gamma^{(1,\mathbf{x}^{(1)})}$ and the term corresponding to $R_i^{(n-1,\mathbf{x})}$ vanishes.
For $n=N-1$, common denominator is $N\mu+\Gamma^{(N-1,\mathbf{x}^{(N-1)})}$ and the term corresponding to $R_i^{(n+1,\mathbf{x})}$ vanishes.

The above expression for expected utility is derived considering that player i obtains a reward of r with probability proportional to its invested computational power if the problem gets solved (the blockchain mining setting). However, it can be noted that the expression holds even if we consider that the obtained reward is proportional to the invested computational power (the classical distributed computing setting).

Note that in connection with a standard MDP, we do not incorporate an explicit discounting factor. However in our case, the utility can be viewed as discounting the future owing to the player exiting the system with a certain probability in any state. We highlight that our analyses are easily generalizable if an explicit discounting factor is incorporated.

2.1.6 Convergence of Expected Utility

Let $\mathbf{R}_i^{(\mathbf{x})}$ be vector whose n^{th} component is $R_i^{(n,\mathbf{x})}$. We show from above recursive expression that $\mathbf{R}_i^{(\mathbf{x})}$ converges for any policy profile $\mathbf{x}$. Let $\mathbf{W}^{(\mathbf{x})}$ be the matrix whose elements are as follows:

TABLE 1
Notation

i	the player under consideration
r	reward offered for solving the problem
c_i	cost incurred by player i when it invests unit power for unit time
λ	arrival rate of players
μ	departure rate of players
n	number of other players in the system
$x_i^{(n)}$	strategy of player i in state n
$\mathbf{x}^{(n)}$	strategy profile of players in state n
$\mathbf{x}$	policy profile
$\Gamma^{(n,\mathbf{x}^{(n)})}$	rate of problem getting solved in state n under strategy profile $\mathbf{x}^{(n)}$
$R_i^{(n,\mathbf{x})}$	expected utility of player i computed in state n under policy profile $\mathbf{x}$
$[n]$	set of other players in state n
$[n\oplus i]$	set of players in state n including player i

For $n=2,\ldots,N-2$:

$$W^{(\mathbf{x})}(n,n+1) = \frac{\lambda}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}} \quad \text{and}$$

$$W^{(\mathbf{x})}(n,n-1) = \frac{n\mu}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}.$$

$$W^{(\mathbf{x})}(1,2) = \frac{\lambda}{\lambda+\mu+\Gamma^{(1,\mathbf{x}^{(1)})}}.$$

$$W^{(\mathbf{x})}(N-1,N-2) = \frac{(N-1)\mu}{N\mu+\Gamma^{(N-1,\mathbf{x}^{(N-1)})}}.$$

All other elements of $\mathbf{W}^{(\mathbf{x})}$ are 0. So we have that $\mathbf{W}^{(\mathbf{x})}$ is a strictly substochastic matrix, that is, the sum of the elements in each of its rows is less than 1.
Let $\mathbf{Z}_i^{(\mathbf{x})}$ be the vector whose n^{th} component is $Z_i^{(n,\mathbf{x})}$, where

For $n=2,\ldots,N-2$:

$$Z_i^{(n,\mathbf{x})} = \left(\frac{\Gamma^{(n,\mathbf{x}^{(n)})}}{\sum_{j\in[n\oplus i]} x_j^{(n)}} r - c_i\right) \frac{x_i^{(n)}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}.$$

$$Z_i^{(1,\mathbf{x})} = \left(\frac{\Gamma^{(n,\mathbf{x}^{(n)})}}{\sum_{j\in[n\oplus i]} x_j^{(n)}} r - c_i\right) \frac{x_i^{(1)}}{\lambda+\mu+\Gamma^{(1,\mathbf{x}^{(1)})}}.$$

$$Z_i^{(N-1,\mathbf{x})} = \left(\frac{\Gamma^{(n,\mathbf{x}^{(n)})}}{\sum_{j\in[n\oplus i]} x_j^{(n)}} r - c_i\right) \frac{x_i^{(N-1)}}{N\mu+\Gamma^{(N-1,\mathbf{x}^{(N-1)})}}.$$

Proposition 1. $\mathbf{R}_i^{(\mathbf{x})} = (\mathbf{I}-\mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Z}_i^{(\mathbf{x})}$

Proof. Let $\mathbf{R}_{i\langle t\rangle}^{(\mathbf{x})} = (R_{i\langle t\rangle}^{(1,\mathbf{x})},\ldots,R_{i\langle t\rangle}^{(N-1,\mathbf{x})})^T$, where t is the iteration number. The iteration for the value of $\mathbf{R}_{i\langle t\rangle}^{(\mathbf{x})}$ starts at $t=0$, and we examine if it converges when $t\to\infty$. Now, the expression for the expected utility in all states can be written in matrix form as

$$\begin{aligned} \mathbf{R}_{i\langle t\rangle}^{(\mathbf{x})} &= \mathbf{W}^{(\mathbf{x})}\mathbf{R}_{i\langle t-1\rangle}^{(\mathbf{x})} + \mathbf{Z}_i^{(\mathbf{x})} \\ &= \mathbf{W}^{(\mathbf{x})}\left(\mathbf{W}^{(\mathbf{x})}\mathbf{R}_{i\langle t-2\rangle}^{(\mathbf{x})} + \mathbf{Z}_i^{(\mathbf{x})}\right) + \mathbf{Z}_i^{(\mathbf{x})} \\ &= \left(\mathbf{W}^{(\mathbf{x})}\right)^2 \mathbf{R}_{i\langle t-2\rangle}^{(\mathbf{x})} + (\mathbf{W}^{(\mathbf{x})}+\mathbf{I})\mathbf{Z}_i^{(\mathbf{x})} \\ &= \left(\mathbf{W}^{(\mathbf{x})}\right)^t \mathbf{R}_{i\langle 0\rangle}^{(\mathbf{x})} + \left(\sum_{\eta=0}^{t-1}\left(\mathbf{W}^{(\mathbf{x})}\right)^\eta\right)\mathbf{Z}_i^{(\mathbf{x})} \end{aligned}$$

Now, since $\mathbf{W}^{(\mathbf{x})}$ is a strictly substochastic matrix (with the sum of each of its rows being strictly less than 1), its spectral radius is less than 1. So when $t \to \infty$, we have $\lim_{t\to\infty}\left(\mathbf{W}^{(\mathbf{x})}\right)^t = \mathbf{0}$. Since $\mathbf{R}^{(\mathbf{x})}_{i\langle 0\rangle}$ is a finite constant, we have $\lim_{t\to\infty}\left(\mathbf{W}^{(\mathbf{x})}\right)^t \mathbf{R}^{(\mathbf{x})}_{i\langle 0\rangle} = \mathbf{0}$. Furthermore, $\lim_{t\to\infty}\sum_{\eta=0}^{t-1}\left(\mathbf{W}^{(\mathbf{x})}\right)^\eta = (\mathbf{I}-\mathbf{W}^{(\mathbf{x})})^{-1}$, an established matrix identity [31]. This implicitly means that $(\mathbf{I}-\mathbf{W}^{(\mathbf{x})})$ is invertible. Hence,

$$\lim_{t\to\infty}\mathbf{R}^{(\mathbf{x})}_{i\langle t\rangle} = \lim_{t\to\infty}\left(\mathbf{W}^{(\mathbf{x})}\right)^t\mathbf{R}^{(\mathbf{x})}_{i\langle 0\rangle} + \left(\sum_{\eta=0}^{\infty}\left(\mathbf{W}^{(\mathbf{x})}\right)^\eta\right)\mathbf{Z}^{(\mathbf{x})}_i$$
$$= \mathbf{0} + (\mathbf{I}-\mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Z}^{(\mathbf{x})}_i \qquad \square$$

Since $\mathbf{W}^{(\mathbf{x})}$ and $\mathbf{I}-\mathbf{W}^{(\mathbf{x})}$ are tridiagonal matrices, a very general analysis would demand the results on the inverse and powers of tridiagonal matrices [32]. In this paper, we consider two special scenarios, which are the most common in a distributed computing system, for which we can circumvent the usage of these intricate results.

2.2 Outline of Our Analyses

In the two scenarios that we study, we first derive SNE (assuming myopic players). We conduct a general analysis for SNE while considering that the players could be heterogeneous, that is, their cost parameters could be different. The SNE for homogeneous players directly follows. As explained earlier, for finding MPE in the stochastic game (considering Markovian players), we consider that the players are homogeneous, as is standard in the literature for analytical tractability. We hence provide insights on our results, while also comparing the strategies of Markovian players with that of myopic players. We follow our theoretical analysis with simulations for quantifying the effects of system parameters on players' utilities.

3 Scenario 1: Rate of Problem Getting Solved is Proportional to the Players' Total Investment

Here, we consider that rate of problem getting solved in state n under strategy profile $\mathbf{x}^{(n)}$ is $\Gamma^{(n,\mathbf{x}^{(n)})} = \gamma\sum_{j\in[n\oplus i]}x_j^{(n)}$, where γ is the (rate) constant of proportionality determined by the distributed computing system.

3.1 Statewise Nash Equilibrium

Let $V_i^{(n,\mathbf{x}^{(n)})}$ be the expected utility of a player i in state n, computed myopically (that is, without foreseeing state transitions). Hence $V_i^{(n,\mathbf{x}^{(n)})}$ can be viewed as $R_i^{(n,\mathbf{x})}$ with $\lambda=\mu=0$ (in Equation (1)). That is,

$$V_i^{(n,\mathbf{x}^{(n)})} = (\gamma r - c_i)\frac{x_i^{(n)}}{\gamma\sum_{j\in[n\oplus i]}x_j^{(n)}} = \left(r-\frac{c_i}{\gamma}\right)\frac{x_i^{(n)}}{\sum_{j\in[n\oplus i]}x_j^{(n)}}$$

So $V_i^{(n,\mathbf{x}^{(n)})}$ is an increasing function of $x_i^{(n)}$ if $c_i<\gamma r$, a decreasing function of $x_i^{(n)}$ if $c_i>\gamma r$, and a constant zero-function if $c_i=\gamma r$. So it is a dominant strategy for any player i to invest its maximum power if $c_i<\gamma r$, no power if $c_i>\gamma r$, and any amount of power if $c_i=\gamma r$. That is, the SNE strategy of a player follows a threshold policy, with a threshold on its cost parameter (whether it is lower than γr) or alternatively, a threshold on the offered reward (whether it is higher than $\frac{c_i}{\gamma}$).

Hence, for homogeneous case where $c_j=c, \forall j\in[n\oplus i]$, all players invest their maximum powers if $c<\gamma r$, no player invests if $c>\gamma r$, and players invest any amount if $c=\gamma r$. So the expected utility in SNE is 0 if $c\geq\gamma r$; while if $c<\gamma r$, considering that the players have a maximum computational power of $\overline{x}$, the expected utility in SNE is

$$\hat{V}_i^{(n,\mathbf{x}^{(n)})} = \left(r-\frac{c}{\gamma}\right)\frac{\overline{x}}{(n+1)\overline{x}} = \frac{1}{n+1}\left(r-\frac{c}{\gamma}\right)$$

3.2 Markov Perfect Equilibrium

Let $\hat{R}_i^{(n,\mathbf{x})}$ be the equilibrium utility of player i in state n, that is, when player i plays the best response strategy to the equilibrium strategies of other players $j\in[n]$ (while foreseeing effects of its actions on the state transitions and resulting utilities). We can determine MPE similar to finding optimal policy in MDP (using policy-value iterations to reach a fixed point which acts as an optimal policy). Here, for maximizing $\hat{R}_i^{(n,\mathbf{x})}$, we assume that we have optimized for other states and use those values to find an optimizing $\mathbf{x}$ for maximizing $\hat{R}_i^{(n,\mathbf{x})}$. In our case, we have derived a closed form expression for vector $\mathbf{R}_i^{(\mathbf{x})}$ in terms of policy $\mathbf{x}$ (Proposition 1); so we could effectively determine the fixed point directly.

A policy is said to be *proper* if from any initial state, the probability of achieving a terminal state is strictly positive. Consider the condition that, there exists at least one proper policy, and for any non-proper policy, there exists at least one state where the corresponding value function is negatively unbounded. It is known that, under this condition, the optimal value function is bounded, and it is the unique fixed point of the optimal Bellman operator [33]. This condition is satisfied by our model, since there does not exist any non-proper policy. Now, from Equation (1), the Bellman equations for each state $n\in\{2,\ldots,N-2\}$ for player i can be written as

$$\hat{R}_i^{(n,\mathbf{x})} = \max_{\mathbf{x}}\left\{\frac{x_i^{(n)}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}\left(\frac{\Gamma^{(n,\mathbf{x}^{(n)})}}{\sum_{j\in[n\oplus i]}x_j^{(n)}}r-c\right)\right.$$
$$\left.+\frac{\lambda}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}\hat{R}_i^{(n+1,\mathbf{x})}+\frac{n\mu}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}\hat{R}_i^{(n-1,\mathbf{x})}\right\}$$

As earlier, for $n=1$, common denominator is $\lambda+\mu+\Gamma^{(1,\mathbf{x}^{(1)})}$ and term corresponding to $\hat{R}_i^{(n-1,\mathbf{x})}$ vanishes. For $n=N-1$, the common denominator is $N\mu+\Gamma^{(N-1,\mathbf{x}^{(N-1)})}$ and term corresponding to $\hat{R}_i^{(n+1,\mathbf{x})}$ vanishes. For this scenario, we have $\Gamma^{(n,\mathbf{x}^{(n)})}=\gamma\sum_{j\in[n\oplus i]}x_j^{(n)}$.

We now derive results leading to the derivation of MPE, the proofs of which are provided in the supplemental file.

Lemma 1. *In Scenario 1, for any state n and policy profile $\mathbf{x}$, $R_i^{(n,\mathbf{x})}<r-\frac{c}{\gamma}$ if $\gamma r>c$, and $R_i^{(n,\mathbf{x})}>r-\frac{c}{\gamma}$ if $\gamma r<c$.*

Proof. Let $\mathbf{V}_i^{(\mathbf{x})}$ be the vector whose n^{th} component is $V_i^{(n,\mathbf{x}^{(n)})}$, the expected utility of player i in state n computed myopically. Let $\mathbf{Z}_i^{(\mathbf{x})}=\mathbf{Y}^{(\mathbf{x})}\mathbf{V}_i^{(\mathbf{x})}$. Note that when $\Gamma^{(n,\mathbf{x}^{(n)})}=\gamma\sum_{j\in[n\oplus i]}x_j^{(n)}$, we have that $\mathbf{Y}^{(\mathbf{x})}$ is a diagonal matrix, with diagonal elements $Y^{(\mathbf{x})}(n,n)=\frac{\gamma\sum_{j\in[n\oplus i]}x_j^{(n)}}{\lambda+(n+1)\mu+\Gamma^{(n,\mathbf{x}^{(n)})}}$ (the denominator is replaced

by $\lambda+\mu+\Gamma^{(1,\mathbf{x}^{(1)})}$ for $n = 1$ and by $N\mu+\Gamma^{(N-1,\mathbf{x}^{(N-1)})}$ for $n = N-1$). When $\Gamma^{(n,\mathbf{x}^{(n)})} = \gamma\sum_{j\in[n\oplus i]} x_j^{(n)}$, we have that $\mathbf{W}^{(\mathbf{x})} + \mathbf{Y}^{(\mathbf{x})}$ is strictly substochastic matrix (the sum of each of its rows is less than 1).

Let $\mathbf{U}^{(\mathbf{x})} = (\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Y}^{(\mathbf{x})}\mathbf{1}$, where $\mathbf{1}$ is the vector whose each element is 1. It is clear that all the elements of $\mathbf{U}^{(\mathbf{x})}$ are non-negative. We will now show that $||\mathbf{U}^{(\mathbf{x})}||_\infty < 1$, that is, the maximum element of the vector $\mathbf{U}^{(\mathbf{x})}$ is less than 1. Let u_{n_0} be the element with the maximum value (one of the maximum, if there are multiple). Suppose $u_{n_0}^{(\mathbf{x})} = ||\mathbf{U}^{(\mathbf{x})}||_\infty \geq 1$. So we would have

$$\begin{aligned}
&\mathbf{U}^{(\mathbf{x})} = (\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Y}^{(\mathbf{x})}\mathbf{1}\\
\implies &\mathbf{U}^{(\mathbf{x})} = \mathbf{W}^{(\mathbf{x})}\mathbf{U}^{(\mathbf{x})} + \mathbf{Y}^{(\mathbf{x})}\mathbf{1}\\
\implies &u_{n_0}^{(\mathbf{x})} = \sum_{j=1}^{N-1} u_j^{(\mathbf{x})} W^{(\mathbf{x})}(n_0, j) + Y^{(\mathbf{x})}(n_0, n_0)\\
\implies &u_{n_0}^{(\mathbf{x})} \leq u_{n_0}^{(\mathbf{x})} \sum_{j=1}^{N-1} W^{(\mathbf{x})}(n_0, j) + u_{n_0}^{(\mathbf{x})} Y^{(\mathbf{x})}(n_0, n_0)\\
&\qquad\qquad (\because \max_j u_j^{(\mathbf{x})} = u_{n_0}^{(\mathbf{x})} \geq 1)\\
\implies &\sum_{j=1}^{N-1} W^{(\mathbf{x})}(n_0, j) + Y^{(\mathbf{x})}(n_0, n_0) \geq 1
\end{aligned}$$

However, this is a contradiction since $\mathbf{W}^{(\mathbf{x})} + \mathbf{Y}^{(\mathbf{x})}$ is strictly substochastic. So, we have shown that $||\mathbf{U}^{(\mathbf{x})}||_\infty = ||(\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Y}^{(\mathbf{x})}\mathbf{1}||_\infty < 1$. That is, $(\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Y}^{(\mathbf{x})}$ is strictly substochastic.

From Proposition 1, $\mathbf{R}_i^{(\mathbf{x})} = (\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Y}^{(\mathbf{x})}\mathbf{V}_i^{(\mathbf{x})}$. Since $(\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Y}^{(\mathbf{x})}$ is strictly substochastic, $R_i^{(n,\mathbf{x})}$ for each n is a linear combination (with weights summing to less than 1) of $V_i^{(n,\mathbf{x}^{(n)})}$ over all n's.

For each n, $V_i^{(n,\mathbf{x}^{(n)})} = \left(r - \frac{c}{\gamma}\right)\frac{x_i^{(n)}}{\sum_{j\in[n\oplus i]} x_j^{(n)}}$. So we have $V_i^{(n,\mathbf{x}^{(n)})} \leq r - \frac{c}{\gamma}$, if $\gamma r > c$, and $V_i^{(n,\mathbf{x}^{(n)})} \geq r - \frac{c}{\gamma}$, if $\gamma r < c$. Since $R_i^{(n,\mathbf{x})}$ for each n is a linear combination (with weights summing to less than 1) of $V_i^{(n,\mathbf{x}^{(n)})}$ over all n's, we have $R_i^{(n,\mathbf{x})} < r - \frac{c}{\gamma}$ if $\gamma r > c$, and $R_i^{(n,\mathbf{x})} > r - \frac{c}{\gamma}$ if $\gamma r < c$. □

Lemma 2. *In Scenario 1, $R_i^{(n,\mathbf{x})}$ is monotone function of $x_i^{(n)}$.*

Proof. Let us define the following for simplifying notation.

$$A_i^{(n,\mathbf{x})} = \begin{cases} \lambda\hat{R}_i^{(n+1,\mathbf{x})} + n\mu\hat{R}_i^{(n-1,\mathbf{x})}, & \text{for } n = 2,\ldots,N-2\\ \lambda\hat{R}_i^{(n+1,\mathbf{x})} & \text{, for } n = 1\\ n\mu\hat{R}_i^{(n-1,\mathbf{x})} & \text{, for } n = N-1 \end{cases}$$

$$E_i^{(n,\mathbf{x}^{(n)})} = \begin{cases} \lambda+(n+1)\mu+\gamma\sum_{j\in[n]} x_j^{(n)} & \text{, for } n = 2,\ldots,N-2\\ \lambda+\mu+\gamma\sum_{j\in[n]} x_j^{(n)} & \text{, for } n = 1\\ (n+1)\mu+\gamma\sum_{j\in[n]} x_j^{(n)} & \text{, for } n = N-1 \end{cases}$$

Hence we can write

$$R_i^{(n,\mathbf{x})} = \frac{A_i^{(n,\mathbf{x})} + (\gamma r - c)x_i^{(n)}}{E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}}$$

$$\begin{aligned}
\frac{dR_i^{(n,\mathbf{x})}}{dx_i^{(n)}} &= \frac{(\gamma r - c)\left(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}\right) - \gamma\left(A_i^{(n,\mathbf{x})} + (\gamma r - c)x_i^{(n)}\right)}{\left(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}\right)^2}\\
&= \frac{(\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma A_i^{(n,\mathbf{x})}}{\left(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}\right)^2}
\end{aligned}$$

The denominator is positive, while the numerator is a constant with respect to $x_i^{(n)}$, since $A_i^{(n,\mathbf{x})}$ and $E_i^{(n,\mathbf{x}^{(n)})}$ do not depend on $x_i^{(n)}$. So $R_i^{(n,\mathbf{x})}$ is a monotone function of $x_i^{(n)}$, and whether it is increasing or decreasing, depends on the sign of the numerator: $(\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma A_i^{(n,\mathbf{x})}$. □

Proposition 2. *In MPE for Scenario 1, each player invests the maximum power if $\gamma r > c$, no power if $\gamma r < c$, and any amount of power if $\gamma r = c$.*

The proof shows that $\frac{dR_i^{(n,\mathbf{x})}}{dx_i^{(n)}}$ has the same sign as $\gamma r - c$, which then results in a consistent solution (or equivalently, a fixed point) to the above Bellman equations.

Proof. Let $W^{(n,\mathbf{x})}$ be the nth row of $\mathbf{W}^{(\mathbf{x})}$.
From the proof of Lemma 2, $\frac{dR_i^{(n,\mathbf{x})}}{dx_i^{(n)}}$ has the same sign as $(\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma A_i^{(n,\mathbf{x})}$. Now,

$$\begin{aligned}
&(\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma A_i^{(n,\mathbf{x})}\\
&= (\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma W^{(n,\mathbf{x})}\hat{R}_i^{(n,\mathbf{x})}(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)})\\
&= (\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma(\hat{R}_i^{(n,\mathbf{x})} - Z_i^{(n,\mathbf{x})})(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)})\\
&= (\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma\hat{R}_i^{(n,\mathbf{x})}(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)})\\
&\quad + \gamma\frac{(\gamma r - c)x_i^{(n)}}{E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}}(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)})\\
&= (\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma\hat{R}_i^{(n,\mathbf{x})}(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}) + \gamma(\gamma r - c)x_i^{(n)}\\
&= (\gamma r - c)E_i^{(n,\mathbf{x}^{(n)})} - \gamma\hat{R}_i^{(n,\mathbf{x})}E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}(\gamma r - c - \gamma\hat{R}_i^{(n,\mathbf{x})})\\
&= E_i^{(n,\mathbf{x}^{(n)})}(\gamma r - c - \gamma\hat{R}_i^{(n,\mathbf{x})}) + \gamma x_i^{(n)}(\gamma r - c - \gamma\hat{R}_i^{(n,\mathbf{x})})\\
&= (\gamma r - c - \gamma\hat{R}_i^{(n,\mathbf{x})})(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)})\\
&= \gamma\left(r - \frac{c}{\gamma} - \hat{R}_i^{(n,\mathbf{x})}\right)(E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)})
\end{aligned}$$

Since $E_i^{(n,\mathbf{x}^{(n)})} + \gamma x_i^{(n)}$ is positive, and the sign of $\left(r - \frac{c}{\gamma} - \hat{R}_i^{(n,\mathbf{x})}\right)$ is the same as that of $(\gamma r - c)$ from Lemma 1, we have that $\frac{dR_i^{(n,\mathbf{x})}}{dx_i^{(n)}}$ has the same sign as $(\gamma r - c)$. Also, note that if $\gamma r = c$, we have $R_i^{(n,\mathbf{x})} = 0, \forall n$ from Proposition 1 when $\Gamma^{(n,\mathbf{x}^{(n)})} = \gamma\sum_{j\in[n\oplus i]} x_j^{(n)}$.

So in any state n, a player invests the maximum power if $\gamma r > c$, no power if $\gamma r < c$, and any amount of power if $\gamma r = c$. Since we consider that the players are homogeneous and the computational power of each player is usually bounded (let the common bound be $\overline{x}$), we can assume that each player invests $\overline{x}$ if $\gamma r > c$.

Hence we have a consistent solution for the Bellman equations that the players invest $\overline{x}$ if $\gamma r > c$, 0 if $\gamma r < c$, and any power in the range $[0, \overline{x}]$ if $\gamma r = c$. □

Note that if all the involved players have practically infinite computational power when $\gamma r > c$, the limit of the expected utility $R_i^{(n,\mathbf{x})}$ computed in any state n is $\frac{1}{n+1}\left(r - \frac{c}{\gamma}\right)$. This is same as the expected utility in SNE. It can also be noted that, since the equilibrium strategy does not depend on the number of players, the assumption of knowing the number of other players can be relaxed.

3.3 Insights

- The policy profiles in SNE and MPE are the same, that is, there is no distinction between myopic and Markovian players with respect to equilibrium strategies. It is noteworthy that, if players have unbounded computational power, expected utilities in MPE and SNE are also same.
- The players follow a thresholding policy in both types of equilibria. $\gamma r < c$ results in a system where it is disadvantageous to invest, while $\gamma r > c$ results in a system where everyone would want to invest their maximum power.
- MPE policy is independent of the number of players in the system; when this number is high, the competition drives each player to invest heavily; when this number is low, each player invests heavily so that the problem gets solved faster (before arrival of additional competing players).
- The players are inclined towards investing a maximal amount of power when $\gamma r > c$; however this is not inefficient since the power would be spent for less time as the problem would get solved faster.
- The expected utility in state n (when there are $n+1$ players in the system) in SNE as well as in MPE (when $\overline{x} \to \infty$) is $\left(\frac{r}{n+1} - \frac{c}{\gamma(n+1)}\right)$. This is intuitive, since the reward is expected to be won by the players with equal probability (hence the term $\frac{r}{n+1}$) and the cost is reduced owing to the joint rate of the problem getting solved (hence the term $\frac{c}{\gamma(n+1)}$).

Insight behind MPE and SNE profiles being same

From Proposition 1, $\mathbf{R}_i^{(\mathbf{x})} = (\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Z}_i^{(\mathbf{x})}$. For $\gamma r > c$, when the invested power x_i increases, $\mathbf{Z}_i^{(\mathbf{x})}$ increases and $(\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}$ decreases. But from the proof of Proposition 2, we know that $\mathbf{R}_i^{(\mathbf{x})}$ increases with x_i when $\gamma r > c$, implying that the rate of increase of $\mathbf{Z}_i^{(\mathbf{x})}$ dominates the rate of decrease of $(\mathbf{I} - \mathbf{W}^{(\mathbf{x})})^{-1}$. So, the effect of $\mathbf{W}^{(\mathbf{x})}$ and hence the state transitions is relatively weak, thus resulting in Markovian players playing SNE strategy itself. Similar argument holds for $\gamma r \leq c$. It would be interesting to study scenarios where the rate of problem getting solved is proportional to a non-linear function of the players' investments. While a linear function is suited to most distributed computing applications including blockchain mining, a non-linear function could possibly see $\mathbf{W}^{(\mathbf{x})}$ having a strong effect leading to MPE being different from SNE.

4 Scenario 2: Rate of Problem Getting Solved is Independent of the Players' Investments

Here, we assume $\Gamma^{(n,\mathbf{x}^{(n)})} = \gamma f(n+1)$, where γ is a constant. $f(n+1)$ is a function of the number of players in the system at state n; $f(n+1)$ could be 1 in which case, the rate of the problem getting solved is constant γ.

4.1 Statewise Nash Equilibrium

The expected utility computed myopically by player i in state n (from Equation (1) with $\lambda = \mu = 0$) is

$$V_i^{(n,\mathbf{x}^{(n)})} = -\frac{c_i x_i^{(n)}}{\gamma f(n+1)} + r\frac{x_i^{(n)}}{\sum_{j\in[n\oplus i]} x_j^{(n)}}$$

It is easy to see that $V_i^{(n,\mathbf{x}^{(n)})}$ is a concave function with respect to $x_i^{(n)}$. The first order condition $\frac{dV_i^{(n,\mathbf{x}^{(n)})}}{dx_i^{(n)}} = 0$ gives

$$0 = -\frac{c_i}{\gamma f(n+1)} + r\frac{\sum_{j\in[n\oplus i]} x_j^{(n)} - x_i^{(n)}}{\left(\sum_{j\in[n\oplus i]} x_j^{(n)}\right)^2}$$

$$\iff x_i^{(n)} = \sum_{j\in[n\oplus i]} x_j^{(n)}\left(1 - \frac{c_i}{\gamma r f(n+1)}\sum_{j\in[n\oplus i]} x_j^{(n)}\right) \quad (2)$$

Summing the above over all players, we get

$$\sum_{j\in[n\oplus i]} x_j^{(n)} = \sum_{j\in[n\oplus i]} x_j^{(n)}\left((n+1) - \frac{\sum_{j\in[n\oplus i]} c_j}{\gamma r f(n+1)}\sum_{j\in[n\oplus i]} x_j^{(n)}\right)$$

So either $\sum_{j\in[n\oplus i]} x_j^{(n)} = 0$, which means $x_j^{(n)} = 0, \forall j \in [n\oplus i]$, or

$$1 = n + 1 - \frac{\sum_{j\in[n\oplus i]} c_j}{\gamma r f(n+1)}\sum_{j\in[n\oplus i]} x_j^{(n)}$$

$$\iff \sum_{j\in[n\oplus i]} x_j^{(n)} = \frac{n\gamma r f(n+1)}{\sum_{j\in[n\oplus i]} c_j}$$

(Note that all players investing 0 is not SNE, since any player can deviate to invest a positive amount of power and obtain the full reward.)

Substituting the above expression in Equation (2), we get

$$x_i^{(n)} = \frac{n\gamma r f(n+1)}{\sum_{j\in[n\oplus i]} c_j}\left(1 - \frac{c_i}{\gamma r f(n+1)}\frac{n\gamma r f(n+1)}{\sum_{j\in[n\oplus i]} c_j}\right)$$

$$= \frac{n\gamma r f(n+1)}{\left(\sum_{j\in[n\oplus i]} c_j\right)^2}\left(\sum_{j\in[n\oplus i]} c_j - nc_i\right)$$

So $x_i^{(n)} \geq 0$ if $\sum_{j\in[n\oplus i]} c_j - nc_i \geq 0$ or equivalently, $c_i \leq \frac{\sum_{j\in[n]} c_j}{n-1}$. So the players which do not satisfy this constraint can be effectively removed from the system. Let T be the set of players who would invest. Players with the cost parameter violating the constraint can hence be iteratively removed, until the constraint $c_j \leq \frac{\sum_{j'\in T} c_{j'}}{|T|-1}$ is satisfied for all players $j \in T$. It can be easily seen that this results in a unique set of players investing, and hence a unique SNE.

For homogeneous case: $c_j = c, \forall j \in [n\oplus i]$, we get

$$x_i^{(n)} = \frac{n\gamma r f(n+1)}{((n+1)c)^2}((n+1)c - nc)$$

$$= \frac{\gamma r}{c}\left(\frac{nf(n+1)}{(n+1)^2}\right)$$

The expected utility in SNE is, thus, $\hat{V}_i^{(n,\mathbf{x}^{(n)})} = \frac{r}{(n+1)^2}$.

4.2 Markov Perfect Equilibrium

Proposition 3. *In MPE for Scenario 2 where the rate of problem getting solved is $\gamma f(n+1)$ in state n, each player invests $\frac{\gamma r}{c}\left(\frac{nf(n+1)}{(n+1)^2}\right)$.*

Proof. Recall that since $\mathbf{W}^{(\mathbf{x})}$ is strictly substochastic, $(\mathbf{I}-\mathbf{W}^{(\mathbf{x})})^{-1} = \lim_{t\to\infty}\sum_{\eta=0}^{t-1}\left(\mathbf{W}^{(\mathbf{x})}\right)^{\eta}$. Since all the elements of $\mathbf{W}^{(\mathbf{x})}$ are non-negative, all the elements of $\left(\mathbf{W}^{(\mathbf{x})}\right)^{\eta}$ also are non-negative for any natural number η, and hence all the elements of $\left(\mathbf{I}-\mathbf{W}^{(\mathbf{x})}\right)^{-1}$ are non-negative. So if we ensure that the components of $\mathbf{Z}_i^{(\mathbf{x})}$ (namely, $Z_i^{(n,\mathbf{x})}$) are individually maximized with respect to $x_i^{(n)}$, we would essentially maximize all the elements of $\mathbf{R}_i^{(\mathbf{x})} = (\mathbf{I}-\mathbf{W}^{(\mathbf{x})})^{-1}\mathbf{Z}_i^{(\mathbf{x})}$, since $\mathbf{W}^{(\mathbf{x})}$ is independent of $x_i^{(n)}$ in this scenario.

It can be easily shown that $Z_i^{(n,\mathbf{x})}$ is concave with respect to $x_i^{(n)}$. The first order condition $\frac{dZ_i^{(n,\mathbf{x})}}{dx_i^{(n)}} = 0$ gives

$$-\frac{c}{\lambda+(n+1)\mu+\gamma f(n+1)}+\frac{\gamma f(n+1)}{\lambda+(n+1)\mu+\gamma f(n+1)}r\frac{1}{\sum_{j\in[n\oplus i]}x_j^{(n)}}$$
$$-\frac{\gamma f(n+1)}{\lambda+(n+1)\mu+\gamma f(n+1)}r\frac{x_i^{(n)}}{\left(\sum_{j\in[n\oplus i]}x_j^{(n)}\right)^2}=0$$
$$\iff x_i^{(n)} = \sum_{j\in[n\oplus i]}x_j^{(n)} - \frac{c}{\gamma r f(n+1)}\left(\sum_{j\in[n\oplus i]}x_j^{(n)}\right)^2$$

(Note that in first order condition, the denominator $\lambda+(n+1)\mu+\gamma f(n+1)$ is replaced by $\lambda+\gamma f(n+1)$ for $n=1$ and by $N\mu+\gamma f(n+1)$ for $n=N-1$; however it vanishes and so the derivation is general for all n).

Replacing player i with any player $j\in[n]$ does not change the above analysis, and the above expression involves the sum of strategies of all the players (which is common to all) and also the common parameters. Hence substituting $x_j^{(n)} = x_i^{(n)}, \forall j\in[n\oplus i]$, we get that either $x_j^{(n)}=0, \forall j\in[n\oplus i]$, or

$$x_i^{(n)} = \frac{\gamma r}{c}\left(\frac{nf(n+1)}{(n+1)^2}\right)$$

(Note that all players investing 0 is not an equilibrium policy, since any player can deviate to invest a positive amount of power and obtain the full reward.) □

For $f(n+1)=n+1$, which means that the rate of problem getting solved is proportional to the number of players, the strategy simplifies to $x_i^{(n)} = \frac{\gamma r}{c}\left(\frac{n}{n+1}\right)$. For $f(n+1)=1$, which means that the rate of problem getting solved is constant, the strategy simplifies to $x_i^{(n)} = \frac{\gamma r}{c}\left(\frac{n}{(n+1)^2}\right)$. It can be shown that for non-negligible number of players, this computational power decreases with the number of players if $f(\cdot)$ is sublinear, else it increases with the number of players.

4.3 Insights

- The policy profiles in SNE and MPE are the same; so myopic and Markovian players act the same in equilibrium.
- In SNE with heterogeneous players, only the players with the low-range cost parameters invest. Usually, small players would have high cost parameters; so if the players are rational, the competition would be among big players who usually have low cost parameters.
- In MPE, players invest proportionally to the reward-cost ratio ($\frac{r}{c}$), and unlike Scenario 1, the policy depends on the number of players in the system. Unlike Scenario 1, it is beneficial for the homogeneous players to invest even if their common cost parameter is very high.
- Since the rate of problem getting solved does not depend on the computational investments, the players could reduce their equilibrium investments proportionally and still obtain the same expected reward with a reduced cost. However, such a strategy is not an equilibrium, since if other players have low investments, a player can deviate and lead the competition, which would then be followed by other players deviating in order to stay competitive.
- The equilibrium power decreases with the number of players (for a non-negligible number of players) if $f(\cdot)$ is sublinear, else it increases.

Insight behind MPE and SNE profiles being same

The transition rates are independent of the players' strategies in this scenario. It can be easily shown that, if $\mathbf{W}^{(\mathbf{x})}$ is constant w.r.t. players' strategies, then the MPE utility in any state ($R_i^{(n,\mathbf{x})}$) is a linear combination (with constant coefficients) of the SNE utilities ($V_i^{(n,\mathbf{x}^{(n)})}$) in all states (the argument goes on similar lines as the first paragraph of the proof of Proposition 3). So, if there is a best strategy in each individual state without considering state transitions, these individual strategies collectively form the best policy (best strategy in each state while considering transitions). So, the SNE strategy profiles in individual states collectively form the MPE policy profile.

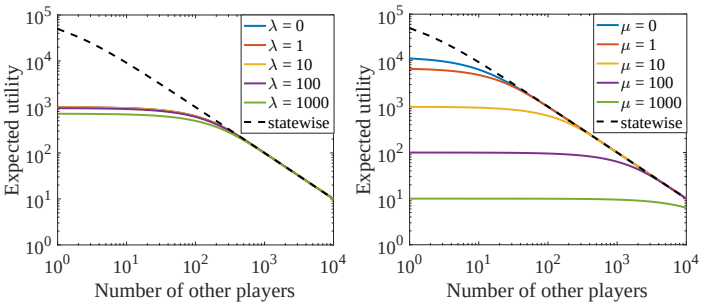
5 Simulation Study

Throughout the paper, we determined the SNE and MPE strategies, which were found to be independent of the arrival and departure rates of the players. In this section, we quantify the effects of these rates on the corresponding utilities using simulations. Furthermore, while our analysis showed that the strategies in SNE and MPE are the same for both the scenarios, the values of the corresponding expected utilities are expected to be different, in general, which we now quantify.

We use numbers from bitcoin mining in our simulations. The current offered reward for successfully mining a block is 12.5 bitcoins. Assuming 1 bitcoin $\approx$ \$8000, the reward translates to $\$10^5$. The bitcoin problem complexity is set such that it takes around 10 minutes on average for a block to get mined. That is, the rate of problem getting solved is 0.1 per minute on average. One of the most powerful ASIC (application-specific integrated circuit) currently available in market is Antminer S9, which performs computations of upto 13 TeraHashes per sec, while consuming about 1.5 kWh in 1 hour, which translates to \$0.18 per hour (at the rate of \$0.12 per kWh), equivalently \$0.003 per minute. As per BitNode (bitnodes.earn.com), a crawler developed to estimate the size of bitcoin network, the number of bitcoin miners is around 10^4. Hence, we consider $r=10^5, \gamma=0.1, c=0.003, N=10^4$.

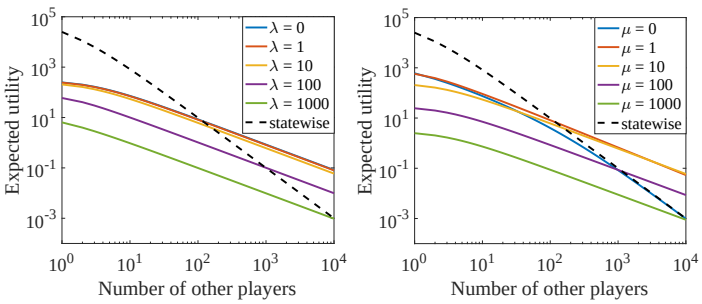
5.1 Simulation Results

As seen from Figures 2 and 3, the curves concerning expected utility generally follow a near-linear curve (of a negative slope) on the log-log scale, with respect to the number of other players



(a) for different λ's ($\mu = 10$) (b) for different μ's ($\lambda = 10$)

Fig. 2. Expected utilities in Scenario 1



(a) for different λ's ($\mu = 10$) (b) for different μ's ($\lambda = 10$)

Fig. 3. Expected utilities in Scenario 2 with constant rate

in system. So the curves actually follow power law; scaling the number of other players by a constant factor would result in proportionate scaling of expected utility.

5.1.1 Scenario 1

Here, we suppose that the maximum computational power available to the players is $\overline{x}$. Figure 2 presents the plots for expected utilities with the analyzed MPE policy for different values of λ and μ, and compares them against the expected utilities in SNE. Following are some insights:

- As we discussed in the section on Scenario 1, the expected utilities in MPE are bounded by $\frac{1}{n+1}\left(r - \frac{c}{\gamma}\right)$, which is same as the expected utility in SNE.
- In Scenario 1, a higher λ results in a higher likelihood of the system having more players, which results in a higher rate of the problem getting solved as well as more competition. This, in turn, reduces the time spent in the system as well as probability of winning for each player, which hence reduces the cost incurred as well as expected reward. Figure 2(a) suggests that, as λ changes, the change in cost incurred balances with the change in expected reward, since the change in expected utility is insignificant.
- For a given μ, if the number of players changes, there is balanced tradeoff between the cost incurred as well as expected reward as above, and so the change in expected utility is insignificant. However, as μ increases, the probability of player exiting the system increases, thus resulting in a lower expected utility (Figure 2(b)).

5.1.2 Scenario 2 with constant rate

As an imitation of bitcoin mining, we consider $f(n+1) = 1$, that is, the rate of problem getting solved is a constant γ. Figure 3 presents the plots for expected utilities with the analyzed MPE policy for different values of λ and μ, and compares them against the expected utilities in SNE. Following are some insights:

- An increase in the number of players increases the competition with no balancing factor (unlike Scenario 1); so probability of winning and hence expected utility reduces.
- For a higher λ, there is a higher likelihood of the system having more players, which thus results in a lower expected utility owing to aforementioned reason. Also, from Figure 3(a), if λ is not very high, an increase in μ is likely to reduce the competition to the extent that, the expected MPE utility in a higher state can exceed the corresponding SNE utility (which would be very low; recall that we quantified it to be $\frac{r}{(n+1)^2}$ in the section on Scenario 2).
- As μ increases, there is a higher likelihood of the competition decreasing, however the probability of the player exiting the system also increases; this results in a tradeoff. Figure 3(b) shows that the effect of probability of the player leaving the system dominates the effect of reduction in competition. For similar reasons as above, the expected MPE utility in a higher state can exceed the corresponding SNE utility.

6 Summary and Future Work

This paper studied a stochastic game framework, highlighting its application to blockchain mining; however the framework is suitable for distributed computing in a very general sense, as explained in the formulation of the utility function. We studied two scenarios depending on whether the rate of problem getting solved depends on the computational power invested by the players. We hence provided detailed insights on our analytical and simulation results for both scenarios.

This work has a number of interesting future directions. One could study a scenario where the rate of problem getting solved depends on the computational powers invested by the players, but not necessarily proportional to the sum of their invested computational powers. One could also study a scenario where the cost increases non-linearly with the computational power invested. It would be interesting to formulate and study the stochastic game involving players solving multiple problems in sequence. One could study a setting where players' departures depend on past outcomes, or some other condition such as the cost incurred thus far exceeding a certain budget. A Stackelberg game could be studied where, the system decides the amount of reward to be offered, and then the miners decide how much computational power to invest based on the offered reward. From the perspective of mechanism design, it would be interesting to elicit the true cost parameters of the miners.

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