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# Local stabilization of analytic systems with $n - 1$ inputs

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**Abstract** In this communication, we investigate the local stabilization of analytic systems  $\dot{x} = f(x) + Bu$  with  $n - 1$  inputs. We state a sufficient condition on the first approximation of  $f$  for which the local asymptotic stabilizability can be achieved; furthermore, we give explicitly the stabilizing feedback.

**Keywords** Feedback, asymptotic stabilization, nonlinear systems, Lyapunov's function.

## 1 Introduction

We consider a nonlinear control system with  $n - 1$  inputs:

$$\begin{cases} \dot{x} = f(x) + Bu \\ x \in \mathbb{R}^n, u \in \mathbb{R}^{n-1} \end{cases} \quad (1)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an analytic mapping,  $B$  is a  $n \times (n - 1)$  matrix of rank  $n - 1$ .

We shall say that system (1) is locally asymptotically stabilizable (L.A.S.) if there exists (locally) a feedback  $u : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  such that  $u(0) = 0$  and the origin is a locally asymptotically state equilibrium point for the closed-loop system  $\dot{x} = f(x) + Bu(x)$ .

Local stabilizability has been investigated by many authors: one can cite Kawski ([7]), Dayawansa-Martin-Knowles ([3],[4]), Boothby-Marino ([1]) who studied planar systems as well as Hermes ([5]).

In a precedent work ([6]), we gave necessary and sufficient conditions for a homogeneous polynomial system to be globally stabilizable. In this paper, we give a sufficient condition for (1) to be L.A.S. and we construct explicitly the stabilizing feedback. As an example, consider the system:

$$\begin{cases} \dot{x}_1 = e^{x_1+x_3} - e^{x_2+x_3} - x_1 - x_2 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = u_3 \end{cases}$$

We shall see that it can be stabilized by means of the following feedback law:

$$\begin{cases} u_2 = -(1 + 2x_1)(e^{x_1} - e^{x_2} - x_1 + x_2) \\ \quad - x_1 \left( 1 - e^{x_2} \frac{\exp(x_1 + x_1^2 + x_2) - 1}{x_1 + x_1^2 + x_2} \right) \\ \quad - x_1 - x_1^2 - x_2 \\ u_3 = -\frac{e^{x_3} - 1}{x_3}(e^{x_1} - e^{x_2}). \\ \quad (x_1 + (x_1 + x_1^2 + x_2)(2x_1 + 1)) \end{cases}$$

The linearized system of (1) is:

$$\dot{x} = Ax + Bu \quad (2)$$

where  $A = \frac{\partial f}{\partial x}(0)$ . It is well known that a sufficient condition for (1) to be L.A.S. is that (2) is also L.A.S. ([2]) but what happens when one has  $A \equiv 0$ ? We can generally write  $f(x) = P_r(x) + o(|x|^r)$  where  $P_r$ , the first approximation of  $f$ , is a homogeneous polynomial of degree  $r$ . In the sequel, we shall give conditions on polynomial  $P_r$  for which system (1) is L.A.S.; in fact we shall see that if the system:

$$\dot{x} = P_r(x) + Bu \quad (3)$$

is stabilizable then so is system (1). The particular case when  $f$  is exactly a homogeneous polynomial was studied in [6].

## 2 Notations and preliminaries

We denote by  $\langle x, y \rangle$  the usual inner product and by  $\det$  the determinant. For  $n - 1$  vectors  $x_1, \dots, x_{n-1}$ ,

we denote by  $x_1 \wedge \dots \wedge x_{n-1}$  the unique vector satisfying:

$$\forall x \in \mathbb{R}^n \quad \det(x_1, \dots, x_{n-1}, x) = \langle x_1 \wedge \dots \wedge x_{n-1}, x \rangle$$

We denote by  $b_2, \dots, b_n$  the columns of matrix  $B$  and by  $b_1$ , the vector  $b_1 = b_2 \wedge \dots \wedge b_n$ ; clearly  $(b_1, b_2, \dots, b_n)$  is a basis of  $\mathbb{R}^n$  in which system (1) can be written:

$$\begin{cases} \dot{x}_1 = f_1(x) \\ \dot{x}_2 = f_2(x) + u_2 \\ \vdots \\ \dot{x}_n = f_n(x) + u_n \end{cases} \quad (4)$$

after the following change in inputs space:

$$u_i = -f_i(x) + v_i \quad i = 2, \dots, n$$

system (4) becomes:

$$\begin{cases} \dot{x}_1 = f_1(x) \\ \dot{x}_2 = v_2 \\ \vdots \\ \dot{x}_n = v_n \end{cases} \quad (5)$$

$f_1$  is an analytic function, so it can be written:

$$f_1(x) = \sum_{i=r}^{\infty} P_i(x)$$

when the  $P_i$  are homogeneous polynomials of degree  $i$ .

### 3 Case where $r$ is even

**Theorem 1** *If the polynomial  $P_r$  takes both positive and negative values, then system (5) is stabilizable.*

**Proof**

Since  $P_r(x)$  changes its sign, there exists  $\tilde{b}_1$  and  $e_2$  such that  $P_r(\tilde{b}_1) \cdot P_r(e_2)$  is negative. Let  $H = \text{span}(b_2, \dots, b_n)$ ,  $H$  has an empty interior so we can suppose that  $\tilde{b}_1$  does not belong to  $H$ . The plane spanned by vectors  $\tilde{b}_1$ , and  $e_2$  intersects  $H$  at  $\tilde{b}_2$  and without loss of generality we can assume that  $(\tilde{b}_2, b_3, \dots, b_n)$  is a basis of  $H$ . The transformation matrix between the basis  $(b_1, b_2, \dots, b_n)$  and  $(\tilde{b}_1, \tilde{b}_2, b_3, \dots, b_n)$  is:

$$T = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ \alpha_2 & \beta_2 & 0 & \dots & 0 \\ \alpha_3 & \beta_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \beta_n & 0 & \dots & 1 \end{pmatrix}$$

$\alpha_1, \dots, \alpha_n, \beta_2, \dots, \beta_n$  being such that  $\tilde{b}_1 = \sum_{i=1}^n \alpha_i b_i$  and  $\tilde{b}_2 = \sum_{i=2}^n \beta_i b_i$

We have:

$$T^{-1} = \begin{pmatrix} 1/\alpha_1 & 0 & 0 & \dots & 0 \\ -\alpha_2/\beta_2\alpha_1 & 1/\beta_2 & 0 & \dots & 0 \\ \star & & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & 0 & \dots & 1 \end{pmatrix}$$

In the new basis  $(\tilde{b}_1, \tilde{b}_2, b_3, \dots, b_n)$  system (5) is given by the set of equations:

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{f}_1(\tilde{x}) \\ \dot{\tilde{x}}_2 = \tilde{f}_2(\tilde{x}) + v_2/\beta_2 \\ \dot{\tilde{x}}_3 = \tilde{f}_3(\tilde{x}) + \gamma_3 v_2 + v_3 \\ \vdots \\ \dot{\tilde{x}}_n = \tilde{f}_n(\tilde{x}) + \gamma_n v_2 + v_n \end{cases} \quad (6)$$

where  $\tilde{f}_1(\tilde{x}) = \frac{1}{\alpha_1} f_1(T\tilde{x})$ ,  $\tilde{f}_1$  can be written:

$$\tilde{f}_1(\tilde{x}) = g(\tilde{x}_1, \tilde{x}_2) + \sum_{i=3}^n \tilde{x}_i g_i(\tilde{x})$$

where  $g_i(\tilde{x}) = \int_0^1 \frac{\partial \tilde{f}}{\partial \tilde{x}_i}(\tilde{x}_1, \tilde{x}_2, t\tilde{x}_3, \dots, t\tilde{x}_n) dt$

and  $g(\tilde{x}_1, \tilde{x}_2) = \tilde{f}_1(\tilde{x}_1, \tilde{x}_2, 0, \dots, 0) = \sum_{i=r}^{\infty} \tilde{P}_i(\tilde{x}_1, \tilde{x}_2)$

the  $\tilde{P}_i$  being homogeneous polynomials of degree  $i$ , and  $\tilde{P}_r$  taking both positive and negative values.

We focus our attention on the system:

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{P}_r(\tilde{x}_1, \tilde{x}_2) + \tilde{P}_{r+1}(\tilde{x}_1, \tilde{x}_2) + \dots \\ \dot{\tilde{x}}_2 = w_2 \end{cases} \quad (7)$$

$\tilde{P}_r(\tilde{x}_1, \tilde{x}_2)$  can be written:

$$\tilde{P}_r(\tilde{x}_1, \tilde{x}_2) = L_1^{r_1} L_2^{r_2} \dots L_q^{r_q} Q_1^{m_1} \dots Q_s^{m_s}$$

where  $L_1, L_2, \dots, L_q$  are linear forms two by two linearly independant and  $Q_1, \dots, Q_s$  are irreducible homogeneous polynomials of degree 2. Since  $\tilde{P}_r$  changes its sign, at least two exponents  $r_i$  and  $r_j$  are odd; we can assume that  $i = 1, j = 2$  and since  $L_1$  and  $L_2$  are linearly independant, we can suppose that  $L_1 = \lambda \tilde{x}_1 + \mu \tilde{x}_2$  with  $\mu \neq 0$ . Let  $h(\tilde{x}_1, \tilde{x}_2, z) = \int_0^1 \frac{\partial g}{\partial \tilde{x}_2}(\tilde{x}_1, \tilde{x}_2 - tz) dt$  and consider the feedback law:

$$w_2(\tilde{x}_1, \tilde{x}_2) = \frac{-\lambda + \gamma(1 + 1/r_1)\tilde{x}_1^{1/r_1}}{\mu} g(\tilde{x}_1, \tilde{x}_2) - \tilde{x}_1 h(\tilde{x}_1, \tilde{x}_2, z) - z$$

together with the positive definite function:

$$V(\tilde{x}_1, \tilde{x}_2) = \frac{1}{2}(\tilde{x}_1^2 + z^2)$$

where we put

$$\begin{aligned} z &= \tilde{x}_2 - k(\tilde{x}_1) \\ k(\tilde{x}_1) &= \frac{-\lambda \tilde{x}_1 + \gamma \tilde{x}_1^{1+1/r_1}}{\mu} \end{aligned}$$

Taking into account that  $g(\tilde{x}_1, \tilde{x}_2) = g(\tilde{x}_1, k(\tilde{x}_1) + zh(\tilde{x}_1, \tilde{x}_2, z))$ , an easy computation shows that  $\dot{V}$ , the derivative of  $V$  along the trajectories of closed-loop system (7), is equal to:

$$\dot{V} = \tilde{x}_1 g(\tilde{x}_1, k(\tilde{x}_1)) - z^2$$

In order to show that 0 is an asymptotically equilibrium point for closed-loop system (7), it suffices to prove that one can choose  $\gamma$  such that:

$$\tilde{x}_1 g(\tilde{x}_1, k(\tilde{x}_1)) < 0 \quad (8)$$

Since for  $i \neq 1$ ,  $L_i$  is linearly independent of  $L_1$  and the  $Q_i$  s are irreducible, one has  $\tilde{P}_r(\tilde{x}_1, k(\tilde{x}_1)) = \gamma^{r_1} A \tilde{x}_1^{r+1} + o(|\tilde{x}_1^{r+1}|)$  with  $A$  a nonzero constant.

For  $s \geq r + 2$ ,  $\tilde{P}_s$  is a homogeneous polynomial of degree  $s$  so  $\tilde{P}_s(\tilde{x}_1, k(\tilde{x}_1)) = o(|\tilde{x}_1^{s-1}|) = o(|\tilde{x}_1^{r+1}|)$ ,  $\tilde{P}_{r+1}$  can be written:

$$\tilde{P}_{r+1}(\tilde{x}_1, \tilde{x}_2) = \sum_{i=0}^{r+1} \alpha_i \tilde{x}_1^i \tilde{x}_2^{r+1-i}$$

so

$$\tilde{P}_{r+1}(\tilde{x}_1, k(\tilde{x}_1)) = \sum_{i=0}^{r+1} \alpha_i \tilde{x}_1^{r+1} \left( \frac{-\lambda + \gamma \tilde{x}_1^{1/r_1}}{\mu} \right)^{r+1-i}$$

therefore, one can write:

$$g(\tilde{x}_1, \tilde{x}_2) = (\gamma^{r_1} A + B) \tilde{x}_1^{r+1} + o(\tilde{x}_1^{r+1})$$

hence if  $\gamma$  is chosen such that  $\gamma^{r_1} A + B < 0$ , inequality (8) holds in a neighborhood of the origin because  $r + 1$  is odd.

Let us return now to system (6) and consider the following feedback law:

$$\begin{cases} v_2(\tilde{x}) &= -\beta_2 \tilde{f}_2(\tilde{x}) + w_2(\tilde{x}_1, \tilde{x}_2) \\ v_3(\tilde{x}) &= \tilde{f}_3(\tilde{x}) - \gamma_3 v_2(\tilde{x}) - \frac{\partial V}{\partial \tilde{x}_1} g_3(\tilde{x}) - \tilde{x}_3 \\ &\vdots \\ v_n(\tilde{x}) &= \tilde{f}_n(\tilde{x}) - \gamma_n v_2(\tilde{x}) - \frac{\partial V}{\partial \tilde{x}_1} g_n(\tilde{x}) - \tilde{x}_n \end{cases}$$

together with the positive definite function:

$$W(\tilde{x}) = V(\tilde{x}_1, \tilde{x}_2) + \frac{1}{2}(\tilde{x}_3^2 + \dots + \tilde{x}_n^2)$$

Clearly the derivative of  $W$  along the trajectories of closed-loop system (6) is:

$$\dot{W} = \tilde{x}_1 g(\tilde{x}_1, k(\tilde{x}_1)) - z^2 - \tilde{x}_3^2 - \dots - \tilde{x}_n^2$$

which is obviously negative definite in a neighborhood of the origin. This proves that the above feedback stabilizes system (6).

## 4 Case where $r$ is odd

**Theorem 2** *If there exists  $(\lambda_2, \dots, \lambda_n) \in \mathbb{R}^{n-1}$  such that  $P_r(1, \lambda_2, \dots, \lambda_n) < 0$  then system (5) is stabilizable.*

**Proof** Consider the following system derived from system (5):

$$\begin{cases} \dot{x}_1 &= P_r(x) \\ \dot{x}_2 &= u_2 \\ &\vdots \\ \dot{x}_n &= u_n \end{cases} \quad (9)$$

We claim that the following feedback law stabilizes system (8):

$$u_i(x) = \lambda_i P_r(x) - x_1 g_i(x) - (x_i - \lambda_i x_1)^r$$

where  $g_i(x) = \int_0^1 \frac{\partial P_r}{\partial x_i}(x_1, t(x_2, \dots, x_n) + x_1(1-t)(\lambda_2, \dots, \lambda_n)) dt$ . Indeed if we introduce the positive definite function:

$$V(x) = \frac{1}{2} \left( x_1^2 + \sum_{i=2}^n (x_i - \lambda_i x_1)^2 \right)$$

taking into account that  $P_r(x) = P_r(x_1, \lambda_2 x_1, \dots, \lambda_n x_1) + \sum_{i=2}^n (x_i - \lambda_i x_1) g_i(x)$ , we obtain that the derivative of  $V$  along the trajectories of closed-loop system (8) is:

$$\dot{V} = x_1^{r+1} P_r(1, \lambda_2, \dots, \lambda_n) - \sum_{i=2}^n (x_i - \lambda_i x_1)^{r+1}$$

which is negative definite. Now, one can remark that the  $u_i$  are homogeneous polynomials of the same degree as  $P_r$ , so a Massera's theorem ([8]) permits to conclude about L.A.S. of closed-loop system (5) with the feedback given above.

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