



# Exponential Stabilization of Nonlinear Systems by an Estimated State Feedback

Abderrahman Iggidr, Gauthier Sallet

## ► To cite this version:

Abderrahman Iggidr, Gauthier Sallet. Exponential Stabilization of Nonlinear Systems by an Estimated State Feedback. Proc. of the 2nd European Control Conference ECC'93, Jun 1993, Groningen, Netherlands. hal-01862880

**HAL Id: hal-01862880**

**<https://inria.hal.science/hal-01862880v1>**

Submitted on 29 Aug 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# EXPONENTIAL STABILIZATION OF NONLINEAR SYSTEMS BY AN ESTIMATED STATE FEEDBACK

A. Iggidr G.Sallet  
 CONGE Project, INRIA Lorraine & University of Metz  
 4, rue Marconi 57 070 METZ –FRANCE  
 e.mail : {iggidr,sallet} @ilm.loria.fr

**Abstract:** In this paper we investigate the stabilizability problem of a class of multi-input multi-output nonlinear systems which linearization at the origin is controllable and observable. Under assumptions on the nonlinear part we prove : (a) the system is globally exponentially stabilizable (G.E.S) by means of linear feedback law. (b) the system can be G.E.S using a state estimation given by an observer.

**Keywords:** exponential stabilization, feedback, nonlinear systems, Lyapunov functions, observer.

## 1 Introduction

We consider a nonlinear system of the form :

$$\begin{cases} \dot{x} = Ax + Bu + g(x, u) \\ x \in \mathbb{R}^n, u \in \mathbb{R}^r, A \in M_{n,n}(\mathbb{R}), B \in M_{n,r}(\mathbb{R}) \end{cases} \quad (1)$$

where  $M_{n,m}(\mathbb{R})$  is the set of matrices with  $n$  rows and  $m$  columns and the map

$$f = (f_1, \dots, f_n)^T : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is Lipschitz continuous such that  $f(0, 0) = 0$ . We assume that the pair  $(A, B)$  is controllable and is in Brunovsky canonical form, i.e.

- A is a block-diagonal matrix of the form

$$A = \begin{pmatrix} A_{k_1} & 0 & . & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 0 \\ 0 & . & . & . & 0 & A_{k_r} \end{pmatrix}$$

where  $A_{k_i}$ ,  $1 \leq i \leq r$ , is a matrix in  $M_{k_i, k_i}(\mathbb{R})$  given by

- B is a block-diagonal matrix of the form

$$B = \begin{pmatrix} b_{k_1} & 0 & . & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 0 \\ 0 & . & . & . & 0 & b_{k_r} \end{pmatrix}$$

where  $b_{k_i}$ ,  $1 \leq i \leq r$ , is a column-vector in  $\mathbb{R}^{k_i}$  given by

$$b_{k_i} = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

In this paper, we study nonlinear systems of the form (1) which have the following property :

**(H1)** *There exists a positive constant  $K$  such that for any  $i = 1, \dots, n$  the following holds :*

$$\begin{cases} |g_i(x, u)| \leq K \|(x_1, \dots, x_i, 0, \dots, 0)\| \\ \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \forall u \in \mathbb{R}^r \end{cases} \quad (2)$$

where  $\| \cdot \|$  is the usual Euclidean norm on  $\mathbb{R}^n$ .

In section 2, we shall see that if condition **(H1)** is satisfied, then system (1) is globally exponentially stabilizable

(G.E.S.) at the origin by means of a linear feedback. This is a generalization of a result of Tsiniias [3] who studied the single input systems, our proof is different from its one and it is based on an idea from [1].

In section 3, we suppose in addition

**(H2)**  $g_i(x, u) = g_i(x_1, \dots, x_i, 0, \dots, 0, u)$

so we can construct an observer for system (1) with the output

$$y = Cx \quad (3)$$

where

$$C = \begin{pmatrix} C_{k_1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C_{k_p} \end{pmatrix}$$

$$C_{k_i} = (1 \quad 0 \quad \dots \quad 0)$$

This observer is of the form :

$$\dot{\hat{x}} = A\hat{x} + Bu + g(\hat{x}, u) - \tilde{E}(C\hat{x} - y) \quad (4)$$

We show that the error will tend to zero with an exponential rate of convergence and we construct a dynamic feedback which globally exponentially stabilizes system (1-3).

## 2 Stabilization

Before given our main result we recall some basic notions about exponential stability. Let us consider a system of ordinary differential equations

$$\dot{x} = X(x), \quad x \in \mathbb{R}^n \quad (5)$$

We suppose that the origin is an equilibrium point for the vector field  $X$  i.e,  $X(0) = 0$  and we denote  $X_t(x)$  the solution of (5) starting from the point  $x$  at  $t = 0$  ( $X_0(x) = x$ ). We say that (5) is *globally exponentially stable* at the origin if there exist positive constants  $M$  and  $\alpha$  such that

$$\|X_t(x)\| < M\|x\|e^{-\alpha t}$$

for any initial condition  $x$  in  $\mathbb{R}^n$  and any  $t > 0$ . To prove exponential stability we shall use the following Lyapunov theorem [2] :

**Theorem** *The solution  $x_t \equiv 0$  of the equation (5) is globally exponentially stable if there exist a Lyapunov function  $V$  and three positive constants  $k_1, k_2, k_3$  such that for any  $x \in \mathbb{R}^n$  one has :*

$$k_1\|x\|^2 \leq V(x) \leq k_2\|x\|^2$$

and

$$\dot{V}(x) = X.V(x) = \langle \nabla V(x), X(x) \rangle \leq -k_3\|x\|^2$$

We shall say that a control system

$$\begin{cases} \dot{x} = f(x, u), & x \in \mathbb{R}^n, u \in \mathbb{R}^r \\ f(0, 0) = 0 \end{cases}$$

is globally exponentially stabilizable (G.E.S) at the origin of  $\mathbb{R}^n$  if there exists a continuous feedback

$$\begin{aligned} u &: \mathbb{R}^n \rightarrow \mathbb{R}^r \\ x &\mapsto u(x) \end{aligned}$$

such that the closed loop system  $\dot{x} = f(x, u(x))$  is globally exponentially stable at the origin.

The aim of this section is to prove that system (1) is G.E.S and to give explicitly the stabilizing feedback provided that assumption **(H1)** is satisfied. To this end let  $\alpha \in \mathbb{R}$ ,  $\alpha > 1$  and introduce the following matrix :

$$\Phi = \begin{pmatrix} \alpha^{-1} & 0 & \dots & 0 \\ 0 & \alpha^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha^{-n} \end{pmatrix}$$

One can write

$$\Phi = \begin{pmatrix} \Phi_{k_1} & 0 & \dots & 0 \\ 0 & \Phi_{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \Phi_{k_r} \end{pmatrix}$$

where  $r$  is the number of the blocs of matrix  $A$  and

$$\Phi_{k_i} = \begin{pmatrix} \alpha^{-(k_{i-1}+1)} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha^{-(k_{i-1}+k_i)} \end{pmatrix}$$

with  $k_0 = 0$

Using the above decomposition, a simple computation can prove the following :

**lemma 1** *Matrices  $A, B$  and  $\Phi$  defined above satisfy :*

$$i) \alpha \Phi^{-1} A \Phi = A$$

$$ii) \forall F \in M_{r,n}(\mathbb{R}) \text{ there exists } \tilde{F} \in M_{r,n}(\mathbb{R})$$

such that

$$B\tilde{F} = \alpha \Phi^{-1} B F \Phi$$

and  $\tilde{F}$  is given by the following formula :

$$\tilde{F} = \alpha B^T \Phi^{-1} B F \Phi$$

$$iii) \forall x \in \mathbb{R}^n : \alpha^{-n} \|x\| \leq \|\Phi x\| \leq \alpha^{-1} \|x\|$$

We can now prove the main result of this section.

**Theorem 1** *If the assumption (H1) holds then the system (1) is G.E.S. at the origin by means of a linear feedback*

$$u = \tilde{F} x$$

where  $\tilde{F}$  is defined by (ii) and  $F$  is such that  $(A + BF)$  has all its eigenvalues with negative real part.

**Proof**

Since the pair  $(A, B)$  is controllable there exists a matrix  $F \in M_{r,n}(\mathbb{R})$  such that  $(A + BF)$  has all its eigenvalues with negative real part. Let  $M = A + BF$ , there exists a symmetric positive definite matrix  $S$  such that  $M^T S + SM = -Q$ ,  $Q$  symmetric positive definite .

Now consider the function :

$$V(x) = x^T \Phi S \Phi x$$

$V$  is positive definite and proper ( $V$  is a quadratic Lyapunov function). Let us evaluate its derivative along the trajectories of the closed-loop system

$$\dot{x} = (A + B\tilde{F})x + g(x, \tilde{F}x) = \tilde{M}x + g(x, \tilde{F}x) \quad (6)$$

$$\dot{V}(x) = x^T (\Phi S \Phi \tilde{M} + \tilde{M}^T \Phi S \Phi)x + 2x^T \Phi S \Phi g(x, \tilde{F}x)$$

where  $\tilde{M} = A + B\tilde{F}$

Taking into account (i) and (ii) we have

$$\tilde{M} = A + B\tilde{F} = \alpha \Phi^{-1} A \Phi + \alpha \Phi^{-1} B F \Phi$$

$$\tilde{M} = \alpha \Phi^{-1} (A + BF) \Phi = \alpha \Phi^{-1} M \Phi$$

Therefore :

$$\begin{aligned} \dot{V}(x) = & \alpha x^T (\Phi S M \Phi + \Phi M^T S \Phi) x \\ & + 2x^T \Phi S \Phi g(x, \tilde{F}x) \end{aligned}$$

$$\dot{V}(x) = -\alpha x^T \Phi Q \Phi x + 2x^T \Phi S \Phi g(x, \tilde{F}x)$$

The matrix  $Q$  is symmetric definite positive so there exists a positive constant  $a$  such that :

$$x^T \Phi Q \Phi x \geq a \|\Phi x\|^2$$

where

$$a = \inf \{ z^T Q z / z \in S^{n-1} \text{ the unit sphere in } \mathbb{R}^n \}$$

And then :

$$\dot{V}(x) \leq -\alpha a \|\Phi x\|^2 + 2 \|\Phi x\| \|\Phi S\| \|\Phi g(x, \tilde{F}x)\|$$

According to (H1) we have

$$\begin{aligned} \|\Phi g(x, \tilde{F}x)\|^2 &= \sum_{i=1}^n \frac{1}{\alpha^{2i}} g_i^2(x, \tilde{F}x) \\ &\leq K^2 \sum_{i=1}^n \frac{1}{\alpha^{2i}} (x_1^2 + \dots + x_i^2) \\ &\leq K^2 \sum_{j=1}^n \left( \frac{x_j}{\alpha^j} \right)^2 \left( 1 + \frac{1}{\alpha^2} + \dots + \frac{1}{\alpha^{2(n-j)}} \right) \\ &\leq nK^2 \|\Phi x\|^2 \end{aligned}$$

So

$$\dot{V}(x) \leq (-\alpha a + 2\sqrt{n} K \|\Phi S\|) \|\Phi x\|^2$$

If we choose  $\alpha > \text{Max} \left( 1, \frac{2\sqrt{n} K \|\Phi S\|}{a} \right)$  then

$$\dot{V} \leq -c \|\Phi x\|^2$$

where  $c$  is a positive constant.

Since  $\|\Phi x\| \geq \frac{1}{\alpha^n} \|x\|$  it follows that :

$$\dot{V}(x) \leq -c' \|x\|^2$$

where  $c'$  is a positive constant , and this completes the proof of theorem 1. ■

**Example 1.** The three dimensional system

$$\begin{cases} \dot{x}_1 = x_2 + x_1 \cos(u(x_2^2 + x_1 + x_3)) \\ \dot{x}_2 = x_3 + \frac{x_1 + x_2}{1 + x_3^2} \\ \dot{x}_3 = \sqrt{x_1^2 + x_2^2 + x_3^2} e^{-u^2} + u \end{cases} \quad (7)$$

has the form (1) with

$$g(x, u) = \begin{pmatrix} x_1 \cos(u(x_2^2 + x_1 + x_3)) \\ \frac{x_1 + x_2}{1 + x_3^2} \\ \sqrt{x_1^2 + x_2^2 + x_3^2} e^{-u^2} \end{pmatrix}$$

which satisfies condition (H1), so system (7) is G.E.S and a stabilizing feedback can be computed according to theorem 1 as follows : We choose  $F = (-1, -3, -3)$ ,  $Q = -Id_{\mathbb{R}^3}$  and we solve  $M^T S + SM = Q$  :

$$S = \begin{pmatrix} \frac{37}{16} & \frac{31}{16} & \frac{1}{2} \\ \frac{31}{16} & \frac{13}{4} & \frac{13}{16} \\ \frac{1}{2} & \frac{13}{16} & \frac{7}{16} \end{pmatrix}$$

We compute  $\tilde{F} = \alpha B^T \Phi^{-1} B F \Phi$ . The feedback law

$$u = -\alpha^3 x_1 - 3\alpha^2 x_2 - 3\alpha x_3$$

globally exponentially stabilizes system (7) if we choose

$$\alpha > \text{Max} \left( 1, \frac{2\sqrt{n} K \|\Phi S\|}{a} \right) \simeq 10\sqrt{3}$$

### 3 Construction of the observer and stabilization using a state estimation

**Theorem 2** *If the condition (H1) and (H2) are satisfied then there exists  $\tilde{E} \in M_{n,p}(\mathbb{R})$  such that the system (4) is an exponential observer for (1-3).*

### Proof

The error  $e = \hat{x} - x$  satisfies the following equation :

$$\dot{e} = (A - \tilde{E}C)e + g(\hat{x}, u) - g(x, u) \quad (8)$$

since  $(A, C)$  is observable, there exists  $E \in M_{n,p}(\mathbb{R})$  such that  $(A - EC)$  has all its eigenvalues with negative real part.

Remark that :

$$|g_i(x, u) - g_i(\hat{x}, u)| \leq K \|p_i(x - \hat{x})\| = K \|(e_1, \dots, e_i)\|$$

where  $p_i : \mathbb{R}^n \rightarrow \mathbb{R}^i$  is the canonical projection.

So if we take

$$\tilde{E} = \alpha \Phi^{-1} E C \Phi C^T$$

then , according to the proof of theorem 1, there exists a quadratic Lyapunov function  $W$  such that for  $\alpha$  large enough :

$$\dot{W}(e) \leq -b \|e\|^2, b > 0$$

This proves that  $\|e(t)\| \leq M \|e_0\| e^{-\alpha t}$  so (4) is an exponential observer for (1-3). ■

Now we use the above results to achieve the stabilization of system (1) with the state estimation given by the observer (4). Consider the following system defined on  $\mathbb{R}^n \times \mathbb{R}^n$  :

$$\begin{cases} \dot{x} = Ax + B\tilde{F}\hat{x} + g(x, \tilde{F}\hat{x}) \\ \dot{e} = (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \end{cases} \quad (9)$$

**Theorem 3** *If the assumption (H1) and (H2) hold then (9) is globally exponentially stable i.e. the closed-loop system (1-3), with the state estimation given by the observer (4) is G.E.S.*

### Proof

Since  $e = \hat{x} - x$ , system (9) becomes :

$$\begin{cases} \dot{x} = (A + B\tilde{F})x + B\tilde{F}e + g(x, \tilde{F}\hat{x}) \\ \dot{e} = (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \end{cases} \quad (10)$$

According to the proofs of theorem 1 and theorem 2, there exist two quadratic Lyapunov functions  $V$  and  $W$  such that :

$$\langle \nabla V(x), (A + B\tilde{F})x + g(x, \tilde{F}\hat{x}) \rangle \leq -c' \|x\|^2$$

$$\langle \nabla W(e), (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \rangle \leq -b \|e\|^2$$

where  $\langle \cdot, \cdot \rangle$  is the usual Euclidean inner product on  $\mathbb{R}^n$ .

Let  $U$  be the function defined on  $\mathbb{R}^n \times \mathbb{R}^n$  by :

$$U(x, e) = \beta V(x) + W(e), \beta > 0$$

$U$  is a quadratic positive definite function and we have :

$$\begin{aligned} \dot{U}(x, e) = & \beta \langle \nabla V(x), (A + B\tilde{F})x + g(x, \tilde{F}\hat{x}) \rangle \\ & + \beta \langle \nabla V(x), B\tilde{F}e \rangle \\ & + \langle \nabla W(e), (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \rangle \end{aligned}$$

So

$$\dot{U}(x, e) \leq -\beta c' \|x\|^2 + 2\beta K' \|x\| \|e\| - b \|e\|^2$$

where  $K'$  is a positive constant defined by :

$$\langle \nabla V(x), B\tilde{F}e \rangle \leq 2K' \|x\| \|e\|$$

If we choose  $\beta$  such that  $0 < \beta < \frac{c' b}{K'^2}$  then  $\dot{U}(x, e)$  is negative definite on  $\mathbb{R}^n \times \mathbb{R}^n$  and so theorem 3 is proved.

## References

- [1] J.P.Gauthier, H.Hammouri, and S.Othman. A simple observer for nonlinear systems, application to bioreactors. *IEEE trans.aut.control*, Vol 36, No 6 (1992) 875–879.
- [2] W. Hahn. *Stability of Motion*. Springer Verlag, (1967).
- [3] J. Tsinias. A theorem on global stabilization. *Systems & Control Letters*, (17):357–362, (1991).