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Mohamed Bensoubaya, Abdelhak Ferfera, Abderrahman Iggidr. Time-varying stabilizing feedback for a class of nonlinear discrete-time control systems. 34th IEEE Conference on Decision and Control, Dec 1995, New Orleans, Louisiana, United States. pp.512–514, 10.1109/cdc.1995.478943 . hal-01862878

**HAL Id: hal-01862878**

**<https://inria.hal.science/hal-01862878>**

Submitted on 30 Aug 2018

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# TIME-VARYING STABILIZING FEEDBACK FOR A CLASS OF NONLINEAR DISCRETE-TIME CONTROL SYSTEMS

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## Abstract

The goal of this paper is to introduce and to use time-varying control for the stabilization of nonlinear discrete-time control systems which can not be stabilized by state feedback.

**Keywords:** stabilization, nonautonomous systems, Lyapunov functions.

## 1. Introduction

The stabilization problem of nonlinear continuous systems has attracted the interest of an increasing number of authors in the last decay (see [8] and references therein). For discrete-time systems there is a few results concerning the stabilization [1, 2, 3, 9]. Recently in [1] we have given the following necessary condition for the existence of a continuous stabilizing state feedback :

**Theorem 1.** *Given a discrete-time nonlinear system*

$$x(k+1) = f(x(k), u(k)) \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and  $f$  is a continuous function in a neighborhood  $A \times U$  of  $(0, 0)$  satisfying  $f(0, 0) = 0$ , a necessary condition for the existence of a continuous feedback  $u(x)$ , with  $u(0) = 0$ , stabilizing the system (1) is that the image of the map  $\gamma : A \times U \rightarrow \mathbb{R}^n$  defined by  $\gamma(x, u) = f(x, u) - x$  contains some neighborhood of the origin.

In this paper we are interested in the stabilization of nonlinear discrete-time systems of the form

$$x(k+1) = x(k) + \sum_{i=1}^m u_i(k)g_i(x(k)) \quad (2)$$

where  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ,  $u = (u_1, \dots, u_m)^T \in \mathbb{R}^m$ , and  $g_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $i = 1, \dots, m$  ( $m \geq 2$ ), are continuous functions on  $\mathbb{R}^n$ . Those systems are not generally stabilizable by continuous state feedback:

**Proposition.** *If  $m < n$  then system (2) can not be stabilized by means of continuous state feedback.*

**Proof:** Since  $\text{rank}\{g_1(0), \dots, g_m(0)\} \leq m < n$ , the map  $\gamma : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined by

$$\gamma(x, u) = f(x, u) - x = \sum_{i=1}^m u_i(k)g_i(x(k))$$

is not locally onto. So the necessary condition given by theorem 1 is violated. ■

As for the continuous systems [4, 6, 7] this impossibility to stabilize by state feedback motivates the introduction of time-varying control.

## 2. Main result

In the sequel, we limit ourself to the class of non linear systems of the form (2) with

$$g_1 = \frac{\partial}{\partial x_1} \quad (3)$$

For all  $x \in \mathbb{R}^n$  and all  $t \in \mathbb{R}$ , set  $\tilde{x} = (x_2, \dots, x_n)^T$  and let

$$V(x, t) = \frac{1}{2} \left[ (x_1 + h(\tilde{x}, t))^2 + \sum_{i=2}^n x_i^2 \right] \quad (4)$$

where  $h$  is any  $\mathcal{C}^2$  function satisfying

$$h(\tilde{x}, t + 2\pi) = h(\tilde{x}, t), \quad \forall (\tilde{x}, t) \quad (5)$$

$$h(0, t) = 0, \quad \forall t \quad (6)$$

A possible choice for  $h$  satisfying (5-6) is

$$h(\tilde{x}, t) = \psi(\tilde{x}) \cos t \quad (7)$$

where  $\psi$  is linear. So  $V$  is  $2\pi$ -periodic with respect to time  $t$ , vanishes if and only if  $x = 0$  and satisfy

$$\{(x, t) \in \mathbb{R}^{n+1} \mid V(x, t) \leq K \text{ and } |t| \leq K\}$$

is bounded  $\forall K > 0$ . Besides, define  $v : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and  $f, \tilde{g}_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  ( $1 \leq i \leq m$ ) by

$$v(x, t) = -\frac{1}{2}(x_1 - h(\tilde{x}, t)) - h(\tilde{x}, t + 1) \quad (8)$$

$$f(x, t) = \begin{pmatrix} x + v(x, t)g_1(x) \\ t + 1 \end{pmatrix} = \begin{pmatrix} x_1 + v(x, t) \\ \tilde{x} \\ t + 1 \end{pmatrix} \quad (9)$$

$$\tilde{g}_i(x, t) = \begin{pmatrix} g_i(x) \\ 0 \end{pmatrix}$$

Now we can state and prove the main theorem of this paper.

**Theorem 2.** Assume that the functions  $\tilde{g}_1, \dots, \tilde{g}_m$  and  $f$  satisfy for any  $(x, t) \in \mathbb{R}^n \times S^1$

$$\text{rank}(\{\tilde{g}_i(f^k(x, t)), k \in \mathbb{N}, 2 \leq i \leq m\}) = n - 1 \quad (10)$$

Then the origin is a globally uniformly asymptotically stable equilibrium point of the time-varying periodic closed-loop system (2) with

$$u(x, t) = \begin{pmatrix} v(x, t) \\ 0 \end{pmatrix} - M^{-1}(x, t)(\nabla V(f(x, t))\tilde{g}(x, t))^T \quad (11)$$

where  $M(x, t) = I + \frac{1}{2}g^T(x)\frac{\partial^2 V}{\partial x^2}(f(x, t))g(x)$ .

**Proof:** The functions  $V$  and  $v$  from  $\mathbb{R}^n \times \mathbb{R}$  to  $\mathbb{R}$  are both  $2\pi$ -periodic with respect to time  $t$ . So they induce naturally some functions from  $\mathbb{R}^n \times S^1$  to  $\mathbb{R}$ . Hence, by taking the change of control

$$\begin{aligned} u_1 &= v(x, t) + \tilde{u}_1 \\ u_i &= \tilde{u}_i, \quad 2 \leq i \leq m \end{aligned}$$

the time-varying closed-loop system can be considered as an autonomous control system on  $\mathbb{R}^n \times S^1$  defined by

$$\begin{cases} x(k+1) = x(k) + v(x(k), \theta(k))g_1(x(k)) \\ \quad + \sum_{i=1}^m \tilde{u}_i(k)g_i(x(k)) \\ \theta(k+1) = \theta(k) + 1 \end{cases} \quad (12)$$

where  $\theta \in S^1$ . The difference of the Lyapunov function  $V$  along the trajectories of system (12) is given by

$$\Delta V(x, \theta) = V(f(x, \theta) + \tilde{g}(x, \theta)\tilde{u}) - V(x, \theta)$$

where  $\tilde{g}(x, \theta) = (\tilde{g}_1(x, \theta), \dots, \tilde{g}_m(x, \theta))$ . If one chooses  $h$  such that  $V(f(x, \theta) + \tilde{g}(x, \theta)\tilde{u})$  is quadratic on  $\tilde{u}$  (of the form (7) for instance), one get

$$\begin{aligned} \Delta V(x, \theta) &= V(f(x, \theta)) - V(x, \theta) \\ &\quad + \tilde{u}^T(\nabla V(f(x, \theta))\tilde{g}(x, \theta))^T \\ &\quad + \frac{1}{2}\tilde{u}^T g^T(x)\frac{\partial^2 V}{\partial x^2}(f(x, \theta))g(x)\tilde{u} \end{aligned}$$

Furthermore, since  $V$  is positive definite on  $\mathbb{R}^n \times S^1$  then, for any  $(x, \theta) \in \mathbb{R}^n \times S^1$ ,

$$N(x, \theta) = g^T(x)\frac{\partial^2 V}{\partial x^2}(f(x, \theta))g(x) \geq 0 \quad (13)$$

As a matter of fact, if there exists  $(x_0, \theta_0) \in \mathbb{R}^n \times S^1$  such that the matrix  $N(x_0, \theta_0)$  has a negative eigenvalue  $\lambda(x_0, \theta_0) < 0$  associated with an eigenvector  $v(x_0, \theta_0) \in \mathbb{R}^m$ , then, for any  $\alpha \in \mathbb{R}$ , one has from the Taylor expansion formula

$$\begin{aligned} V(f(x_0, \theta_0) + \tilde{g}(x_0, \theta_0)\alpha v(x_0, \theta_0)) &= \\ V(f(x_0, \theta_0)) + \nabla V(f(x_0, \theta_0))\tilde{g}(x_0, \theta_0)v(x_0, \theta_0)\alpha & \\ + \frac{1}{2}\lambda(x_0, \theta_0)\|v(x_0, \theta_0)\|^2\alpha^2 & \end{aligned}$$

It turns out that

$$\lim_{\alpha \rightarrow \infty} V(f(x_0, \theta_0) + \tilde{g}(x_0, \theta_0)\alpha v(x_0, \theta_0)) = -\infty$$

that provides a contradiction with the fact that  $V$  is positive definite on  $\mathbb{R}^n \times S^1$ , and so (13) is established. Hence, the matrix

$$M(x, \theta) = I + \frac{1}{2}g^T(x)\frac{\partial^2 V}{\partial x^2}(f(x, \theta))g(x)$$

is invertible. It follows from (4), (8) and (9) that with the feedback law defined on  $\mathbb{R}^n \times S^1$  by

$$\tilde{u}(x, \theta) = -M^{-1}(x, \theta)(\nabla V(f(x, \theta))\tilde{g}(x, \theta))^T \quad (14)$$

the difference of the Lyapunov function  $V$  along the trajectories of the closed-loop system (12-14) becomes

$$\begin{aligned} \Delta V(x, \theta) &= V(f(x, \theta)) - V(x, \theta) - \tilde{u}^T(x, \theta)\tilde{u}(x, \theta) \\ &= -\frac{3}{8}(x_1 + h(\tilde{x}, \theta))^2 - \tilde{u}^T(x, \theta)\tilde{u}(x, \theta) \leq 0 \end{aligned}$$

which implies that  $V$  decreases along the solutions of the closed-loop system (12-14) and then the origin is Lyapunov stable. To show that zero is uniformly globally asymptotically stable equilibrium point of the closed-loop system (2-11) or that the invariant set  $\{0\} \times S^1$  is globally attractive, set

$$\begin{aligned} E &= \{(x, \theta) \in \mathbb{R}^n \times S^1 \mid \Delta V(x, \theta) = 0\} \\ &= \{(x, \theta) \mid V(f(x, \theta)) - V(x, \theta) = 0, \tilde{u}(x, \theta) = 0\} \end{aligned} \quad (15)$$

According to LaSalle's invariance principle all the solutions of the closed-loop system (12-14) tend to  $\Omega$  the largest invariant set contained in  $E$ . In order to prove the global attractivity let us show that  $\Omega = \{0\} \times S^1$ . Using (14) one has

$$\tilde{u}(x, \theta) = 0 \Rightarrow \nabla V(f(x, \theta))\tilde{g}(x, \theta) = 0 \quad (16)$$

Let  $(x(k), \theta(k))$  be a solution of the closed-loop system with  $(x(0), \theta(0)) = (x, \theta) \in \Omega$ . Since  $\Omega$  is invariant for the closed-loop system we have  $(x(k), \theta(k)) \in \Omega$  for all  $k \geq 0$ . But,  $\tilde{u}(x, \theta)$  vanishing on  $\Omega$ , one has  $(x(k), \theta(k)) = f^k(x, \theta)$  and so, by (15) and (16)

$$V(f^{k+1}(x, \theta)) - V(f^k(x, \theta)) = 0, \quad \forall k \in \mathbb{N} \quad (17)$$

and

$$\nabla V(f^{k+1}(x, \theta))\tilde{g}(f^k(x, \theta)) = 0, \quad \forall k \in \mathbb{N} \quad (18)$$

Notice that, by (4),

$$\nabla V(x, \theta) = \begin{pmatrix} x_1 + h(\tilde{x}, \theta) \\ \tilde{x} + (x_1 + h(\tilde{x}, \theta))\frac{\partial h}{\partial \tilde{x}}(\tilde{x}, \theta) \\ (x_1 + h(\tilde{x}, \theta))\frac{\partial h}{\partial \theta}(\tilde{x}, \theta) \end{pmatrix}$$

and, by (8) and (9), for any  $k \in \mathbb{N}$

$$f^k(x, \theta) = \begin{pmatrix} \frac{1}{2^k}(x_1 + h(\tilde{x}, \theta)) - h(\tilde{x}, \theta + k) \\ \tilde{x} \\ \theta + k \end{pmatrix}$$

so that

$$\nabla V(f^k(x, \theta)) = \begin{pmatrix} \frac{1}{2^k}(x_1 + h(\tilde{x}, \theta)) \\ \tilde{x} + \frac{1}{2^k}(x_1 + h(\tilde{x}, \theta))\frac{\partial h}{\partial \tilde{x}}(\tilde{x}, \theta + k) \\ \frac{1}{2^k}(x_1 + h(\tilde{x}, \theta))\frac{\partial h}{\partial \theta}(\tilde{x}, \theta + k) \end{pmatrix}$$

Now, a simple computation shows that (17) implies

$$x_1 + h(\tilde{x}, \theta) = 0 \quad (19)$$

and so

$$\nabla V(f^{k+1}(x, \theta))\tilde{g}(f^k(x, \theta)) =$$

$$\left\langle \left\langle \begin{pmatrix} 0 \\ \tilde{x} \\ 0 \end{pmatrix}, \tilde{g}_1(f^k(x, \theta)) \right\rangle, \dots, \left\langle \begin{pmatrix} 0 \\ \tilde{x} \\ 0 \end{pmatrix}, \tilde{g}_m(f^k(x, \theta)) \right\rangle \right\rangle$$

It turns out from (3) and (18) that  $(0, \tilde{x}^T, 0)^T$  belongs to

$$(\text{Span}(\{\tilde{g}_2(f^k(x, \theta)), \dots, \tilde{g}_m(f^k(x, \theta)), k \in \mathbb{N}\}))^\perp$$

Hence, it follows from (10) that  $\tilde{x} = 0$ , and so, by (19),  $x_1 = -h(0, \theta) = 0$  which finally implies  $x = 0$ . ■

**Example.** Consider the system evolving in  $\mathbb{R}^3$  defined by

$$\begin{cases} x(k+1) = x(k) + u_1(k) \\ y(k+1) = y(k) + x(k)u_2(k) \\ z(k+1) = z(k) + u_2(k) \end{cases}$$

This system doesn't satisfy the necessary stabilizability condition of theorem 1. So the best we can do is to design a time-varying feedback stabilizer. The assumptions of theorem 2 are satisfied with the Lyapunov function

$$V(x, y, z, t) = (x + y \cos(t))^2 + y^2 + z^2$$

So, one may easily check that this system can be stabilized thanks to the following periodic time-varying feedback:

$$u_1(x(k), y(k), z(k), k) = -\frac{1}{2}[x(k) - y(k) \cos(k) + 2y(k) \cos(k+1)] + \frac{\varphi_1(x(k), y(k), z(k), k)}{4 + 2x^2(k) + x^2(k)\cos^2(k+1)}$$

$$u_2(x(k), y(k), z(k), k) = \frac{\varphi_2(x(k), y(k), z(k), k)}{4 + 2x^2(k) + x^2(k)\cos^2(k+1)}$$

with

$$\varphi_1(x, y, z, t) = -2x - x^3 - 6y \cos(t+1) - x^2y \cos(t+1) + 2xz \cos(t+1) + 4y \cos(t+2) + 2x^2y \cos(t+2)$$

$$\varphi_2(x, y, z, t) = -4xy - 4z - x^2 \cos(t+1) - 3xyz \cos^2(t+1) + 2xy \cos(t+1) \cos(t+2)$$

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