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# Stabilization of Periodic Discrete-Time Nonlinear Systems

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## Abstract

This paper provides sufficient conditions for stabilization of periodic discrete systems. These conditions are derived from a discrete version of a Theorem due to Krasovski for continuous-time systems. This tool allows to give a systematic design of time-varying stabilizing control for autonomous discrete systems that can not be stabilized by time-invariant feedback.

**Keywords:** Lyapunov function, stability, stabilization, time-varying feedback, periodic discrete systems.

## 1 Introduction

The aim of this article is to design stabilizer feedback for a class of periodic discrete-time nonlinear control systems. More precisely, we consider systems that can be written  $x_{k+1} = F(x_k, u_k, k)$ ,  $F$  being time-periodic and the unforced system being Lyapunov stable. It is also assumed that there exists a positive definite periodic function  $V(x, k)$  in such a way  $V(F(x, 0, k), k+1) - V(x, k) \leq 0$ . The paper is organized as follows. In Section 2, we give a discrete version of Krasovski theorem [6] and a sufficient condition under which the attractivity of the equilibrium point is guaranteed. In Section 3, this will be used to derive a stabilization result of periodic systems. As an application, we derive a sufficient condition for time-varying feedback stabilization for a class of autonomous systems that can not be stabilized by time-invariant feedback [1, 2, 3, 5]. Due to the lack of space, the proofs are not reproduced here, but can be found in [4].

## 2 Stability of periodic systems

Let us consider a discrete nonlinear system

$$x_{k+1} = f(x_k, k), \quad f(0, k) = 0, \quad x(k) \in \mathbb{R}^n, \quad k \in \mathbb{N} \quad (1)$$

We assume that  $f$  is continuous and periodic with respect to  $k$ , i.e. there exists  $T \in \mathbb{N}$  such that  $f(x, k+T) = f(x, k)$ ,  $\forall (x, k) \in \mathbb{R}^n \times \mathbb{N}$ . We shall denote by  $x(k, x_{k_0}, k_0)$  the solution of (1) with initial state  $x_{k_0}$  at initial time  $k_0$ . Let  $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a continuous function. We shall say that  $a$  belongs to  $\mathcal{K}$  if it is strictly increasing, vanishes at zero and  $\lim_{r \rightarrow +\infty} a(r) = +\infty$ .

**Theorem 1** *Suppose there exists a continuous function  $V : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}$ , periodic in  $k$  with period  $T \geq 2$  such that for some  $a \in \mathcal{K}$ ,  $\forall x \in \mathbb{R}^n$ ,  $\forall k \in \mathbb{N}$ :*

$$(1) \quad V(x, k) \geq a(\|x\|), \quad V(0, k) = 0.$$

$$(2) \quad \Delta V(x, k) = V(f(x, k), k+1) - V(x, k) \leq 0.$$

(3) *The set  $\{(x, k) \in \mathbb{R}^n \times \mathbb{N} : \Delta V(x, k) = 0\}$  contains no complete solution of (1) except the trivial one.*

*Then the origin is globally uniformly asymptotically stable.*

For the sequel, we define for all  $0 \leq k \leq T$  a function  $\hat{f}_k$  in the following way :  $\hat{f}_k^k(x) = x$  and by induction  $\hat{f}_k^{p+1}(x) = f(\hat{f}_k^p(x), p)$  for all  $p \geq k$ . Actually  $\hat{f}_k^p(x)$  is nothing but the value at time  $p$  of the solution of (1) with initial state  $x$  at initial time  $k$ . With this notation we can prove the following result which gives a sufficient condition to get the condition (3) of the above theorem.

**Proposition 1** *Suppose (1) and (2) hold. If the set  $\{x \in \mathbb{R}^n : V(\hat{f}_k^{p+1}(x), p+1) - V(\hat{f}_k^p(x), p) = 0, p \geq k\}$  is reduced to the origin for all  $k = 0, \dots, T$ , then assumption (3) of Theorem (1) is satisfied and so the origin is globally uniformly asymptotically stable.*

## 3 Stabilization

Consider a discrete-time nonlinear control system

$$x_{k+1} = F(x_k, u_k, k), \quad F(0, 0, k) = 0, \quad k \in \mathbb{N} \quad (2)$$

where  $x_k \in \mathbb{R}^n$  is the state at time  $k$  and  $u \in \mathbb{R}^m$  is the control.  $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}^n$  is assumed to be continuous in  $(x, u)$  and periodic with respect to time  $k$  with period  $T \geq 2$ . The problem addressed here is the following: does a periodic feedback  $u(x, k)$ ,  $u(0, k) = 0$ , exists such that the origin is a globally asymptotically stable equilibrium point for the closed-loop system

$$x_{k+1} = F(x_k, u(x_k, k), k)$$

We will give a sufficient condition under which such a feedback exists. To do this we need the following notations. For a Lyapunov function  $V(x, k)$  which is  $C^2$  in  $x$  and  $T$ -periodic in  $k$  let  $\tilde{V} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}$  be defined by

$$\tilde{V}(x, u, k) = V(F(x, u, k), k+1) \quad (3)$$

$$\varphi(x, u, v, k) = \int_0^1 (1-t) v^T \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu, k) v \, dt \quad (4)$$

Set  $F_0(x, k) = F(x, 0, k)$ . For a fixed  $\eta > 0$ , let  $K_1(x, k)$  and  $K_2(x, k)$  be any  $T$ -periodic and continuous real valued functions satisfying for all  $(x, k) \in \mathbb{R}^n \times \mathbb{N}$ ,  $K_1(x, k) + K_2(x, k) \neq 0$  and

$$K_1(x, k) \geq \sup_{\|u\| \leq \eta, \|v\|=1} |\varphi(x, u, v, k)| \quad (5)$$

$$K_2(x, k) \geq \left\| \frac{\partial V}{\partial x}(F_0(x, k), k+1) \frac{\partial F}{\partial u}(x, 0, k) \right\| \quad (6)$$

and set

$$K(x, k) = \frac{\eta}{\eta K_1(x) + K_2(x)} \quad (7)$$

One can notice that  $K$  is  $T$ -periodic with respect to  $k$ . Now, we can state the main result of this section :

**Theorem 2** Assume that  $x_{k+1} = F_0(x_k, k)$  is stable and there exists a  $C^2$  Lyapunov function  $V(x, k)$   $T$ -periodic in  $k$  such that  $V$  and  $F_0$  satisfy the assumptions (1) and (2) of Theorem (1). If for all  $k \in \{0, \dots, T\}$  the set

$$\{x \in \mathbb{R}^n : V(\hat{F}_{0k}^{p+1}(x), p+1) = V(\hat{F}_{0k}^p(x), p), \\ \frac{\partial V}{\partial x}(\hat{F}_{0k}^{p+1}(x), p+1) \frac{\partial F}{\partial u}(\hat{F}_{0k}^p(x), 0, p) = 0, p \geq k\}$$

is reduced to the origin then, for any positive constant  $\eta$ , system (2) is globally asymptotically stabilizable by means of the continuous feedback law

$$u = -K(x, k) \left( \frac{\partial V}{\partial x}(F_0(x, k), k+1) \frac{\partial F}{\partial u}(x, 0, k) \right)^T$$

which satisfies

$$\begin{cases} \|u(x, k)\| \leq \eta & \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}, \\ u(0, k) = 0 & \forall k \in \mathbb{N}, \\ u(x, k+T) = u(x, k) & \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}. \end{cases}$$

Now, we apply the above theorem to the following nonlinear control system

$$x_{k+1} = x_k + \Phi(x_k, u_k) \quad (8)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $\Phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a  $C^2$  function satisfying  $\Phi(x, 0) = 0$ ,  $\forall x \in \mathbb{R}^n$ . It is known that if the map  $\Phi$  fails to be locally onto then (8) cannot be stabilized by means of continuous state feedback  $u_k = u(x_k)$ . In what follows we give a sufficient condition for system (8) to be stabilizable by a time-varying feedback  $u_k = u(x_k, k)$ .

**Theorem 3** Assume that for all  $1 \leq i, j \leq m$

$$\frac{\partial^2 \Phi_i}{\partial u_1 \partial u_j}(x, u) = 0, \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \quad (9)$$

and that there exist a  $C^2$  Lyapunov function  $V(x, k)$  and a continuous real valued function  $\alpha(x, k)$  which are  $T$ -periodic with respect to time  $k$  such that  $V$  and the function  $F_0$  defined by  $F_0(x, k) = x + \alpha(x, k)g_1(x)$  satisfy the conditions of theorem 2. Then, for any positive constant  $\eta$ , system (8) is globally asymptotically stabilizable by means of the continuous feedback

$$u(x, k) = \nu(x, k) + \tilde{u}(x, k) \quad (10)$$

$$\nu(x, k) = [\alpha(x, k), 0, \dots, 0]^T$$

$$\tilde{u}(x, k) = -K(x, k) \left[ \frac{\partial V}{\partial x}(F_0(x, k), k+1) \frac{\partial \Phi}{\partial u}(x, \nu(x, k)) \right]^T$$

where  $K(x, k)$  is got from (3)-(7) with

$$\tilde{V}(x, u, k) = V(x + \Phi(x, \nu(x, k) + u), k+1)$$

Since the conditions and the design of the control laws in theorem 3 make use of the functions  $V$  and  $\alpha$ , it is natural to look for particular systems of the form (8) for which  $V$  and  $\alpha$  can be explicitated. Setting

$$g(x) = (g_1(x), \dots, g_m(x)) = \frac{\partial \Phi}{\partial u}(x, 0)$$

it turns out, as in continuous-time (see [7]), that if

$$g_1 = \frac{\partial \Phi}{\partial u_1}(x, 0) = \frac{\partial}{\partial x_1} \quad (11)$$

it is actually possible to give an explicit design of  $V$  and  $\alpha$ , and by the way of the control laws.

Hereafter, for all  $x \in \mathbb{R}^n$ , set  $\bar{x} = (x_2, \dots, x_n)^T$  and for a fixed integer  $T \geq 2$  let

$$V(x, k) = \frac{1}{2} ([x_1 + h(\bar{x}, k)]^2 + \|\bar{x}\|^2)$$

$$\alpha(x, k) = -\frac{1}{2} [x_1 - h(\bar{x}, k)] - h(\bar{x}, k+1)$$

where  $h$  is a  $C^2$  function satisfying  $h(\bar{x}, k+T) = h(\bar{x}, k)$ ,  $\forall (\bar{x}, k) \in \mathbb{R}^{n-1} \times \mathbb{N}$  and  $h(0, k) = 0$ ,  $\forall k \in \mathbb{N}$ . A possible choice for  $h$  is  $h(\bar{x}, k) = \psi(\bar{x})H(k)$  where  $\psi$  is a definite function with respect to  $\bar{x}$ , and  $H(k)$  is a time periodic function with period  $T$ . So  $V$  and  $\alpha$  are  $T$ -periodic with respect to time  $k$ , and  $V$  vanishes if and only if  $x = 0$ . With these notations we have:

**Proposition 2** Assume that (9) and (11) hold. If there exists a function  $h$  as specified above such that for all  $\bar{x} \in \mathbb{R}^{n-1}$ ,

$$\text{rank}\{\tilde{g}_i(h(\bar{x}, k), \bar{x}), 2 \leq i \leq m, 0 \leq k \leq T-1\} = n-1$$

where  $\tilde{g}_i(x) = (0, g_i^2(x), \dots, g_i^n(x))$ , then, for any positive constant  $\eta$ , system (8) is globally asymptotically stabilizable by means of the feedback (10).

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