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# ON THE CONTROL OF AN EXPLOITED POPULATION OF FISH \*

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## Abstract

The aim of this paper is to show how some tools of control theory can be helpful to regulate the exploitation of a population of fish.

## 1 Introduction

In this paper, we are interested in the stabilization of an exploited population of fish around a non trivial steady state.

The dynamic of the population is supposed to be described by a discrete-time system of the form

$$x(t+1) = F(x(t), u(t)), \quad (1)$$

where  $x(t)$  is the state variable at time  $k = 0, 1, 2, \dots$  and  $u(t)$  is the control (here it is the fishing effort).

The problem addressed here is how to compute the fishing effort (as a feedback control)  $u(x)$  in such a way, that for a given state  $x^0 \neq 0$ , one has

- (i)  $F(x^0, u(x^0)) = x^0$  ( $x^0$  is an equilibrium point).
- (ii)  $x^0$  is a globally asymptotically stable equilibrium point for the closed-loop system

$$x(t+1) = F(x(t), u(x(t))).$$

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More precisely, we consider a density-dependent model of a population of exploited fish which is structured in  $n$  age classes [liu, mag, ric] :

$$\begin{cases} x_1(t+1) = f(\sum_{i=1}^n b_i x_i(t)) \\ x_2(t+1) = x_1(t) \exp(-M_1 - q_1 u(t)) \\ x_3(t+1) = x_2(t) \exp(-M_2 - q_2 u(t)) \\ \vdots \\ x_n(t+1) = x_{n-1}(t) \exp(-M_{n-1} - q_{n-1} u(t)) \end{cases} \quad (2)$$

Where:

- $b_i \geq 0$  is the number of individuals produced by individuals of the  $i^{th}$  age class.
- $M_i \geq 0$  is the natural mortality of individuals of age  $i$ .
- $q_i \geq 0$  is the catchability of individuals of age  $i$ .
- $u(t)$  is the fishing effort at time  $t$  and is regarded as an input.
- $f$  is the stock-recruitment function. It is a continuous function satisfying  $f(0) = 0$ .

Several authors have proposed different kind of functions  $f$  (see [bev, mag, ric]). We shall use in this paper the expression of  $f$  used in Beverton and Holt model [bev]

$$f(x) = \frac{x}{1 + \beta x}, \quad \beta > 0.$$

To construct the stabilizing feedback law, we shall use and adapt a machinery developed in [ben, igg].

## 2 Main result

For the sake of simplicity we shall give the result for  $n = 3$  (the calculus are exactly the same for an arbitrary  $n$  but the expression of the feedback is longer). We also suppose that only individuals of age  $n$  and over are reproductive ( $b_1 = b_2 = 0$ ). So that, we consider the following system

$$\begin{cases} x_1(t+1) = f(b_3 x_3(t)) \\ x_2(t+1) = x_1(t) \exp(-M_1 - q_1 u(t)) \\ x_3(t+1) = x_2(t) \exp(-M_2 - q_2 u(t)) \end{cases} \quad (3)$$

For a constant fishing effort  $u^0$ , system (3) has a non trivial equilibrium state

$$x_1^0 = \frac{b_3 a_1 a_2 - 1}{\beta b_3 a_1 a_2} = \frac{b_3 x_3^0}{1 + \beta b_3 x_3^0}, \quad x_2^0 = a_1 x_1^0, \quad x_3^0 = a_1 a_2 x_1^0$$

Where  $a_i = \exp(-M_i - q_i u^0)$ .

This steady state belongs to  $\Omega = \mathbb{R}_+^3$  provided that

$$b_3 > \frac{1}{a_1 a_2}.$$

**Theorem 2.1** *for any positive constant  $\eta \leq u^0$ , system (3) is globally asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = u^0 + v(x) \quad (4)$$

which satisfies

$$\|v(x)\| \leq \eta, \quad \forall x \in \Omega.$$

**Proof.** Let  $V$  be the following candidate Lyapunov function

$$V(x) = (x_1 - x_1^0)^2 + \frac{(x_2 - x_2^0)^2}{a_1^2} + \frac{(x_3 - x_3^0)^2}{(a_1 a_2)^2}$$

and define

$$\tilde{F}(x) = F(x, u^0). \quad (5)$$

$\tilde{V} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and

$$\tilde{V}(x, u) = V(F(x, u)), \quad (6)$$

We evaluate the variation of  $V$  along the closed-loop system (3-4):

$$\begin{aligned} \Delta V(x) &= V(F(x, u(x))) - V(x) \\ &= \tilde{V}(x, u(x)) - V(x) \\ &= \tilde{V}(x, u^0 + v(x)) - V(x) \\ &= \tilde{V}(x, u^0) - V(x) + \frac{\partial \tilde{V}}{\partial u}(x, u^0) v(x) \\ &\quad + \int_0^1 (1-t) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + t v(x)) v^2(x) dt. \end{aligned}$$

Notice that

$$\tilde{V}(x, u^0) = V(F(x, u^0)), \quad (7)$$

and

$$\frac{\partial \tilde{V}}{\partial u}(x, u^0) = \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0).$$

So,

$$\begin{aligned} \Delta V(x) &= V(F(x, u^0)) - V(x) \\ &\quad + \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0) v(x) \\ &\quad + \int_0^1 (1-t) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + t v(x)) v^2(x) dt. \end{aligned} \quad (8)$$

Now we shall construct a feedback control  $v(x)$  in order to get  $\Delta V(x) \leq 0$  for all  $x \in \Omega$ . To this end, we introduce some notations. Let  $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$\varphi(x, v, w) = \int_0^1 (1-t) v^T(x) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + t v(x)) w^2(x) dt. \quad (9)$$

For a fixed number  $\eta$  satisfying  $0 < \eta < u^0$ , let  $K_1(x)$  and  $K_2(x)$  be any nonnegative continuous real valued functions satisfying  $K_1(x) + K_2(x) \neq 0, \quad \forall x \in \Omega$  and

$$K_1(x) \geq \sup_{|v| \leq \eta, |w|=1} |\varphi(x, v, w)|, \quad \forall x \in \Omega. \quad (10)$$

$$K_2(x) \geq \left| \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0) \right|, \quad \forall x \in \Omega. \quad (11)$$

and set

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)} > 0 \quad \forall x \in \Omega. \quad (12)$$

We construct the feedback control according to the following formula :

$$v(x) = -K(x) \left( \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0) \right) \quad (13)$$

which satisfies

$$\|v(x)\| \leq \eta, \quad \forall x \in \Omega. \quad (14)$$

Tacking into account (8-13-9), the variation of  $V$  along the solutions of the closed-loop system can be written :

$$\begin{aligned} \Delta V(x) &= V(F(x, u^0)) - V(x) \\ &\quad - \frac{1}{K(x)} v^2(x) + \varphi(x, v(x), v(x)). \end{aligned} \quad (15)$$

On the one hand, we have

$$\begin{aligned} V(F(x, u^0)) - V(x) &= (f(b_3 x_3) - x_1^0)^2 - \frac{(x_3 - x_3^0)^2}{(a_1 a_2)^2} \\ &= \left( \frac{b_3 x_3}{1 + \beta b_3 x_3} - x_1^0 \right)^2 - \left( \frac{x_3 - x_3^0}{a_1 a_2} \right)^2 \\ &= \left( \frac{b_3 x_3}{1 + \beta b_3 x_3} - \frac{b_3 x_3^0}{1 + \beta b_3 x_3^0} \right)^2 - \left( \frac{x_3 - x_3^0}{a_1 a_2} \right)^2 \end{aligned}$$

Hence

$$\begin{aligned}
 V(F(x, u^0)) - V(x) &= \\
 &= \left( \frac{b_3(x_3 - x_3^0)}{(1 + \beta b_3 x_3)(1 + \beta b_3 x_3^0)} \right)^2 - \left( \frac{x_3 - x_3^0}{a_1 a_2} \right)^2 \\
 &= \frac{(b_3 a_1 a_2)^2 - (1 + \beta b_3 x_3)^2 (1 + \beta b_3 x_3^0)^2}{(a_1 a_2)^2 (1 + \beta b_3 x_3)^2 (1 + \beta b_3 x_3^0)^2} (x_3 - x_3^0)^2
 \end{aligned}$$

We have  $1 + \beta b_3 x_3^0 = 1 + \beta b_3 \frac{b_3 a_1 a_2 - 1}{\beta b_3 a_1 a_2} a_1 a_2 = b_3 a_1 a_2$ . This yields,

$$\frac{V(F(x, u^0)) - V(x)}{(1 + \beta b_3 x_3)^2 (1 + \beta b_3 x_3^0)^2} b_3^2 (x_3 - x_3^0)^2 \leq 0 \quad (16)$$

On the other hand,  $\varphi(x, v, w)$  being homogeneous of degree 2 with respect to  $w$ , we have for all  $x \in \Omega$  such that  $v(x) \neq 0$ :

$$\varphi(x, v(x), v(x)) = v^2(x) \varphi(x, v(x), \frac{v(x)}{|v(x)|})$$

From this and (15-16), we get

$$\Delta V(x) = V(F(x, u^0)) - V(x) \leq 0 \text{ if } v(x) = 0.$$

And for  $v(x) \neq 0$ ,

$$\begin{aligned}
 \Delta V(x) &= V(F(x, u^0)) - V(x) \\
 &= -v^2(x) \left( \frac{1}{K(x)} - \varphi(x, v(x), \frac{v(x)}{|v(x)|}) \right).
 \end{aligned}$$

Thanks to the construction of  $v(x)$  and  $K(x)$ , we have  $\frac{1}{K(x)} > \varphi(x, v(x), \frac{v(x)}{|v(x)|})$ . This allows to conclude that

$$\Delta V(x) \leq 0 \quad \forall x \in \Omega.$$

The closed-loop system is then Lyapunov stable. On the other hand,

$$\Delta V(x) = 0 \Leftrightarrow x_3 = x_3^0 \text{ and } v(x) = 0.$$

It is easy to show that the largest invariant set contained in

$$\{x \in \Omega \mid \Delta V(x) = V(F(x, u(x))) - V(x) = 0\}$$

is reduced to  $\{x^0\}$  so, by Lasalle Invariance Principle [las], the equilibrium  $x^0$  is a globally asymptotically stable equilibrium point for the closed-loop system.

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