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Jean-Claude Vivalda

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# On the regulation of the state of a harvested fish population <sup>1</sup>

M. El Bagdouri<sup>‡</sup>   A. Iggidr<sup>‡b2</sup>   A. Ouahbi<sup>‡</sup>   J.C. Vivalda<sup>‡b</sup>

<sup>b</sup>CONGE Project

INRIA Lorraine & CNRS UPRESA 7035

I.S.G.M.P. Bat. A, Ile du Saulcy, 57045 Metz cedex 01, FRANCE.

e-mail: {iggidr, vivalda}@loria.fr

<sup>‡</sup>Laboratoire d'Automatique et d'Etude des Procédés

Faculté des Sciences-Semlalia, Université Cadi Ayyad, B.P. S/15, Marrakesh, MOROCCO.

e-mail: elbagdouri@ucam.ac.ma

## Abstract

The goal of this paper is to apply some tools of control theory to an age-structured fishery model in order to stabilize an exploited fish population around a nontrivial steady state.

## 1 Introduction

In this paper, we consider a population of exploited fish which is structured in  $n$  age classes ( $n \geq 2$ ); under some assumptions on the population, we can represent the dynamics of the population by the following system of difference equations [6]:

$$\begin{cases} x_1(t+1) = f(\sum_{i=1}^n b_i x_i(t)), \\ x_2(t+1) = x_1(t) \exp(-M_1 - q_1 u(t)), \\ \vdots \\ x_n(t+1) = x_{n-1}(t) \exp(-M_{n-1} - q_{n-1} u(t)). \end{cases} \quad (1)$$

Where:  $b_i$  is the number of individuals produced by individuals of the  $i^{\text{th}}$  age class,  $M_i$  is the natural mortality of the individuals of the  $i^{\text{th}}$  age class,  $q_i$  is the catchability of the individuals of the  $i^{\text{th}}$  age class,  $u(t)$  is the fishing effort at time  $t$  and is regarded as an input, and  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is the stock-recruitment function. It is a continuous map such that  $f(0) = 0$ . We shall use in this paper the expression of  $f$  used in Beverton and Holt model [2]  $f(x) = \frac{x}{1+\beta x}$ ,  $\beta > 0$ . The dynamic of the population can be written as a discrete-time control system of the form

$$\begin{cases} x(t+1) = F(x(t), u(t)), \\ u(t) \in \mathbb{R}_+, \\ x(t) \in \Omega = \mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0\}. \end{cases} \quad (2)$$

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<sup>2</sup>Corresponding author.

where  $x(t)$  is the state variable at time  $t = 0, 1, 2, \dots$ ,  $u(t)$  is the control (here it is the fishing effort).  $F: \Omega \times \mathbb{R}_+ \rightarrow \Omega$  is a continuous function. We focus attention on the following problem: Is it possible to compute the fishing effort (as a feedback control)  $u(x)$  in such a way, that for a given state  $x^0 \neq 0$ , one has :

- (i)  $F(x^0, u(x^0)) = x^0$  ( $x^0$  is an equilibrium point).
- (ii)  $x^0$  is a globally asymptotically stable equilibrium point for the closed-loop system  $x(t+1) = F(x(t), u(x(t)))$ .

The considered system is nonlinear so the classical linear control techniques can not be used (unless to get local results but here we are interested in global stabilization by means of a nonnegative feedback control). Thus, we shall use a discrete version of Jurdjevic-Quinn theorem [3] that has been derived in ([1],[5]).

## 2 Main result

For a constant fishing effort  $u^0$ , system (1) has a non trivial equilibrium state  $x^0$  whose coordinates are:

$$x_1^0 = \frac{K-1}{\beta K}, \text{ and for } j \geq 2, x_j^0 = \left( \prod_{i=1}^{j-1} a_i \right) x_1^0.$$

where  $K = b_1 + a_1 b_2 + a_1 a_2 b_3 + \dots + a_1 a_2 \dots a_{n-1} b_n$  and  $a_i = e^{-M_i - q_i u^0}$ .

This steady state belongs to  $\Omega$  provided that  $K > 1$ .

**Proposition 2.1** *for any positive constant  $\eta \leq u^0$ , system (1) is globally asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = u^0 + v(x) \quad (3)$$

*which satisfies*

$$\|v(x)\| \leq \eta, \quad \forall x \in \Omega.$$

**Remark 2.1** The condition  $\eta \leq u^0$  is needed only to ensure that  $u(x) \geq 0$  for all  $x \in \Omega$ . The feedback  $v(x)$  will be explicitly constructed in the proof.

**Proof.** For the sake of clarity, we shall prove this result for  $n = 3$  (the proof is the same for arbitrary  $n$  but the calculus are longer and more complicated).

Let  $V$  be the following candidate Lyapunov function  $V(x) = (x_1 - x_1^0)^2 + \frac{K-b_1}{K} \left( \frac{x_2-x_2^0}{a_1} \right)^2 + \frac{a_1 a_2 b_3}{K} \left( \frac{x_3-x_3^0}{a_1 a_2} \right)^2$ .  $V$  satisfies:  $V(x^0) = 0$  and  $V(x) > 0$  for all  $x \neq x^0$ . Let  $\tilde{F}(x) = F(x, u^0)$ ,

$$V(\tilde{F}(x)) = (f(\sum b_i x_i) - x_1^0)^2 + \frac{K-b_1}{K} (x_1 - x_1^0)^2 + \frac{a_1 a_2 b_3}{K} \left( \frac{x_2-x_2^0}{a_1} \right)^2.$$

We have

$$f(\sum b_i x_i) - x_1^0 = \frac{\sum b_i (x_i - x_i^0)}{(1 + \beta \sum b_i x_i)(1 + \beta \sum b_i x_i^0)}$$

Since  $1 + \beta \sum b_i x_i \geq 1$  and  $1 + \beta \sum b_i x_i^0 = K$ , we get

$$|f(\sum b_i x_i) - x_1^0| \leq \frac{|\sum b_i (x_i - x_i^0)|}{K}$$

Hence,

$$\begin{aligned} V(\tilde{F}(x)) &\leq \left( \frac{\sum b_i (x_i - x_i^0)}{K} \right)^2 \\ &\quad + \frac{K-b_1}{K} (x_1 - x_1^0)^2 + \frac{a_1 a_2 b_3}{K} \left( \frac{x_2 - x_2^0}{a_1} \right)^2 \\ &\leq \left( \frac{b_1}{K} (x_1 - x_1^0) + \frac{a_1 b_2}{K} \frac{x_2 - x_2^0}{a_1} + \frac{a_1 a_2 b_3}{K} \frac{x_3 - x_3^0}{a_1 a_2} \right)^2 \\ &\quad + \frac{K-b_1}{K} (x_1 - x_1^0)^2 + \frac{a_1 a_2 b_3}{K} \left( \frac{x_2 - x_2^0}{a_1} \right)^2. \end{aligned}$$

Using the convexity of the map  $x \mapsto x^2$  and the fact that  $K = b_1 + a_1 b_2 + a_1 a_2 b_3$ , we get  $V(\tilde{F}(x)) \leq V(x)$ . Hence, the equilibrium  $x^0$  is Lyapunov stable for the system  $x(t+1) = F(x(t), u^0) = \tilde{F}(x(t))$ . Now, using ([4], Th. 4.1), we shall compute  $v(x)$ . We define  $\tilde{V}(x, u) = V(F(x, u))$ . Using the Taylor expansion formula, we evaluate the variation of  $V$  along the solutions of the closed-loop system  $x(t+1) = F(x(t), u^0 + v(x))$  in the following way:

$$\begin{aligned} \Delta V(x) &= V(F(x, u^0 + v(x))) - V(x) \\ &= V(\tilde{F}(x)) - V(x) + v(x) \frac{\partial V}{\partial x}(\tilde{F}(x)) \frac{\partial F}{\partial u}(x, u^0) \\ &\quad + v^2(x) \underbrace{\int_0^1 (1-t) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + tv(x)) dt}_{\varphi(x, v)}. \end{aligned}$$

For a fixed number  $\eta$  satisfying  $0 < \eta \leq u^0$ , let  $\lambda_1(x)$  and  $\lambda_2(x)$  be any nonnegative continuous real valued functions satisfying  $\lambda_1(x) + \lambda_2(x) \neq 0$ ,  $\forall x \in \Omega$  and

$$\lambda_1(x) \geq \sup_{|v| \leq \eta} |\varphi(x, v)|, \lambda_2(x) \geq \left| \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0) \right|$$

and set  $\lambda(x) = \frac{\eta}{\eta \lambda_1(x) + \lambda_2(x)}$ . We construct the feedback control according to the following formula :

$$v(x) = -\lambda(x) \left( \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0) \right) \quad (4)$$

which satisfies

$$|v(x)| \leq \eta, \quad \forall x \in \Omega. \quad (5)$$

With this feedback and the expression of  $\varphi$ , the variation of  $V$  along the solutions of the closed-loop system can be written :

$$\Delta V = V(\tilde{F}(x)) - V(x) - v^2(x) \left( \frac{1}{\lambda(x)} - \varphi(x, v(x)) \right). \quad (6)$$

Taking into account the definition of  $\lambda$ ,  $\lambda_1$ , and  $\lambda_2$  we have

$$\frac{1}{\lambda(x)} - \varphi(x, v(x)) \geq \frac{\lambda_2(x)}{\eta} \geq 0.$$

Hence,

$$\Delta V(x) = V(F(x, u^0 + v(x))) - V(x) \leq 0 \quad \forall x \in \Omega.$$

The closed-loop system is then Lyapunov stable and since  $V$  is proper, all the solutions are bounded. It remains to show the attractivity of the equilibrium  $x^0$ . Let  $L$  be the largest invariant set contained in  $\{x \in \Omega \mid \Delta V(x) = V(F(x, u(x))) - V(x) = 0\}$ . By Lasalle Invariance Principle, all the solutions tend to  $L$ . Here, we have

$$\Delta V(x) = 0 \Rightarrow v(x) = 0 \Rightarrow x_1 = x_1^0 \text{ and } x_2 = x_2^0.$$

It is easy to see that  $L = \{x^0\}$  which ends the proof.

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