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# MULTIGRADED CAYLEY-CHOW FORMS

BRIAN OSSERMAN AND MATTHEW TRAGER

ABSTRACT. We introduce a theory of multigraded Cayley-Chow forms associated to subvarieties of products of projective spaces. Two new phenomena arise: first, the construction turns out to require certain inequalities on the dimensions of projections; and second, in positive characteristic the multigraded Cayley-Chow forms can have higher multiplicities. The theory also provides a natural framework for understanding multifocal tensors in computer vision.

## 1. INTRODUCTION

Let  $X \subseteq \mathbb{P}^n$  be a projective variety of dimension  $r$  and degree  $d$ . The set of all linear spaces of dimension  $n - r - 1$  meeting  $X$  is a hypersurface  $Z_X$  in the Grassmannian  $\text{Gr}(n - r - 1, n)$ . Any such hypersurface can be written as the zero set inside the Grassmannian of a polynomial  $H_X$  in the Plücker coordinates, which turns out to also be of degree  $d$ . The polynomial  $H_X$  is known as the “Chow form” or “Cayley form” of  $X$ , and we will refer to it as the **Cayley-Chow form**. From the Cayley-Chow form  $H_X$  we immediately recover the hypersurface  $Z_X$ , and one can then recover  $X$  as the set of points  $P \in \mathbb{P}^n$  such that every  $(n - r - 1)$ -dimensional linear space containing  $P$  corresponds to an element of  $Z_X$ . Thus, the Cayley-Chow form can be used to encode subvarieties of projective space, and this classical construction has played an important role in moduli space theory, especially in the guise of Chow varieties, but also for instance in Grothendieck’s original construction of Quot and Hilbert schemes [Gro61]. See §2 of Chapter 3 of [GKZ94] for a presentation of this material.

The purpose of this paper is to generalize this classical theory to the case of subvarieties of products of projective spaces. We find that the generalization displays some interesting properties, particularly that certain dimensional inequalities have to be satisfied in order for it to work. We give necessary and sufficient conditions for the generalized theory to go through. Moreover, we explain that in cases where the necessary inequalities are not satisfied, the failure of the theory can in fact shed light on previously observed phenomena in computer vision. In addition, positive-characteristic phenomena arise in our more general setting, causing the Cayley-Chow form to sometimes have higher multiplicities.

Before we state our main results, we need to recall the notion of **multidegree** of a subvariety  $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ . If  $X$  has codimension  $c$ , we can represent its

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multidegree as a homogeneous polynomial

$$\sum_{\gamma=(\gamma_1, \dots, \gamma_k)} a_\gamma t_1^{\gamma_1} \cdots t_k^{\gamma_k}$$

of degree  $c$ , where  $a_\gamma$  is determined as the number of points of intersection (counting multiplicity) of  $X$  with  $L_1 \times \cdots \times L_k$ , where each  $L_i$  is a general linear space of dimension  $\gamma_i$ . The multidegree is also equivalent to the Chow class of  $X$ , although we will not need this.

We can summarize our main results in the following theorem:

**Theorem 1.1.** *Let  $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$  be a projective variety of dimension  $r$ , and suppose that  $X$  is not of the form  $X' \times \prod_{i \notin I} \mathbb{P}^{n_i}$  for any  $I \subsetneq \{1, \dots, k\}$  and  $X' \subseteq \prod_{i \in I} \mathbb{P}^{n_i}$ . Given also  $\beta = (\beta_1, \dots, \beta_k)$  with  $0 \leq \beta_i \leq n_i$  for  $i = 1, \dots, k$  and  $\sum_{i=1}^k \beta_i = r + 1$ , write  $\alpha_i = n_i - \beta_i$  for each  $i$ .*

*Consider the closed subset*

$$Z_{X,\beta} = \{(L_1, \dots, L_k) : X \cap (L_1 \times \cdots \times L_k) \neq \emptyset\} \subseteq \text{Gr}(\alpha_1, n_1) \times \cdots \times \text{Gr}(\alpha_k, n_k).$$

*Then  $Z_{X,\beta}$  is a hypersurface determining  $X$  if and only if for every nonempty  $I \subsetneq \{1, \dots, k\}$  we have*

$$\dim p_I(X) \geq \sum_{i \in I} \beta_i,$$

*where  $p_I(X)$  denotes the projection of  $X$  onto  $\prod_{i \in I} \mathbb{P}^{n_i}$ .*

*Assuming the above inequalities are satisfied, we have that  $Z_{X,\beta}$  is the zero set of a single multihomogeneous polynomial  $F_X$  under the product of the Plücker embeddings of the  $\text{Gr}(\alpha_i, n_i)$ . There is a multiplicity  $\epsilon_{X,\beta} \geq 1$  such that if we write*

$$H_{X,\beta} := F_X^{\epsilon_{X,\beta}},$$

*and  $X$  has multidegree*

$$\sum_{\gamma} a_\gamma t_1^{\gamma_1} \cdots t_k^{\gamma_k},$$

*then the multidegree of  $H_{X,\beta}$  (as a multihomogeneous polynomial) is given by*

$$(a_{\alpha_1+1, \alpha_2, \dots, \alpha_k}, \dots, a_{\alpha_1, \dots, \alpha_{k-1}, \alpha_k+1}).$$

*Finally, in characteristic 0, we always have  $\epsilon_{X,\beta} = 1$ .*

In fact, we first show in Proposition 3.1 that  $Z_{X,\beta}$  is a hypersurface if and only if the slightly weaker inequalities (3.1) are satisfied, and then in Proposition 3.6 that when  $Z_{X,\beta}$  is a hypersurface, it determines  $X$  uniquely if and only if the above inequalities are satisfied. The condition that  $X$  not be a product with any of the projective spaces is just to simplify the statement; see Example 3.5 below. The multiplicity  $\epsilon_{X,\beta}$  is defined naturally in Definition 5.1 as the degree of the map from an incidence correspondence to  $Z_{X,\beta}$ , and in positive characteristic, it may be strictly greater than 1; see Example 5.13 below. The remaining statements of the theorem are proved in Corollary 4.2, Proposition 5.3, and Corollary 5.10. An interesting aspect of Proposition 3.6 in comparison to the classical case is that if we define the set  $S_Z$  to consist of points  $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$  with the property that every  $L_1 \times \cdots \times L_k$  containing  $P$  must lie in  $Z_{X,\beta}$ , then  $X \subseteq S_Z$  always, but we will have in general that  $S_Z$  contains additional components.

In addition, while we have stated the fundamental inequalities in terms of the dimensions of projections of  $X$ , according to work of Castillo, Li and Zhang [CLZ]

this may be reinterpreted with an equivalent formulation in terms of the multidegree of  $X$ . We carry this out in Corollary 5.11 below. Thus, while we do require non-trivial conditions on  $X$  in order for our theory to apply, when it applies it does so uniformly to all  $X$  of given multidegree.

It is natural to wonder how our construction compares to applying a Segre embedding together with the classical construction; we discuss this briefly in Remark 5.12 below.

We next make the following observation.

*Remark 1.2.* The inequalities given in Theorem 1.1 can only be satisfied if  $k \leq r+1$ , since otherwise the set of  $i$  such that  $\beta_i \neq 0$  is necessarily proper in  $\{1, \dots, k\}$ , and will violate the necessary inequality.

We can now explain the relationship to computer vision, and specifically to the reconstruction of a configuration of unknown cameras. A basic model for a (positioned) camera is as a linear projection from the three-dimensional world to the two-dimensional film/sensor plane, which we consider as a  $\mathbb{P}^3$  and a  $\mathbb{P}^2$ , respectively. Thus, a  $k$ -tuple of cameras corresponds to an  $k$ -tuple of linear projections, which together induce a rational map

$$\mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^k.$$

The closure of the image of this map is then a three-dimensional subvariety of  $(\mathbb{P}^2)^k$ , which is called the “multiview variety.” This can be thought of as describing which  $k$ -tuples of points in  $\mathbb{P}^2$  could come from a single point in  $\mathbb{P}^3$ . Knowing the multiview variety is equivalent to knowing the camera configuration, at least up to change of ‘world coordinates’ in  $\mathbb{P}^3$ . It is well known in computer vision that for  $k = 2, 3, 4$ , there exists a  $k$ -tensor which determines the camera configuration, called the “multifocal tensor.” It is equally well known that this construction does not extend to  $k > 4$ .

The multifocal tensor is described in terms of incidences with  $k$ -tuples of linear spaces, and in fact this inspired our construction. On the other hand, we can reinterpret the theory of multifocal tensors in terms of Theorem 1.1 as follows. First, Aholt, Sturmfels and Thomas showed in Corollary 3.5 of [AST13] that all the coefficients of the multidegree of a multiview variety are equal to 1, so we see that when our Cayley-Chow form construction applies, the result is a multilinear polynomial in  $k$  variable sets, which is to say, a  $k$ -tensor. It is routine to check that for  $k \leq 4$ , our construction does apply for suitable choice of  $\beta$ , and the Cayley-Chow form coming from the multiview variety is precisely the multifocal tensor. Conversely, if  $k \geq 5$ , then Remark 1.2 implies that no analogous construction exists for any choice of  $\beta$ . See Examples 5.14 and 5.15 for further details. Beyond giving a new point of view on these known constructions, we also hope that Theorem 1.1 will provide new applications in computer vision, in the context of generalized cameras. Recent work of Ponce, Sturmfels and the second author [PST17], and of Escobar and Knutson [EK17] develops a theory of configurations of such cameras, including multidegree-type formulas, and we expect that Theorem 1.1 will provide a generalization of multifocal tensors to this setting, where the tensors will be replaced with higher-degree forms.

*Remark 1.3.* A different connection is to the notion of circuit polynomials in matroid theory, which we now describe. First observe that when all  $n_i$  are equal to 1, we

must have  $\beta_i = 1$  for some subset  $S$  of  $I$  of size  $r + 1$ , and  $\beta_i = 0$  for  $i \notin S$ . The inequalities of Theorem 1.1 are never satisfied except in the trivial case that  $r = k - 1$ , and  $\beta_i = 1$  for all  $i$ , so that  $H_{X,\beta}$  will simply recover the defining polynomial of  $X$ . However, the weaker inequalities (3.1) will be satisfied more generally: specifically, whenever we have  $\dim p_I(X) = r$  for the  $I$  as above. Thus, in this case we can still define our multigraded Cayley-Chow form, although it will not suffice to recover  $X$ . This special case of our construction turns out to be connected to algebraic matroids.

Specifically, one approach to algebraic matroids is as follows: in order to construct a matroid on  $\{1, \dots, k\}$  of rank  $r$ , choose a variety  $X \subseteq \mathbb{A}^k = (\mathbb{A}^1)^k$  of dimension  $r$ , with prime ideal  $\mathfrak{p} \subseteq K[x_1, \dots, x_k]$ . Define independent sets by algebraic independence of the images of the  $x_i$  modulo  $\mathfrak{p}$ . In this context, a circuit  $C \subseteq \{1, \dots, k\}$  will have the property that the closure of the projection of  $X$  to  $\prod_{i \in C} \mathbb{A}^1$  has codimension 1, and hence is cut out by a single polynomial  $H$  in  $r + 1$  variables. This polynomial is called the ‘‘circuit polynomial,’’ and precisely cuts out the closure of the locus of  $(P_i)_{i \in C} \in (\mathbb{A}^1)^{r+1}$  such that there exists  $(P_i)_{i \notin C}$  with  $(P_1, \dots, P_k) \in X$ . See §5 of Király-Rosen-Theren [KRT13]. Replacing  $X$  by its closure in  $(\mathbb{P}^1)^k$  and  $H$  by its multihomogenization, we recover the Cayley-Chow form construction with  $\beta_i = 1$  for  $i \in C$  and  $\beta_i = 0$  otherwise (at least, up to omission of  $\epsilon_{X,\beta}$ ).

Finally, we mention that the inequalities arising both in Theorem 1.1 and in the condition for  $Z_{X,\beta}$  to be a hypersurface (see Proposition 3.1 below) are closely related to concepts arising in polymatroid theory; see Remark 2.4 below.

**Acknowledgements.** We would like to thank Bernd Sturmfels for bringing to our attention various connections to the literature, particularly the notion of matroid circuit polynomials.

**Conventions.** We work throughout over an algebraically closed field  $K$ . A variety is always assumed irreducible.

Given  $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{Z}^k$ , we will write  $|\beta| := \sum_{i=1}^k \beta_i$ , and for  $I \subseteq \{1, \dots, k\}$ , we will write  $\beta_I := (\beta_i)_{i \in I}$ , and  $|\beta_I| := \sum_{i \in I} \beta_i$ . We also write  $I^c := \{1, \dots, k\} \setminus I$ .

## 2. MULTIDEGREES AND DIMENSIONS OF PROJECTIONS

We begin by collecting some background results on the relationship between multidegree and dimensions of projections. If we have a subvariety  $X$  of  $\prod_{i=1}^k \mathbb{P}^{n_i}$  with multidegree

$$\sum_{\gamma} a_{\gamma} t_1^{\gamma_1} \cdots t_k^{\gamma_k},$$

we say the **support** of the multidegree is the set of  $\gamma$  for which  $a_{\gamma} \neq 0$ . Note that by definition, this is contained in the subset of  $\gamma$  with  $\gamma_i \geq 0$  for all  $i$ , and  $\sum_i \gamma_i = \text{codim } X$ .

The main theorem of [CLZ] asserts:

**Theorem 2.1** (Castillo-Li-Zhang). *If*

$$X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

is irreducible, the support of its multidegree is

$$\left\{ \gamma : \sum_i (n_i - \gamma_i) = \dim X, \text{ and } \sum_{i \in I} (n_i - \gamma_i) \leq \dim p_I(X) \quad \forall I \subsetneq \{1, \dots, k\} \right\},$$

where  $p_I$  denotes projection onto the product of the subset of the  $\mathbb{P}^n$ s indexed by  $I$ .

Moreover, the function  $\delta(I) = \dim p_I X$  satisfies the following conditions:

- $\delta(\emptyset) = 0$ ;
- for  $I \subseteq J$  we have  $\delta(I) \leq \delta(J)$ ;
- and for any  $I, J$  we have  $\delta(I \cap J) + \delta(I \cup J) \leq \delta(I) + \delta(J)$ .

In fact, they treat the case that all  $n_i$  are equal, but one reduces immediately to this case by linearly embedding each  $\mathbb{P}^{n_i}$  into a larger projective space of fixed dimension. Note that the first part of the theorem is equivalent to saying that a general choice of  $L_i$  of dimension  $\gamma_i$  will yield  $X \cap (L_1 \times \dots \times L_k) \neq \emptyset$  if and only if  $\sum_{i \in I} (n_i - \gamma_i) \leq \dim p_I(X)$  for all  $I \subsetneq \{1, \dots, k\}$ . The second part of the theorem connects multidegrees to polymatroid theory, and says in particular that the function  $\delta$  is “submodular.”

The first part of the theorem implies that the dimensions of the projections determine the support of the multidegree. We see using some standard facts in polymatroid theory that the converse also holds.

**Corollary 2.2.** *Given*

$$X \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$$

*irreducible, the data of the support of the multidegree of  $X$  is equivalent to the data of the dimensions of  $p_I(X)$  for all  $I \subseteq \{1, \dots, k\}$ .*

*Proof.* Let  $\mathcal{P}$  be the polytope cut out by the inequalities of Theorem 2.1, and  $\bar{\mathcal{P}}$  the face of  $\mathcal{P}$  cut out by the hyperplane  $\sum_i (n_i - \gamma_i) = \dim X$ . Then  $\mathcal{P}$  is known as a “polymatroid”, and  $\bar{\mathcal{P}}$  is the set of “bases;” see §1 of [HH02]. Moreover, from Proposition 1.3 of [HH02] we see that the vertices of  $\bar{\mathcal{P}}$  are integral, and for every  $I$  there is some vertex lying in the corresponding bounding hyperplane (in their notation, we take any  $\pi$  such that  $I = \{i_1, \dots, i_{|I|}\}$ ). It follows that we can recover the  $\dim p_I(X)$  from the integral points of  $\bar{\mathcal{P}}$ . Since Theorem 2.1 says that the support of the multidegree is equal to the set of lattice points in  $\bar{\mathcal{P}}$ , we conclude that it determines the  $\dim p_I(X)$ , as desired.  $\square$

The following standard fact from polymatroid theory will also be helpful. Since the proof is quite short, we include it.

**Proposition 2.3.** *Suppose we have a function  $\delta$  from subsets of  $\{1, \dots, k\}$  to  $\mathbb{Z}_{\geq 0}$  satisfying the conditions in Theorem 2.1, and write  $r = \delta(\{1, \dots, k\})$ . Suppose also that we are given  $\beta = (\beta_1, \dots, \beta_k) \in (\mathbb{Z}_{\geq 0})^k$  with  $|\beta| = r + 1$ , and satisfying that for all  $I \subseteq \{1, \dots, k\}$ , we have*

$$(2.1) \quad |\beta_I| \leq \delta(I) + 1.$$

*Then there exists a nonempty  $J \subseteq \{1, \dots, k\}$  such that for all  $I \subseteq \{1, \dots, k\}$ , we have*

$$(2.2) \quad |\beta_I| = \delta(I) + 1$$

*if and only if  $I \supseteq J$ .*

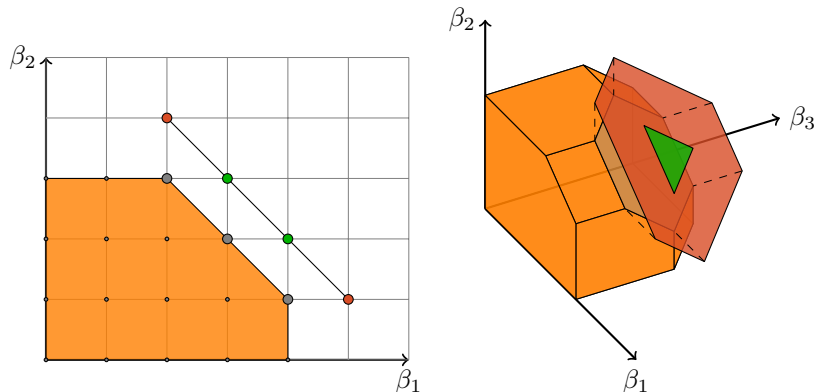


FIGURE 1. Two polymatroids. The sets of bases (corresponding to our multidegree supports) are in gray; while the sets of circuits and of non-circuit 1-deficient vectors (both satisfying  $|\beta| = r + 1$ ) are in green and red, respectively.

*Proof.* First note that if such a  $J$  exists, it is necessarily nonempty, since we have assumed  $\delta(\emptyset) = 0$ . Now, if we have  $I_1$  and  $I_2$  satisfying (2.2), then we see that

$$\begin{aligned} \delta(I_1 \cap I_2) &\leq \delta(I_1) + \delta(I_2) - \delta(I_1 \cup I_2) \\ &\leq |\beta_{I_1}| - 1 + |\beta_{I_2}| - 1 - |\beta_{I_1 \cup I_2}| - 1 \\ &= |\beta_{I_1 \cap I_2}| - 1, \end{aligned}$$

so  $I_1 \cap I_2$  also satisfies (2.2). The result follows.  $\square$

*Remark 2.4.* In polymatroid theory, there is a notion of “1-deficient” vectors, and among vectors  $\beta = (\beta_1, \dots, \beta_k)$  satisfying the conditions that  $|\beta| = r + 1$ , those satisfying (2.1) are precisely the 1-deficient vectors. This is exactly the condition that arises for us in order for  $Z_{X, \beta}$  to be a hypersurface – see Proposition 3.1 below. Now, given such a  $\beta$ , note that the  $J$  of Proposition 2.3 is equal to all of  $\{1, \dots, k\}$  if and only if  $\beta$  satisfies the stronger inequalities of Theorem 1.1. This requires that all  $\beta_i$  are strictly positive, and if we restrict our attention to  $\beta$  with all  $\beta_i > 0$ , then the condition that  $J = \{1, \dots, k\}$  is equivalent in the polymatroid language to saying that  $\beta$  is a “circuit.” Thus, among the vectors  $\beta$  with  $|\beta| = r + 1$  and all  $\beta_i$  strictly positive, the set of vectors satisfying the inequalities of Theorem 1.1 is exactly the set of circuits of the polymatroid determined by the multidegree of  $X$ . See Figure 1 for examples, and §1.2 of [MPS07] for more on the polymatroid terminology.

### 3. SLICING BY PRODUCTS OF LINEAR SPACES

In this section, we carry out our fundamental analysis of the behavior of slicing with products of general linear spaces. We begin with the following.

**Proposition 3.1.** *Let  $X \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$  be a projective variety of dimension  $r$ , and suppose we have  $\beta = (\beta_1, \dots, \beta_k)$  with  $0 \leq \beta_i \leq n_i$  for  $i = 1, \dots, k$  and  $|\beta| = r + 1$ . Write  $\alpha_i = n_i - \beta_i$  for each  $i$ .*

Consider the closed subset

$$Z_{X,\beta} = \{(L_1, \dots, L_k) : X \cap (L_1 \times \dots \times L_k) \neq \emptyset\} \subseteq \text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k).$$

Then  $Z_{X,\beta}$  is a hypersurface if and only if for every nonempty  $I \subseteq \{1, \dots, k\}$  we have

$$(3.1) \quad \dim p_I(X) \geq |\beta_I| - 1,$$

where  $p_I(X)$  denotes the projection of  $X$  onto  $\prod_{i \in I} \mathbb{P}^{n_i}$ .

*Remark 3.2.* Since  $|\beta| = r + 1$ , we have that (3.1) is equivalent to having  $r - \dim p_I(X) \leq |\beta_{I^c}|$ , where  $I^c = \{1, \dots, k\} \setminus I$ . Hence, when the conditions from the previous Proposition are satisfied, we have that  $r - \dim p_{I^c}(X) \leq |\beta_I| \leq \dim p_I(X) + 1$ . The former inequality has the geometric interpretation that the generic fiber of  $X$  under  $p_{I^c}$  has dimension at most  $|\beta_I|$ .

*Proof.* Define the incidence correspondence  $V_X \subseteq X \times \text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k)$  given by

$$(3.2) \quad V_X = \{(P, L_1, \dots, L_k) : P \in L_1 \times \dots \times L_k\}$$

Thus,  $Z_{X,\beta}$  is the image of  $V_X$  under projection to the product of Grassmannians. Considering the projection of  $V_X$  to  $X$ , we see that  $V_X$  is irreducible of dimension  $d_1 = r + \sum_{i=1}^k \alpha_i(n_i - \alpha_i)$ . In particular,  $Z_{X,\beta}$  is automatically irreducible. The dimension of the product of Grassmannians is  $d_2 = \sum_{i=1}^k (\alpha_i + 1)(n_i - \alpha_i)$ . Because  $d_2 - d_1 = 1$ , we will have that  $Z_{X,\beta}$  is a hypersurface if and only if  $V_X$  has generic fiber dimension 0 under the projection to the product of Grassmannians, or equivalently, if there exist  $L_1, \dots, L_k$  such that  $X \cap (L_1 \times \dots \times L_k)$  is finite and nonempty.

First suppose that we have  $\dim p_I(X) < |\beta_I| - 1$  for some  $I \subseteq \{1, \dots, k\}$ . Let  $L_1, \dots, L_k$  be such that  $X \cap (L_1 \times \dots \times L_k)$  is not empty, and write  $L_I = \prod_{i \in I} L_i$ , and similarly for  $L_{I^c}$ , so that  $X \cap (L_1 \times \dots \times L_k) = (X \cap p_I^{-1} L_I) \cap p_{I^c}^{-1} L_{I^c}$ . Since  $X \cap p_I^{-1} L_I$  is not empty, its dimension is at least the dimension of the generic fiber of  $X$  under  $p_I$ , that is  $r - \dim p_I(X)$ . Hence,  $\dim(X \cap p_I^{-1} L_I) \geq r - \dim p_I(X) > r + 1 - |\beta_I| = |\beta_{I^c}|$ , while  $\text{codim}(p_{I^c}^{-1} L_{I^c}) = |\beta_{I^c}|$ . We conclude that  $X \cap (L_1 \times \dots \times L_k)$  has positive dimension, and thus that all the inequalities are necessary in order for  $Z_{X,\beta}$  to be a hypersurface.

Conversely, suppose that the stated inequalities are satisfied, and fix a point  $P = (P_1, \dots, P_k) \in X$  such that for all  $I \subsetneq \{1, \dots, k\}$ , the fiber  $p_I^{-1}(p_I(P)) \cap X$  has dimension less than or equal to  $\sum_{i \notin I} (n_i - \alpha_i)$ . Note that this is always possible since our inequalities are equivalent to assuming that the minimal fiber dimension  $r - \dim p_I(X)$  is less than or equal to the desired value. Then we claim that a general choice of  $L_1, \dots, L_k$  with  $P_i \in L_i$  for each  $i$  will have  $X \cap (L_1 \times \dots \times L_k)$  nonempty and finite. We prove this by considering the incidence correspondence  $Y \subseteq X \times \text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k)$  consisting of  $(Q, L_1, \dots, L_k)$  with  $L_i \ni P_i$  for all  $i$ , and  $Q \in L_1 \times \dots \times L_k$ . Considering the case  $Q = P$ , we see that the image of  $Y$  under projection to  $\text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k)$  is exactly the tuples of  $L_i$  containing  $P_i$ , which is itself a product of Grassmannians, having dimension  $\sum_i \alpha_i(n_i - \alpha_i)$ . The finiteness statement we want amounts to showing that  $Y$  has generically finite fibers under this projection, or equivalently, that the dimension of  $Y$  is no larger than the dimension of its image. We do this by decomposing  $Y$  into locally closed subsets  $Y_I$ , defined to be the subset of  $Y$  on which  $Q_i = P_i$  precisely



when  $i \in I$  (here we allow  $I = \emptyset$ ). We then consider the projection of  $Y_I$  onto  $X$ . First,  $Y_I \neq \emptyset$  if and only if  $\alpha_i \geq 1$  for all  $i \notin I$ . In this case,  $Y_I$  maps into  $p_I^{-1}(p_I(P)) \cap X \subseteq X$ , and every fiber will have dimension equal to

$$\sum_{i \in I} \alpha_i(n_i - \alpha_i) + \sum_{i \notin I} (\alpha_i - 1)(n_i - \alpha_i).$$

We conclude that  $Y_I$  has dimension less than or equal to

$$\sum_{i \notin I} (n_i - \alpha_i) + \sum_{i \in I} \alpha_i(n_i - \alpha_i) + \sum_{i \notin I} (\alpha_i - 1)(n_i - \alpha_i) = \sum_i \alpha_i(n_i - \alpha_i),$$

as desired.  $\square$

**Example 3.3.** Consider a product of varieties  $X = X_1 \times \cdots \times X_k$ , so that each  $X_i \subseteq \mathbb{P}^{n_i}$  has dimension  $r_i$ , and  $\sum_i r_i = r$ . If  $\beta$  is such that  $|\beta| = r + 1$ , and  $Z_{X,\beta} \subseteq \text{Gr}(\alpha_1, n_1) \times \cdots \times \text{Gr}(\alpha_k, n_k)$  is as defined above, then Proposition 3.1 states that  $Z_{X,\beta}$  has codimension 1 if and only if  $\sum_{i \in I} r_i \leq |\beta_I| \leq \sum_{i \in I} r_i + 1$  for all  $I$ . This can occur if and only if  $\beta_j = r_j + 1$  for one index and  $\beta_i = r_i$  otherwise (a condition that can also be deduced directly from  $Z_{X,\beta}$ ). In this case,  $Z_{X,\beta} = \text{Gr}(\alpha_1, n_1) \times \cdots \times Z_{X_j} \times \cdots \times \text{Gr}(\alpha_k, n_k)$ , where  $Z_{X_j}$  is the hypersurface arising in the classical Cayley-Chow construction.

**Example 3.4.** Consider a variety  $X_0 \subseteq \mathbb{P}^{n_0}$ , of dimension  $r_0$ , and let  $X$  be the image of  $X_0$  in the diagonal embedding  $\mathbb{P}^{n_0} \rightarrow (\mathbb{P}^{n_0})^k$ . In this case, the conditions in Proposition 3.1 require that  $0 \leq |\beta_I| \leq r_0 + 1$ . Hence,  $Z_{X,\beta}$  is a hypersurface for any choice of  $(\beta_1, \dots, \beta_k)$  summing to  $r_0 + 1$ . More specifically, we have a surjective rational map

$$\text{Gr}(\alpha_1, n_1) \times \cdots \times \text{Gr}(\alpha_k, n_k) \dashrightarrow \text{Gr}(n_0 - r_0 - 1, n_0)$$

given by intersection of linear spaces, and it is clear that on the open subset where this map is defined, we have that  $Z_{X,\beta}$  is the preimage of the hypersurface  $Z_{X_0}$  arising in the classical Cayley-Chow construction. By irreducibility of  $Z_{X,\beta}$ , we see that it must simply be the closure of the preimage of  $Z_{X_0}$ .

**Example 3.5.** Suppose that for some  $I \subsetneq \{1, \dots, k\}$ , we have  $X' \subseteq \prod_{i \in I} \mathbb{P}^{n_i}$  such that  $X = X' \times \prod_{i \notin I} \mathbb{P}^{n_i}$  (equivalently,  $X = p_I^{-1}(p_I(X))$ ). Write  $r'$  for the dimension of  $X'$ , and suppose we have  $\beta$  satisfying (3.1). Then we have  $|\beta_I| \leq \dim p_I(X) + 1 = r' + 1$ , so we must have  $|\beta_{I^c}| \geq \sum_{i \in I^c} n_i$ , and then it follows that  $\beta_i = n_i$  for all  $i \in I^c$ , and also that  $|\beta_I| = r' + 1$ . In particular, the inequalities of Theorem 1.1 are violated. However, one easily verifies that

$$Z_{X,\beta} = Z_{X',\beta_I} \times \prod_{i \notin I} \text{Gr}(\alpha_i, n_i),$$

so the study of  $Z_{X,\beta}$  in this case reduces to the study of  $Z_{X',\beta_I}$ , and in particular  $X$  can be recovered from  $Z_{X,\beta}$  if and only if  $X'$  can be recovered from  $Z_{X',\beta_I}$ .

We now analyze when  $X$  can be recovered from  $Z_{X,\beta}$ .

**Proposition 3.6.** *In the situation of Proposition 3.1 (and in particular assuming (3.1)), we have that  $X$  is uniquely determined by  $Z_{X,\beta}$  if*

$$(3.3) \quad \dim p_I(X) \geq |\beta_I|$$

for all  $I \subsetneq \{1, \dots, k\}$ .

Conversely, if  $X$  is not of the form of Example 3.5, and  $X$  is uniquely determined by  $Z_{X,\beta}$ , then (3.3) is satisfied.

*Proof.* Let  $S_Z$  be the set of points  $(P_1, \dots, P_k)$  with the property that every  $L_1 \times \dots \times L_k$  containing  $(P_1, \dots, P_k)$  has  $X \cap (L_1 \times \dots \times L_k) \neq \emptyset$ . Then obviously  $X \subseteq S_Z$ . We claim that if (3.3) is satisfied, then

$$(3.4) \quad S_Z \subseteq X \cup \bigcup_{I \subsetneq \{1, \dots, k\}} p_I^{-1}(X_I)$$

where  $X_I \subseteq p_I(X)$  is the closed subset over which the fibers of  $X$  under  $p_I$  have dimension greater than or equal to  $\sum_{i \notin I} (n_i - \alpha_i)$ . Note that (3.3) implies that  $X_I \neq p_I(X)$ . From the claim we see that – given the multidegree of  $X$  – we can recover  $X$  from  $S_Z$ : indeed, by Corollary 2.2 the multidegree of  $X$  determines  $\dim p_I(X)$  for all  $I$ , and we see that every other potential component of  $S_Z$  has dimension strictly smaller than  $X$  under at least one projection.

To prove the claim, fix  $P = (P_1, \dots, P_k) \notin X$  and with  $p_I(P) \notin X_I$  for all  $I \subsetneq \{1, \dots, k\}$ . We wish to show that  $P \notin S_Z$ , or equivalently, that there exist  $L_1, \dots, L_k$  with  $P_i \in L_i$  for all  $i$ , and with  $X \cap (L_1 \times \dots \times L_k) = \emptyset$ . The proof is similar to the proof of Proposition 3.1: consider the incidence correspondence  $Y \subseteq X \times \text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k)$  consisting of  $(Q, L_1, \dots, L_k)$  with  $L_i \ni P_i$  for all  $i$ , and  $Q \in L_1 \times \dots \times L_k$ . In this case, we wish to show that the image of  $Y$  under projection to  $\text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k)$  does not contain all tuples of  $L_i$  containing  $P_i$ , so it will suffice to show that  $Y$  has dimension strictly smaller than  $\sum_i \alpha_i (n_i - \alpha_i)$ . We decompose  $Y$  into the subsets  $Y_I$  as before. For  $I \subsetneq \{1, \dots, k\}$ , just as before  $Y_I$  has image contained in  $p_I^{-1}(p_I(P)) \cap X \subseteq X$ , with every fiber having dimension equal to

$$\sum_{i \in I} \alpha_i (n_i - \alpha_i) + \sum_{i \notin I} (\alpha_i - 1)(n_i - \alpha_i).$$

We conclude that  $Y_I$  has dimension less than or equal to

$$\sum_{i \notin I} (n_i - \alpha_i) - 1 + \sum_{i \in I} \alpha_i (n_i - \alpha_i) + \sum_{i \notin I} (\alpha_i - 1)(n_i - \alpha_i) = \sum_i \alpha_i (n_i - \alpha_i) - 1,$$

as desired.

Conversely, suppose that we have some  $I \subsetneq \{1, \dots, k\}$  such that  $|\beta_I| > \dim p_I(X)$ . Then we have that for a general choice of  $L_i$  for  $i \in I$ , the intersection  $X \cap \bigcap_{i \in I} p_i^{-1} L_i$  is empty. Thus,  $p_I(Z_{X,\beta})$  is a proper subset of  $\prod_{i \in I} \text{Gr}(\alpha_i, n_i)$ . Since  $Z_{X,\beta}$  is a hypersurface, it must be equal to  $p_I(Z_{X,\beta}) \times \prod_{i \notin I} \text{Gr}(\alpha_i, n_i)$ . We conclude that  $Z_{X,\beta}$  is invariant under applying automorphisms of  $\mathbb{P}^{n_i}$  for  $i \notin I$ , and since we have assumed that  $X$  is not of the form of Example 3.5, it follows that  $X$  cannot be recovered from  $Z_{X,\beta}$ .  $\square$

Note that (unlike in the classical case), it may really be the case that the  $S_Z$  in the above proof contains components other than  $X$ ; see Example 5.15 below.

*Remark 3.7.* In the case that the inequalities (3.1) are satisfied, we can understand the hypersurface  $Z_{X,\beta}$  as follows: according to Proposition 2.3, there a nonempty  $J \subseteq \{1, \dots, k\}$  which is minimal – in the strong sense – satisfying

$$|\beta_J| = \dim p_J(X) + 1.$$

In this case,  $X' = p_J(X)$  will satisfy our setup with the stronger inequalities (3.3), so  $X'$  can be recovered from  $Z_{X',\beta_J}$ . Furthermore, we will have  $Z_{X,\beta}$  equal to the product of  $Z_{X',\beta_J}$  with all the  $\text{Gr}(\alpha_i, n_i)$  for  $i \notin J$ , so the information in  $Z_{X,\beta}$  is exactly equal to the data of  $p_J(X)$ .

#### 4. TENSOR PRODUCTS OF UNIQUE FACTORIZATION DOMAINS

In the classical setting, the fact that the hypersurface  $Z_X$  is the zero set of a single polynomial  $F_X$  in Plücker coordinates is a consequence of the fact that the homogeneous coordinate ring of a Grassmannian in Plücker coordinates is a unique factorization domain (UFD). We will make use of this to conclude the same statement in our case, but this requires a certain amount of care, as the condition of being a UFD is not very stable (for instance, there are examples where  $R$  is a UFD, but the power series ring  $R[[t]]$  is not). We address this with the following proposition, which states that under relatively mild additional hypotheses, Gauss' argument for unique factorization in a polynomial ring over a UFD extends to more general tensor products.

For the following proposition, we temporarily drop the hypothesis that we are working over a field  $K$ .

**Proposition 4.1.** *Let  $A$  be a ring,  $B$  and  $C$  algebras over  $A$ , and suppose that  $B$  is a Noetherian UFD,  $C$  is flat and finitely generated over  $A$ , and for every field  $K$  over  $A$ , we have that  $K \otimes_A C$  is a UFD, with unit group equal to  $K^\times$ . Then  $B \otimes_A C$  is a UFD.*

*Proof.* First observe that under our hypotheses, if  $B'$  is an  $A$ -algebra with fraction field  $K'$ , then we have injections  $K' \rightarrow K' \otimes_A C$  and  $B' \rightarrow B' \otimes_A C \rightarrow K' \otimes_A C$ . Indeed, injectivity of the last map follows from flatness of  $C$ , while injectivity of the first is a consequence of the implicit hypothesis that  $K' \otimes_A C$ , being a domain, is not the zero ring. Injectivity of the map  $B' \rightarrow B' \otimes_A C$  follows from the injectivity of the first map. Note also that  $B \otimes_A C$  is finitely generated over a Noetherian ring, hence Noetherian, so it suffices to show that every irreducible element is prime.

Our first claim is that if  $x \in B$  is a prime element, then  $x \otimes 1$  is prime in  $B \otimes_A C$ , and given  $y \in B$  we have that  $y$  is a multiple of  $x$  if and only if  $y \otimes 1$  is a multiple of  $x \otimes 1$  in  $B \otimes_A C$ . Indeed, if we let  $B' = B/(x)$  and apply the above, the second statement follows immediately from the injectivity of  $B/(x) \rightarrow B/(x) \otimes_A C = (B \otimes_A C)/(x \otimes 1)$ , while the injection  $B/(x) \otimes_A C \rightarrow K' \otimes_A C$  together with the hypothesis that  $K' \otimes_A C$  is an integral domain implies that  $x \otimes 1$  is prime. From the second part of the claim, we can conclude that if  $K$  is the fraction field of  $B$ , then the intersection of  $K$  with  $B \otimes_A C$  inside of  $K \otimes_A C$  is equal to  $B$ . We also see that conversely if  $x \otimes 1$  is irreducible in  $B \otimes_A C$  then  $x$  must be irreducible in  $B$ . In this case,  $x$  is prime in  $B$ , and hence  $x \otimes 1$  is prime in  $B \otimes_A C$ .

Now, suppose  $x$  is irreducible in  $B \otimes_A C$ , and consider the image of  $x$  in  $K \otimes_A C$ . If  $x$  becomes a unit in  $K \otimes_A C$ , then by hypothesis it is of the form  $y \otimes 1$  for some  $y \in K^\times$ , so we have from the above that  $y \in B$ , so  $x = y \otimes 1$  is prime in  $B \otimes_A C$ .

Next, we see that any nonzero element of  $K \otimes_A C$  can be written uniquely up to  $B^\times$  in the form  $\alpha f$ , for  $\alpha \in K^*$ , and  $f \in B \otimes_A C$  with the property that  $f$  is not a multiple of any non-unit in  $B$ . Indeed, every element of  $K \otimes_A C$  can be multiplied by an element of  $B$  to clear denominators, so is of the form  $\alpha f$  where  $f \in B \otimes_A C$ . Obviously, if  $f$  is a multiple of a non-unit in  $B$ , we can absorb it into  $\alpha$ , so we

can write the element in the desired form. Uniqueness follows by taking two such representations, clearing denominators, and using that primes in  $B$  remain prime in  $B \otimes_A C$ .

It then follows that if we have an irreducible element of  $B \otimes_A C$  which does not become a unit in  $K \otimes_A C$ , then it must remain irreducible in  $K \otimes_A C$ . Indeed, a nontrivial factorization in  $K \otimes_A C$  can be represented as  $(\alpha_1 f_1)(\alpha_2 f_2)$  as above, and then we see that since  $f_1$  and  $f_2$  are not multiples of any non-units of  $B$ , the same is true of  $f_1 f_2$ . Thus, the hypothesis that  $\alpha_1 \alpha_2 f_1 f_2 \in B \otimes_A C$  implies that  $\alpha_1 \alpha_2$  can't have any denominators, and then  $(\alpha_1 \alpha_2 f_1) f_2$  gives a nontrivial factorization in  $B \otimes_A C$ . Finally, we conclude that any such irreducible element must be prime: it is prime in  $K \otimes_A C$  by hypothesis, so if it divides a product in  $B \otimes_A C$ , it divides one of the factors in  $K \otimes_A C$ . But again using the above representation, we see that it must also divide the same factor in  $B \otimes_A C$ . We thus conclude that every irreducible element of  $B \otimes_A C$  is prime, and hence that  $B \otimes_A C$  is a UFD.  $\square$

Returning to varieties over  $K$ , we then conclude the desired statement on hypersurfaces in products of Grassmannians.

**Corollary 4.2.** *Let  $G := G_1 \times \cdots \times G_k \subseteq \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_k}$  be a product of Plücker embeddings of Grassmannians, and let  $Z \subseteq G$  be a hypersurface. Then  $Z = Z(F)$  for some multihomogeneous form  $F$ .*

*Proof.* First, we have that  $Z$  corresponds to a multihomogeneous prime ideal of height 1 in  $S(G)$ , the multihomogeneous coordinate ring of  $G$ . Now, we claim that  $S(G)$  is a UFD. Since  $S(G) = S(G_1) \otimes_K \cdots \otimes_K S(G_k)$ , we will prove this by induction on  $k$ , using Proposition 4.1. The base case is exactly the classical case; see Proposition 2.1 of Chapter 3 of [GKZ94]. Thus, we need only observe that  $S(G_k)$  over  $K$  satisfies the hypotheses of the proposition, most of which are immediate: flatness is automatic over  $K$ , the hypothesis on the units comes from the fact that  $S(G_k)$  is the homogeneous coordinate ring of a projective variety, and the hypothesis that for any field extension  $K'$  over  $K$  we have that  $K' \otimes_K S(G_k)$  is a UFD also follows from the classical case, since  $K' \otimes_K S(G_k)$  is simply the homogeneous coordinate ring of the Plücker embedding over  $K'$ . We thus conclude that  $S(G)$  is a UFD, and therefore that  $Z = Z(F)$  for some  $F \in S(G)$ . Finally,  $F$  must be multihomogeneous, or it could not generate a multihomogeneous prime ideal.  $\square$

## 5. (MULTI)DEGREES AND CAYLEY-CHOW FORMS

We are now ready to define multigraded Cayley-Chow forms. In order to obtain good behavior of multidegrees, there is one additional twist to consider. Namely, unlike in the classical case, it is possible that the Cayley-Chow form naturally has a multiplicity greater than 1.

**Definition 5.1.** In the situation of Proposition 3.1, suppose also that (3.1) is satisfied. Let  $\epsilon_{X,\beta}$  be the degree of the map  $V_X \rightarrow Z_{X,\beta}$ , where  $V_X$  is as in (3.2).

Then let  $F_X$  be a multihomogeneous polynomial in multi-Plücker coordinates with  $Z(F_X) = Z_{X,\beta}$  (Corollary 4.2), and define the **multigraded Cayley-Chow form** of  $X$  to be

$$H_{X,\beta} := F_X^{\epsilon_{X,\beta}}.$$

Recall that  $V_X$  is the incidence correspondence consisting of a point of  $X$  together with a tuple of linear spaces containing the coordinates of the point. Then the map to  $Z_{X,\beta}$  is simply the map forgetting the point of  $X$ .

Without further hypotheses, it may certainly be the case that  $\epsilon_{X,\beta} > 1$ .

**Example 5.2.** Let  $X = C_1 \times C_2 \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ , where each  $C_i$  is the curve defined by a homogeneous form  $F_i$  of degree  $d_i$ . In order for (3.1) to be satisfied, we need to have either  $\beta = (1, 2)$  or  $\beta = (2, 1)$ . In the first case, we see that if we have fixed  $L, P$  with  $X \cap (L \times P) \neq \emptyset$ , then in fact  $X \cap (L \times P)$  contains  $d_1$  points, so  $\epsilon_{X,\beta} = d_1$ . Meanwhile  $Z_{X,\beta}$  depends only on  $P$ , and the  $F_X$  of Definition 5.1 is simply  $F_2$ . Thus,  $H_{X,\beta} = F_2^{d_1}$ . Similarly, if  $\beta = (2, 1)$ , we find  $H_{X,\beta} = F_1^{d_2}$ .

For a more interesting example in positive characteristic, see Example 5.13 below. However, we have the following.

**Proposition 5.3.** *In the situation of Definition 5.1, suppose further that the inequalities of (3.3) are satisfied. Then the map  $V_X \rightarrow Z_{X,\beta}$  is generically injective, and if  $K$  has characteristic 0, we have  $\epsilon_{X,\beta} = 1$ .*

*Proof.* The generic injectivity amounts to saying that if  $P = (P_1, \dots, P_k) \in X$  is general, then general choices of  $L_i$  containing  $P_i$  will have  $X \cap (L_1 \times \dots \times L_k) = \{P\}$ . We recall that (3.3) implies that for all  $I \subsetneq \{1, \dots, k\}$ , the generic fiber dimension of  $X$  under  $p_I$  is strictly less than  $|\beta_{I^c}|$ . We prove by induction on  $k$  the following slightly more general statement: suppose that  $Y \subseteq \prod_i \mathbb{P}^{n_i}$  is a pure-dimensional algebraic set such that for every  $I \subsetneq \{1, \dots, k\}$ , every component of  $Y$  has generic fiber dimension under  $p_I$  strictly less than  $|\beta_{I^c}|$ . Then there exists a dense open subset  $U$  of  $Y$  such that for every  $P = (P_1, \dots, P_k) \in U$ , a general choice of  $L_i$  containing  $P_i$  will have  $Y \cap (L_1 \times \dots \times L_k) = \{P\}$ . Note that we allow  $I = \emptyset$  in our hypotheses, which says simply that  $Y$  has dimension strictly smaller than  $|\beta|$ .

We first observe that the desired statement reduces to the case that  $Y$  is irreducible. Indeed, if  $Y_1, \dots, Y_n$  are the components of  $Y$ , and if we construct  $S_{Z,i}$  from each  $Y_i$  as in the proof of Proposition 3.6 (with  $Y_i$  in place of  $X$ ), then (3.4) together with our hypotheses on the  $Y_i$  implies that we cannot have  $Y_i \subseteq S_{Z,i}$  for any distinct  $i, j$ . To see this, if we write  $Y_{I,i}$  in place of  $X_I$ , and write  $r = \dim Y = \dim Y_i$ , we see that each  $Y_{I,i}$  must have dimension strictly less than  $r - |\beta_{I^c}|$ , while our hypotheses imply that  $\dim p_I(Y_j) > r - |\beta_{I^c}|$ , so we cannot have  $p_I(Y_j) \subseteq Y_{I,i}$ . Thus, if we suppose that  $U_i$  satisfies the desired conditions for each  $Y_i$  separately, we then see that

$$\bigcup_i \left( U_i \setminus \bigcup_{j \neq i} S_{Z,j} \right)$$

satisfies the desired condition for all of  $Y$ . Consequently, for simplicity we will henceforth assume  $Y$  is irreducible.

Then the base case our of induction is  $k = 1$ , which is exactly the classical situation, and is proved by considering projection from any point in  $Y$ . For induction, we claim that the general fiber of  $Y$  under  $p_k$  satisfies our hypotheses for  $k - 1$ . First, since  $Y$  is irreducible the general fiber will be pure-dimensional. Next, for any fixed  $P_k \in p_k(Y)$ , and  $I' \subsetneq \{1, \dots, k - 1\}$ , if  $Y_{P_k}$  denotes the fiber of  $Y$  over  $P_k$ , and if we set  $I = I' \cup \{k\}$ , we see that for any  $(P_i)_{i \in I'} \in p_{I'}(Y_{P_k})$ , the fiber of  $Y_{P_k}$  over  $(P_i)_{i \in I'}$  is equal to the fiber of  $Y$  over  $(P_i)_{i \in I}$ . Note that  $|\beta_{I^c}|$  is the desired dimension bound also for  $k - 1$ , since now we take the complement of  $I'$

in  $\{1, \dots, k-1\}$ . By hypothesis, there is an open subset  $U_I \subseteq p_I(Y)$  such that the fiber dimension of  $Y$  over any point of  $U_I$  is strictly less than  $|\beta_{I^c}|$ . Then we observe that there is an open subset  $V_I$  of  $p_k(Y)$  such that  $p_I^{-1}(U_I)$  is dense in every fiber  $Y \cap p_k^{-1}(Q)$  for  $Q \in V_I$ : indeed, this follows from constructibility of images, together with semicontinuity of fiber dimension, since the only way that  $p_I^{-1}(U_I)$  can fail to be dense in the fiber over some  $Q \in p_k(Y)$  is if the dimension of  $Y \setminus p_I^{-1}(U_I)$  over  $Q$  is strictly larger than the generic fiber dimension of  $Y \setminus p_I^{-1}(U_I)$  over  $p_k(Y)$ . Then a general  $P_k$  will not only yield  $Y_{P_k}$  pure-dimensional, but will also lie in every  $V_I$ , so we see that every component of  $Y_{P_k}$  necessarily satisfies the desired generic fiber dimension bound.

Now, we note that the set of  $(P, L_1, \dots, L_k) \in Y \times \text{Gr}(\alpha_1, n_1) \times \dots \times \text{Gr}(\alpha_k, n_k)$  such that  $Y \cap (L_1 \times \dots \times L_k) = \{P\}$  is constructible: indeed, this follows from semicontinuity of fiber dimension, properness, and constructibility of connected fibers (see Theorem 9.7.7 of [GD66]).<sup>1</sup> Thus, to prove the desired statement, it suffices to show that this set is Zariski dense inside the locus of points satisfying  $P \in L_1 \times \dots \times L_k$ . If this were not the case, it would be contained in a proper Zariski closed subset  $Z$ . We would then have that a dense open subset of  $Y$  has the property that  $Z$  does not fully contain any of the fibers over that subset: this is, there would be a dense open subset of points  $P \in Y$  such that the choices of  $L_i$  containing  $P_i$  and having  $Y \cap (L_1 \times \dots \times L_k) = \{P\}$  are contained in a proper closed subset.

On the other hand, by the above claim and the induction hypothesis, if we fix a general  $P_k \in p_k(Y)$ , and let  $Y_{P_k}$  be the corresponding fiber, then we know that for  $(P_1, \dots, P_{k-1})$  general in  $Y_{P_k}$ , and  $L_i$  general containing  $P_i$  for  $i = 1, \dots, k-1$ , we have  $Y_{P_k} \cap (L_1 \times \dots \times L_{k-1}) = \{(P_1, \dots, P_{k-1})\}$ . We also know that

$$\dim p_k(Y \cap (L_1 \times \dots \times L_{k-1} \times \mathbb{P}^{n_k})) < \beta_k,$$

so  $Y \cap (L_1 \times \dots \times L_{k-1} \times \mathbb{P}^{n_k})$  will meet a general  $L_k$  containing  $P_k$  only in  $P_k$ . Thus, we will have  $Y \cap (L_1 \times \dots \times L_k) = \{P\}$ . Given the generality of  $P_i$  and  $L_i$ , this proves the desired statement.

We have thus proved the generic injectivity statement in general. To prove that  $\epsilon_{X, \beta} = 1$  in characteristic 0, we use the Bertini theorem given as Corollary 5 of [Kle74]: a general divisor in a basepoint-free linear system is smooth at all smooth points of the ambient scheme. In particular, intersecting a generically reduced scheme with the preimage of a general hyperplane in any of the  $\mathbb{P}^{n_i}$  will yield another generically reduced scheme. Applying this inductively, if we fix general  $L_1, \dots, L_{k-1}$ , and general  $L'_k$  of dimension  $\alpha_k + 1$ , then

$$X \cap (L_1 \times \dots \times L_{k-1} \times L'_k)$$

will consist of a finite number of reduced points, and the same will still be true if we further intersect with any  $L_k$  of codimension 1 in  $L'_k$ . Since we have already shown that such an intersection consists of a single point, we conclude that it is in fact a single reduced point, and  $\epsilon_{X, \beta} = 1$ .  $\square$

**Example 5.4.** The following example demonstrates the delicacy of the inductive statement proved in Proposition 5.3: given  $d_1, d_2 > 1$ , let  $S_1, S_2$  be surfaces of

<sup>1</sup>Here we are considering only classical points; the correct statement for schemes involves geometrically connected fibers, but since we work over an algebraically closed field and consider only classical points, this amounts to the same thing.

degrees  $d_1, d_2$  respectively in  $\mathbb{P}^3$ , and  $C_1, C_2$  curves in  $S_1, S_2$ . Set

$$Y = (C_1 \times S_2) \cup (S_1 \times C_2),$$

and  $\beta_1 = \beta_2 = 2$ . Then  $p_1(Y)$  and  $p_2(Y)$  are both irreducible, and the “generic fibers” of both  $p_i$  have dimension 1 (which is strictly smaller than either  $\beta_i$ ), in the sense that there are open dense subsets of each  $p_i(Y)$  on which the fiber dimension is 1. Thus,  $Y$  satisfies a slight variant of the induction statement, but we see that any  $L_1 \times L_2$  meeting  $Y$  has to contain at least  $\min(d_1, d_2)$  points.

In order to complete the proofs of the basic properties of our generalized Cayley-Chow construction, it will be helpful to extend it to cycles. The main reason for this is that even if  $X$  is irreducible, its intersections with general products of linear spaces are not always irreducible (Bertini theorems imply that they usually are, but they will not be if for instance some projection has 1-dimensional image). Some preliminary notation is the following.

*Notation 5.5.* If  $X \subseteq Y$  is a pure-dimensional closed subscheme of a smooth variety, denote by  $[X]$  the associated cycle. If  $\Xi, \Xi'$  are cycles on  $Y$  which meet in the expected codimension, write  $\Xi \cdot \Xi'$  for the induced intersection cycle (see for instance Serre’s definition of intersection multiplicity on p. 427 of [Har77]).

Note that we do not work up to rational (or other) equivalence; the point of introducing the notation is that it is not always true that  $[X] \cdot [X'] = [X \cap X']$ , even when  $X$  and  $X'$  intersect in the expected dimension. In our situations, we will have  $[X] \cdot [X'] = [X \cap X']$  due to generality hypotheses, but we will have to justify this point.

**Definition 5.6.** Given  $n_1, \dots, n_k, r$  and  $\beta = (\beta_1, \dots, \beta_k)$  with  $|\beta| = r + 1$  and  $0 \leq \beta_i \leq n_i$  for all  $i$ , linearly extend the construction  $X \mapsto \epsilon_{X, \beta}[Z_{X, \beta}]$  to effective  $r$ -cycles on  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$  by using our previous construction for those components satisfying (3.1), and extending by zero for any additional components. Extend the resulting Cayley-Chow form construction multiplicatively. For an effective  $r$ -cycle  $\Xi$ , denote the resulting constructions by  $Z_{\Xi, \beta}$  and  $H_{\Xi, \beta}$  respectively.

Note that we are incorporating the multiplicities into our new notation, so that if  $\Xi = [X]$ , we have  $Z_{\Xi, \beta} = \epsilon_{X, \beta}[Z_{X, \beta}]$ .

*Remark 5.7.* Observe that we can rephrase Definition 5.1 as saying that we are taking the form cutting out  $\epsilon_{X, \beta}[Z_{X, \beta}]$ , and the latter is  $\pi_*[V_X]$ , where  $\pi$  is the projection to the product of Grassmannians. Definition 5.6 allows us to extend this as follows: if  $\Xi$  is an effective  $r$ -cycle, we can construct an incidence correspondence cycle  $V_{\Xi}$  on  $(\prod_{i \in I^c} \mathbb{P}^{n_i}) \times (\prod_{i \in I} \text{Gr}(\alpha_i, n_i))$  by linearly extending our previous construction, and our extension by zero in Definition 5.6 means that we will still have  $Z_{\Xi, \beta} = \pi_* V_{\Xi}$ . Indeed, according to the proof of Proposition 3.1, a component of  $\Xi$  fails to satisfy (3.1) precisely when the corresponding component of  $V_{\Xi}$  drops dimension under  $\pi$ .

We then have the following description of the behavior of multigraded Cayley-Chow forms under partial evaluation.

**Proposition 5.8.** *Let  $\Xi$  be an effective  $r$ -cycle as in Definition 5.6, and  $H_{\Xi, \beta}$  its associated multigraded Cayley-Chow form. For any  $I \subsetneq \{1, \dots, k\}$ , given general  $(L_i)_{i \in I}$  of codimensions  $\beta_i$ , set  $L_I := \prod_{i \in I} L_i$ . Then the partial evaluation of*

$H_{\Xi, \beta}$  at the  $L_i$  for  $i \in I$  yields the multigraded Cayley-Chow form associated to  $p_{I^c*}([p_I^{-1}(L_I)] \cdot \Xi)$  and  $\beta_{I^c}$ .

In the above, if  $p_I^{-1}(L_I)$  does not meet the support of  $\Xi$ , we should interpret the associated multigraded Cayley-Chow form to be constant.

*Remark 5.9.* Note that if  $X$  is a subvariety, the generality of  $L_I$  implies that  $p_I^{-1}(L_I) \cap X$  is necessarily pure-dimensional, of codimension  $|\beta_I|$  in  $X$ , so it is reasonable to pass to the associated cycle, and apply pushforward of cycles. If  $X$  is not Cohen-Macaulay, then even when  $p_I^{-1}(L_I)$  meets  $X$  in the expected codimension, we could *a priori* have that  $[p_I^{-1}(L_I) \cap X] \neq [p_I^{-1}(L_I)] \cdot [X]$ , so we have to be slightly careful with our arguments. However, we see that with  $L_I$  general, this will not be the case: the non-smooth locus of  $X$  is strictly smaller-dimensional, so again using generality of  $L_I$ , we see that every component of  $p_I^{-1}(L_I) \cap X$  must have a dense open subset inside the smooth locus of  $X$ . But  $p_I^{-1}(L_I)$  is also smooth, so we conclude that in this case, the intersection multiplicities of every component of  $p_I^{-1}(L_I) \cap X$  are simply determined by the lengths of the intersected scheme, which is to say that  $[p_I^{-1}(L_I) \cap X] = [p_I^{-1}(L_I)] \cdot [X]$ .

We also note that the proof of Proposition 3.1 shows that if  $X$  satisfies (3.1), and if  $p_I^{-1}(L_I) \cap X \neq \emptyset$  (still assuming  $L_I$  general), then we will have that  $p_I^{-1}(L_I) \cap X$  has generically finite fibers under  $p_{I^c}$ , so that  $\dim p_{I^c}(p_I^{-1}(L_I) \cap X) = \dim p_I^{-1}(L_I) \cap X$ . Indeed, if  $p_I^{-1}(L_I)$  is nonempty for a general  $L_I$ , this means that  $V_X$  maps dominantly to  $\prod_{i \in I} \text{Gr}(\alpha_i, n_i)$  under  $p_I \circ \pi$ , where  $\pi$  is projection to the product of Grassmannians. On the other hand, the proof of Proposition 3.1 implies that there is a dense open subset  $U \subseteq V_X$  on which projection to  $\prod_i \text{Gr}(\alpha_i, n_i)$  is finite. Thus, a general  $L_I$  is in the image of  $U$ , meaning that there exist  $L_i$  for  $i \in I^c$  such that  $X \cap (L_1 \times \cdots \times L_k)$  is (nonempty and) finite. In particular, we must have that  $p_I^{-1}(L_I) \cap X$  has finite fiber over  $(L_i)_{i \in I^c}$ , as claimed.

*Proof of Proposition 5.8.* Both sides being multiplicative, the desired identity reduces immediately to the case that  $\Xi = [X]$ , with  $X$  a subvariety. Let  $\tilde{Y} = p_I^{-1}(L_I) \cap X$ , let  $\Psi = p_{I^c*}[\tilde{Y}]$ , and let  $\bar{L}_I \in \prod_{i \in I} \text{Gr}(\alpha_i, n_i)$  be the point determined by the  $L_i$ . Then on the level of underlying sets, we see that  $Z_{\Psi, \beta_{I^c}}$  is given by the fiber of  $Z_{X, \beta}$  over  $\bar{L}_I$ , which in turn is the vanishing cycle of partial evaluation of  $H_{X, \beta}$  at  $\bar{L}_I$ . Thus, we need to see that the associated multiplicities behave as expected. Let  $V_X$  be the incidence correspondence in  $(\prod_i \mathbb{P}^{n_i}) \times (\prod_i \text{Gr}(\alpha_i, n_i))$ , and  $V_\Psi$  be the incidence correspondence cycle as in Remark 5.7, so that we have  $Z_{[X], \beta} = \pi_* V_X$  and  $Z_{\Psi, \beta_{I^c}} = \pi_* V_\Psi$  (although note that the two maps  $\pi$  are onto different products of Grassmannians).

Write  $\tilde{L}_I \subseteq \prod_i \text{Gr}(\alpha_i, n_i)$  for the fiber  $p_I^{-1}(\bar{L}_I)$ , and let  $V_{\tilde{Y}} \subseteq (\prod_i \mathbb{P}^{n_i}) \times (\prod_{i \in I^c} \text{Gr}(\alpha_i, n_i))$  be the scheme-theoretic incidence correspondence, which we will consider as lying in  $(\prod_i \mathbb{P}^{n_i}) \times (\prod_i \text{Gr}(\alpha_i, n_i))$  using the point  $\bar{L}_I$ . One then checks easily that  $V_{\tilde{Y}} = V_X \cap \pi^{-1}(\tilde{L}_I)$ , for instance by comparing the functors of points. Next, we note that because of the generality of  $L_I$ , we have  $[V_X \cap \pi^{-1}(\tilde{L}_I)] = [V_X] \cdot [\pi^{-1}(\tilde{L}_I)]$ . Indeed,  $V_X \cap \pi^{-1}(\tilde{L}_I)$  is simply the fiber of  $V_X$  over a general point of  $\prod_{i \in I} \text{Gr}(\alpha_i, n_i)$ , and the non-Cohen-Macaulay locus of  $V_X$  is a proper algebraic subset, hence of strictly smaller dimension. Semicontinuity of fiber dimension then



implies that no component of a general fiber is entirely contained in the non-Cohen-Macaulay locus of  $V_X$ , and we obtain the desired identity as in Remark 5.9. The same argument shows that  $[Z_{X,\beta} \cap \tilde{L}_I] = [Z_{X,\beta}] \cdot [\tilde{L}_I]$ .

We next claim that  $V_\Psi = (p_{I^c} \times p_{I^c})_*[V_{\tilde{Y}}]$ . This is clear again on the level of underlying sets, so we just need to verify that the multiplicities agree. By construction,  $V_{\tilde{Y}}$  is smooth over  $\tilde{Y}$ , so inherits the same multiplicities, and the two pushforwards under  $p_{I^c}$  visibly have the same fibers, so the claim follows.

We then have

$$\begin{aligned} Z_{\Psi,\beta_{I^c}} &= \pi_* V_\Psi = \pi_*(p_{I^c} \times p_{I^c})_*[V_{\tilde{Y}}] = p_{I^c*} \pi_*[V_X \cap \pi^{-1}(\tilde{L}_I)] \\ &= p_{I^c*} \pi_*([V_X] \cdot [\pi^{-1}(\tilde{L}_I)]) = p_{I^c*}((\pi_*[V_X]) \cdot [\tilde{L}_I]) \\ &= p_{I^c*}([Z_{[X],\beta}] \cdot [\tilde{L}_I]) = p_{I^c*}[Z_{[X],\beta} \cap \tilde{L}_I], \end{aligned}$$

where the fifth equality is the projection formula on the level of cycles; see Proposition 8.1.1(c) of [Ful98]. Note that in the final expression, we are applying  $p_{I^c*}$  to a cycle already supported in a fiber of  $p_I$ , so this is just a formality, and we obtain the desired expression.  $\square$

We now conclude the desired assertion on multidegrees of Cayley-Chow forms. We can extend multidegree linearly to cycles, so we state the result in that context.

**Corollary 5.10.** *Under the hypotheses of Proposition 5.8, suppose that  $\Xi$  has multidegree  $\sum_\gamma a_\gamma t_1^{\gamma_1} \cdots t_k^{\gamma_k}$ . Then given  $\beta$ , the Cayley-Chow form  $H_{\Xi,\beta}$  has multidegree*

$$(a_{\alpha_1+1,\alpha_2,\dots,\alpha_k}, \dots, a_{\alpha_1,\dots,\alpha_{k-1},\alpha_k+1}).$$

*Proof.* By linearity, it suffices to treat the case that  $\Xi = [X]$  for a subvariety  $X$ . For each  $j \in \{1, \dots, k\}$ , we wish to show that the degree of  $H_{X,\beta}$  in the  $j$ th set of variables is equal to

$$(5.1) \quad a_{\alpha_1,\dots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\dots,\alpha_k}.$$

Setting  $I = \{1, \dots, k\} \setminus \{j\}$  and applying Proposition 5.8, it suffices to show that the classical Cayley-Chow form of  $p_{j*}(p_I^{-1}(L_I) \cap X)$  has degree given by (5.1) (note that it necessarily has the expected dimension; see Remark 5.9). By the classical theory (Proposition 2.1 and 2.2 of [GKZ94]), we thus want to show that  $p_{j*}(p_I^{-1}(L_I) \cap X)$  has degree in  $\mathbb{P}^{n_j}$  given by (5.1). But this follows from the definitions and the projection formula in intersection theory (see for instance p. 426 of [Har77]).  $\square$

Finally, using the work of Castillo, Li and Zhang [CLZ], we can also translate the inequalities (3.1) and (3.3) into multidegree-based criteria as follows.

**Corollary 5.11.** *If  $X$  has multidegree  $\sum_\gamma a_\gamma t_1^{\gamma_1} \cdots t_k^{\gamma_k}$ , then the  $Z_{X,\beta}$  associated to  $(\beta_1, \dots, \beta_k) = (n_1 - \alpha_1, \dots, n_k - \alpha_k)$  is a hypersurface if and only if*

$$(5.2) \quad a_{\alpha_1+1,\dots,\alpha_k} t_1 + \cdots + a_{\alpha_1,\dots,\alpha_k+1} t_k$$

*is not identically zero. Moreover,  $Z_{X,\beta}$  determines  $X$  if and only if every term of (5.2) is nonzero.*

*Proof.* If some  $n$ -tuple  $(\alpha_1, \dots, \alpha_j+1, \dots, \alpha_k)$  is in the support of the multi-degree, then Theorem 2.1 implies that  $|\beta_I| = \sum_{i \in I} (n_i - \alpha_i) \leq \dim p_I(X) + 1$  for all  $I$ , so according to Proposition 3.1 we have that  $Z_{X,\beta}$  is a hypersurface. Conversely, if

$|\beta_I| \leq \dim p_I(X) + 1$  for all  $I$ , then we claim that there exists  $j$  such that for any  $I$  with

$$|\beta_I| = \dim p_I(X) + 1,$$

we necessarily have  $j \in I$ . Indeed, this follows immediately from Proposition 2.3, by choosing any  $j \in J$ . We then have that  $(\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_k)$  is in the support of the multidegree of  $X$ .

Next, if  $|\beta_I| \leq \dim p_I(X)$  for all  $I \subsetneq \{1, \dots, k\}$ , it is clear that for all  $j$  and  $I$ , we will have  $\sum_{i \in I} (n_i - \gamma_i) \leq \dim p_I(X)$ , where as before  $(\gamma_1, \dots, \gamma_k) = (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_k)$ . Thus, for each  $j$  we have  $\gamma$  in the support of the multidegree of  $X$ . Conversely, if for some  $I \subseteq \{1, \dots, k\}$  we have  $|\beta_I| > \dim p_I(X)$ , then for any  $j \notin I$ , with  $\gamma$  as above we will have  $\sum_{i \in I} (n_i - \gamma_i) = |\beta_I| > \dim p_I(X)$ , so  $\gamma$  is not in the support of the multidegree of  $X$ .  $\square$

*Remark 5.12.* Given our basic setup, it is of course possible to re-embed  $X$  into a high-dimensional projective space via the Segre embedding, and then apply the classical Cayley-Chow construction. This works canonically and unconditionally to characterize  $X$ , but it doesn't reflect the geometry of the embedding of  $X$  into the original product of projective spaces, and it will typically require a great deal more data. For instance, if a 3-fold is embedded in  $\mathbb{P}^2 \times \mathbb{P}^2$ , then our construction will give a bihomogeneous form in two sets of three variables. On the other hand, the Segre embedding gives a 3-fold in  $\mathbb{P}^8$ , so the relevant Grassmannian will be  $\text{Gr}(4, 8)$ , whose Plücker embedding lands in  $\mathbb{P}^{125}$ . Thus, the classical Cayley-Chow form is in 126 variables in this case!

We conclude with further examples. The first shows that in positive characteristic, our multigraded Cayley-Chow form may indeed come with multiplicity strictly greater than 1.

**Example 5.13.** Let  $K$  have characteristic  $p$ , and let  $X \subseteq \mathbb{P}^2 \times \mathbb{P}^2$  be the graph of the Frobenius morphism  $\varphi$ . Then  $X$  has multidegree  $p^2 t_1^2 + p t_1 t_2 + t_2^2$ . If  $\beta = (2, 1)$ , then

$$Z_{X, \beta} = \{(P, L) : \varphi(P) \in L\} \subseteq \mathbb{P}^2 \times (\mathbb{P}^2)^*.$$

If  $L = Z(G)$  for a linear form  $G$  and  $P = (u_0, u_1, u_2)$ , then  $\varphi(P) \in L$  if and only if  $G(u_0^p, u_1^p, u_2^p) = 0$ , so we see that  $Z_{X, \beta}$  is cut out by a  $(p, 1)$ -form, as it should be.

On the other hand, if  $\beta = (1, 2)$ , then

$$Z_{X, \beta} = \{(L, P) : P \in \varphi(L)\} \subseteq (\mathbb{P}^2)^* \times \mathbb{P}^2.$$

If  $L = Z(G)$  and  $P = (u_0, u_1, u_2)$  as above, then we observe that  $\varphi(L)$  is cut out by  $\widehat{G}$ , the linear form obtained from  $G$  by raising the coefficients to the  $p$ th power. Then  $P \in \varphi(L)$  if and only if  $\widehat{G}(u_0, u_1, u_2) = 0$ , so in this case  $Z_{X, \beta}$  is still cut out by a  $(p, 1)$ -form. Thus, in order to get the right degree, we have to take the cycle  $pZ_{X, \beta}$  in place of  $Z_{X, \beta}$ . We see this geometrically by observing that if we intersect  $X$  with  $p_2^{-1}(Y)$  for any point  $Y$ , we get a length- $p^2$  subscheme which can be identified under the first projection with the fiber of  $\varphi$  over  $Y$ . Intersecting with a line in  $\mathbb{P}^2$  will then reduce the length to  $p$ , but cannot reduce it to 1. Thus, the forgetful map from the incidence correspondence has degree  $p$  in this case.

The following examples come from computer vision.

**Example 5.14.** [Multifocal tensors] We expand on the discussion of multiview varieties from the introduction. Given  $k \geq 2$  linear projections  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  (viewed as positioned pinhole cameras, with the centers of projection being camera centers), we let  $X \subseteq (\mathbb{P}^2)^k$  be the closure of the image of the induced rational map. This is called the “multiview variety” associated to the camera configuration, and determines the configuration (up to linear change of coordinates on  $\mathbb{P}^3$ ). If we assume that the camera centers are all distinct, then  $\dim p_I(X) = 3$  whenever  $|I| \geq 2$ , so Proposition 3.1 guarantees that  $Z_{X,\beta}$  is a hypersurface for all  $(\beta_1, \dots, \beta_k)$  such that  $|\beta| = 4$ .

On the other hand,  $Z_{X,\beta}$  determines  $X$  if and only if  $|\beta_I| \leq 3$  for all  $I \subseteq \{1, \dots, k\}$ , that is, if and only if  $\beta_i \neq 0$  for all  $i$ . This clearly requires  $k \leq 4$ , and we see that if  $k = 2$  the vector  $\beta$  must be  $(2, 2)$ ; if  $k = 3$  then it is a permutation of  $(2, 1, 1)$ ; if  $k = 4$  then it is  $(1, 1, 1, 1)$ . Moreover, the multidegree of  $X$  is computed in Corollary 3.5 of Aholt-Sturmfels-Thomas [AST13] to be

$$t_1^2 \cdots t_k^2 \left( \sum_{1 \leq i_1 < i_2 < i_3 \leq k} \frac{1}{t_{i_1} t_{i_2} t_{i_3}} + \sum_{1 \leq i_1, i_2 \leq k} \frac{1}{t_{i_1}^2 t_{i_2}} \right),$$

so according to Corollary 5.10, we find that in the allowed cases,  $H_{X,\beta}$  is multilinear, so it is associated with a tensor.

We thus recover the multifocal tensor construction when  $k \leq 4$ ; these are known as the “fundamental matrix,” the “trifocal tensor,” and the “quadrifocal tensor,” respectively. On the other hand, for  $k \geq 5$ , we see that the constructed form never suffices to recover  $X$ .

**Example 5.15.** Consider the  $k = 3$  (i.e., trifocal) case of the previous example, and let  $P_i$  be the centers of projection. It is helpful to observe that we can think of  $X$  as consisting of triples  $(\ell_1, \ell_2, \ell_3)$  where each  $\ell_i$  is a line through  $P_i$  in  $\mathbb{P}^3$ , and  $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$  (see Proposition 2.1 of [Li]). To avoid having to discuss too many cases, we will assume that the  $P_i$  are not collinear (and in particular are distinct). As discussed in the previous example, we will have to have up to permutation that  $\beta = (2, 1, 1)$ , so that in  $\mathbb{P}^2$ , we will have  $L_1$  a point, and  $L_2$  and  $L_3$  lines. In the ambient  $\mathbb{P}^3$ , they will correspond to lines and planes containing  $P_i$ , respectively. We will analyze the set  $S_Z$  from the proof of Proposition 3.6. In fact, the second author, Hebert and Ponce give a description of  $S_Z$  in Proposition 9 of [THP15], observing that it does indeed contain extra components beyond  $X$  itself.

To describe  $S_Z$ , suppose we have fixed  $(\ell_1, \ell_2, \ell_3)$ , not necessarily in  $X$ , so that we want to know under what conditions every  $L_1, L_2, L_3$  containing  $\ell_1, \ell_2, \ell_3$  must meet  $X$ , or equivalently, under what conditions every  $L_2, L_3$  contain some choice of  $\ell'_2, \ell'_3$  such that  $\ell_1 \cap \ell'_2 \cap \ell'_3 \neq \emptyset$  (note that  $L_1 = \ell_1$  necessarily). Obviously, this is the case if  $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$  already, so that  $(\ell_1, \ell_2, \ell_3) \in X$ . However, there are two other cases in which this occurs: if  $\ell_1 = \ell_2 = \overline{P_1 P_2}$ , or  $\ell_1 = \ell_3 = \overline{P_1 P_3}$ . Indeed, in the former case, we have that any plane  $L_3$  must meet the line  $\ell_1 = \ell_2$ , yielding a choice of  $\ell'_3$ , and similarly for the latter case. On the other hand, one can check directly that in any other situation, we can always find  $L_2, L_3$  such that no  $\ell'_2, \ell'_3$  will have nonempty simultaneous intersection with  $\ell_1$ . Indeed, we will be always be able to choose  $L_2$  and  $L_3$  so that  $L_1 \cap L_2$  and  $L_1 \cap L_3$  are distinct points, and  $L_1 \cap L_2 \cap L_3$  is therefore empty. We thus find that  $S_Z$  consists of  $X$  together with two additional 2-dimensional components.

To compare this to the containment in (3.4), we describe the geometry of the projections of  $X$ ; by symmetry, it suffices to look at  $p_1$  and  $p_{\{1,2\}}$ . We have that  $p_1$  is surjective, and if we fix  $\ell_1$ , the fiber of  $X$  over  $\ell_1$  consists of pairs of lines  $\ell_2, \ell_3$  which intersect  $\ell_1$  in a common point. If  $\ell_1 \neq \overline{P_1P_2}, \overline{P_1P_3}$ , then each of  $\ell_2$  and  $\ell_3$  can meet  $\ell_1$  in only a single point, so the choice of  $\ell_2$  is determined by the choice of a point on  $\ell_1$ , and then  $\ell_3$  is determined as well. Thus, on this set the fibers are 1-dimensional. However, if  $\ell_1 = \overline{P_1P_2}$  (so that  $P_3 \notin \ell_1$ ), then every choice of  $\ell_2$  meets  $\ell_1$ , and as long as  $\ell_2 \neq \ell_1$ , then  $\ell_1 \cap \ell_2 = \{P_2\}$ , and  $\ell_3 = \overline{P_2P_3}$  is uniquely determined. On the other hand, we could also have  $\ell_2 = \ell_1$ , in which case we have a 1-dimensional set of choices of  $\ell_3$ . Thus, in this case the fiber has two components, one of dimension 2, and one of dimension 1. The same holds if  $\ell_1 = \overline{P_1P_3}$ . We conclude that the general fiber is 1-dimensional, but there are two fibers which are (non-purely) 2-dimensional, corresponding to  $\overline{P_1P_2}$  and  $\overline{P_1P_3}$ , respectively. The analogous description holds for  $p_2$  and  $p_3$ .

Next, the image of  $p_{\{1,2\}}$  is precisely the set of pairs  $(\ell_1, \ell_2)$  which have nonempty intersection, which forms a 3-dimensional set. Provided  $\ell_1 \neq \ell_2$ , we will have that  $\ell_1 \cap \ell_2$  is a single point. If this point is not  $P_3$ , then  $\ell_3$  is uniquely determined, and thus  $p_{\{1,2\}}$  is injective over such pairs. If  $\ell_1 = \ell_2 = \overline{P_1P_2}$ , so that  $P_3$  is not on  $\ell_1$  or  $\ell_2$ , then  $\ell_3$  is determined by a choice of point of  $\ell_1$ , and we have a 1-dimensional fiber. Finally, if  $\ell_1 \neq \ell_2$  but both go through  $P_3$  (so that they are necessarily  $\overline{P_1P_3}$  and  $\overline{P_2P_3}$  respectively), then any choice of  $\ell_3$  is valid, and we obtain a 2-dimensional fiber. To summarize, the general fiber is 0-dimensional, but the fiber corresponding to  $(\overline{P_1P_2}, \overline{P_1P_2})$  is 1-dimensional, and the fiber corresponding to  $(\overline{P_1P_3}, \overline{P_2P_3})$  is 2-dimensional. The analogous description holds for  $p_{\{2,3\}}$  and  $p_{\{1,3\}}$ .

We compare this to the proof of Proposition 3.6 as follows: if  $I = \{i\}$ , the set  $X_I$  is where the fibers of  $p_i$  have dimension at least 2 if  $i = 1$ , and at least 3 if  $i = 2, 3$ . The latter two cases give the empty set, but the former consists of the two points where  $\ell_1$  is either  $\overline{P_1P_2}$  or  $\overline{P_1P_3}$ . Note that the two extra components of  $S_Z$  we have identified are contained in  $p_1^{-1}(X_{\{1\}})$ , but strictly. On the other hand, if  $I = \{i, j\}$ , the set  $X_I$  is where the fibers of  $p_{\{i,j\}}$  have dimension at least 2 if  $I = \{2, 3\}$ , and at least 1 otherwise. In the first case, we get that  $X_I$  is the single point where  $\ell_2 = \overline{P_1P_2}$  and  $\ell_3 = \overline{P_1P_3}$ . In this case, we already have that  $p_{\{2,3\}}^{-1}(\ell_2, \ell_3)$  is contained in  $X$ , so we do not get any new component of  $S_Z$ . For  $I = \{1, 2\}$ , we have the same behavior over  $(\overline{P_1P_3}, \overline{P_2P_3})$ , but  $X_I$  also includes the point  $(\overline{P_1P_2}, \overline{P_1P_2})$ , and one additional component of  $S_Z$  is equal to the fiber of  $p_{\{1,2\}}$  over this point. Considering  $I = \{1, 3\}$  gives the other additional component of  $S_Z$ . Thus, in this case we have strict containment in (3.4), although we obtain equality if we restrict  $I$  to two-element sets.

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