



HAL
open science

A Note on the Ei Function and a Useful Sum-Inequality

Lilian Besson

► **To cite this version:**

| Lilian Besson. A Note on the Ei Function and a Useful Sum-Inequality. 2018. hal-01847480

HAL Id: hal-01847480

<https://inria.hal.science/hal-01847480v1>

Submitted on 23 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Note on the Ei Function and a Useful Sum-Inequality

Lilian Besson

LILIAN.BESSON@CENTRALESUPELEC.FR

CentraleSupélec (campus of Rennes), IETR, SCEE Team,

Avenue de la Boulaie – CS 47601, F-35576 Cesson-Sévigné, France

Abstract

This short note defines formally the Ei function, and gives some interesting inequalities and integration using it. I illustrate the inequalities and detail what is still to be proved.

Keywords: Analysis; Inequalities; Primitive.

1. Introduction and Motivation

Take $a > 1$, $b > 1$ and $\gamma > 0$, and for an integer $L > 0$ consider the sum $\sum_{i=1}^L (a^{b^i})^\gamma$. We want to bound it, and the goal is to show that it is bounded by a constant times its last term. A first naive bound is $\sum_{i=1}^L (a^{b^i})^\gamma \leq (L+1)(a^{b^L})^\gamma$ which is too brutal as soon as $L \rightarrow \infty$.

I first remind and prove two useful elementary results, and then we define and study the Ei function, to finally prove the desired inequality.

2. Lemma and Proof

Lemma 1 For any $n \in \mathbb{N}^*$, $a > 1$, $b > 1$ and $\gamma > 0$, we have

$$\sum_{i=0}^n (a^{b^i})^\gamma \leq a^\gamma + \left(1 + \frac{1}{(\log(a))(\log(b^\gamma))}\right) (a^{b^n})^\gamma = \mathcal{O}\left((a^{b^n})^\gamma\right). \quad (1)$$

Proof We first isolate both the first and last term in the sum and focus on the from $i = 1$ sum up to $i = n - 1$. As the function $t \mapsto (a^{b^t})^\gamma$ is increasing for $t \geq 1$, we use a sum-integral inequality, and then the change of variable $u := \gamma b^t$, of Jacobian $dt = \frac{1}{\log b} \frac{du}{u}$, gives

$$\sum_{i=1}^{n-1} (a^{b^i})^\gamma \leq \int_1^n a^{\gamma b^t} dt \leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} \frac{a^u}{u} du$$

Now for $u \geq 1$, observe that $\frac{a^u}{u} \leq a^u$, and as $\gamma b > 1$, we have

$$\leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} a^u du \leq \frac{1}{\log(b^\gamma)} \frac{1}{\log(a)} a^{\gamma b^n} = \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma.$$

Finally, we obtain as desired, $\sum_{i=0}^n (a^{b^i})^\gamma \leq a^\gamma + (a^{b^n})^\gamma + \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma$. ■

3. Elementary Results

3.1. Integration by Part

The Integration by Part is a basic but useful result to establish inequalities, *e.g.*, for Lemma 10 using Lemma 3, and to prove the existence of finite integrals, *e.g.*, for Lemma 5 using two chained IP.

Lemma 2 (Integration by Part (IP)) *Let $x, y \in \mathbb{R}$, $x \leq y$, and u, v two functions of class¹ \mathcal{C}^1 , and with this notation $[uv]_x^y := u(y)v(y) - u(x)v(x)$, then*

$$\int_x^y u(t)v'(t) dt = [uv]_x^y - \int_x^y u'(t)v(t) dt. \quad (2)$$

Proof The two integrals and the two evaluations are well defined by the \mathcal{C}^1 hypothesis on both u and v (u and v are continuous at x and y and $u'v$ is continuous so integrable on the interval $[x, y]$).

The product function uv is differentiable, and $(uv)' = u'v + uv'$, so $[uv]_x^y = \int_x^y (uv)'(t) dt = \int_x^y u(t)v'(t) dt + \int_x^y u'(t)v(t) dt$ as wanted, by the linearity of the integral. ■

Lemma 3 (IP Inequality) *If both u, v are non-negative, and non-decreasing, then*

$$\int_x^y u(t)v'(t) dt \leq u(y)v(y). \quad (3)$$

Proof The non-negativeness gives that $-u(x)v(x) \leq 0$ and the monotony hypothesis gives that $u(t)v'(t)$ is non-negative on the interval $[x, y]$, and so $-\int_x^y u(t)v'(t) dt \leq 0$, so an Integration by Part gives the desired inequality. ■

3.2. Sum-Integral Inequality

A well known result is the following, which bound a discrete sum $\sum_{i=x}^y f(i)$ by two integrals for non-decreasing functions, and it is used for Lemma 10.

Lemma 4 *For any $x, y \in \mathbb{N}^*$, $x \leq y$, and f a non-decreasing function on $[0, +\infty)$, then*

$$\int_{x-1}^y f(t) dt \leq \sum_{i=x}^y f(i) \leq \int_x^{y+1} f(t) dt, \quad (4)$$

and

$$f(x) + \int_x^y f(t) dt \leq \sum_{i=x}^y f(i) \leq f(y) + \int_x^y f(t) dt. \quad (5)$$

Proof For the first inequality, both parts comes from the monotony of f and monotony and additivity of the integral. On any interval $[i, i + 1]$, $f(i) = \min_{t \in [i, i+1]} f(t) \leq \int_i^{i+1} f(t) dt$, and $f(i) =$

$\max_{t \in [i-1, i]} f(t) \geq \int_{i-1}^i f(t) dt$, And so, if we sum these terms from $i = x$ to y , we get

$$\sum_{i=x}^y f(i) \leq \sum_{i=x}^y \int_i^{i+1} f(t) dt = \int_x^{y+1} f(t) dt.$$

1. A function of class \mathcal{C}^1 is continuous, differentiable and of continuous derivative on its interval of definition.

as well as

$$\sum_{i=x}^y f(i) \geq \sum_{i=x}^y \int_{i-1}^i f(t) dt = \int_{x-1}^y f(t) dt.$$

The two sides of second inequality are immediate by isolating the first (or last) term of the sum $f(y)$ (or $f(x)$), and applying the first inequality to $x - 1$ instead of x (or $y - 1$ instead of y). ■

4. The Exponential Integral Ei Function

This last Subsection is rather long, and actually not required to obtain the Lemma 1. But I find this Ei function to be quite interesting, so I wanted to write down these proofs. We define the Ei function (Weisstein, 2017; Collective, 2017), by carefully justifying its existence, and then we give two results using it, to obtain the non-trivial Lemma 10.

Lemma 5 For any $\varepsilon > 0$, $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$ exists and is finite, it satisfies this identity

$$I(\varepsilon) = (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du. \quad (6)$$

Additionally, it stays finite when $\varepsilon \rightarrow 0$, and $\lim_{\varepsilon \rightarrow 0} I(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du = 0$.

Proof Fix $\varepsilon > 0$, and let $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$.

Roughly, one just needs to observe² that for u close to 0, $\frac{e^u}{u} \sim \frac{1}{u}$ as $e^u \sim 1$, and $\frac{1}{u}$ can be integrated on $[-\varepsilon, \varepsilon]$, even if it is not defined at 0, because it is odd: $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} du = \lim_{t \rightarrow 0} (\int_{-\varepsilon}^t \frac{1}{u} du + \int_t^{\varepsilon} \frac{1}{u} du)$ (as Cauchy's principal values), and $\int_{-\varepsilon}^t \frac{1}{u} du = -\int_t^{\varepsilon} \frac{1}{v} dv$ with the change of variable $v = -u$. So $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} du = 0$ for any $\varepsilon \geq 0$.

But we have to justify more properly that $I(\varepsilon)$ exists for any $\varepsilon > 0$ and that $I(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$. A first Integration by Part (Lemma 2) with $a(u) = e^u$ and $b'(u) = \frac{1}{u}$, that is $a'(u) = e^u$ and by choosing $b(u) = \log |u|$, gives

$$\begin{aligned} I(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du \\ &= [e^u \log |u|]_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} e^u \log |u| du \\ &= (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - \int_0^\varepsilon (e^u + e^{-u}) \log |u| du. \end{aligned}$$

Let $I_2(\varepsilon) := \int_0^\varepsilon (e^u + e^{-u}) \log u du$. A second Integration by Part (Lemma 2) with $a(u) = e^u + e^{-u}$ and $b'(u) = \log u$, that is $a'(u) = e^u - e^{-u}$ and $b(u) = u \log u - u$ (\mathcal{C}^1 on $(0, \varepsilon]$), gives

$$\begin{aligned} I_2(\varepsilon) &= \int_0^\varepsilon (e^u + e^{-u}) \log u du \\ &= [(e^u + e^{-u})(u \log u - u)]_0^\varepsilon - \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du. \end{aligned}$$

2. The notation $f(u) \sim g(u)$ for $u \rightarrow u_0$ means that $g(u) \neq 0$ and $f(u)/g(u) \rightarrow 1$ for $u \rightarrow u_0$.

Indeed, $b(u) = u \log u - u$ is well defined for $u \rightarrow 0$, as $u \log u \rightarrow 0$, so we can define $b(0) = 0$ to have b of class \mathcal{C}^1 on $[0, \varepsilon]$. Therefore, $I_2(\varepsilon)$ exists, and we have, as wanted, the following identity

$$I(\varepsilon) = (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du.$$

The last integral is well defined and finite, as the integrated function is continuous and finite for all u , even at 0. So this proves that $I(\varepsilon)$ is finite for any $\varepsilon > 0$.

Now, taking $\varepsilon \rightarrow 0$ gives, for each of the three terms in $I(\varepsilon)$,

$$\begin{cases} (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon \sim ((1 + \varepsilon) - (1 - \varepsilon)) \log \varepsilon = 2\varepsilon \log \varepsilon \rightarrow 0 \\ (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) \sim 2b(\varepsilon) \rightarrow 0 \\ \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du \rightarrow 0, \end{cases}$$

so $I(\varepsilon) \rightarrow 0$, as wanted. ■

Lemma 6 For any $0 < \varepsilon \leq 1$, $I(\varepsilon)$ satisfies $I(\varepsilon) \leq e^\varepsilon - e^{-\varepsilon}$. In particular, $I(1) \leq e - e^{-1}$.

Proof For $0 < \varepsilon \leq 1$, $b(1) \geq -1$, and $(e^\varepsilon + e^{-\varepsilon}) \log(\varepsilon) \leq 0$, and so the identity (6) gives $I(\varepsilon) \leq (e^\varepsilon - e^{-\varepsilon}) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du$, but $(e^u - e^{-u})(u \log u - u) \leq 0$ for all $u \in [0, 1]$, so $I(\varepsilon) \leq (e^\varepsilon - e^{-\varepsilon})$ as wanted. In particular, $I(1) \leq (e - e^{-1})$. ■

Definition 7 The Exponential Integral Ei function is defined for $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ by

$$\text{Ei}(x) := \int_{-\infty}^x \frac{e^u}{u} du, \tag{7}$$

where the Cauchy's principal value of the integral is taken.

Proof This integral exists and is finite for $x < 0$, as the function $u \mapsto \frac{e^u}{u}$ is of class \mathcal{C}^1 on $(-\infty, 0)$.

For $x \rightarrow 0$ (from above or from below), $\text{Ei}(x) \rightarrow -\infty$.

And for $x > 0$, let $\varepsilon > 0$, and observe that we can write the integral from $-\infty$ to x as three terms, $\text{Ei}(x) = \text{Ei}(-\varepsilon) + \int_{-\varepsilon}^\varepsilon \frac{e^u}{u} du + \int_\varepsilon^x \frac{e^u}{u} du$. $\text{Ei}(-\varepsilon)$ and the last integral both exist and are finite, thanks to the first case of $x < 0$ and as the function $u \mapsto \frac{e^u}{u}$ is of class \mathcal{C}^1 on $(\varepsilon, +\infty)$. And thanks to Lemma 5, $\int_{-\varepsilon}^\varepsilon \frac{e^u}{u} du$ is finite. So all the three terms in the decomposition of $\text{Ei}(x)$ exist and are finite, therefore $\text{Ei}(x)$ is well defined. ■

A few properties of Ei worth noting include the following: it has a unique zero (located at $x_0 \simeq 0.327$), it is negative for $x < x_0$ and in particular for $x < 0$, it is positive for $x > x_0$, and it is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$. Ei is also concave on $(-\infty, 0)$ and $(0, 1)$, and convex on $(1, +\infty)$.

Illustration We can plot this function³:

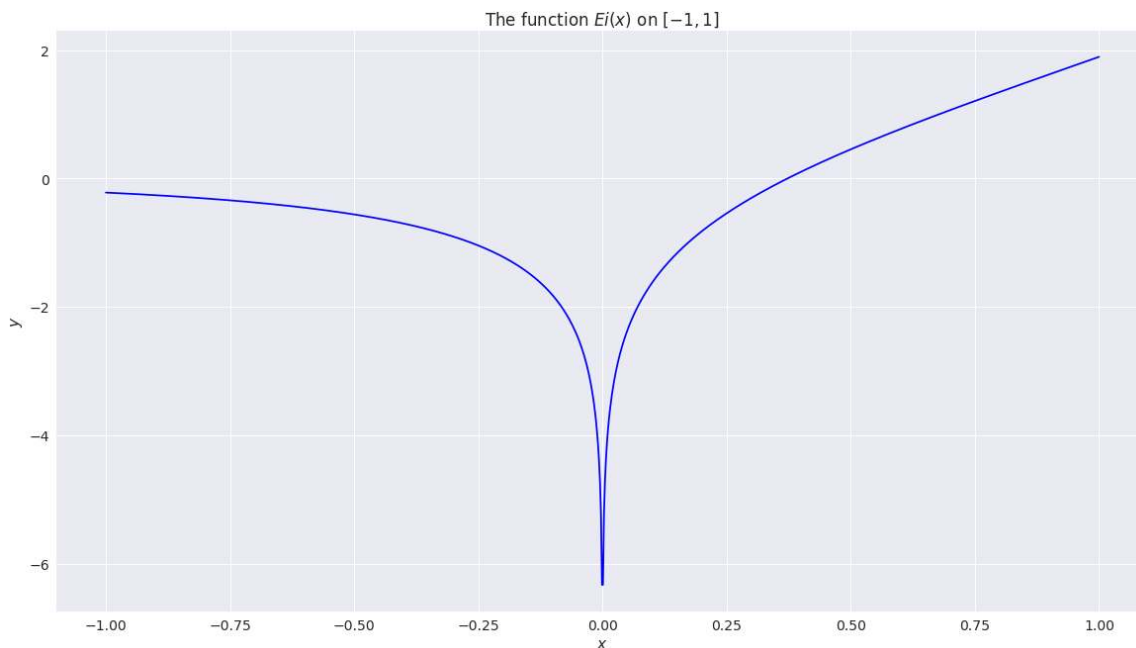


Figure 1: The Ei function on $[-1, 1]$.

5. Using Ei to compute primitives

Lemma 8 For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $a, b > 1$, $\gamma > 0$ and $L > 0$,

$$\int_0^L (a^{bt})^\gamma dt = \frac{1}{\log b} \left(\text{Ei} \left(\gamma \log(a^{bL}) \right) - \text{Ei} \left(\gamma \log(a) \right) \right). \quad (8)$$

Proof A first change of variable with $u := b^t$ gives $dt = \frac{1}{\log b} \frac{1}{u} du$ ($\log b > 0$ as $b > 1$), and so

$$\int_0^L (a^{bt})^\gamma dt = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^u)^\gamma du = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^\gamma)^u du$$

And a second change of variable with $v := \log(a^\gamma)u = \gamma \log(a)u$ gives $\frac{1}{u} du = \frac{1}{v} dv$ (and no change in the order of the integral's bounds, as $\log a > 0$ as $a > 1$), and so

$$\begin{aligned} &= \frac{1}{\log b} \int_{\gamma \log(a)}^{\gamma \log(a) b^L} \frac{e^v}{v} dv = \frac{1}{\log b} \left[\text{Ei}(v) \right]_{\gamma \log(a)}^{\gamma \log(a) b^L} \\ &= \frac{1}{\log b} \left(\text{Ei} \left(\gamma \log \left(a^{b^L} \right) \right) - \text{Ei} \left(\gamma \log(a) \right) \right). \end{aligned}$$

3. See for instance, the `scipy.special.expi` function, on <https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.expi.html>, if you use Python and SciPy (Foundation, 2017; Jones et al., 2001–).

■

6. Inequalities for Ei

Lemma 9 (First Inequalities Using Ei) For any $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\text{Ei}(x) \leq e^x$.

Moreover, for $x \geq 1$, we also have $\text{Ei}(x) \geq \text{Ei}(1) + \frac{e^x - e}{x} \geq -1 + \frac{e^x}{x}$.

A useful consequence is that for any $y \geq 1$ and $0 \leq \gamma \leq 1$,

$$\text{Ei}(\log(y^\gamma)) = \text{Ei}(\gamma \log(y)) \leq y^\gamma. \quad (9)$$

Proof Let $x \in \mathbb{R}$. First, if $x < 0$, then clearly $\text{Ei}(x) \leq 0 < e^x$.

If $0 < x < 1$, we can split the integral defining $\text{Ei}(x)$ in two terms, and as $I(x) = \int_{-x}^x \frac{e^u}{u} du \leq e^x - e^{-x}$ (see Lemma 6),

$$\begin{aligned} \text{Ei}(x) &= \underbrace{\int_{-\infty}^{-x} \frac{e^u}{u} du}_{=\text{Ei}(-x) \leq 0} + \underbrace{\int_{-x}^x \frac{e^u}{u} du}_{=I(x)} \leq I(x) \\ &\leq e^x - e^{-x} \leq e^x. \end{aligned}$$

If $x > 1$, we do the same with three terms, and by using $\text{Ei}(-1) = \int_{-\infty}^{-1} \frac{e^u}{u} du \leq 0$, and $I(1) = \int_{-1}^1 \frac{e^u}{u} du \leq e - e^{-1}$ (see Lemma 6), we have

$$\begin{aligned} \text{Ei}(x) &= \underbrace{\int_{-\infty}^{-1} \frac{e^u}{u} du}_{=\text{Ei}(-1) \leq 0} + \underbrace{\int_{-1}^1 \frac{e^u}{u} du}_{=I(1)} + \underbrace{\int_1^x \frac{e^u}{u} du}_{\leq e^x - e^{-1}} \leq I(1) + e^x - e \\ &\leq e - e^{-1} + e^x - e = e^x - e^{-1} \leq e^x. \end{aligned}$$

Now for the lower bound, let $x \geq 1$, and we use the same splitting. For $I(1)$, we use conversely that $I(1) \geq 0$ (see Lemma 6), and for the integral we have $\int_1^x \frac{e^u}{u} du \geq \frac{1}{x}(e^x - e)$. So $\text{Ei}(x) \geq \text{Ei}(1) + \frac{e^x - e}{x}$. We also have $-\frac{e}{x} \geq -e$ and numerically, $\text{Ei}(1) - e \geq -1$ (as $\text{Ei}(1) \simeq 1.895$), so $\text{Ei}(x) \geq -1 + \frac{e^x}{x}$.

Finally, if $x = \log(y^\gamma)$ and $y \geq 0$, then $e^x = y^\gamma$, so $\text{Ei}(x) = \text{Ei}(\gamma \log(y)) \leq e^x = y^\gamma$. ■

Illustration We can check this inequality $Ei(x) \leq e^x$ graphically, as well as a tighter inequality $Ei(x) \leq Ei(-1) - \frac{1}{e} + e^x$.

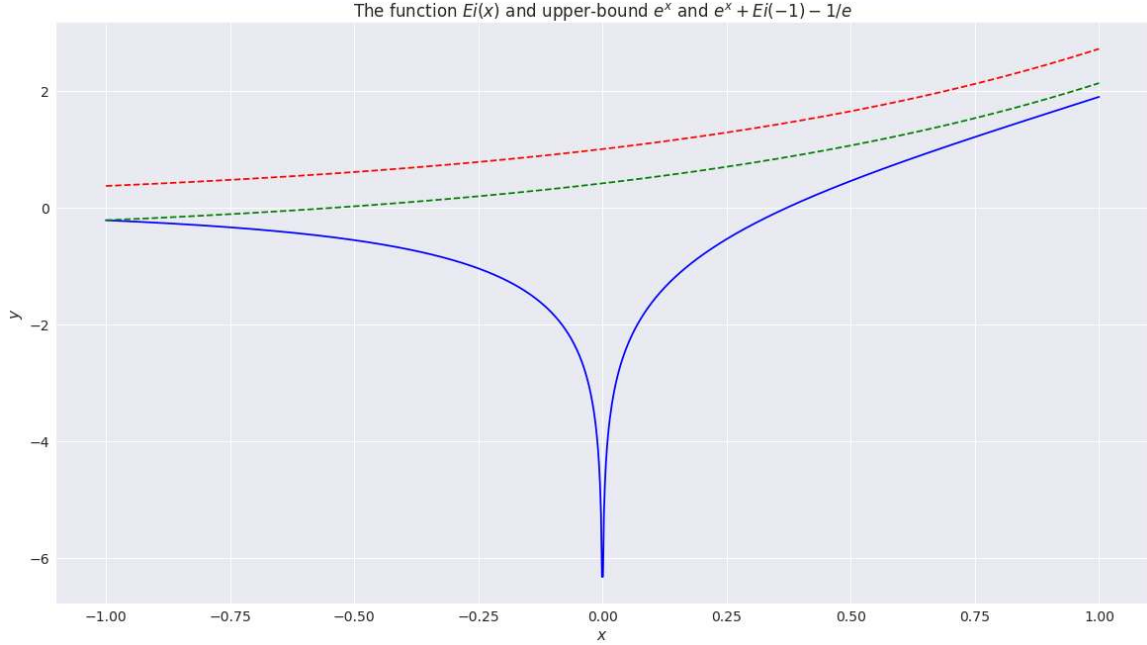


Figure 2: The Ei function and two upper-bounds valid respectively on \mathbb{R} and $[1, +\infty)$.

This last sum-inequality is the result we were looking for.

Lemma 10 (Sum Inequality Using Ei) For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $a, b > 1$, $\gamma > 0$ and $L > 0$, and if $Ei(\gamma \log(a)) \geq 0$, then

$$\sum_{i=0}^{L-1} (a^{b^i})^\gamma \leq \frac{1}{\log b} (a^{b^L})^\gamma. \quad (10)$$

And by isolating the last term, we also have

$$\sum_{i=0}^L (a^{b^i})^\gamma \leq \left(1 + \frac{1}{\log b}\right) (a^{b^L})^\gamma \quad (11)$$

Proof Using the sum-integral inequality (Lemma 4) and then Lemma 8, we have directly that

$$\begin{aligned} \sum_{i=0}^{L-1} (a^{b^i})^\gamma &\leq \int_0^L (a^{b^t})^\gamma dt \\ &\leq \frac{1}{\log b} Ei\left(\gamma \log(a^{b^L})\right) \leq \frac{1}{\log b} (a^{b^L})^\gamma. \end{aligned}$$



In particular, this inequality (11) holds as soon as $a \geq e^{0.373/\gamma}$, as $\gamma \log(a) \geq 0.373 > x_0 \implies \text{Ei}(\gamma \log(a)) \geq \text{Ei}(x_0) > 0$ and $x_0 \simeq 0.372507 \geq 0.373$. For instance, $\gamma = 1/2$ gives $a \geq e^{0.373/\gamma} = e^{0.746} \simeq 2.107$, close to the simplest value $a = 2$.

And if $\gamma = 0$, then $\text{Ei}(\gamma \log(a))$ cannot be ≥ 0 , but the sum in (10) is constant and equals to L .

7. Conclusion

This small note defines and studies a useful non-canonical function called the “exponential integral” function, Ei, and we use it to find a bound on any sum of the form $\sum_{i=0}^L (a^{b^i})^\gamma$.

Note: the simulation code used for the experiments is using Python 3, (Foundation, 2017), and Matplotlib (Hunter, 2007) for plotting, as well as SciPy (Jones et al., 2001–). It is open-sourced at github.com/Naareen/notebooks/blob/master/Exponential_Integral_Python.ipynb. This document is also distributed under the open-source MIT License, and is available online at perso.crans.org/besson/publis/A_note_on_the_Ei_function.pdf.

References

- Collective. Exponential Integral. https://en.wikipedia.org/wiki/Exponential_integral, 2017. From Wikipedia, The Free Encyclopedia.
- Python Software Foundation. Python language reference, version 3.6. Online at: www.python.org, October 2017. URL <https://www.python.org>.
- John D. Hunter. Matplotlib: A 2D graphics environment. *Computing In Science & Engineering*, 9(3):90–95, 2007. doi: 10.1109/MCSE.2007.55.
- Eric Jones, Travis E. Oliphant, Pearu Peterson, et al. SciPy: Open source scientific tools for Python. Online at: www.scipy.org, 2001–. URL <http://www.scipy.org>.
- E. W. Weisstein. Exponential Integral. <http://mathworld.wolfram.com/ExponentialIntegral.html>, 2017. From MathWorld – A Wolfram Web Resource.