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A Note on the Ei Function and a Useful Sum-Inequality

Lilian Besson

LILIAN.BESSON@CENTRALESUPELEC.FR

CentraleSupélec (campus of Rennes), IETR, SCEE Team, Avenue de la Boulaie – CS 47601, F-35576 Cesson-Sévigné, France

Abstract

This short note defines formally the Ei function, and gives some interesting inequalities and integration using it. I illustrate the inequalities and detail what is still to be proved.

Keywords: Analysis; Inequalities; Primitive.

1. Introduction and Motivation

Take $a>1,\,b>1$ and $\gamma>0$, and for an integer L>0 consider the sum $\sum_{i=1}^L (a^{b^i})^\gamma$. We want to bound it, and the goal is to show that it is bounded by a constant times its last term. A first naive bound is $\sum_{i=1}^L (a^{b^i})^\gamma \leq (L+1)(a^{b^L})^\gamma$ which is too brutal as soon as $L\to\infty$.

I first remind and prove two useful elementary results, and then we define and study the Ei function, to finally prove the desired inequality.

2. Lemma and Proof

Lemma 1 For any $n \in \mathbb{N}^*$, a > 1, b > 1 and $\gamma > 0$, we have

$$\sum_{i=0}^{n} (a^{b^{i}})^{\gamma} \le a^{\gamma} + \left(1 + \frac{1}{(\log(a))(\log(b^{\gamma}))}\right) (a^{b^{n}})^{\gamma} = \mathcal{O}\left((a^{b^{n}})^{\gamma}\right). \tag{1}$$

Proof We first isolate both the first and last term in the sum and focus on the from i=1 sum up to i=n-1. As the function $t\mapsto (a^{b^t})^{\gamma}$ is increasing for $t\geq 1$, we use a sum-integral inequality, and then the change of variable $u:=\gamma b^t$, of Jacobian $\mathrm{d} t=\frac{1}{\log b}\frac{\mathrm{d} u}{u}$, gives

$$\sum_{i=1}^{n-1} (a^{b^i})^{\gamma} \le \int_1^n a^{\gamma b^t} dt \le \frac{1}{\log(b^{\gamma})} \int_{\gamma b}^{\gamma b^n} \frac{a^u}{u} du$$

Now for $u \geq 1$, observe that $\frac{a^u}{u} \leq a^u$, and as $\gamma b > 1$, we have

$$\leq \frac{1}{\log(b^{\gamma})} \int_{\gamma b}^{\gamma b^n} a^u \, \mathrm{d}u \leq \frac{1}{\log(b^{\gamma})} \frac{1}{\log(a)} a^{\gamma b^n} = \frac{1}{(\log(a))(\log(b^{\gamma}))} (a^{b^n})^{\gamma}.$$

Finally, we obtain as desired,
$$\sum_{i=0}^n (a^{b^i})^{\gamma} \leq a^{\gamma} + (a^{b^n})^{\gamma} + \frac{1}{(\log(a))(\log(b^{\gamma}))} (a^{b^n})^{\gamma}$$
.

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3. Elementary Results

3.1. Integration by Part

The Integration by Part is a basic but useful result to establish inequalities, *e.g.*, for Lemma 10 using Lemma 3, and to prove the existence of finite integrals, *e.g.*, for Lemma 5 using two chained IP.

Lemma 2 (Integration by Part (IP)) Let $x, y \in \mathbb{R}$, $x \leq y$, and u, v two functions of class¹ C^1 , and with this notation $[uv]_x^y := u(y)v(y) - u(x)v(x)$, then

$$\int_{x}^{y} u(t)v'(t) dt = \left[uv\right]_{x}^{y} - \int_{x}^{y} u'(t)v(t) dt.$$
 (2)

Proof The two integrals and the two evaluations are well defined by the C^1 hypothesis on both u and v (u and v are continuous at x and y and u'v is continuous so integrable on the interval [x, y]).

The product function uv is differentiable, and (uv)' = u'v + uv', so $[uv]_x^y = \int_x^y (uv)'(t) dt = \int_x^y u(t)v'(t) dt + \int_x^y u'(t)v(t) dt$ as wanted, by the linearity of the integral.

Lemma 3 (IP Inequality) If both u, v are non-negative, and non-decreasing, then

$$\int_{x}^{y} u(t)v'(t) dt \le u(y)v(y). \tag{3}$$

Proof The non-negativeness gives that $-u(x)v(x) \leq 0$ and the monotony hypothesis gives that u(t)v'(t) is non-negative on the interval [x,y], and so $-\int_x^y u(t)v'(t) \, \mathrm{d}t \leq 0$, so an Integration by Part gives the desired inequality.

3.2. Sum-Integral Inequality

A well known result is the following, which bound a discrete sum $\sum_{i=x}^{y} f(i)$ by two integrals for non-decreasing functions, and it is used for Lemma 10.

Lemma 4 For any $x, y \in \mathbb{N}^*$, $x \leq y$, and f a non-decreasing function on $[0, +\infty)$, then

$$\int_{x-1}^{y} f(t) dt \le \sum_{i=x}^{y} f(i) \le \int_{x}^{y+1} f(t) dt,$$
 (4)

and

$$f(x) + \int_{x}^{y} f(t) dt \le \sum_{i=x}^{y} f(i) \le f(y) + \int_{x}^{y} f(t) dt.$$
 (5)

Proof For the first inequality, both parts comes from the monotony of f and monotony and additivity of the integral. On any interval [i,i+1], $f(i) = \min_{t \in [i,i+1]} f(t) \le \int_i^{i+1} f(t) \, dt$, and $f(i) = \int_i^{i+1} f(t) \, dt$

 $\max_{t \in [i-1,i]} f(t) \ge \int_{i-1}^i f(t) \; \mathrm{d}t$, And so, if we sum these terms from i=x to y, we get

$$\sum_{i=x}^{y} f(i) \le \sum_{i=x}^{y} \int_{i}^{i+1} f(t) dt = \int_{x}^{y+1} f(t) dt.$$

^{1.} A function of class \mathcal{C}^1 is continuous, differentiable and of continuous derivative on its interval of definition.

as well as

$$\sum_{i=x}^{y} f(i) \ge \sum_{i=x}^{y} \int_{i-1}^{i} f(t) dt = \int_{x-1}^{y} f(t) dt.$$

The two sides of second inequality are immediate by isolating the first (or last) term of the sum f(y) (or f(x)), and applying the first inequality to x-1 instead of x (or y-1 instead of y).

4. The Exponential Integral Ei Function

This last Subsection is rather long, and actually not required to obtain the Lemma 1. But I find this Ei function to be quite interesting, so I wanted to write down these proofs. We define the Ei function (Weisstein, 2017; Collective, 2017), by carefully justifying its existence, and then we give two results using it, to obtain the non-trivial Lemma 10.

Lemma 5 For any $\varepsilon > 0$, $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$ exists and is finite, it satisfies this identity

$$I(\varepsilon) = (e^{\varepsilon} - e^{-\varepsilon})\log \varepsilon - (e^{\varepsilon} + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^{\varepsilon} (e^u - e^{-u})(u \log u - u) du.$$
 (6)

Additionally, it stays finite when $\varepsilon \to 0$, and $\lim_{\varepsilon \to 0} I(\varepsilon) = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u} \, \mathrm{d}u = 0$.

Proof Fix $\varepsilon > 0$, and let $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$.

Roughly, one just needs to observe² that for u close to 0, $\frac{\mathrm{e}^u}{u} \sim \frac{1}{u}$ as $\mathrm{e}^u \sim 1$, and $\frac{1}{u}$ can be integrated on $[-\varepsilon, \varepsilon]$, even if it is not defined at 0, because it is odd: $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} \, \mathrm{d}u = \lim_{t \to 0} (\int_{-\varepsilon}^{t} \frac{1}{u} \, \mathrm{d}u + \int_{t}^{\varepsilon} \frac{1}{u} \, \mathrm{d}u)$ (as Cauchy's principal values), and $\int_{-\varepsilon}^{t} \frac{1}{u} \, \mathrm{d}u = -\int_{t}^{\varepsilon} \frac{1}{v} \, \mathrm{d}v$ with the change of variable v = -u. So $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} \, \mathrm{d}u = 0$ for any $\varepsilon \geq 0$.

But we have to justify more properly that $I(\varepsilon)$ exists for any $\varepsilon > 0$ and that $I(\varepsilon) \to 0$ for $\varepsilon \to 0$. A first Integration by Part (Lemma 2) with $a(u) = \mathrm{e}^u$ and $b'(u) = \frac{1}{u}$, that is $a'(u) = \mathrm{e}^u$ and by choosing $b(u) = \log |u|$, gives

$$I(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} \frac{e^{u}}{u} du$$

$$= \left[e^{u} \log |u| \right]_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} e^{u} \log |u| du$$

$$= \left(e^{\varepsilon} - e^{-\varepsilon} \right) \log \varepsilon - \int_{0}^{\varepsilon} (e^{u} + e^{-u}) \log |u| du.$$

Let $I_2(\varepsilon) := \int_0^{\varepsilon} (e^u + e^{-u}) \log u \, du$. A second Integration by Part (Lemma 2) with $a(u) = e^u + e^{-u}$ and $b'(u) = \log u$, that is $a'(u) = e^u - e^{-u}$ and $b(u) = u \log u - u \, (\mathcal{C}^1 \text{ on } (0, \varepsilon))$, gives

$$I_2(\varepsilon) = \int_0^{\varepsilon} (e^u + e^{-u}) \log u \, du$$
$$= \left[(e^u + e^{-u})(u \log u - u) \right]_0^{\varepsilon} - \int_0^{\varepsilon} (e^u - e^{-u})(u \log u - u) \, du.$$

^{2.} The notation $f(u) \sim q(u)$ for $u \to u_0$ means that $q(u) \neq 0$ and $f(u)/q(u) \to 1$ for $u \to u_0$.

Indeed, $b(u) = u \log u - u$ is well defined for $u \to 0$, as $u \log u \to 0$, so we can define b(0) = 0 to have b of class C^1 on $[0, \varepsilon]$. Therefore, $I_2(\varepsilon)$ exists, and we have, as wanted, the following identity

$$I(\varepsilon) = (e^{\varepsilon} - e^{-\varepsilon})\log \varepsilon - (e^{\varepsilon} + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^{\varepsilon} (e^u - e^{-u})(u \log u - u) du.$$

The last integral is well defined and finite, as the integrated function is continuous and finite for all u, even at 0. So this proves that $I(\varepsilon)$ is finite for any $\varepsilon > 0$.

Now, taking $\varepsilon \to 0$ gives, for each of the three terms in $I(\varepsilon)$,

$$\begin{cases} (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}) \log \varepsilon \sim ((1+\varepsilon) - (1-\varepsilon)) \log \varepsilon = 2\varepsilon \log \varepsilon \to 0 \\ (\mathrm{e}^{\varepsilon} + \mathrm{e}^{-\varepsilon}) (\varepsilon \log \varepsilon - \varepsilon) \sim 2b(\varepsilon) \to 0 \\ \int_{0}^{\varepsilon} (\mathrm{e}^{u} - \mathrm{e}^{-u}) (u \log u - u) \, \mathrm{d}u \to 0, \end{cases}$$

so $I(\varepsilon) \to 0$, as wanted.

Lemma 6 For any $0 < \varepsilon \le 1$, $I(\varepsilon)$ satisfies $I(\varepsilon) \le e^{\varepsilon} - e^{-\varepsilon}$. In particular, $I(1) \le e - e^{-1}$.

Proof For $0 < \varepsilon \le 1$, $b(1) \ge -1$, and $(e^{\varepsilon} + e^{-\varepsilon})\log(\varepsilon) \le 0$, and so the identity (6) gives $I(\varepsilon) \le (e^{\varepsilon} - e^{-\varepsilon}) + \int_0^{\varepsilon} (e^u - e^{-u})(u\log u - u) \ du$, but $(e^u - e^{-u})(u\log u - u) \le 0$ for all $u \in [0,1]$, so $I(\varepsilon) \le (e^{\varepsilon} - e^{-\varepsilon})$ as wanted. In particular, $I(1) \le (e - e^{-1})$.

Definition 7 The Exponential Integral Ei function is defined for $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ by

$$\operatorname{Ei}(x) := \int_{-\infty}^{x} \frac{\mathrm{e}^{u}}{u} \, \mathrm{d}u,\tag{7}$$

where the Cauchy's principal value of the integral is taken.

Proof This integral exists and is finite for x < 0, as the function $u \mapsto \frac{e^u}{u}$ is of class \mathcal{C}^1 on $(-\infty, 0)$. For $x \to 0$ (from above or from below), $\operatorname{Ei}(x) \to -\infty$.

And for x>0, let $\varepsilon>0$, and observe that we can write the integral from $-\infty$ to x as three terms, $\mathrm{Ei}(x)=\mathrm{Ei}(-\varepsilon)+\int_{-\varepsilon}^{\varepsilon}\frac{\mathrm{e}^{u}}{u}\;\mathrm{d}u+\int_{\varepsilon}^{x}\frac{\mathrm{e}^{u}}{u}\;\mathrm{d}u.$ $\mathrm{Ei}(-\varepsilon)$ and the last integral both exists and are finite, thanks to the first case of x<0 and as the function $u\mapsto\frac{\mathrm{e}^{u}}{u}$ is of class \mathcal{C}^{1} on $(\varepsilon,+\infty)$. And thanks to Lemma 5, $\int_{-\varepsilon}^{\varepsilon}\frac{\mathrm{e}^{u}}{u}\;\mathrm{d}u$ is finite. So all the three terms in the decomposition of $\mathrm{Ei}(x)$ exist and are finite, therefore $\mathrm{Ei}(x)$ is well defined.

A few properties of Ei worth noting include the following: it has a unique zero (located at $x_0 \simeq 0.327$), it is negative for $x < x_0$ and in particular for x < 0, it is positive for $x > x_0$, and it is decreasing on $(-\infty,0)$ and increasing on $(0,+\infty)$. Ei is also concave on $(-\infty,0)$ and (0,1), and convex on $(1,+\infty)$.

Illustration We can plot this function³:

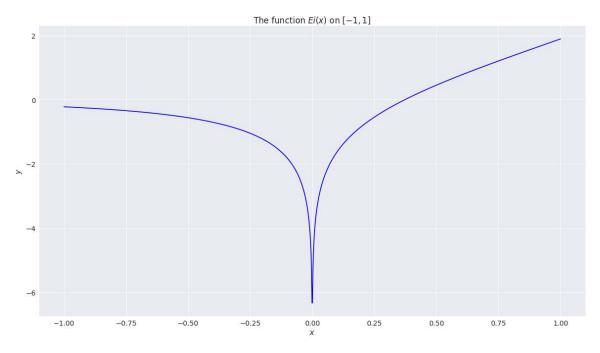


Figure 1: The Ei function on [-1, 1].

5. Using Ei to compute primitives

Lemma 8 For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that a, b > 1, $\gamma > 0$ and L > 0,

$$\int_0^L (a^{b^t})^{\gamma} dt = \frac{1}{\log b} \left(\operatorname{Ei} \left(\gamma \log(a^{b^L}) \right) - \operatorname{Ei} \left(\gamma \log(a) \right) \right). \tag{8}$$

Proof A first change of variable with $u:=b^t$ gives $dt=\frac{1}{\log b}\frac{1}{u}\ du\ (\log b>0\ \text{as}\ b>1)$, and so

$$\int_0^L (a^{b^t})^{\gamma} dt = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^u)^{\gamma} du = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^{\gamma})^u du$$

And a second change of variable with $v := \log(a^{\gamma})u = \gamma \log(a)u$ gives $\frac{1}{u} du = \frac{1}{v} dv$ (and no change in the order of the integral's bounds, as $\log a > 0$ as a > 1), and so

$$= \frac{1}{\log b} \int_{\gamma \log(a)}^{\gamma \log(a)b^L} \frac{e^v}{v} dv = \frac{1}{\log b} \left[\operatorname{Ei}(v) \right]_{\gamma \log(a)}^{\gamma \log(a^{b^L})}$$
$$= \frac{1}{\log b} \left(\operatorname{Ei} \left(\gamma \log \left(a^{b^L} \right) \right) - \operatorname{Ei} \left(\gamma \log(a) \right) \right).$$

^{3.} See for instance, the scipy.special.expi function, on https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.expi.html, if you use Python and SciPy (Foundation, 2017; Jones et al., 2001-).

6. Inequalities for Ei

Lemma 9 (First Inequalities Using Ei) For any $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\operatorname{Ei}(x) \leq \operatorname{e}^x$. Moreover, for $x \geq 1$, we also have $\operatorname{Ei}(x) \geq \operatorname{Ei}(1) + \frac{\operatorname{e}^x - \operatorname{e}}{x} \geq -1 + \frac{\operatorname{e}^x}{x}$. A useful consequence is that for any $y \geq 1$ and $0 \leq \gamma \leq 1$,

$$\operatorname{Ei}(\log(y^{\gamma})) = \operatorname{Ei}(\gamma \log(y)) \le y^{\gamma}.$$
 (9)

Proof Let $x \in \mathbb{R}$. First, if x < 0, then clearly $\text{Ei}(x) \le 0 < e^x$.

If 0 < x < 1, we can split the integral defining Ei(x) in two terms, and as $I(x) = \int_{-x}^{x} \frac{e^{u}}{u} du \le e^{x} - e^{-x}$ (see Lemma 6),

$$\operatorname{Ei}(x) = \underbrace{\int_{-\infty}^{-x} \frac{e^{u}}{u} du}_{=\operatorname{Ei}(-x) \le 0} + \underbrace{\int_{-x}^{x} \frac{e^{u}}{u} du}_{=I(x)} \le I(x)$$
$$\le e^{x} - e^{-x} \le e^{x}.$$

If x>1, we do the same with three terms, and by using $\mathrm{Ei}(-1)=\int_{-\infty}^{-1}\frac{\mathrm{e}^u}{u}\;\mathrm{d}u\leq 0$, and $I(1)=\int_{-1}^{1}\frac{\mathrm{e}^u}{u}\;\mathrm{d}u\leq \mathrm{e}-\mathrm{e}^{-1}$ (see Lemma 6), we have

$$\operatorname{Ei}(x) = \underbrace{\int_{-\infty}^{-1} \frac{e^{u}}{u} du}_{=\operatorname{Ei}(-1) \le 0} + \underbrace{\int_{-1}^{1} \frac{e^{u}}{u} du}_{=I(1)} + \underbrace{\int_{1}^{x} \frac{e^{u}}{u} du}_{\le e^{x} - e^{1}} \le e^{x} - e^{-1} \le e^{x}.$$

Now for the lower bound, let $x \ge 1$, and we use the same splitting. For I(1), we use conversely that $I(1) \ge 0$ (see Lemma 6), and for the integral we have $\int_1^x \frac{\mathrm{e}^u}{u} \, \mathrm{d}u \ge \frac{1}{x} (\mathrm{e}^x - \mathrm{e})$. So $\mathrm{Ei}(x) \ge \mathrm{Ei}(1) + \frac{\mathrm{e}^x - \mathrm{e}}{x}$. We also have $-\frac{\mathrm{e}}{x} \ge -\mathrm{e}$ and numerically, $\mathrm{Ei}(1) - \mathrm{e} \ge -1$ (as $\mathrm{Ei}(1) \simeq 1.895$), so $\mathrm{Ei}(x) \ge -1 + \frac{\mathrm{e}^x}{x}$.

Finally, if $x = \log(y^{\gamma})$ and $y \ge 0$, then $e^x = y^{\gamma}$, so $\text{Ei}(x) = \text{Ei}(\gamma \log(y)) \le e^x = y^{\gamma}$.

Illustration We can check this inequality $\text{Ei}(x) \leq e^x$ graphically, as well as a tighter inequality $\text{Ei}(x) \leq \text{Ei}(-1) - \frac{1}{e} + e^x$.

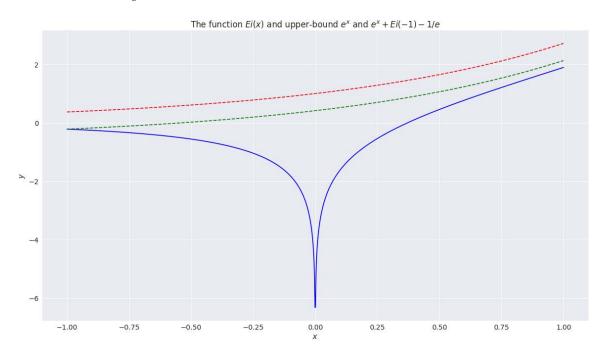


Figure 2: The Ei function and two upper-bounds valid respectively on \mathbb{R} and $[1, +\infty)$.

This last sum-inequality is the result we were looking for.

Lemma 10 (Sum Inequality Using Ei) *For any* $a,b,\gamma \in \mathbb{R}$ *and* $L \in \mathbb{N}$ *such that* a,b>1, $\gamma>0$ *and* L>0, *and if* Ei $(\gamma \log(a)) \geq 0$, *then*

$$\sum_{i=0}^{L-1} (a^{b^i})^{\gamma} \le \frac{1}{\log b} (a^{b^L})^{\gamma}. \tag{10}$$

And by isolating the last term, we also have

$$\sum_{i=0}^{L} (a^{b^i})^{\gamma} \le \left(1 + \frac{1}{\log b}\right) (a^{b^L})^{\gamma} \tag{11}$$

Proof Using the sum-integral inequality (Lemma 4) and then Lemma 8, we have directly that

$$\sum_{i=0}^{L-1} (a^{b^i})^{\gamma} \le \int_0^L (a^{b^t})^{\gamma} dt$$

$$\le \frac{1}{\log b} \operatorname{Ei} \left(\gamma \log(a^{b^L}) \right) \le \frac{1}{\log b} (a^{b^L})^{\gamma}.$$

In particular, this inequality (11) holds as soon as $a \ge e^{0.373/\gamma}$, as $\gamma \log(a) \ge 0.373 > x_0 \Longrightarrow \text{Ei}\left(\gamma \log(a)\right) \ge \text{Ei}(x_0) > 0$ and $x_0 \simeq 0.372507 \ge 0.373$. For instance, $\gamma = 1/2$ gives $a \ge e^{0.373/\gamma} = e^{0.746} \simeq 2.107$, close to the simplest value a = 2.

And if $\gamma = 0$, then $\text{Ei}(\gamma \log(a))$ cannot be ≥ 0 , but the sum in (10) is constant and equals to L.

7. Conclusion

This small note defines and studies a useful non-canonical function called the "exponential integral" function, Ei, and we use it to find a bound on any sum of the form $\sum_{i=0}^{L} (a^{b^i})^{\gamma}$.

Note: the simulation code used for the experiments is using Python 3, (Foundation, 2017), and Matplotlib (Hunter, 2007) for plotting, as well as SciPy (Jones et al., 2001–). It is open-sourced at github.com/Naereen/notebooks/blob/master/Exponential_Integral_Python.ipynb. This document is also distributed under the open-source MIT License, and is available online at perso.crans.org/besson/publis/A_note_on_the_Ei_function.pdf.

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