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# A Note on the Ei Function and a Useful Sum-Inequality

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## Abstract

This short note defines formally the Ei function, and gives some interesting inequalities and integration using it. I illustrate the inequalities and detail what is still to be proved.

**Keywords:** Analysis; Inequalities; Primitive.

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## 1. Introduction and Motivation

Take  $a > 1$ ,  $b > 1$  and  $\gamma > 0$ , and for an integer  $L > 0$  consider the sum  $\sum_{i=1}^L (a^{b^i})^\gamma$ . We want to bound it, and the goal is to show that it is bounded by a constant times its last term. A first naive bound is  $\sum_{i=1}^L (a^{b^i})^\gamma \leq (L+1)(a^{b^L})^\gamma$  which is too brutal as soon as  $L \rightarrow \infty$ .

I first remind and prove two useful elementary results, and then we define and study the Ei function, to finally prove the desired inequality.

## 2. Lemma and Proof

**Lemma 1** *For any  $n \in \mathbb{N}^*$ ,  $a > 1$ ,  $b > 1$  and  $\gamma > 0$ , we have*

$$\sum_{i=0}^n (a^{b^i})^\gamma \leq a^\gamma + \left(1 + \frac{1}{(\log(a))(\log(b^\gamma))}\right) (a^{b^n})^\gamma = \mathcal{O}\left((a^{b^n})^\gamma\right). \quad (1)$$

**Proof** We first isolate both the first and last term in the sum and focus on the from  $i = 1$  sum up to  $i = n - 1$ . As the function  $t \mapsto (a^{b^t})^\gamma$  is increasing for  $t \geq 1$ , we use a sum-integral inequality, and then the change of variable  $u := \gamma b^t$ , of Jacobian  $dt = \frac{1}{\log b} \frac{du}{u}$ , gives

$$\sum_{i=1}^{n-1} (a^{b^i})^\gamma \leq \int_1^n a^{\gamma b^t} dt \leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} \frac{a^u}{u} du$$

Now for  $u \geq 1$ , observe that  $\frac{a^u}{u} \leq a^u$ , and as  $\gamma b > 1$ , we have

$$\leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} a^u du \leq \frac{1}{\log(b^\gamma)} \frac{1}{\log(a)} a^{\gamma b^n} = \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma.$$

Finally, we obtain as desired,  $\sum_{i=0}^n (a^{b^i})^\gamma \leq a^\gamma + (a^{b^n})^\gamma + \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma$ . ■

### 3. Elementary Results

#### 3.1. Integration by Part

The Integration by Part is a basic but useful result to establish inequalities, *e.g.*, for Lemma 10 using Lemma 3, and to prove the existence of finite integrals, *e.g.*, for Lemma 5 using two chained IP.

**Lemma 2 (Integration by Part (IP))** *Let  $x, y \in \mathbb{R}$ ,  $x \leq y$ , and  $u, v$  two functions of class<sup>1</sup>  $\mathcal{C}^1$ , and with this notation  $[uv]_x^y := u(y)v(y) - u(x)v(x)$ , then*

$$\int_x^y u(t)v'(t) dt = [uv]_x^y - \int_x^y u'(t)v(t) dt. \quad (2)$$

**Proof** The two integrals and the two evaluations are well defined by the  $\mathcal{C}^1$  hypothesis on both  $u$  and  $v$  ( $u$  and  $v$  are continuous at  $x$  and  $y$  and  $u'v$  is continuous so integrable on the interval  $[x, y]$ ).

The product function  $uv$  is differentiable, and  $(uv)' = u'v + uv'$ , so  $[uv]_x^y = \int_x^y (uv)'(t) dt = \int_x^y u(t)v'(t) dt + \int_x^y u'(t)v(t) dt$  as wanted, by the linearity of the integral. ■

**Lemma 3 (IP Inequality)** *If both  $u, v$  are non-negative, and non-decreasing, then*

$$\int_x^y u(t)v'(t) dt \leq u(y)v(y). \quad (3)$$

**Proof** The non-negativeness gives that  $-u(x)v(x) \leq 0$  and the monotony hypothesis gives that  $u(t)v'(t)$  is non-negative on the interval  $[x, y]$ , and so  $-\int_x^y u(t)v'(t) dt \leq 0$ , so an Integration by Part gives the desired inequality. ■

#### 3.2. Sum-Integral Inequality

A well known result is the following, which bound a discrete sum  $\sum_{i=x}^y f(i)$  by two integrals for non-decreasing functions, and it is used for Lemma 10.

**Lemma 4** *For any  $x, y \in \mathbb{N}^*$ ,  $x \leq y$ , and  $f$  a non-decreasing function on  $[0, +\infty)$ , then*

$$\int_{x-1}^y f(t) dt \leq \sum_{i=x}^y f(i) \leq \int_x^{y+1} f(t) dt, \quad (4)$$

and

$$f(x) + \int_x^y f(t) dt \leq \sum_{i=x}^y f(i) \leq f(y) + \int_x^y f(t) dt. \quad (5)$$

**Proof** For the first inequality, both parts comes from the monotony of  $f$  and monotony and additivity of the integral. On any interval  $[i, i+1]$ ,  $f(i) = \min_{t \in [i, i+1]} f(t) \leq \int_i^{i+1} f(t) dt$ , and  $f(i) =$

$\max_{t \in [i-1, i]} f(t) \geq \int_{i-1}^i f(t) dt$ . And so, if we sum these terms from  $i = x$  to  $y$ , we get

$$\sum_{i=x}^y f(i) \leq \sum_{i=x}^y \int_i^{i+1} f(t) dt = \int_x^{y+1} f(t) dt.$$

1. A function of class  $\mathcal{C}^1$  is continuous, differentiable and of continuous derivative on its interval of definition.

as well as

$$\sum_{i=x}^y f(i) \geq \sum_{i=x}^y \int_{i-1}^i f(t) dt = \int_{x-1}^y f(t) dt.$$

The two sides of second inequality are immediate by isolating the first (or last) term of the sum  $f(y)$  (or  $f(x)$ ), and applying the first inequality to  $x - 1$  instead of  $x$  (or  $y - 1$  instead of  $y$ ). ■

#### 4. The Exponential Integral Ei Function

This last Subsection is rather long, and actually not required to obtain the Lemma 1. But I find this Ei function to be quite interesting, so I wanted to write down these proofs. We define the Ei function (Weisstein, 2017; Collective, 2017), by carefully justifying its existence, and then we give two results using it, to obtain the non-trivial Lemma 10.

**Lemma 5** For any  $\varepsilon > 0$ ,  $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$  exists and is finite, it satisfies this identity

$$I(\varepsilon) = (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du. \quad (6)$$

Additionally, it stays finite when  $\varepsilon \rightarrow 0$ , and  $\lim_{\varepsilon \rightarrow 0} I(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du = 0$ .

**Proof** Fix  $\varepsilon > 0$ , and let  $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$ .

Roughly, one just needs to observe<sup>2</sup> that for  $u$  close to 0,  $\frac{e^u}{u} \sim \frac{1}{u}$  as  $e^u \sim 1$ , and  $\frac{1}{u}$  can be integrated on  $[-\varepsilon, \varepsilon]$ , even if it is not defined at 0, because it is odd:  $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} du = \lim_{t \rightarrow 0} (\int_{-\varepsilon}^t \frac{1}{u} du + \int_t^{\varepsilon} \frac{1}{u} du)$  (as Cauchy's principal values), and  $\int_{-\varepsilon}^t \frac{1}{u} du = -\int_t^{\varepsilon} \frac{1}{v} dv$  with the change of variable  $v = -u$ . So  $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} du = 0$  for any  $\varepsilon \geq 0$ .

But we have to justify more properly that  $I(\varepsilon)$  exists for any  $\varepsilon > 0$  and that  $I(\varepsilon) \rightarrow 0$  for  $\varepsilon \rightarrow 0$ . A first Integration by Part (Lemma 2) with  $a(u) = e^u$  and  $b'(u) = \frac{1}{u}$ , that is  $a'(u) = e^u$  and by choosing  $b(u) = \log |u|$ , gives

$$\begin{aligned} I(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du \\ &= [e^u \log |u|]_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} e^u \log |u| du \\ &= (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - \int_0^\varepsilon (e^u + e^{-u}) \log |u| du. \end{aligned}$$

Let  $I_2(\varepsilon) := \int_0^\varepsilon (e^u + e^{-u}) \log u du$ . A second Integration by Part (Lemma 2) with  $a(u) = e^u + e^{-u}$  and  $b'(u) = \log u$ , that is  $a'(u) = e^u - e^{-u}$  and  $b(u) = u \log u - u$  ( $\mathcal{C}^1$  on  $(0, \varepsilon]$ ), gives

$$\begin{aligned} I_2(\varepsilon) &= \int_0^\varepsilon (e^u + e^{-u}) \log u du \\ &= [(e^u + e^{-u})(u \log u - u)]_0^\varepsilon - \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du. \end{aligned}$$

2. The notation  $f(u) \sim g(u)$  for  $u \rightarrow u_0$  means that  $g(u) \neq 0$  and  $f(u)/g(u) \rightarrow 1$  for  $u \rightarrow u_0$ .

Indeed,  $b(u) = u \log u - u$  is well defined for  $u \rightarrow 0$ , as  $u \log u \rightarrow 0$ , so we can define  $b(0) = 0$  to have  $b$  of class  $\mathcal{C}^1$  on  $[0, \varepsilon]$ . Therefore,  $I_2(\varepsilon)$  exists, and we have, as wanted, the following identity

$$I(\varepsilon) = (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du.$$

The last integral is well defined and finite, as the integrated function is continuous and finite for all  $u$ , even at 0. So this proves that  $I(\varepsilon)$  is finite for any  $\varepsilon > 0$ .

Now, taking  $\varepsilon \rightarrow 0$  gives, for each of the three terms in  $I(\varepsilon)$ ,

$$\begin{cases} (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon \sim ((1 + \varepsilon) - (1 - \varepsilon)) \log \varepsilon = 2\varepsilon \log \varepsilon \rightarrow 0 \\ (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) \sim 2b(\varepsilon) \rightarrow 0 \\ \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du \rightarrow 0, \end{cases}$$

so  $I(\varepsilon) \rightarrow 0$ , as wanted. ■

**Lemma 6** For any  $0 < \varepsilon \leq 1$ ,  $I(\varepsilon)$  satisfies  $I(\varepsilon) \leq e^\varepsilon - e^{-\varepsilon}$ . In particular,  $I(1) \leq e - e^{-1}$ .

**Proof** For  $0 < \varepsilon \leq 1$ ,  $b(1) \geq -1$ , and  $(e^\varepsilon + e^{-\varepsilon}) \log(\varepsilon) \leq 0$ , and so the identity (6) gives  $I(\varepsilon) \leq (e^\varepsilon - e^{-\varepsilon}) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du$ , but  $(e^u - e^{-u})(u \log u - u) \leq 0$  for all  $u \in [0, 1]$ , so  $I(\varepsilon) \leq (e^\varepsilon - e^{-\varepsilon})$  as wanted. In particular,  $I(1) \leq (e - e^{-1})$ . ■

**Definition 7** The Exponential Integral Ei function is defined for  $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$  by

$$\text{Ei}(x) := \int_{-\infty}^x \frac{e^u}{u} du, \quad (7)$$

where the Cauchy's principal value of the integral is taken.

**Proof** This integral exists and is finite for  $x < 0$ , as the function  $u \mapsto \frac{e^u}{u}$  is of class  $\mathcal{C}^1$  on  $(-\infty, 0)$ .

For  $x \rightarrow 0$  (from above or from below),  $\text{Ei}(x) \rightarrow -\infty$ .

And for  $x > 0$ , let  $\varepsilon > 0$ , and observe that we can write the integral from  $-\infty$  to  $x$  as three terms,  $\text{Ei}(x) = \text{Ei}(-\varepsilon) + \int_{-\varepsilon}^\varepsilon \frac{e^u}{u} du + \int_\varepsilon^x \frac{e^u}{u} du$ .  $\text{Ei}(-\varepsilon)$  and the last integral both exist and are finite, thanks to the first case of  $x < 0$  and as the function  $u \mapsto \frac{e^u}{u}$  is of class  $\mathcal{C}^1$  on  $(\varepsilon, +\infty)$ . And thanks to Lemma 5,  $\int_{-\varepsilon}^\varepsilon \frac{e^u}{u} du$  is finite. So all the three terms in the decomposition of  $\text{Ei}(x)$  exist and are finite, therefore  $\text{Ei}(x)$  is well defined. ■

A few properties of Ei worth noting include the following: it has a unique zero (located at  $x_0 \simeq 0.327$ ), it is negative for  $x < x_0$  and in particular for  $x < 0$ , it is positive for  $x > x_0$ , and it is decreasing on  $(-\infty, 0)$  and increasing on  $(0, +\infty)$ . Ei is also concave on  $(-\infty, 0)$  and  $(0, 1)$ , and convex on  $(1, +\infty)$ .

**Illustration** We can plot this function<sup>3</sup>:

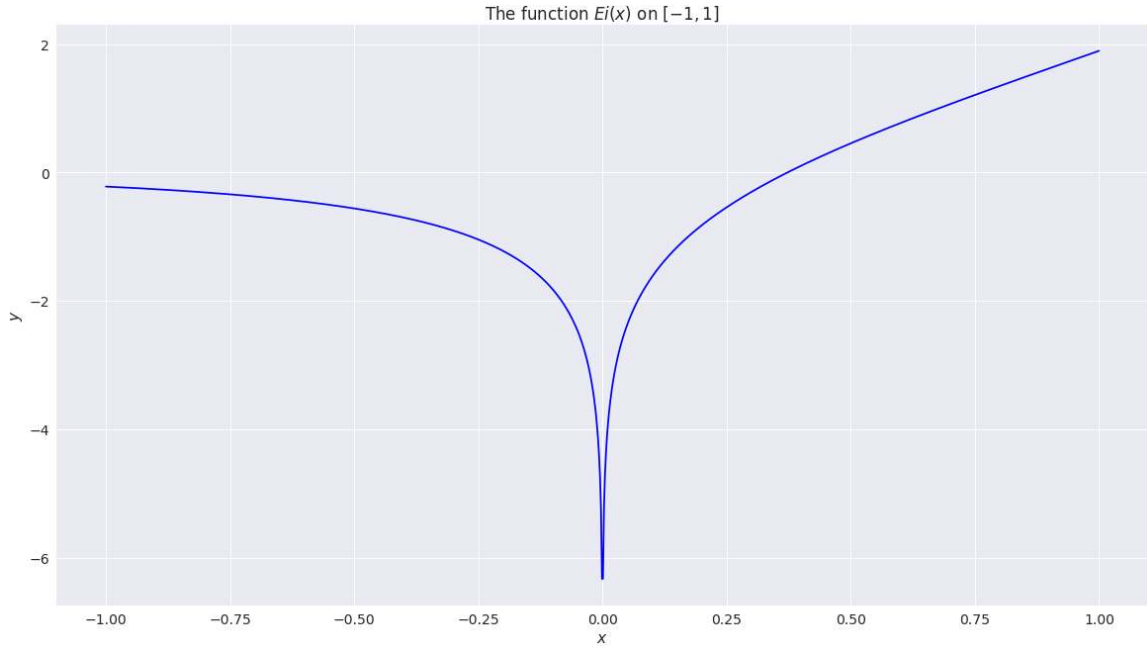


Figure 1: The Ei function on  $[-1, 1]$ .

## 5. Using Ei to compute primitives

**Lemma 8** For any  $a, b, \gamma \in \mathbb{R}$  and  $L \in \mathbb{N}$  such that  $a, b > 1$ ,  $\gamma > 0$  and  $L > 0$ ,

$$\int_0^L (a^{bt})^\gamma dt = \frac{1}{\log b} \left( \text{Ei} \left( \gamma \log(a^{b^L}) \right) - \text{Ei} \left( \gamma \log(a) \right) \right). \quad (8)$$

**Proof** A first change of variable with  $u := b^t$  gives  $dt = \frac{1}{\log b} \frac{1}{u} du$  ( $\log b > 0$  as  $b > 1$ ), and so

$$\int_0^L (a^{bt})^\gamma dt = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^u)^\gamma du = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^\gamma)^u du$$

And a second change of variable with  $v := \log(a^\gamma)u = \gamma \log(a)u$  gives  $\frac{1}{u} du = \frac{1}{v} dv$  (and no change in the order of the integral's bounds, as  $\log a > 0$  as  $a > 1$ ), and so

$$\begin{aligned} &= \frac{1}{\log b} \int_{\gamma \log(a)}^{\gamma \log(a) b^L} \frac{e^v}{v} dv = \frac{1}{\log b} \left[ \text{Ei}(v) \right]_{\gamma \log(a)}^{\gamma \log(a) b^L} \\ &= \frac{1}{\log b} \left( \text{Ei} \left( \gamma \log(a^{b^L}) \right) - \text{Ei} \left( \gamma \log(a) \right) \right). \end{aligned}$$

3. See for instance, the `scipy.special.expi` function, on <https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.expi.html>, if you use Python and SciPy (Foundation, 2017; Jones et al., 2001–).

■

## 6. Inequalities for Ei

**Lemma 9 (First Inequalities Using Ei)** *For any  $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ ,  $\text{Ei}(x) \leq e^x$ .*

*Moreover, for  $x \geq 1$ , we also have  $\text{Ei}(x) \geq \text{Ei}(1) + \frac{e^x - e}{x} \geq -1 + \frac{e^x}{x}$ .*

*A useful consequence is that for any  $y \geq 1$  and  $0 \leq \gamma \leq 1$ ,*

$$\text{Ei}(\log(y^\gamma)) = \text{Ei}(\gamma \log(y)) \leq y^\gamma. \quad (9)$$

**Proof** Let  $x \in \mathbb{R}$ . First, if  $x < 0$ , then clearly  $\text{Ei}(x) \leq 0 < e^x$ .

If  $0 < x < 1$ , we can split the integral defining  $\text{Ei}(x)$  in two terms, and as  $I(x) = \int_{-x}^x \frac{e^u}{u} du \leq e^x - e^{-x}$  (see Lemma 6),

$$\begin{aligned} \text{Ei}(x) &= \underbrace{\int_{-\infty}^{-x} \frac{e^u}{u} du}_{=\text{Ei}(-x) \leq 0} + \underbrace{\int_{-x}^x \frac{e^u}{u} du}_{=I(x)} \leq I(x) \\ &\leq e^x - e^{-x} \leq e^x. \end{aligned}$$

If  $x > 1$ , we do the same with three terms, and by using  $\text{Ei}(-1) = \int_{-\infty}^{-1} \frac{e^u}{u} du \leq 0$ , and  $I(1) = \int_{-1}^1 \frac{e^u}{u} du \leq e - e^{-1}$  (see Lemma 6), we have

$$\begin{aligned} \text{Ei}(x) &= \underbrace{\int_{-\infty}^{-1} \frac{e^u}{u} du}_{=\text{Ei}(-1) \leq 0} + \underbrace{\int_{-1}^1 \frac{e^u}{u} du}_{=I(1)} + \underbrace{\int_1^x \frac{e^u}{u} du}_{\leq e^x - e^1} \leq I(1) + e^x - e \\ &\leq e - e^{-1} + e^x - e = e^x - e^{-1} \leq e^x. \end{aligned}$$

Now for the lower bound, let  $x \geq 1$ , and we use the same splitting. For  $I(1)$ , we use conversely that  $I(1) \geq 0$  (see Lemma 6), and for the integral we have  $\int_1^x \frac{e^u}{u} du \geq \frac{1}{x}(e^x - e)$ . So  $\text{Ei}(x) \geq \text{Ei}(1) + \frac{e^x - e}{x}$ . We also have  $-\frac{e}{x} \geq -e$  and numerically,  $\text{Ei}(1) - e \geq -1$  (as  $\text{Ei}(1) \simeq 1.895$ ), so  $\text{Ei}(x) \geq -1 + \frac{e^x}{x}$ .

Finally, if  $x = \log(y^\gamma)$  and  $y \geq 0$ , then  $e^x = y^\gamma$ , so  $\text{Ei}(x) = \text{Ei}(\gamma \log(y)) \leq e^x = y^\gamma$ . ■

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**Illustration** We can check this inequality  $\text{Ei}(x) \leq e^x$  graphically, as well as a tighter inequality  $\text{Ei}(x) \leq \text{Ei}(-1) - \frac{1}{e} + e^x$ .

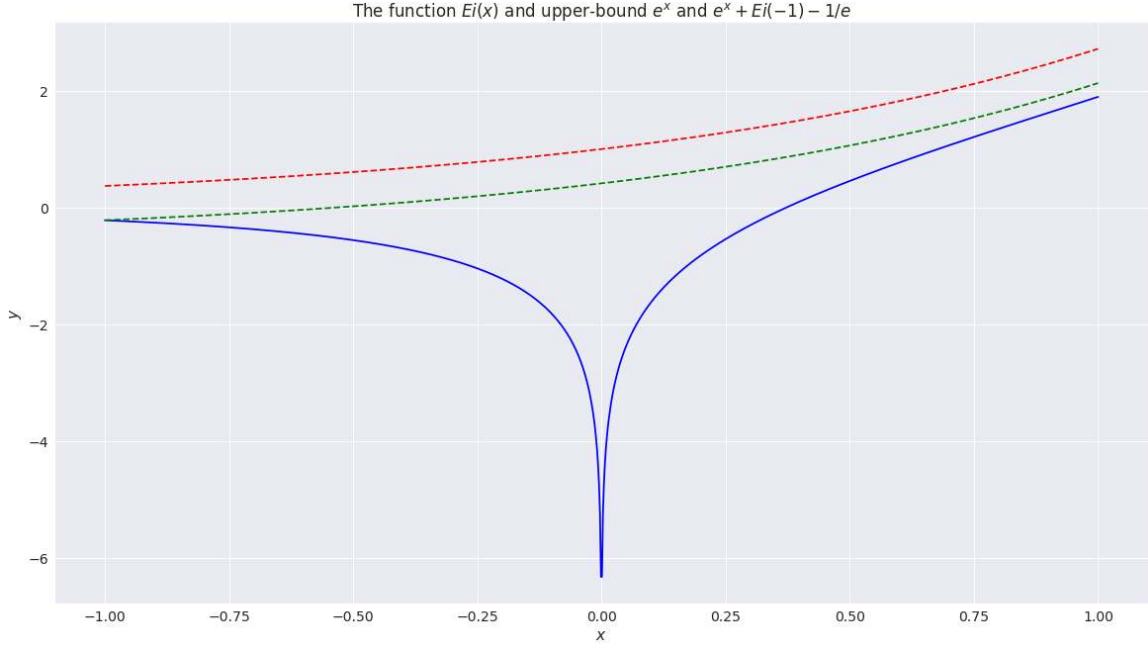


Figure 2: The Ei function and two upper-bounds valid respectively on  $\mathbb{R}$  and  $[1, +\infty)$ .

This last sum-inequality is the result we were looking for.

**Lemma 10 (Sum Inequality Using Ei)** For any  $a, b, \gamma \in \mathbb{R}$  and  $L \in \mathbb{N}$  such that  $a, b > 1$ ,  $\gamma > 0$  and  $L > 0$ , and if  $\text{Ei}(\gamma \log(a)) \geq 0$ , then

$$\sum_{i=0}^{L-1} (a^{b^i})^\gamma \leq \frac{1}{\log b} (a^{b^L})^\gamma. \quad (10)$$

And by isolating the last term, we also have

$$\sum_{i=0}^L (a^{b^i})^\gamma \leq \left(1 + \frac{1}{\log b}\right) (a^{b^L})^\gamma \quad (11)$$

**Proof** Using the sum-integral inequality (Lemma 4) and then Lemma 8, we have directly that

$$\begin{aligned} \sum_{i=0}^{L-1} (a^{b^i})^\gamma &\leq \int_0^L (a^{b^t})^\gamma dt \\ &\leq \frac{1}{\log b} \text{Ei}(\gamma \log(a^{b^L})) \leq \frac{1}{\log b} (a^{b^L})^\gamma. \end{aligned}$$





In particular, this inequality (11) holds as soon as  $a \geq e^{0.373/\gamma}$ , as  $\gamma \log(a) \geq 0.373 > x_0 \implies \text{Ei}(\gamma \log(a)) \geq \text{Ei}(x_0) > 0$  and  $x_0 \simeq 0.372507 \geq 0.373$ . For instance,  $\gamma = 1/2$  gives  $a \geq e^{0.373/\gamma} = e^{0.746} \simeq 2.107$ , close to the simplest value  $a = 2$ .

And if  $\gamma = 0$ , then  $\text{Ei}(\gamma \log(a))$  cannot be  $\geq 0$ , but the sum in (10) is constant and equals to  $L$ .

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## 7. Conclusion

This small note defines and studies a useful non-canonical function called the “exponential integral” function,  $\text{Ei}$ , and we use it to find a bound on any sum of the form  $\sum_{i=0}^L (a^{b^i})^\gamma$ .

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*Note:* the simulation code used for the experiments is using Python 3, (Foundation, 2017), and Matplotlib (Hunter, 2007) for plotting, as well as SciPy (Jones et al., 2001–). It is open-sourced at [github.com/Naeereen/notebooks/blob/master/Exponential\\_Integral\\_Python.ipynb](https://github.com/Naeereen/notebooks/blob/master/Exponential_Integral_Python.ipynb). This document is also distributed under the open-source MIT License, and is available online at [perso.crans.org/besson/publis/A\\_note\\_on\\_the\\_Ei\\_function.pdf](https://perso.crans.org/besson/publis/A_note_on_the_Ei_function.pdf).

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