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► **To cite this version:**

| Lilian Besson. A Note on the Ei Function and a Useful Sum-Inequality. 2018. hal-01847480

HAL Id: hal-01847480

<https://inria.hal.science/hal-01847480v1>

Submitted on 23 Jul 2018

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A Note on the Ei Function and a Useful Sum-Inequality

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Abstract

This short note defines formally the Ei function, and gives some interesting inequalities and integration using it. I illustrate the inequalities and detail what is still to be proved.

Keywords: Analysis; Inequalities; Primitive.

1. Introduction and Motivation

Take $a > 1$, $b > 1$ and $\gamma > 0$, and for an integer $L > 0$ consider the sum $\sum_{i=1}^L (a^{b^i})^\gamma$. We want to bound it, and the goal is to show that it is bounded by a constant times its last term. A first naive bound is $\sum_{i=1}^L (a^{b^i})^\gamma \leq (L+1)(a^{b^L})^\gamma$ which is too brutal as soon as $L \rightarrow \infty$.

I first remind and prove two useful elementary results, and then we define and study the Ei function, to finally prove the desired inequality.

2. Lemma and Proof

Lemma 1 For any $n \in \mathbb{N}^*$, $a > 1$, $b > 1$ and $\gamma > 0$, we have

$$\sum_{i=0}^n (a^{b^i})^\gamma \leq a^\gamma + \left(1 + \frac{1}{(\log(a))(\log(b^\gamma))}\right) (a^{b^n})^\gamma = \mathcal{O}\left((a^{b^n})^\gamma\right). \quad (1)$$

Proof We first isolate both the first and last term in the sum and focus on the from $i = 1$ sum up to $i = n - 1$. As the function $t \mapsto (a^{b^t})^\gamma$ is increasing for $t \geq 1$, we use a sum-integral inequality, and then the change of variable $u := \gamma b^t$, of Jacobian $dt = \frac{1}{\log b} \frac{du}{u}$, gives

$$\sum_{i=1}^{n-1} (a^{b^i})^\gamma \leq \int_1^n a^{\gamma b^t} dt \leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} \frac{a^u}{u} du$$

Now for $u \geq 1$, observe that $\frac{a^u}{u} \leq a^u$, and as $\gamma b > 1$, we have

$$\leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} a^u du \leq \frac{1}{\log(b^\gamma)} \frac{1}{\log(a)} a^{\gamma b^n} = \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma.$$

Finally, we obtain as desired, $\sum_{i=0}^n (a^{b^i})^\gamma \leq a^\gamma + (a^{b^n})^\gamma + \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma$. ■

3. Elementary Results

3.1. Integration by Part

The Integration by Part is a basic but useful result to establish inequalities, *e.g.*, for Lemma 10 using Lemma 3, and to prove the existence of finite integrals, *e.g.*, for Lemma 5 using two chained IP.

Lemma 2 (Integration by Part (IP)) *Let $x, y \in \mathbb{R}$, $x \leq y$, and u, v two functions of class¹ \mathcal{C}^1 , and with this notation $[uv]_x^y := u(y)v(y) - u(x)v(x)$, then*

$$\int_x^y u(t)v'(t) dt = [uv]_x^y - \int_x^y u'(t)v(t) dt. \quad (2)$$

Proof The two integrals and the two evaluations are well defined by the \mathcal{C}^1 hypothesis on both u and v (u and v are continuous at x and y and $u'v$ is continuous so integrable on the interval $[x, y]$).

The product function uv is differentiable, and $(uv)' = u'v + uv'$, so $[uv]_x^y = \int_x^y (uv)'(t) dt = \int_x^y u(t)v'(t) dt + \int_x^y u'(t)v(t) dt$ as wanted, by the linearity of the integral. ■

Lemma 3 (IP Inequality) *If both u, v are non-negative, and non-decreasing, then*

$$\int_x^y u(t)v'(t) dt \leq u(y)v(y). \quad (3)$$

Proof The non-negativeness gives that $-u(x)v(x) \leq 0$ and the monotony hypothesis gives that $u(t)v'(t)$ is non-negative on the interval $[x, y]$, and so $-\int_x^y u(t)v'(t) dt \leq 0$, so an Integration by Part gives the desired inequality. ■

3.2. Sum-Integral Inequality

A well known result is the following, which bound a discrete sum $\sum_{i=x}^y f(i)$ by two integrals for non-decreasing functions, and it is used for Lemma 10.

Lemma 4 *For any $x, y \in \mathbb{N}^*$, $x \leq y$, and f a non-decreasing function on $[0, +\infty)$, then*

$$\int_{x-1}^y f(t) dt \leq \sum_{i=x}^y f(i) \leq \int_x^{y+1} f(t) dt, \quad (4)$$

and

$$f(x) + \int_x^y f(t) dt \leq \sum_{i=x}^y f(i) \leq f(y) + \int_x^y f(t) dt. \quad (5)$$

Proof For the first inequality, both parts comes from the monotony of f and monotony and additivity of the integral. On any interval $[i, i + 1]$, $f(i) = \min_{t \in [i, i+1]} f(t) \leq \int_i^{i+1} f(t) dt$, and $f(i) =$

$\max_{t \in [i-1, i]} f(t) \geq \int_{i-1}^i f(t) dt$, And so, if we sum these terms from $i = x$ to y , we get

$$\sum_{i=x}^y f(i) \leq \sum_{i=x}^y \int_i^{i+1} f(t) dt = \int_x^{y+1} f(t) dt.$$

1. A function of class \mathcal{C}^1 is continuous, differentiable and of continuous derivative on its interval of definition.

as well as

$$\sum_{i=x}^y f(i) \geq \sum_{i=x}^y \int_{i-1}^i f(t) dt = \int_{x-1}^y f(t) dt.$$

The two sides of second inequality are immediate by isolating the first (or last) term of the sum $f(y)$ (or $f(x)$), and applying the first inequality to $x - 1$ instead of x (or $y - 1$ instead of y). \blacksquare

4. The Exponential Integral Ei Function

This last Subsection is rather long, and actually not required to obtain the Lemma 1. But I find this Ei function to be quite interesting, so I wanted to write down these proofs. We define the Ei function (Weisstein, 2017; Collective, 2017), by carefully justifying its existence, and then we give two results using it, to obtain the non-trivial Lemma 10.

Lemma 5 For any $\varepsilon > 0$, $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$ exists and is finite, it satisfies this identity

$$I(\varepsilon) = (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du. \quad (6)$$

Additionally, it stays finite when $\varepsilon \rightarrow 0$, and $\lim_{\varepsilon \rightarrow 0} I(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du = 0$.

Proof Fix $\varepsilon > 0$, and let $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du$.

Roughly, one just needs to observe² that for u close to 0, $\frac{e^u}{u} \sim \frac{1}{u}$ as $e^u \sim 1$, and $\frac{1}{u}$ can be integrated on $[-\varepsilon, \varepsilon]$, even if it is not defined at 0, because it is odd: $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} du = \lim_{t \rightarrow 0} (\int_{-\varepsilon}^t \frac{1}{u} du + \int_t^{\varepsilon} \frac{1}{u} du)$ (as Cauchy's principal values), and $\int_{-\varepsilon}^t \frac{1}{u} du = -\int_t^{\varepsilon} \frac{1}{v} dv$ with the change of variable $v = -u$. So $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u} du = 0$ for any $\varepsilon \geq 0$.

But we have to justify more properly that $I(\varepsilon)$ exists for any $\varepsilon > 0$ and that $I(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$. A first Integration by Part (Lemma 2) with $a(u) = e^u$ and $b'(u) = \frac{1}{u}$, that is $a'(u) = e^u$ and by choosing $b(u) = \log |u|$, gives

$$\begin{aligned} I(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u} du \\ &= [e^u \log |u|]_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} e^u \log |u| du \\ &= (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - \int_0^\varepsilon (e^u + e^{-u}) \log |u| du. \end{aligned}$$

Let $I_2(\varepsilon) := \int_0^\varepsilon (e^u + e^{-u}) \log u du$. A second Integration by Part (Lemma 2) with $a(u) = e^u + e^{-u}$ and $b'(u) = \log u$, that is $a'(u) = e^u - e^{-u}$ and $b(u) = u \log u - u$ (\mathcal{C}^1 on $(0, \varepsilon]$), gives

$$\begin{aligned} I_2(\varepsilon) &= \int_0^\varepsilon (e^u + e^{-u}) \log u du \\ &= [(e^u + e^{-u})(u \log u - u)]_0^\varepsilon - \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du. \end{aligned}$$

2. The notation $f(u) \sim g(u)$ for $u \rightarrow u_0$ means that $g(u) \neq 0$ and $f(u)/g(u) \rightarrow 1$ for $u \rightarrow u_0$.

Indeed, $b(u) = u \log u - u$ is well defined for $u \rightarrow 0$, as $u \log u \rightarrow 0$, so we can define $b(0) = 0$ to have b of class \mathcal{C}^1 on $[0, \varepsilon]$. Therefore, $I_2(\varepsilon)$ exists, and we have, as wanted, the following identity

$$I(\varepsilon) = (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon - (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du.$$

The last integral is well defined and finite, as the integrated function is continuous and finite for all u , even at 0. So this proves that $I(\varepsilon)$ is finite for any $\varepsilon > 0$.

Now, taking $\varepsilon \rightarrow 0$ gives, for each of the three terms in $I(\varepsilon)$,

$$\begin{cases} (e^\varepsilon - e^{-\varepsilon}) \log \varepsilon \sim ((1 + \varepsilon) - (1 - \varepsilon)) \log \varepsilon = 2\varepsilon \log \varepsilon \rightarrow 0 \\ (e^\varepsilon + e^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) \sim 2b(\varepsilon) \rightarrow 0 \\ \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du \rightarrow 0, \end{cases}$$

so $I(\varepsilon) \rightarrow 0$, as wanted. ■

Lemma 6 For any $0 < \varepsilon \leq 1$, $I(\varepsilon)$ satisfies $I(\varepsilon) \leq e^\varepsilon - e^{-\varepsilon}$. In particular, $I(1) \leq e - e^{-1}$.

Proof For $0 < \varepsilon \leq 1$, $b(1) \geq -1$, and $(e^\varepsilon + e^{-\varepsilon}) \log(\varepsilon) \leq 0$, and so the identity (6) gives $I(\varepsilon) \leq (e^\varepsilon - e^{-\varepsilon}) + \int_0^\varepsilon (e^u - e^{-u})(u \log u - u) du$, but $(e^u - e^{-u})(u \log u - u) \leq 0$ for all $u \in [0, 1]$, so $I(\varepsilon) \leq (e^\varepsilon - e^{-\varepsilon})$ as wanted. In particular, $I(1) \leq (e - e^{-1})$. ■

Definition 7 The Exponential Integral Ei function is defined for $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ by

$$\text{Ei}(x) := \int_{-\infty}^x \frac{e^u}{u} du, \tag{7}$$

where the Cauchy's principal value of the integral is taken.

Proof This integral exists and is finite for $x < 0$, as the function $u \mapsto \frac{e^u}{u}$ is of class \mathcal{C}^1 on $(-\infty, 0)$.

For $x \rightarrow 0$ (from above or from below), $\text{Ei}(x) \rightarrow -\infty$.

And for $x > 0$, let $\varepsilon > 0$, and observe that we can write the integral from $-\infty$ to x as three terms, $\text{Ei}(x) = \text{Ei}(-\varepsilon) + \int_{-\varepsilon}^\varepsilon \frac{e^u}{u} du + \int_\varepsilon^x \frac{e^u}{u} du$. $\text{Ei}(-\varepsilon)$ and the last integral both exist and are finite, thanks to the first case of $x < 0$ and as the function $u \mapsto \frac{e^u}{u}$ is of class \mathcal{C}^1 on $(\varepsilon, +\infty)$. And thanks to Lemma 5, $\int_{-\varepsilon}^\varepsilon \frac{e^u}{u} du$ is finite. So all the three terms in the decomposition of $\text{Ei}(x)$ exist and are finite, therefore $\text{Ei}(x)$ is well defined. ■

A few properties of Ei worth noting include the following: it has a unique zero (located at $x_0 \simeq 0.327$), it is negative for $x < x_0$ and in particular for $x < 0$, it is positive for $x > x_0$, and it is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$. Ei is also concave on $(-\infty, 0)$ and $(0, 1)$, and convex on $(1, +\infty)$.

Illustration We can plot this function³:

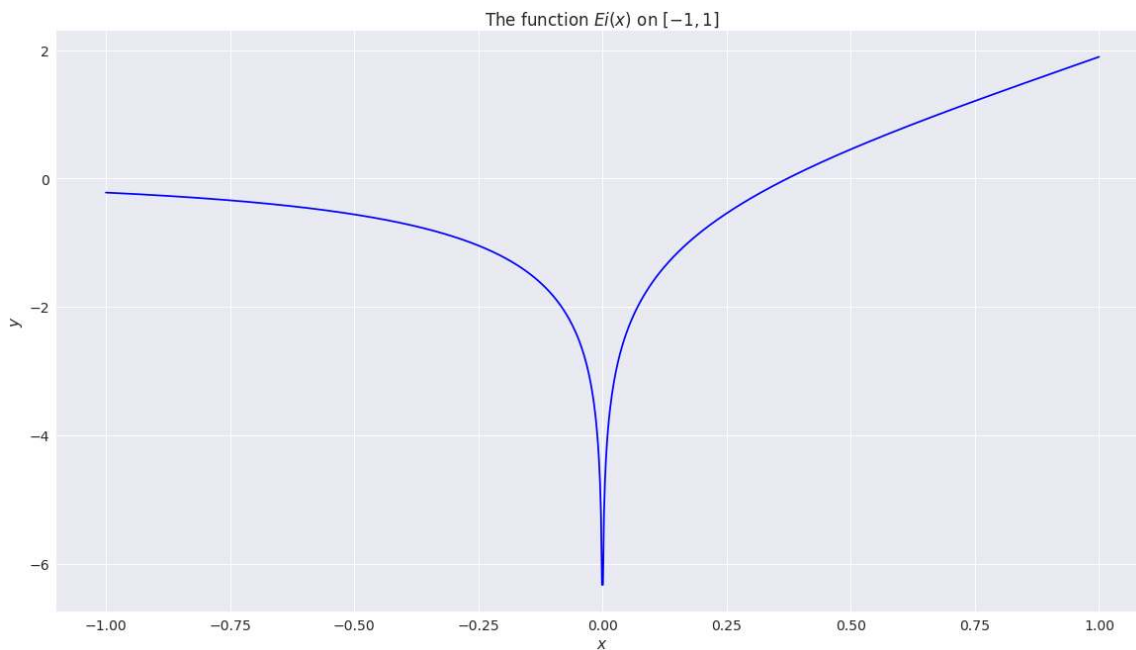


Figure 1: The Ei function on $[-1, 1]$.

5. Using Ei to compute primitives

Lemma 8 For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $a, b > 1$, $\gamma > 0$ and $L > 0$,

$$\int_0^L (a^{bt})^\gamma dt = \frac{1}{\log b} \left(\text{Ei} \left(\gamma \log(a^{bL}) \right) - \text{Ei} \left(\gamma \log(a) \right) \right). \quad (8)$$

Proof A first change of variable with $u := b^t$ gives $dt = \frac{1}{\log b} \frac{1}{u} du$ ($\log b > 0$ as $b > 1$), and so

$$\int_0^L (a^{bt})^\gamma dt = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^u)^\gamma du = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^\gamma)^u du$$

And a second change of variable with $v := \log(a^\gamma)u = \gamma \log(a)u$ gives $\frac{1}{u} du = \frac{1}{v} dv$ (and no change in the order of the integral's bounds, as $\log a > 0$ as $a > 1$), and so

$$\begin{aligned} &= \frac{1}{\log b} \int_{\gamma \log(a)}^{\gamma \log(a)b^L} \frac{e^v}{v} dv = \frac{1}{\log b} \left[\text{Ei}(v) \right]_{\gamma \log(a)}^{\gamma \log(a)b^L} \\ &= \frac{1}{\log b} \left(\text{Ei} \left(\gamma \log \left(a^{b^L} \right) \right) - \text{Ei} \left(\gamma \log(a) \right) \right). \end{aligned}$$

3. See for instance, the `scipy.special.expi` function, on <https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.expi.html>, if you use Python and SciPy (Foundation, 2017; Jones et al., 2001–).

■

6. Inequalities for Ei

Lemma 9 (First Inequalities Using Ei) For any $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\text{Ei}(x) \leq e^x$.

Moreover, for $x \geq 1$, we also have $\text{Ei}(x) \geq \text{Ei}(1) + \frac{e^x - e}{x} \geq -1 + \frac{e^x}{x}$.

A useful consequence is that for any $y \geq 1$ and $0 \leq \gamma \leq 1$,

$$\text{Ei}(\log(y^\gamma)) = \text{Ei}(\gamma \log(y)) \leq y^\gamma. \quad (9)$$

Proof Let $x \in \mathbb{R}$. First, if $x < 0$, then clearly $\text{Ei}(x) \leq 0 < e^x$.

If $0 < x < 1$, we can split the integral defining $\text{Ei}(x)$ in two terms, and as $I(x) = \int_{-x}^x \frac{e^u}{u} du \leq e^x - e^{-x}$ (see Lemma 6),

$$\begin{aligned} \text{Ei}(x) &= \underbrace{\int_{-\infty}^{-x} \frac{e^u}{u} du}_{=\text{Ei}(-x) \leq 0} + \underbrace{\int_{-x}^x \frac{e^u}{u} du}_{=I(x)} \leq I(x) \\ &\leq e^x - e^{-x} \leq e^x. \end{aligned}$$

If $x > 1$, we do the same with three terms, and by using $\text{Ei}(-1) = \int_{-\infty}^{-1} \frac{e^u}{u} du \leq 0$, and $I(1) = \int_{-1}^1 \frac{e^u}{u} du \leq e - e^{-1}$ (see Lemma 6), we have

$$\begin{aligned} \text{Ei}(x) &= \underbrace{\int_{-\infty}^{-1} \frac{e^u}{u} du}_{=\text{Ei}(-1) \leq 0} + \underbrace{\int_{-1}^1 \frac{e^u}{u} du}_{=I(1)} + \underbrace{\int_1^x \frac{e^u}{u} du}_{\leq e^x - e} \leq I(1) + e^x - e \\ &\leq e - e^{-1} + e^x - e = e^x - e^{-1} \leq e^x. \end{aligned}$$

Now for the lower bound, let $x \geq 1$, and we use the same splitting. For $I(1)$, we use conversely that $I(1) \geq 0$ (see Lemma 6), and for the integral we have $\int_1^x \frac{e^u}{u} du \geq \frac{1}{x}(e^x - e)$. So $\text{Ei}(x) \geq \text{Ei}(1) + \frac{e^x - e}{x}$. We also have $-\frac{e}{x} \geq -e$ and numerically, $\text{Ei}(1) - e \geq -1$ (as $\text{Ei}(1) \simeq 1.895$), so $\text{Ei}(x) \geq -1 + \frac{e^x}{x}$.

Finally, if $x = \log(y^\gamma)$ and $y \geq 0$, then $e^x = y^\gamma$, so $\text{Ei}(x) = \text{Ei}(\gamma \log(y)) \leq e^x = y^\gamma$. ■

Illustration We can check this inequality $Ei(x) \leq e^x$ graphically, as well as a tighter inequality $Ei(x) \leq Ei(-1) - \frac{1}{e} + e^x$.

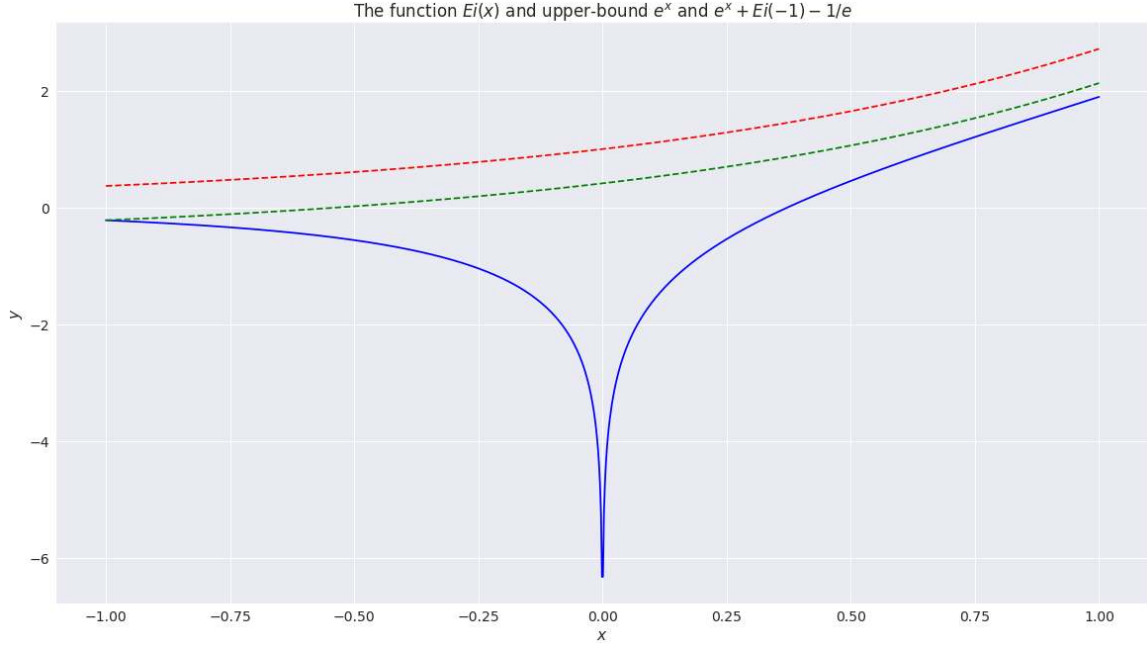


Figure 2: The Ei function and two upper-bounds valid respectively on \mathbb{R} and $[1, +\infty)$.

This last sum-inequality is the result we were looking for.

Lemma 10 (Sum Inequality Using Ei) For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $a, b > 1$, $\gamma > 0$ and $L > 0$, and if $Ei(\gamma \log(a)) \geq 0$, then

$$\sum_{i=0}^{L-1} (a^{b^i})^\gamma \leq \frac{1}{\log b} (a^{b^L})^\gamma. \quad (10)$$

And by isolating the last term, we also have

$$\sum_{i=0}^L (a^{b^i})^\gamma \leq \left(1 + \frac{1}{\log b}\right) (a^{b^L})^\gamma \quad (11)$$

Proof Using the sum-integral inequality (Lemma 4) and then Lemma 8, we have directly that

$$\begin{aligned} \sum_{i=0}^{L-1} (a^{b^i})^\gamma &\leq \int_0^L (a^{b^t})^\gamma dt \\ &\leq \frac{1}{\log b} Ei\left(\gamma \log(a^{b^L})\right) \leq \frac{1}{\log b} (a^{b^L})^\gamma. \end{aligned}$$



In particular, this inequality (11) holds as soon as $a \geq e^{0.373/\gamma}$, as $\gamma \log(a) \geq 0.373 > x_0 \implies \text{Ei}(\gamma \log(a)) \geq \text{Ei}(x_0) > 0$ and $x_0 \simeq 0.372507 \geq 0.373$. For instance, $\gamma = 1/2$ gives $a \geq e^{0.373/\gamma} = e^{0.746} \simeq 2.107$, close to the simplest value $a = 2$.

And if $\gamma = 0$, then $\text{Ei}(\gamma \log(a))$ cannot be ≥ 0 , but the sum in (10) is constant and equals to L .

7. Conclusion

This small note defines and studies a useful non-canonical function called the “exponential integral” function, Ei , and we use it to find a bound on any sum of the form $\sum_{i=0}^L (a^{b^i})^\gamma$.

Note: the simulation code used for the experiments is using Python 3, (Foundation, 2017), and Matplotlib (Hunter, 2007) for plotting, as well as SciPy (Jones et al., 2001–). It is open-sourced at github.com/Naareen/notebooks/blob/master/Exponential_Integral_Python.ipynb. This document is also distributed under the open-source MIT License, and is available online at perso.crans.org/besson/publis/A_note_on_the_Ei_function.pdf.

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