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The MHM method for the second-order elastodynamic model

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Joint work with

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Outline

Introduction

Hybridization procedure

The MHM method

Numerical results

- An analytical problem

- Multiscale problem

Final conclusions and remarks

Introduction

Elastodynamic problem

Equation of motion and the constitutive equation

$$\rho \partial_{tt} \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}, \quad \boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} := \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}$$

with $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$, $\partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ and $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$.

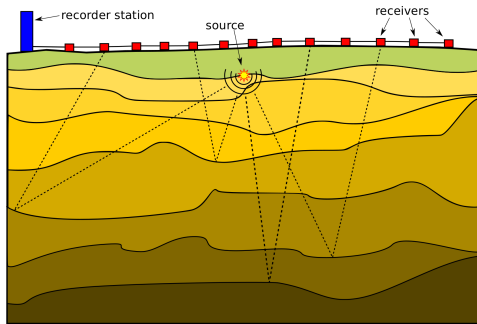


Figure : Seismic imaging technique.

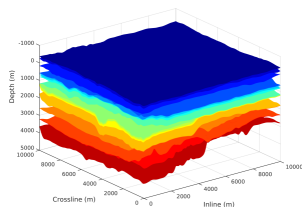


Figure : Layered media.

MHM - Multiscale Hybrid-Mixed method ¹

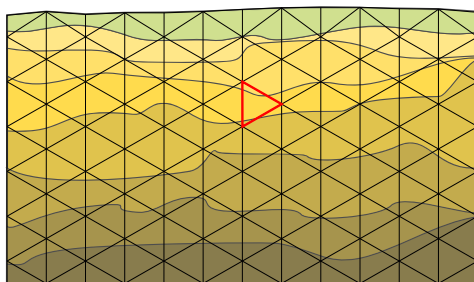


Figure : \mathcal{T}_H with elements K .

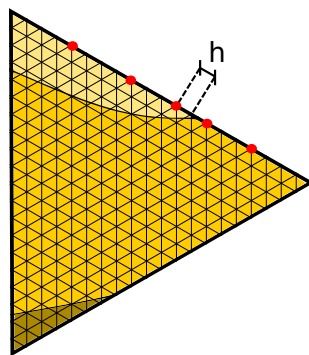


Figure : Local mesh \mathcal{T}_h^K .

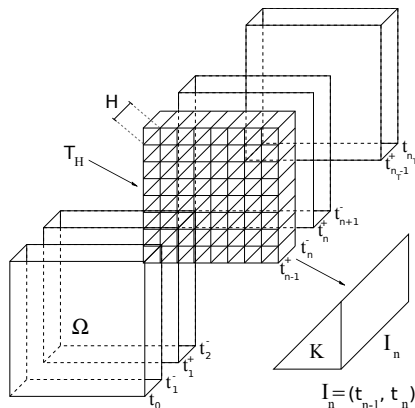
¹C. Harder, D. Paredes, and F. Valentin. A family of Multiscale Hybrid-Mixed finite element methods for the Darcy equation with rough coefficients. *J. Comput. Phys.*, 245:107–130, 2013

Hybridization procedure

1. Partition the domain $\Omega \times [0, T]$

$$\mathbf{V} := \{ \mathbf{w} \in \mathbf{L}^2(\Omega) : \mathbf{w}|_K \in \mathbf{H}^1(K) \quad \forall K \in \mathcal{T}_H \},$$

$$\mathbf{\Lambda} := \{ \boldsymbol{\tau} \mathbf{n}^K|_{\partial K}, \forall K \in \mathcal{T}_H : \boldsymbol{\tau} \in H(\mathbf{div}; \Omega; \mathbb{S}) \}.$$



Notation:

- $(\cdot, \cdot)_K$ - $\mathbf{L}^2(K)$ product;
- $\langle \cdot, \cdot \rangle_{\partial K}$ - dual product $\mathbf{H}^{-\frac{1}{2}}(\partial K) \times \mathbf{H}^{\frac{1}{2}}(\partial K)$;
- $(\cdot, \cdot)_{\partial \mathcal{T}_H} := \sum_{K \in \mathcal{T}_H} \langle \cdot, \cdot \rangle_{\partial K}$;
- $\mathbf{u}^{(n)} := \mathbf{u}|_{I_n}$.

2. A hybrid variational problem

Find $(\mathbf{u}, \boldsymbol{\lambda})$ that, for $n = 1, \dots, n_T$, satisfies

$$\left\{ \begin{array}{l} d_{tt}(\rho \mathbf{u}^{(n)}, \mathbf{w})_K + (\boldsymbol{\sigma}(\mathbf{u}^{(n)}), \boldsymbol{\varepsilon}(\mathbf{w}))_K - \langle \boldsymbol{\lambda}^{(n)}, \mathbf{w} \rangle_{\partial K} = (\mathbf{f}, \mathbf{w})_K, \quad \forall \mathbf{w} \in \mathbf{H}^1(K), \\ (\boldsymbol{\mu}, \mathbf{u}^{(n)})_{\partial \mathcal{T}_H} = 0, \quad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \\ \mathbf{u}^{(n)}(t_{n-1}) = \mathbf{u}^{(n-1)}(t_{n-1}), \\ d_t \mathbf{u}^{(n)}(t_{n-1}) = d_t \mathbf{u}^{(n-1)}(t_{n-1}), \end{array} \right.$$

where $\mathbf{u}_n := \mathbf{u}^{(n)}(t_n) = \mathbf{u}(t_n)$ and $\mathbf{v}_n := d_t \mathbf{u}^{(n)}(t_n) = d_t \mathbf{u}(t_n)$.

Theorem

The problem is well-defined for

$\mathbf{f} \in H^1(0, T; \mathbf{L}^2(\Omega))$, $\mathbf{u}_0, \mathbf{v}_0 \in \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{\sigma}(\mathbf{u}_0) \in H(\mathbf{div}; \Omega; \mathbb{S})$.

The solution is unique in

$$C(0, T; \mathbf{V}) \cap C^1(0, T; \mathbf{L}^2(\Omega)) \times L^2(0, T; \boldsymbol{\Lambda}).$$

Moreover, \mathbf{u} solves the original problem and

$$\boldsymbol{\lambda} = \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^K \quad \text{on } \partial K, \quad \forall K \in \mathcal{T}_H.$$

3. Split the solution

$$\mathbf{u}^{(n)} = T_n(\boldsymbol{\lambda}^{(n)}) + \hat{T}_n(\mathbf{z}_n), \quad \mathbf{z}_n := (\mathbf{u}_{n-1}, \mathbf{v}_{n-1}, \mathbf{f}).$$

where T_n and \hat{T}_n are linear operators

$$T_n : L^2(I_n; \boldsymbol{\Lambda}) \rightarrow L^2(I_n; \mathbf{L}^2(\Omega)),$$

$$\hat{T}_n : \mathbf{V} \times \mathbf{L}^2(\Omega) \times L^2(I_n; \mathbf{L}^2(\Omega)) \rightarrow C(I_n; \mathbf{V}) \cap C^1(I_n; \mathbf{L}^2(\Omega)).$$

For each $K \times I_n$, $\mathbf{u}_\lambda^{(n)} := T_n(\boldsymbol{\lambda}^{(n)})$ is the solution of

$$\begin{cases} d_{tt}(\rho \mathbf{u}_\lambda^{(n)}, \mathbf{w})_K + (\boldsymbol{\sigma}(\mathbf{u}_\lambda^{(n)}), \boldsymbol{\varepsilon}(\mathbf{w}))_K = \langle \boldsymbol{\lambda}^{(n)}, \mathbf{w} \rangle_{\partial K}, \quad \forall \mathbf{w} \in \mathbf{H}^1(K), \\ \mathbf{u}_\lambda^{(n)}(t_{n-1}) = d_t \mathbf{u}_\lambda^{(n)}(t_{n-1}) = \mathbf{0}, \end{cases}$$

and $\mathbf{u}_z^{(n)} := \hat{T}_n(\mathbf{z}_n)$ is the solution of

$$\begin{cases} d_{tt}(\rho \mathbf{u}_z^{(n)}, \mathbf{w})_K + (\boldsymbol{\sigma}(\mathbf{u}_z^{(n)}), \boldsymbol{\varepsilon}(\mathbf{w}))_K = (\mathbf{f}, \mathbf{w})_K, \quad \forall \mathbf{w} \in \mathbf{H}^1(K), \\ \mathbf{u}_z^{(n)}(t_{n-1}) = \mathbf{u}_{n-1}, \\ d_t \mathbf{u}_z^{(n)}(t_{n-1}) = \mathbf{v}_{n-1}. \end{cases}$$

Global-Local formulation ²

Global-Local formulation

Find λ that, for $n = 1, \dots, n_T$, satisfies

$$(\mu, T_n(\lambda^{(n)}))_{\partial T_H} = -(\mu, \hat{T}_n(\mathbf{z}_n))_{\partial T_H}, \quad \forall \mu \in \mathbf{\Lambda},$$

where

$$\mathbf{z}_n := (\mathbf{u}_{n-1}, \mathbf{v}_{n-1}, \mathbf{f}),$$

$$\mathbf{u}_n := T_n(\lambda^{(n)})|_{t_n} + \hat{T}_n(\mathbf{z}_n)|_{t_n},$$

$$\mathbf{v}_n := d_t T_n(\lambda^{(n)})|_{t_n} + d_t \hat{T}_n(\mathbf{z}_n)|_{t_n},$$

This is equivalent to the hybrid problem using

$$\mathbf{u}^{(n)} := T_n(\lambda^{(n)}) + \hat{T}_n(\mathbf{z}_n).$$

²A. T. Gomes, D. Paredes, W. Pereira, R. Souto, and F. Valentin. [A Multiscale Hybrid-Mixed Method for the Elastodynamic Model with Rough Coefficients](#). In *XXXVIII CILAMCE*, 2017

The MHM method

One-level MHM method

Global discretization

- $\mathbf{\Lambda}_H \subset \mathbf{\Lambda}$ and basis $\{\psi_i\}_{i=1}^{m_\Lambda}$.
- $\mathbb{P}_0(I_n; \mathbf{\Lambda}_H) \subset L^2(I_n; \mathbf{\Lambda})$.

$$\mathbf{u}_H^{(n)} := \sum_{j=1}^{m_\Lambda} \alpha_{nj} T_n(\psi_j) + \hat{T}_n(\mathbf{z}_{H,n})$$

For $n = 1, \dots, n_T$, find $\alpha_n \in \mathbb{R}^{m_\Lambda}$ such that

$$\sum_{j=1}^{m_\Lambda} \alpha_{nj} (\psi_i, T_n(\psi_j)|_{t_n})_{\partial\mathcal{T}_H} = -(\psi_i, \hat{T}_n(\mathbf{z}_{H,n})|_{t_n})_{\partial\mathcal{T}_H}, \quad \forall i = 1, \dots, m_\Lambda,$$

where

$$\mathbf{z}_{H,n} := (\mathbf{u}_{H,n-1}, \mathbf{v}_{H,n-1}, \mathbf{f}), \quad \mathbf{u}_{H,n} := \mathbf{u}_H^{(n)}(t_n), \quad \mathbf{v}_{H,n} := d_t \mathbf{u}_H^{(n)}(t_n)$$

and initial conditions

$$\mathbf{u}_{H,0} := \mathbf{u}_0 \quad \text{and} \quad \mathbf{v}_{H,0} := \mathbf{v}_0.$$

Two-level MHM method

Discretization in $K \times I_n$:

- $\mathbf{V}_h(K) \subset \mathbf{H}^1(K)$, $\forall K \in \mathcal{T}_H$;
- Newmark- (γ, β) in I_n , $\forall n = 1, \dots, n_t$.

$$\boldsymbol{\eta}_j^{(n)} \approx T_n(\boldsymbol{\psi}_j)|_{t_n}, \quad \boldsymbol{\eta}_0^{(n)} \approx \hat{T}_n(\mathbf{z}_{h,n})|_{t_n}, \quad \dot{\boldsymbol{\eta}}_j^{(n)} \approx d_t T_n(\boldsymbol{\psi}_j)|_{t_n}, \quad \dot{\boldsymbol{\eta}}_0^{(n)} \approx d_t \hat{T}_n(\mathbf{z}_{h,n})|_{t_n},$$

$$\mathbf{u}_{h,n|K} := \sum_{j=1}^{m_\Lambda} \alpha_{nj} \boldsymbol{\eta}_j^{(n)} + \boldsymbol{\eta}_0^{(n)}, \quad \mathbf{v}_{h,n|K} := \sum_{j=1}^{m_\Lambda} \alpha_{nj} \dot{\boldsymbol{\eta}}_j^{(n)} + \dot{\boldsymbol{\eta}}_0^{(n)}, \quad \boldsymbol{\sigma}_{h,n|\tau} := \boldsymbol{\sigma}(\mathbf{u}_{h,n})|_\tau.$$

MHM method

For $n = 1, \dots, n_T$, find $\alpha_n \in \mathbb{R}^{m_\Lambda}$ such that

$$\sum_{j=1}^{m_\Lambda} \alpha_{nj} (\boldsymbol{\psi}_i, \boldsymbol{\eta}_j^{(n)})_{\partial\mathcal{T}_H} = -(\boldsymbol{\psi}_i, \boldsymbol{\eta}_0^{(n)})_{\partial\mathcal{T}_H}, \quad \forall i = 1, \dots, m_\Lambda,$$

where $\mathbf{z}_{h,n} := (\mathbf{u}_{h,n-1}, \mathbf{v}_{h,n-1}, \mathbf{f})$, $\mathbf{u}_{h,0} \approx \mathbf{u}_0$ and $\mathbf{v}_{h,0} \approx \mathbf{v}_0$.

Two-level MHM method

- $\mathbf{u}_{h,n} \in \mathbf{H}^1(\mathcal{T}_H)$, $\forall n = 1, \dots, n_T$ ($\mathbf{H}^1(\Omega)$ -non-conforming)
- Strong symmetric σ_h
- **High-order** local approximations in space $\mathbf{V}_h(K)$
- Use of different local parameters (h, p) and $(\Delta\tau = \frac{\Delta t}{n_\tau}, \gamma, \beta)$ to different $K \times I_n$
- Energy-conservative local method \Rightarrow energy-conservative MHM

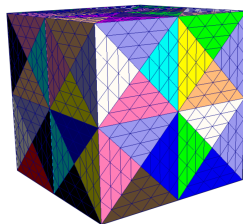


Numerical results

An analytical problem

Isotropic homogeneous media with exact solution $\mathbf{u} = [u_1, u_2, u_3]^T$:

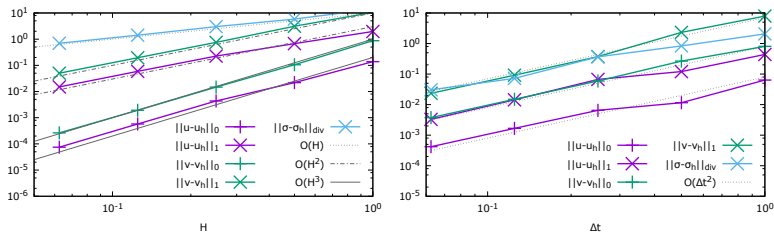
$$\begin{cases} u_1(t, x, y, z) &= \frac{1}{2}(1 - \cos(\omega t)) \sin(2\pi x) \sin(2\pi y) \sin(2\pi z), \\ u_2(t, x, y, z) &= -\frac{1}{2}(1 - \cos(\omega t)) \sin(2\pi x) \sin(2\pi y) \sin(2\pi z), \\ u_3(t, x, y, z) &= \frac{1}{2}(1 - \cos(\omega t)) \sin(\pi x) \sin(\pi y) \sin(\pi z). \end{cases}$$



$$\begin{aligned} \Lambda_H &:= \{ \boldsymbol{\mu}_H \in \Lambda : \boldsymbol{\mu}_H|_F \in [\mathbb{P}_\ell(F)]^2 \text{ for all } F \in \partial\mathcal{T}_H \} \\ \mathbf{V}_h(K) &:= \{ \mathbf{v}_h \in C(\overline{K}) : \mathbf{v}_h|_\tau \in [\mathbb{P}_k(\tau)]^3 \text{ for all } \tau \in \mathcal{T}_h^K \} \end{aligned}$$

Convergence results

Figure : Experiments with $\ell = 3$ and $k = 5$.



Error convergence observed numerically:

- $O(\Delta t^2)$ in all norms.
- $O(H^{\ell+2})$ in the $L^2(\Omega)$ -norm, $O(H^{\ell+1})$ in the $H^1(\mathcal{T}_H)$ -norm and $O(H^\ell)$ in the div -norm.

Multiscale scenario ³

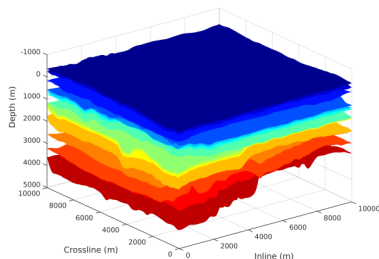


Figure : The physical domain divided in layers.

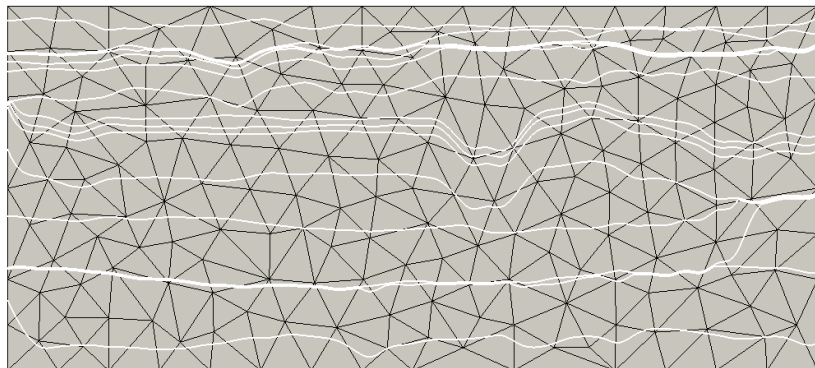
Problem:

- Flat case.
- Single shot.
- Two-dimensional problem at a slice (inline = 0 m).
- Free surface boundary condition.

³J. de la Puente. [HPC4E Seismic Test Suite, 2016](#). Copyright Josep de la Puente (Barcelona Supercomputing Center) 2016. Licenced under the Creative-Commons Attribution-ShareAlike 4.0 International License

MHM configuration

- Edges divided in 4 equally spaced parts.
- \mathbb{P}^1 approximation in each part of edge.
- Local meshes with 4096 \mathbb{P}^3 elements ($h = H/64$).



MHM solution

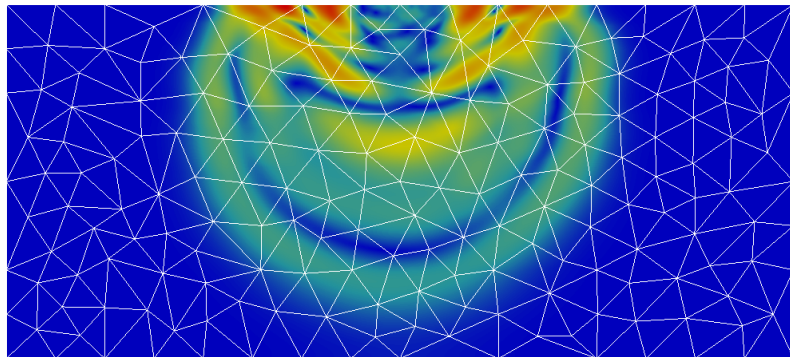
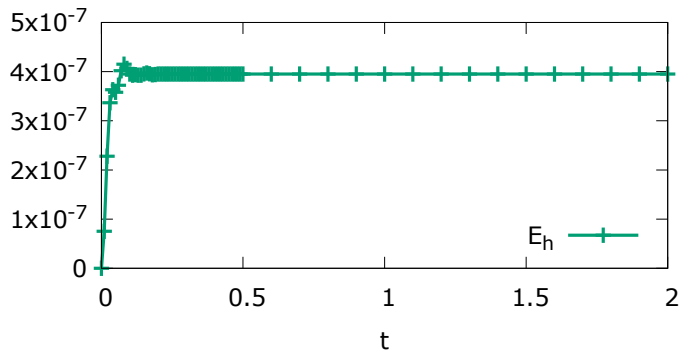


Figure : Isolines of $|\mathbf{u}_h|$ solution at $t = 0.15$.

Global energy

$$E_h := \frac{1}{2} \sum_{K \in \mathcal{T}_H} (\rho \mathbf{v}_h, \mathbf{v}_h)_K + (\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}(\mathbf{u}_h))_K$$



Comparison with a Galerkin solution

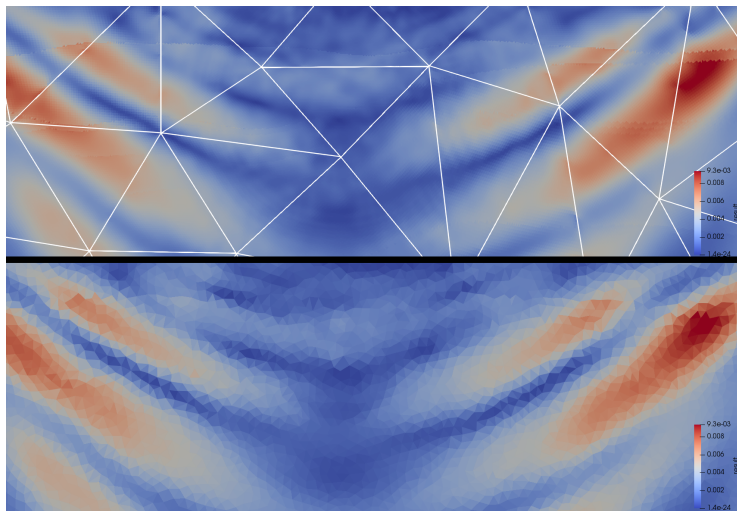


Figure : Isovalues of the von Mises stress for MHM (top) and a reference solution (bottom), at $t = 0.15$.

Final conclusions and remarks

- The MHM method provides a natural level of parallelism.
- It allows local mesh refinement and high-order approximations.
- Optimal convergence in time and super convergence in space for displacement, velocity and stress.
- It conserves energy depending on the local level.
- It is robust on non-aligned mesh with interfaces.

Ongoing work:

- Local time step validation.
- Error and stability analysis.

Thank you!

Funding and resources:



Referências I

- [1] J. de la Puente. HPC4E Seismic Test Suite, 2016. Copyright Josep de la Puente (Barcelona Supercomputing Center) 2016. Licenced under the Creative-Commons Attribution-ShareAlike 4.0 International License.
- [2] A. T. Gomes, D. Paredes, W. Pereira, R. Souto, and F. Valentin. A Multiscale Hybrid-Mixed Method for the Elastodynamic Model with Rough Coefficients. In *XXXVIII CILAMCE*, 2017.
- [3] C. Harder, A. L. Madureira, and F. Valentin. A Hybrid-Mixed Method for Elasticity. *ESAIM: M2AN*, 50(2):311–336, 2016.
- [4] C. Harder, D. Paredes, and F. Valentin. A family of Multiscale Hybrid-Mixed finite element methods for the Darcy equation with rough coefficients. *J. Comput. Phys.*, 245:107–130, 2013.

Referências II

- [5] C. Harder and F. Valentin. *Foundations of the MHM method*, volume 114 of *Lecture Notes in Computational Science and Engineering*, chapter Building Bridges: Connections and Challenges in Modern Approaches to Numerical Partial Differential Equations, pages 401–433. Springer, 2016.
- [6] W. S. Pereira and F. Valentin. A Locking-Free MHM Method for Elasticity. In *Proceeding Series of the Brazilian Society of Computational and Applied Mathematics*, volume 5, 2017.
- [7] P. A. Raviart, J. M. Thomas, P. G. Ciarlet, and J.-L. Lions. *Introduction à l'analyse numérique des équations aux dérivées partielles*. Masson, Paris, 1983.