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On the choice of weight functions for linear representations of persistence diagrams

Vincent Divol^{*}, Wolfgang Polonik[†]

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Abstract

Persistence diagrams are efficient descriptors of the topology of a point cloud. As they do not naturally belong to a Hilbert space, standard statistical methods cannot be directly applied to them. Instead, feature maps (or representations) are commonly used for the analysis. A large class of feature maps, which we call *linear*, depends on some weight functions, the choice of which is a critical issue. An important criterion to choose a weight function is to ensure stability of the feature maps with respect to Wasserstein distances on diagrams. We improve known results on the stability of such maps, and extend it to general weight functions. We also address the choice of the weight function by considering an asymptotic setting; assume that \mathbb{X}_n is an i.i.d. sample from a density on $[0, 1]^d$. For the Čech and Rips filtrations, we characterize the weight functions for which the corresponding feature maps converge as n approaches infinity, and by doing so, we prove laws of large numbers for the total persistences of such diagrams. Both approaches lead to the same simple heuristic for tuning weight functions: if the data lies near a d -dimensional manifold, then a sensible choice of weight function is the persistence to the power α with $\alpha \geq d$.

1 Introduction

Topological data analysis, or TDA (see [11] for a survey) is a recent field at the intersection of computational geometry, statistics and probability theory which has

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been successfully applied to various scientific areas, including biology [31], chemistry [24], material science [22] or the study of time series [28]. It consists of an array of techniques aimed at understanding the topology of a d -dimensional manifold based on an approximating point cloud \mathbb{X} . For instance, clustering can be seen as the estimation of the connected components of a given manifold. Persistence diagrams are one of the tools used most often in TDA. They are efficient descriptors of the topology of a point cloud, consisting in a multiset Dgm of points in $\mathbb{R}_>^2 = \{r = (r_1, r_2) \in \mathbb{R}^2, r_1 < r_2\}$ (see Section 2 for a more precise definition). The space \mathcal{D} of persistence diagrams is not naturally endowed with a Hilbert space structure, making statistical inference rather awkward. A common scheme to overcome this issue is to use a *representation* or *feature map* $\Phi : \mathcal{D} \rightarrow \mathcal{B}$, where \mathcal{B} is some Banach space: classical machine learning techniques are then applied to $\Phi(\text{Dgm})$ instead of Dgm , where it is assumed that an entire set (or sample) of persistence diagrams is observed. A natural way to create such feature maps is to consider a function $\phi : \mathbb{R}_>^2 \rightarrow \mathcal{B}$ and to define

$$\Phi(\text{Dgm}) := \sum_{r \in \text{Dgm}} \phi(r). \quad (1)$$

A multiset can equivalently be seen as a measure. Therefore we let Dgm also denote the measure $\sum_{r \in \text{Dgm}} \delta_r$ with δ_r denoting Dirac measure in r . With this notation, $\Phi(\text{Dgm})$ is equal to $\text{Dgm}(\phi)$, the integration of ϕ against the measure Dgm . Representations as in (1) are called *linear* as they define linear maps from the space of finite signed measures to the Banach space \mathcal{B} . In the following, a representation will always be considered linear. Many linear representations exist in the literature, including persistence images and its variants [12, 27, 21, 1], persistence silhouettes [10] or accumulated persistence function [2]. Notable non-linear representations include persistence landscapes [6], and sliced Wasserstein kernels [7].

In machine learning, a common way to circumvent the so-called "curse of dimensionality" is to assume that the data lies near some low-dimensional manifold M . Under this assumption, the persistence diagram of the data set (built with the Čech filtration, for instance) is made of two different types of points: points Dgm_{true} far away from the diagonal, which estimate the diagram of the manifold M , and points Dgm_{noise} close to the diagonal, which are generally considered to be "topological noise" (see Figure 1). This interpretation is a consequence of the stability theorem for persistence diagrams; see [13]. If the relevant information lies in the structure of the manifold, then the topological noise indeed represents true noise, and representations of the form $\text{Dgm}(\phi)$ are bound to fail if $\text{Dgm}_{noise}(\phi)$ is dominating $\text{Dgm}_{true}(\phi)$. A way to avoid such behaviour is to weigh the points in diagrams by means of a weighting function $w : \mathbb{R}_>^2 \rightarrow \mathbb{R}$. If w is chosen properly,

i.e. small enough close to the diagonal, then one can hope that $\text{Dgm}_{\text{true}}(w\phi)$ can be separated from $\text{Dgm}_{\text{noise}}(w\phi)$. The weight functions w are typically chosen as functions of the persistence $\text{pers}(\mathbf{r}) = r_2 - r_1$, a choice which will be made here also. Of course, it is not clear what "small enough" really means, and there are several ways to address the issue.

A first natural answer is to look at the problem from a stability point of view. Indeed, as data are intrinsically noisy, a statistical method has to be stable with respect to some metric in order to be meaningful. Standard metrics on the space of diagrams \mathcal{D} are Wasserstein distances W_p , which are, under mild assumptions, known to be stable with respect to the data on which diagrams are built. The task therefore becomes to find the representations $\text{Dgm}(w\phi)$ which are continuous with respect to some Wasserstein distance. A recent work from [20] shows that when sampling from a d -dimensional manifold, a weight function of the form $w(\mathbf{r}) = \arctan(A \cdot \text{pers}(\mathbf{r})^\alpha)$ with $\alpha > d + 1$ ensures that so-called persistence images are Lipschitz. Our first contribution is to show that, for a general class of weight functions, a choice of $\alpha > d$ is enough to make *all* linear representations continuous (even Hölderian of exponent $\alpha - d$).

Our second (and main) contribution is to answer this question from an asymptotic point of view. Assume that the diagram Dgm_n is built on a data set of size n . For which weight functions does $\text{Dgm}_{n,\text{noise}}(w\phi)$ not diverge? Of course, for the question to make sense, a model for the data set has to be specified. A simple model is given by a Poisson (or binomial) process \mathbb{X}_n of intensity n in a cube of dimension d . We denote the corresponding diagrams built on a filtration \mathcal{K} by diagrams $\text{Dgm}[\mathcal{K}(\mathbb{X}_n)]$. A precise definition is given below in Section 2. In this setting, there are no "true" topological features (other than the trivial topological feature of $[0, 1]^d$ being connected), and thus the diagram based on the sampled data is uniquely made of topological noise. A first promising result is the vague convergence of the measure $\mu_n := n^{-1}\text{Dgm}[\mathcal{K}(n^{1/d}\mathbb{X}_n)]$, which was recently proven in [17] for a homogeneous Poisson process. However, vague convergence is not enough for our purpose, as neither ϕ nor w have good reasons to have compact support. Our main theorem, Theorem 6, extend the result of [17] to a stronger convergence, allowing test functions both to have non-compact support (but to converge to 0 near the diagonal) and to have a polynomial growth. As a corollary of this general result, the convergence of the α -th total persistence, which plays an important role in TDA, are shown. The α -th total persistence is defined as $\text{Pers}_\alpha(\text{Dgm}) := \text{Dgm}(\text{pers}^\alpha) = \sum_{\mathbf{r} \in \text{Dgm}} \text{pers}(\mathbf{r})^\alpha$.

Theorem 1. *Let κ be a density on $[0, 1]^d$ such that $0 < \inf \kappa \leq \sup \kappa < \infty$. Let \mathbb{X}_n be either a binomial process with parameters n and κ or a Poisson process of*

intensity $n\kappa$ in the cube $[0, 1]^d$. Define $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)]$ the persistence diagram of \mathbb{X}_n for q -dimensional homology, built with either the Rips or the Čech filtration. Then, with probability one,

$$n^{\frac{\alpha}{d}-1} \text{Pers}_\alpha(\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)]) \rightarrow \mu_q^\kappa(\text{pers}^\alpha) < \infty \quad (2)$$

for some non-degenerate Radon measure μ_q^κ on $\mathbb{R}_{>}^2$.

Therefore, the quantity $\text{Dgm}_{n,\text{noise}}(w)$, which can be expected to be close to $\text{Pers}_\alpha(\text{Dgm}[\mathcal{K}(\mathbb{X}_n)])$, is shown to converge to 0 if and only if the weight function $w = \text{pers}^\alpha$ is such that $\alpha \geq d$. The same heuristic is found through both approaches: a weight function of the form pers^α with $\alpha \geq d$ is sensible if the data lies near a d -dimensional object.

Further properties on the process are also shown, namely non-asymptotic rate of decays for the number of points in said diagrams, and the absolute continuity of the marginals of μ_q^κ with respect to the Lebesgue measure on \mathbb{R} .

1.1 Related work

Techniques used to derive the large sample results indicated above are closely related to the field of geometric probability, which is the study of geometric quantities arising naturally from point processes in \mathbb{R}^d . A classical result in this field [29] proves the convergence of the total length of the minimum spanning tree built on n i.i.d. points in the cube. This pioneering work can be seen as a 0-dimensional special case of our general results about persistence diagrams built for homology of dimension q . This type of result has been extended to a large class of functionals with the works of J. E. Yukich and M. Penrose (see for instance [23, 34, 26] and [25] or [35] for monographs on the subject). The study of higher dimensional properties of such processes is much more recent. Known results include convergence of the Betti numbers for various models and under various asymptotics [18, 19, 33, 4]. Few results exist about the persistence homology of random point clouds. [3] finds bounds on the persistence of cycles in random complexes and [17] proves limit theorems for persistence diagrams. Note that our results constitutes a natural extension of the latter.

1.2 Notation

- $\|\cdot\|$ is the Euclidean distance on \mathbb{R}^D .
- $\|\cdot\|_\infty$ is the the ∞ -norm of a function.

- $B(x, r)$ is the open ball of radius r centered at x .
- $\text{diam}(\mathbb{X})$ is the diameter of a set \mathbb{X} , defined as $\sup_{x, y \in \mathbb{X}} \|x - y\|$.
- $|\cdot|$ is the total variation of a measure.
- $\#$ is the cardinality of a set.

The rest of the paper is organized as follows. In Section 2, some background on persistent homology is briefly described. The stability results are then discussed in Section 3 whereas the convergence results related to the asymptotic behavior of the sample-based linear representations are stated in Section 4. Section 5 presents some discussion. Proofs can be found in Section 6.

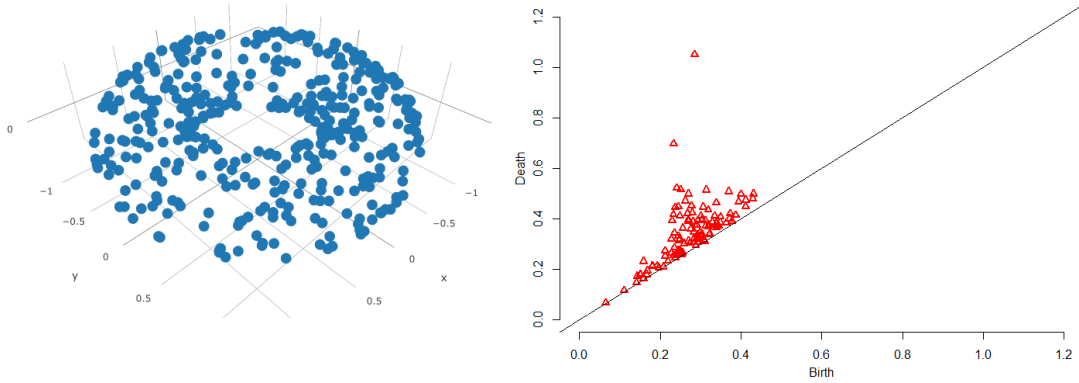


Figure 1: The persistence diagram for homology of degree 1 of the Rips filtration of 2000 i.i.d. points uniformly sampled on a torus. There are two distinct points in the diagram, corresponding to the two equivalence classes of one-dimensional holes of the torus.

2 Background on persistence diagrams

Persistent homology deals with the evolution of homology through a sequence of topological spaces. We use \mathbb{F}_2 , the field of two elements to build the homology groups. A filtration $\mathcal{K} = (K^r)_{r \geq 0}$ is an increasing right-continuous sequence of topological spaces : $K^{r'} \subset K^r$ if $r' \leq r$ and $K^r = \bigcap_{r' < r} K^{r'}$. For any $q \geq 0$, the inclusion of spaces give rise to linear maps between corresponding homology groups $H_q(K^r)$.

The persistence diagram $\text{Dgm}_q[\mathcal{K}]$ of the filtration is a succinct way to summarize the evolution of the homology groups. It is a multiset of points in $\mathbb{R}_{>}^2 = \{\mathbf{r} = (r_1, r_2) \in \mathbb{R}^2, r_1 < r_2\}$ ¹, so that each point $\mathbf{r} = (r_1, r_2)$ corresponds informally to a q -dimensional "hole" in the filtration \mathcal{K} that appears (or is born) at r_1 and disappears (or dies) at r_2 . The persistence $\text{pers}(\mathbf{r})$ of \mathbf{r} is defined as $r_2 - r_1$ and is understood as the lifetime of the corresponding hole. Persistence diagrams are known to exist given mild assumptions on the filtration (see [8] for a monograph on the subject). Some basic descriptors of persistence diagrams include the α -th total persistence of a diagram, defined as

$$\text{Pers}_\alpha(\text{Dgm}) := \text{Dgm}(\text{pers}^\alpha) = \sum_{\mathbf{r} \in \text{Dgm}} \text{pers}(\mathbf{r})^\alpha, \quad \alpha > 0, \quad (3)$$

and the persistent Betti numbers, defined as

$$\beta^{r,s}(\text{Dgm}) := \text{Dgm}([0, r] \times (s, \infty)) = \sum_{\mathbf{r} \in \text{Dgm}} \mathbf{1}\{\mathbf{r} \in [0, r] \times (s, \infty)\}, \quad 0 \leq r \leq s. \quad (4)$$

Also, for $M \geq 0$, define

$$\text{Pers}_\alpha(\text{Dgm}, M) := \text{Dgm}(\text{pers}^\alpha \mathbf{1}\{\text{pers} \geq M\}). \quad (5)$$

Given a subset \mathbb{X} of a metric space (\mathbb{Y}, d) , standard constructions of filtrations are the Čech filtration $\mathcal{C}(\mathbb{X}) = (C^r(\mathbb{X}))_{r \geq 0}$ and the Rips filtration $\mathcal{R}(\mathbb{X}) = (R^r(\mathbb{X}))_{r \geq 0}$:

$$C^r(\mathbb{X}) = \left\{ \text{finite } \sigma \subset \mathbb{X}, \bigcap_{x \in \sigma} B(x, r) \neq \emptyset \right\} \quad \text{and} \quad (6)$$

$$R^r(\mathbb{X}) = \{ \text{finite } \sigma \subset \mathbb{X}, \text{diam}(\sigma) \leq r \}, \quad (7)$$

where the abstract simplicial complexes on the right are identified with their geometric realizations. The dimension of a simplex σ is equal to $\#\sigma - 1$. If K is a simplicial complex, the set of its simplexes of dimension q is denoted by K_q .

The space of persistence diagrams \mathcal{D} is the set of all finite multisets in $\mathbb{R}_{>}^2$. Wasserstein distances are standard distances on \mathcal{D} . For $p \geq 1$, they are defined as:

$$W_p(\text{Dgm}, \text{Dgm}') := \min_{\gamma} \left(\sum_{\mathbf{r} \in \text{Dgm} \cup \Delta} \|\mathbf{r} - \gamma(\mathbf{r})\|^p \right)^{1/p}, \quad (8)$$

¹Persistence diagrams are in all generality multiset of points in $\{\mathbf{r} = (r_1, r_2), -\infty \leq r_1 < r_2 \leq \infty\}$. We only consider diagrams which do not contain points "at infinity" throughout the paper.

where Δ is the diagonal of \mathbb{R}^2 and $\gamma : \text{Dgm} \cup \Delta \rightarrow \text{Dgm}' \cup \Delta$ is a bijection. The definition is extended to $p = \infty$ by

$$W_\infty(\text{Dgm}, \text{Dgm}') = \min_{\gamma} \max_{r \in \text{Dgm} \cup \Delta} \|r - \gamma(r)\|, \quad (9)$$

which is called the *bottleneck distance*.

The use of Wasserstein distances is motivated by crucial stability properties they satisfy. Let $f, g : \mathbb{Y} \rightarrow \mathbb{R}$ be two continuous functions on a triangulable space \mathbb{Y} . Under a mild technical assumptions, called *tameness*, the persistence diagrams $\text{Dgm}_q[f]$ and $\text{Dgm}_q[g]$ of the filtrations defined by the sublevel sets of f and g exist. The stability property (see [13]) asserts that $W_\infty(\text{Dgm}_q[f], \text{Dgm}_q[g]) \leq \|f - g\|_\infty$, i.e. the diagrams are stable with respect to the functions they are built with. The functions f and g have to be thought of as representing the data: for instance, if the Čech filtration is built on a data set $\mathbb{X}_n = \{X_1, \dots, X_n\}$, then $\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)] = \text{Dgm}_q[f]$ where f is the distance function to \mathbb{X}_n , i.e. $f(x) = d(x, \mathbb{X}_n)$. When $p < \infty$, similar stability results have been proved under more restrictive conditions on the ambient space \mathbb{X} , which we now detail.

Definition 1. *A metric space \mathbb{X} is said to have bounded m -th total persistence if there exists a constant $C_{\mathbb{X},m}$ such that for all tame 1-Lipschitz functions $f : \mathbb{X} \rightarrow \mathbb{R}$ and for all $q \geq 1$, $\text{Pers}_m(\text{Dgm}_q[f]) \leq C_{\mathbb{X},m}$.*

This assumption is, for instance, verified for a d -dimensional manifold \mathbb{X} when $m > d$ with

$$C_{\mathbb{X},m} = \frac{m}{m-d} C_{\mathbb{X}} \text{diam}(\mathbb{X})^{m-d}, \quad (10)$$

$C_{\mathbb{X}}$ being a constant depending only on \mathbb{X} (see [14]). The stability theorem for the p -th Wasserstein distances claims:

Theorem 2 (chapter 3 of [14]). *Let \mathbb{X} be a compact triangulable metric space which has bounded m -th total persistence for some $m \geq 1$. Let $f, g : \mathbb{X} \rightarrow \mathbb{R}$ be two tame Lipschitz functions. Then, for $q \geq 0$,*

$$W_p(\text{Dgm}_q[f], \text{Dgm}_q[g]) \leq C_0^{\frac{1}{p}} \|f - g\|_\infty^{1 - \frac{m}{p}}, \quad (11)$$

for $p \geq m$, where $C_0 = C_{\mathbb{X},m} \max\{\text{Lip}(f)^m, \text{Lip}(g)^m\}$.

3 Stability results for linear representations

In [20], representations of diagrams are shown to be Lipschitz with respect to the p -th Wasserstein distances for weight functions of the form $w(r) = \arctan(A \cdot \text{pers}(r)^\alpha)$

with $\alpha > m + 1$, provided the diagrams are built with the sublevels of functions defined on a space \mathbb{X} having bounded m -th total persistence. The stability result is proved for a particular function $\phi : \mathbb{R}_>^2 \rightarrow \mathcal{B}$ defined by $\phi(\mathbf{r}) = K(\mathbf{r}, \cdot)$, with K a Gaussian kernel and \mathcal{B} the associated RKHS. We present a generalization of the stability result to (i) general weight functions w , (ii) any bounded Lipschitz function ϕ and (iii) for $\alpha > m$.

Consider weight functions $w : \mathbb{R}_>^2 \rightarrow \mathbb{R}_+$ of the form $w(\mathbf{r}) = \tilde{w}(\text{pers}(\mathbf{r}))$ for $\tilde{w} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a differentiable function with $\tilde{w}(0) = 0$, satisfying for some $A > 0$, $\alpha \geq 1$,

$$\forall u \geq 0, |\tilde{w}'(u)| \leq Au^{\alpha-1}. \quad (12)$$

Examples of such functions include $w(\mathbf{r}) = \arctan(B \cdot \text{pers}(\mathbf{r})^\alpha)$ for $A = B\alpha'$ and $w(\mathbf{r}) = \text{pers}(\mathbf{r})^\alpha$. We denote the class of such weight functions by $\mathbb{W}(\alpha, A)$. In contrast to [20], the function ϕ does not necessarily take its values in a RKHS, but simply in a Banach space (so that its Bochner integral [16] is well defined).

Theorem 3. *Let $A > 0$, $\alpha \geq 1$ and $\phi : \mathbb{R}_>^2 \rightarrow (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Lipschitz continuous function with Lipschitz constant $\text{Lip}(\phi)$. For $w \in \mathbb{W}(\alpha, A)$, let $\Phi_w(\text{Dgm}) := \text{Dgm}(w\phi)$ and let Dgm_1 and Dgm_2 be two persistence diagrams. Define for $q \geq 0$, $G\{t\} := \max\{\text{Pers}_t(\text{Dgm}_1), \text{Pers}_t(\text{Dgm}_2)\}$. Then, for $1 \leq p \leq \infty$, and $a \in [0, 1]$ (and using the conventions $0/\infty = 0$ and $\infty/\infty = 1$), we have*

$$\begin{aligned} \|\Phi_w(\text{Dgm}_1) - \Phi_w(\text{Dgm}_2)\|_{\mathcal{B}} &\leq \text{Lip}(\phi) \frac{A}{\alpha} \left(G \left\{ p \frac{\alpha}{p-1} \right\} \right)^{1-\frac{1}{p}} W_p(\text{Dgm}_1, \text{Dgm}_2) \quad (13) \\ &+ \|\phi\|_{\infty} A \left(2G \left\{ p \frac{\alpha-a}{p-a} \right\} \right)^{1-\frac{a}{p}} W_p(\text{Dgm}_1, \text{Dgm}_2)^a. \end{aligned}$$

The quantity $G\{q\}$ can often be controlled. This is, for instance, the case when the diagrams are built with Lipschitz continuous functions $f : \mathbb{X} \rightarrow \mathbb{R}$ and \mathbb{X} is a space having bounded m -th total persistence.

Corollary 1. *Let $A > 0$, $\alpha \geq 1$, and let \mathbb{X} be a compact triangulable metric space having bounded m -th total persistence for some $m \geq 1$. Assume $f, g : \mathbb{X} \rightarrow \mathbb{R}$ are two tame Lipschitz continuous functions, and let $w \in \mathbb{W}(\alpha, A)$. Let $a \in [0, 1]$. Then, for $m \leq p \leq \infty$ such that $\alpha \geq a + m \left(1 - \frac{a}{p}\right) \geq 0$, we have with $C_0 = C_{\mathbb{X}, m} \max\{\text{Lip}(f)^m, \text{Lip}(g)^m\}$ and ℓ the maximum persistence in the two diagrams:*

$$\begin{aligned}
& \|\Phi_w(\text{Dgm}_q[f]) - \Phi_w(\text{Dgm}_q[g])\|_{\mathcal{B}} \\
& \leq \text{Lip}(\phi) \frac{A}{\alpha} \ell^{\alpha-m(1-\frac{1}{p})} C_0^{1-\frac{1}{p}} W_p(\text{Dgm}_q[f], \text{Dgm}_q[g]) \\
& \quad + \|\phi\|_{\infty} A \ell^{\alpha-a-m(1-\frac{a}{p})} (2C_0)^{1-\frac{a}{p}} W_p(\text{Dgm}_q[f], \text{Dgm}_q[g])^a, \quad (14)
\end{aligned}$$

If $\alpha > m + 1$ and $p = \infty$, then the result is similar to Theorem 3.3 in [20]. The novelty lies in that Corollary 1 implies that the representations are still continuous (actually Hölder continuous) when $\alpha \in (m, m + 1]$. Indeed, for such an α , one can always choose a small enough and p large enough such that the stability result (14) holds. The proofs of Theorem 3 and Corollary 1 consist of an adaptation of similar proofs in [20]. They can be found in Section 6.

Remark 1. (a) *One cannot expect to obtain an inequality of the form (13) without quantities $G\{t\}$ (or other quantities depending on the diagrams) appearing on the right-hand side. Indeed, in the case $p = \infty$, it is clear that adding an arbitrary number of points near the diagonal will not change the bottleneck distance between the diagram, whereas the distance between representations can become arbitrarily large.*

(b) *Laws of large numbers stated in the next section, and specifically Theorem 1 already stated in the introduction, show that Theorem 3 is optimal: take $w = \text{pers}^{\alpha}$ and $\phi \equiv 1$. If \mathbb{X}_n is a sample on the d -dimensional cube $[0, 1]^d$ (which has bounded m -th total persistence for $m > d$), then $\Phi_w(\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)]) = \text{Pers}_{\alpha}(\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)])$. The quantity $\text{Pers}_{\alpha}(\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)])$ does not converge to 0 for $\alpha \leq d$ (it even diverges if $\alpha < d$), whereas the bottleneck distance between $\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)]$ and 0 does converge to 0.*

As a corollary to the stability result, and as a first approach to the asymptotic study of next section, we give rates of convergence of representations in a random setting. Let $\mathbb{X}_n = \{X_1, \dots, X_n\}$ be a n -sample of i.i.d. points from a distribution with density κ with respect to the d -dimensional Hausdorff measure on some compact d -dimensional manifold \mathbb{X} . Assume that $0 < \inf \kappa \leq \sup \kappa < \infty$. We are interested in the convergence of representations $\Phi(\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)])$ to the representations $\Phi(\text{Dgm}_q[\mathcal{C}(\mathbb{X})])$. The nerve theorem asserts that for any subspace $\mathbb{X}' \subset \mathbb{X}$, $\text{Dgm}_q[\mathcal{C}(\mathbb{X}')] = \text{Dgm}_q[f]$ where $f(x)$ is the distance from x to \mathbb{X}' . We obtain the following corollary, whose proof is found in Section 6:

Corollary 2. *Under the previous set of assumptions, for n large enough,*

$$E \left[\|\Phi(\text{Dgm}_q[\mathcal{C}(\mathbb{X}_n)]) - \Phi(\text{Dgm}_q[\mathcal{C}(\mathbb{X})])\|_{\mathcal{B}} \right] \leq C \|\phi\|_{\infty} \frac{\alpha}{\alpha - d} \left(\frac{\ln n}{n} \right)^{\frac{\alpha}{d} - 1}, \quad (15)$$

for some constant C depending on \mathbb{X} , A and the density κ .

4 Convergence of total persistence

Consider again the i.i.d. model: let $\mathbb{X}_n = \{X_1, \dots, X_n\}$ be i.i.d. observations of density κ with respect to the d -dimensional Hausdorff measure on some d -dimensional manifold \mathbb{X} . The general question we are addressing in this section is the convergence of the observed diagrams $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)]$ to $\text{Dgm}_q[\mathcal{K}(\mathbb{X})]$, with \mathcal{K} either the Rips or the Čech filtration. Of course, the question has already been answered in some sense. Indeed, Theorem 2 affirms that the sequence of observed diagrams will always converge to $\text{Dgm}_q[\mathcal{K}(\mathbb{X})]$ for the bottleneck distance. However, this is not informative with respect to the convergence of the representations introduced in the previous section, which is related to a weak convergence of measure: For which functions ϕ does $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)](\phi)$ converge to $\text{Dgm}_q[\mathcal{K}(\mathbb{X})](\phi)$?

The stability theorem for the bottleneck distance asserts that, for small $\varepsilon > 0$, and for n large enough, $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)]$ can be decomposed into two separate sets of points: a set of fixed size $\text{Dgm}_{\text{true},q}[\mathcal{K}(\mathbb{X}_n)]$ that is ε -close to points in $\text{Dgm}_q[\mathcal{K}(\mathbb{X})]$ and the remaining part of the diagram, $\text{Dgm}_{\text{noise},q}[\mathcal{K}(\mathbb{X}_n)]$, usually consisting of a large number of points, which have persistence smaller than ε , i.e. these are the points lie close to the diagonal. A Taylor expansion of ϕ shows that the difference between $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)](\phi)$ and $\text{Dgm}_q[\mathcal{K}(\mathbb{X})](\phi)$ is of the order of $\text{Dgm}_{\text{noise},q}[\mathcal{K}(\mathbb{X}_n)](\text{pers}^\alpha)$ for some $\alpha \geq 0$. The latter quantities are therefore of utmost interest to achieve our goal. Instead of directly studying $\text{Dgm}_{\text{noise},q}[\mathcal{K}(\mathbb{X}_n)](\text{pers}^\alpha)$ for \mathbb{X}_n on a d -dimensional manifold, we focus on the study of the quantity $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)](\text{pers}^\alpha)$ for \mathbb{X}_n in a cube $[0, 1]^d$.

Contributions to the study of quantities of the form $\text{Dgm}_q[\mathcal{K}(\mathbb{S}_n)](\phi)$ have been made in [17], where \mathbb{S}_n is considered to be the restriction of a stationary process to a box of volume n in \mathbb{R}^d . Specifically, [17] shows the vague convergence of the rescaled diagram $n^{-1}\text{Dgm}_q[\mathcal{K}(\mathbb{S}_n)]$ to some Radon measure μ . Vague convergence deals with continuous functions ϕ with compact support, whereas we are interested in functions of the type pers^α , which are not even bounded. Our contributions to the matter are two-folds. First, in the Čech and Rips setting, we extend the results of [17] to non-homogeneous models: both non-homogeneous binomial and Poisson process on the cube $[0, 1]^d$ are considered. Second, we prove a stronger convergence, allowing test functions to have non-compact support and polynomial growth. As a gentle introduction to the formalism used later, we first recall some known results from geometric probability on the study of Betti numbers, and we also detail relevant

results of [17].

4.1 Prior work

In the following, \mathcal{K} refers to either the Čech or the Rips filtration. Let κ be a density on $[0, 1]^d$ such that:

$$0 < \inf \kappa \leq \sup \kappa < \infty. \quad (16)$$

Note that the cube $[0, 1]^d$ could be replaced by any compact convex body (i.e. the boundary of an open bounded convex set). However, the proofs (especially geometric arguments of Section 6.4) becomes much more involved in this greater generality. To keep the main ideas clear, we therefore restrict ourselves to the case of the cube. We indicate, however, when challenges arise in the more general setting.

Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables sampled from density κ and let $(N_i)_{i \geq 1}$ be an independent sequence of Poisson variables with parameter i . In the following \mathbb{X}_n denotes either $\{X_1, \dots, X_n\}$, a binomial process of intensity κ and of size n , or $\{X_1, \dots, X_{N_n}\}$, a Poisson process of intensity $n\kappa$. The fact that the binomial and Poisson processes are built in this fashion is not important for weak laws of large numbers (only the law of the variables is of interest), but it is crucial for strong laws of large numbers to make sense.

The persistent Betti numbers $\beta_q^{r,s}(\text{Dgm}_q[\mathcal{K}]) = \text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)](\mathbf{1}_{[r,\infty)} \times \mathbf{1}_{[s,\infty)})$ are denoted more succinctly by $\beta_q^{r,s}(\mathcal{K})$. When $r = s$, we use the notation $\beta_q^r(\mathcal{K})$.

Theorem 4 (Theorem 4.6 in [33]). *Let $r > 0$ and $q \geq 0$. Then, with probability one:*

$$n^{-1} \left(\beta_q^r(\mathcal{K}(n^{1/d}\mathbb{X}_n)) - E[\beta_q^r(\mathcal{K}(n^{1/d}\mathbb{X}_n))] \right) \xrightarrow[n \rightarrow \infty]{} 0. \quad (17)$$

The theorem is originally stated with the Čech filtration but its generalization to the Rips filtration (or even to more general filtrations considered in [17]) is straightforward. The proof of this theorem is based on a simple, yet useful geometric lemma, which still holds for the persistent Betti numbers, as proven in [17].

Lemma 1 (Lemma 2.11 in [17]). *Let $\mathbb{X} \subset \mathbb{Y}$ be two subsets of \mathbb{R}^d . Then*

$$|\beta_q^{r,s}(\mathcal{K}(\mathbb{X})) - \beta_q^{r,s}(\mathcal{K}(\mathbb{Y}))| \leq \sum_{j=q}^{q+1} |K_j^s(\mathbb{Y}) \setminus K_j^s(\mathbb{X})|. \quad (18)$$

In [17], this lemma was used to prove the convergence of expectations of diagrams of stationary point processes. A point process \mathbb{S} is said to be stationary if $\mathbb{S} + x$ has the same law as \mathbb{S} for all $x \in \mathbb{R}^d$. Define $\mathbb{S}_n := \mathbb{S} \cap [0, n^{1/d}]^d$. The following theorem

is the main theorem of [17] and deals with vague convergence of measures. Let $C_c(\mathbb{R}_>^2)$ be the set of functions $\phi : \mathbb{R}_>^2 \rightarrow \mathbb{R}$ with compact support. We say that a sequence $(\mu_n)_{n \geq 0}$ of measures on $\mathbb{R}_>^2$ converges *C_c -vaguely* to μ if $\forall \phi \in C_c(\mathbb{R}_>^2)$, $\mu_n(\phi) \xrightarrow[n \rightarrow \infty]{} \mu(\phi)$. Note that this does not include the function $\phi = 1$ or the function $\phi = \text{pers}$. Vague convergence is denoted by \xrightarrow{vc} . We refer to [17] for further definitions appearing in the following theorem.

Theorem 5 (Theorem 1.5 of [17]). *Assume that \mathbb{S} is a stationary point process on \mathbb{R}^d having all finite moments. Then, for $q \geq 0$, there exists a unique Radon measure μ_q on $\mathbb{R}_>^2$ such that*

$$n^{-1} E[\text{Dgm}_q[\mathcal{K}(\mathbb{S}_n)]] \xrightarrow[n \rightarrow \infty]{vc} \mu_q. \quad (19)$$

Assume further that \mathbb{S} is ergodic. Then,

$$n^{-1} \text{Dgm}_q[\mathcal{K}(\mathbb{S}_n)] \xrightarrow[n \rightarrow \infty]{vc} \mu_q \text{ a.s.} \quad (20)$$

The measure μ_q is called the persistence diagram of the process \mathbb{S} .

The convergence (20) thus holds for a homogeneous Poisson process, which is ergodic. However, when \mathbb{S} is a homogeneous Poisson process on \mathbb{R}^d , the *joint* laws of the processes $(\{X_1, \dots, X_{N_n}\})_{n \geq 0}$ and $(\mathbb{S}_n)_{n \geq 0}$ are different, even if $\kappa \equiv 1$. Therefore, the strong law of large numbers for our model of Poisson process is not yet shown.

4.2 Main results

A function $\phi : \mathbb{R}_>^2 \rightarrow \mathbb{R}$ is said to vanish on the diagonal if

$$\lim_{\varepsilon \rightarrow 0} \sup_{\text{pers}(\mathbf{r}) \leq \varepsilon} |\phi(\mathbf{r})| = 0. \quad (21)$$

Denote by $C_0(\mathbb{R}_>^2)$ the set of all such functions. The weight functions of Section 3 all lie in $C_0(\mathbb{R}_>^2)$. We say that a function $\phi : \mathbb{R}_>^2 \rightarrow \mathbb{R}$ has polynomial growth if there exist two constants $A, \alpha > 0$, such that

$$|\phi(\mathbf{r})| \leq A(1 + \text{pers}(\mathbf{r})^\alpha). \quad (22)$$

The class $C_{\text{poly}}(\mathbb{R}_>^2)$ of functions in $C_0(\mathbb{R}_>^2)$ with polynomial growth constitutes a reasonable class of functions $w \cdot \phi$ one may want to build a representation with. Our goal is to extend the convergence of Theorem 5 to this larger class of functions. Convergence of measures μ_n to μ with respect to $C_{\text{poly}}(\mathbb{R}_>^2)$, i.e. $\forall \phi \in C_c(\mathbb{R}_>^2)$, $\mu_n(\phi) \xrightarrow[n \rightarrow \infty]{} \mu(\phi)$, is denoted by \xrightarrow{vp} . Note that this class of functions is standard: it is for instance known to characterize p -th Wasserstein convergence in optimal transport (see [30]). Denote by μ_q^n the random measure $n^{-1} \text{Dgm}_q[\mathcal{K}(n^{1/d} \mathbb{X}_n)]$.

Theorem 6. (i) For $q \geq 0$, there exists a unique Radon measure μ_q^κ such that $E[\mu_q^n] \xrightarrow[n \rightarrow \infty]{v_p} \mu_q^\kappa$ and, with probability one, $\mu_q^n \xrightarrow[n \rightarrow \infty]{v_p} \mu_q^\kappa$. The measure μ_q^κ is called the q -th persistence diagram of intensity κ for the filtration \mathcal{K} . It does not depend on whether \mathbb{X}_n is a Poisson or a binomial process, and is of positive finite mass.

(ii) The convergence also holds pointwise for the L_p distance: $\forall \phi \in C_{poly}(\mathbb{R}_>^2), \forall p \geq 1, \mu_q^n(\phi) \xrightarrow[n \rightarrow \infty]{L_p} \mu_q^\kappa(\phi)$. In particular, $|\mu_q^\kappa(\phi)| < \infty$.

(iii) The measure μ_q^κ is the Radon measure given by

$$\mu_q^\kappa(\phi) = E \left[\int_{\mathbb{R}_>^2} \phi(r\kappa(X)^{-1/d}) d\mu_q(r) \right] \quad \forall \phi \in C_c(\mathbb{R}_>^2), \quad (23)$$

where $\mu_q = \mu_q^1$ is the q -th persistence diagram of uniform density on $[0, 1]^d$, appearing in Theorem 5.

Remark 2. (a) Assume $q = 0$ and $d = 1$. Then, the persistence diagram $\text{Dgm}_0[\mathcal{K}(\mathbb{X}_n)]$ is simply the collection of the intervals $(X_{(i+1)} - X_{(i)})$ where $X_{(1)} < \dots < X_{(n)}$ is the order statistics of \mathbb{X}_n . The measure $E[\mu_0^n]$ can be explicitly computed: it converges to a measure having density $u \mapsto E[\exp(-u\kappa(X))\kappa(X)]$ with respect to the Lebesgue measure on \mathbb{R}_+ . Take κ the uniform density on $[0, 1]$: one sees that this is coherent with the basic fact that the spacings of a homogeneous Poisson process on \mathbb{R} are distributed according to an exponential distribution. Moreover, the expression (23) is found again in this special case.

(b) It is proven in [32] that, for homogeneous Poisson processes, the Betti numbers $\beta_q^r(\mathcal{K}(\mathbb{X}_n))$ have a non-degenerate limit. Using equation (23), this implies in particular that μ_q^κ has a positive mass.

(c) Theorem 1 is a direct corollary of Theorem 6. Indeed, we have

$$n^{\frac{\alpha}{d}-1} \text{Pers}(\text{Dgm}_q[\mathbb{X}_n]) = n^{\frac{\alpha}{d}-1} \text{Dgm}_q[\mathbb{X}_n](\text{pers}^\alpha) = \frac{1}{n} \text{Dgm}_q[n^{\frac{1}{d}}\mathbb{X}_n](\text{pers}^\alpha) = \mu_n^\kappa(\text{pers}^\alpha),$$

a quantity which converges. The relevance of Theorem 1 is illustrated in Figure 2, where Čech complexes are computed on random samples on the torus.

The core of the proof of Theorem 6 consists in a control of the number of points appearing in diagrams. This bound is obtained thanks to geometric properties satisfied by the Čech and Rips filtrations. Finding good requirements to impose on a filtration \mathcal{K} for this control to hold is an interesting question. The following states some non-asymptotic controls of the number of points in diagrams which are interesting by themselves.

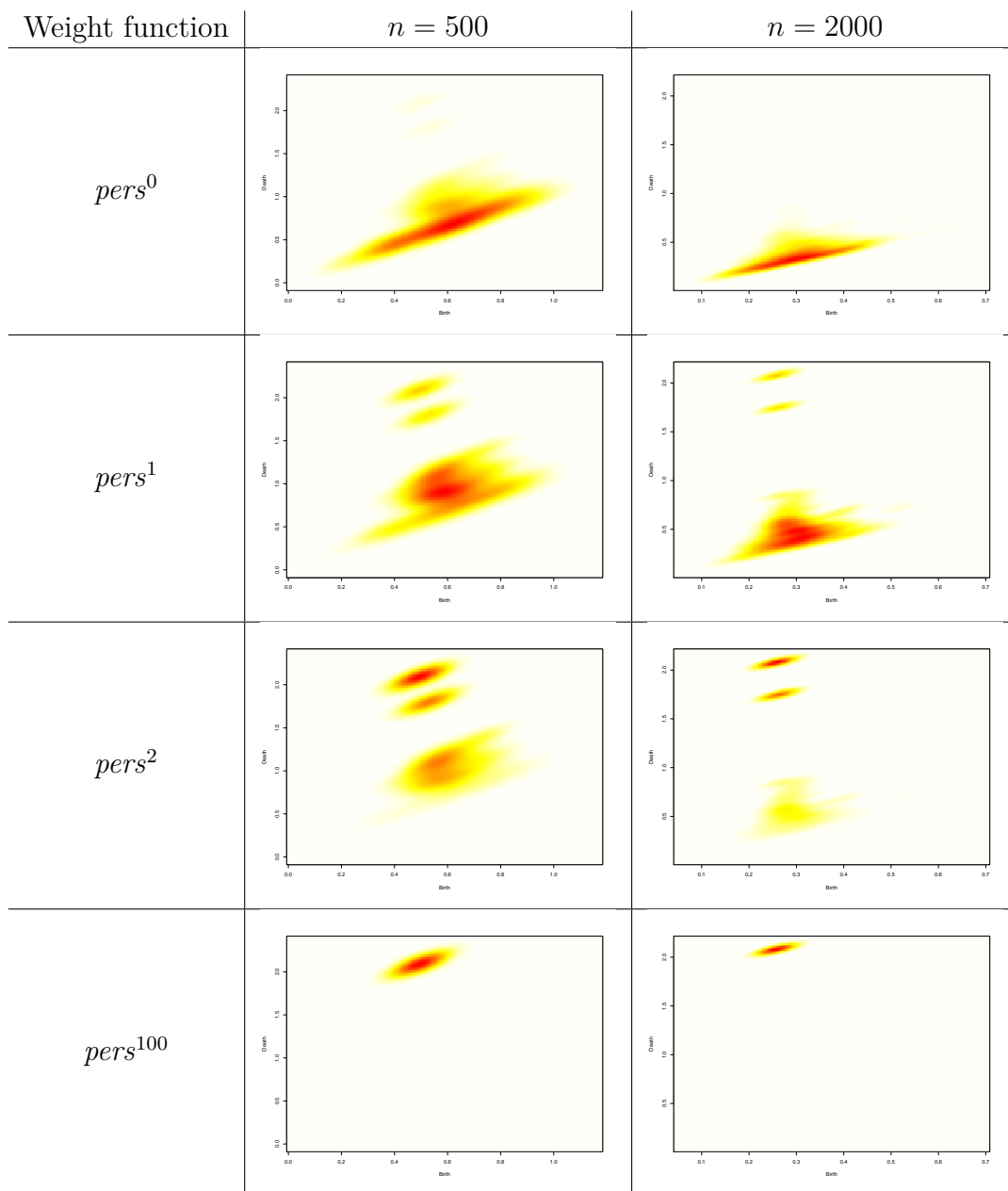


Figure 2: For $n = 500$ or 2000 points uniformly sampled on the torus, persistence images [1] for different weight functions are displayed. For $\alpha < 2$, the mass of the topological noise is far larger than the mass of the true signal, the latter being comprised by the two points with high-persistence. For $\alpha = 2$, the two points with high-persistence are clearly distinguishable. For $\alpha = 100$, the noise has also disappeared, but so has one of the point with high-persistence.

Proposition 1. *Let $M \geq 0$ and define $U_M = \mathbb{R} \times [M, \infty)$. Then, there exists constants $c_1, c_2 > 0$ (which can be made explicit) depending on κ and q , such that, for any $t > 0$,*

$$P(\mu_q^n(U_M) > t) \leq c_1 \exp(-c_2(M^d + t^{1/(q+1)})). \quad (24)$$

As an immediate corollary, the moments of the total mass $|\mu_n|$ are uniformly bounded. However, the proof of the almost sure finiteness of $\sup_n |\mu_n|$ is much more intricate. Indeed, we are unable to control directly this quantity, and we prove that a majorant of $|\mu_n|$ satisfies concentration inequalities. The majorant arises as the number of simplicial complexes of a simpler process, whose expectation is also controlled.

It is natural to wonder whether μ_q^κ has some density with respect to the Lebesgue measure on $\mathbb{R}_>^2$: it is the case for the for $d = 1$, and [9] show that $E[\mu_q^n]$ also has a density. Even if those elements are promising, it is not clear whether the limit μ_q^κ has a density in a general setting. However, we are able to prove that the marginals of μ_q^κ have densities.

Proposition 2. *Let π_1 (resp. π_2) be the projection on the x -axis (resp. y -axis). Then, for $q > 0$, the pushforwards $\pi_1^*(\mu_q^\kappa)$ and $\pi_2^*(\mu_q^\kappa)$ have densities with respect to the Lebesgue measure on \mathbb{R} . For $q = 0$, $\pi_2^*(\mu_q^\kappa)$ has a density.*

5 Discussion

The tuning of the weight functions in the representations of persistence diagrams is a critical issue in practice. When the statistician has good reasons to believe that the data lies near a d -dimensional structure, we give, through two different approaches, an heuristic to tune this weight function: a weight of the form pers^α with $\alpha \geq d$ is sensible. The study carried out in this paper allowed us to show new results on the asymptotic structure of random persistence diagrams. While the existence of a limiting measure in a weak sense was already known, we strengthen the convergence, allowing a much larger class of test functions. Some results about the properties of the limit are also shown, namely that it has a finite mass, finite moments, and that its marginals have densities with respect to the Lebesgue measure. Challenging open questions include:

- Convergence of the rescaled diagrams μ_n with respect to some transport metric: The main issue consists in showing that one can extend, in a meaningful way, the distance W_p to general Radon measures. This is the topic of an on-going work.

- Existence of a density for the limiting measure: An option to obtain this result would be to control the numbers of points of a diagram in some square $[r_1, r_2] \times [s_1, s_2]$.
- Convergence of the number of points in the diagrams: The number of points in the diagrams is a quantity known to be not stable (motivating the use of bottleneck distances, which is blind to them). However, experiments show that this number, conveniently rescaled, converges in this setting. An analog of Lemma 1 for the number of points in the diagrams with small persistence would be crucial to attack this problem.
- Generalization to manifolds: All our results are stated for i.i.d. points in the cube $[0, 1]^d$. While a generalization to convex bodies seems achievable without too much extra work, similar proofs for samples on Riemannian manifolds would require much more refined mathematical machinery. Techniques from [4] could probably be helpful to achieve this goal.
- Dimension estimation: We have proved that the total persistence of a diagram built on a given point cloud depends crucially on the intrinsic dimension of such a point cloud. Inferring the dependence of the total persistence with respect to the size of the point cloud (through subsampling) leads to estimators of this intrinsic dimension. Studying the properties of such estimators is the topic of an on-going work of Henry Adams and co-authors (personal communication).

6 Proofs

6.1 Proof of Theorem 3

We only treat the case $p < \infty$, the proof being easily adapted to the case $p = \infty$. Fix two persistence diagrams Dgm_1 and Dgm_2 . Denote $\mu = \text{Dgm}_1(w \cdot)$ (resp. $\nu = \text{Dgm}_2(w \cdot)$) the measure having density w with respect to Dgm_1 (resp. Dgm_2). For γ a matching attaining the p -th Wasserstein distance between Dgm_1 and Dgm_2 , denote $\tilde{\mu} = \sum_{r \in \text{Dgm}_1 \cup \Delta} w(\gamma(r))\delta_r$. We have

$$\begin{aligned} \|\Phi_w(\text{Dgm}_1) - \Phi_w(\text{Dgm}_2)\|_{\mathcal{B}} &= \|\mu(\phi) - \nu(\phi)\|_{\mathcal{B}} \leq \|\mu(\phi) - \tilde{\mu}(\phi)\|_{\mathcal{B}} + \|\tilde{\mu}(\phi) - \nu(\phi)\|_{\mathcal{B}} \\ &\leq \|\phi\|_{\infty} |\mu - \tilde{\mu}| + \text{Lip}(\phi) MK(\tilde{\mu}, \nu), \end{aligned} \tag{25}$$

where MK is the Monge-Kantorovitch distance between measures, the cost being measured by the L_2 norm on $\mathbb{R}_{>}^2$. We bound the two terms in the sum separately. Let

us first bound $MK(\tilde{\mu}, \nu)$. Define q such that $\frac{1}{p} + \frac{1}{q} = 1$. As condition (12) implies that $\|w(\mathbf{r})\| \leq \frac{A}{\alpha} \text{pers}(\mathbf{r})^\alpha$, the distance $MK(\tilde{\mu}, \nu)$ is bounded by

$$\begin{aligned} \sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} w(\gamma(\mathbf{r})) \|r - \gamma(r)\| &\leq \left(\sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} w(\gamma(\mathbf{r}))^q \right)^{1/q} \left(\sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} \|r - \gamma(r)\|^p \right)^{1/p} \\ &\leq \frac{A}{\alpha} \left(\sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} \text{pers}(\gamma(\mathbf{r}))^{q\alpha} \right)^{1/q} W_p(\text{Dgm}_1, \text{Dgm}_2) \\ &\leq \frac{A}{\alpha} (G \{q\alpha\})^{1/q} W_p(\text{Dgm}_1, \text{Dgm}_2). \end{aligned} \quad (26)$$

We now treat the first part of the sum in (25). For $\mathbf{r}_1, \mathbf{r}_2$, in $\mathbb{R}_>^2$ with $\text{pers}(\mathbf{r}_1) \leq \text{pers}(\mathbf{r}_2)$, define the path with unit speed $h : [\text{pers}(\mathbf{r}_1), \text{pers}(\mathbf{r}_2)] \rightarrow \mathbb{R}_>^2$ by

$$h(t) = \mathbf{r}_2 \frac{t - \text{pers}(\mathbf{r}_1)}{\text{pers}(\mathbf{r}_2) - \text{pers}(\mathbf{r}_1)} + \mathbf{r}_1 \frac{\text{pers}(\mathbf{r}_2) - t}{\text{pers}(\mathbf{r}_2) - \text{pers}(\mathbf{r}_1)},$$

so that it satisfies $\text{pers}(h(t)) = t$. The quantity $|w(\mathbf{r}_1) - w(\mathbf{r}_2)|$ is bounded by

$$\begin{aligned} \int_{\text{pers}(\mathbf{r}_1)}^{\text{pers}(\mathbf{r}_2)} |\nabla w(h(t)) \cdot h'(t)| dt &\leq \int_{\text{pers}(\mathbf{r}_1)}^{\text{pers}(\mathbf{r}_2)} A \text{pers}(h(t))^{\alpha-1} dt \\ &\leq \int_{\text{pers}(\mathbf{r}_1)}^{\text{pers}(\mathbf{r}_2)} A t^{\alpha-1} dt \\ &= \frac{A}{\alpha} (\text{pers}(\mathbf{r}_2)^\alpha - \text{pers}(\mathbf{r}_1)^\alpha). \end{aligned}$$

For $0 < y < x$ and $0 \leq a \leq 1$, using the convexity of $t \mapsto t^\alpha$, it is easy to see that $x^\alpha - y^\alpha \leq \alpha(x - y)^a x^{\alpha-a}$. Define $p' = \frac{p}{a}$, $q' = \frac{p'}{p'-1}$ and $M(\mathbf{r}) := \max(\text{pers}(\mathbf{r}), \text{pers}(\gamma(\mathbf{r})))$. We have,

$$\begin{aligned} |\mu - \tilde{\mu}| &= \sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} |w(\mathbf{r}) - w(\gamma(\mathbf{r}))| \\ &\leq A \sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} |\text{pers}(\mathbf{r}) - \text{pers}(\gamma(\mathbf{r}))|^a M(\mathbf{r})^{\alpha-a} \\ &\leq A \left(\sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} |\text{pers}(\mathbf{r}) - \text{pers}(\gamma(\mathbf{r}))|^{ap'} \right)^{1/p'} \left(\sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} M(\mathbf{r})^{q'(\alpha-a)} \right)^{1/q'} \\ &\leq AW_{ap'}(\text{Dgm}_1, \text{Dgm}_2)^a \left(\sum_{\mathbf{r} \in \text{Dgm}_1 \cup \Delta} (\text{pers}(\mathbf{r})^{q'(\alpha-a)} + \text{pers}(\gamma(\mathbf{r}))^{q'(\alpha-a)}) \right)^{1/q'} \end{aligned}$$

$$\leq AW_{ap'}(\text{Dgm}_1, \text{Dgm}_2)^a 2^{1/q'} (G\{q'(\alpha - a)\})^{1/q'} \quad (27)$$

Combining equations (25), (26) and (27) concludes the proof.

6.2 Proof of Corollary 1

Corollary 1 is immediate. It relies on the definition of a space having bounded m -th total persistence and on the inequality $G\{t_1 + t_2\} \leq \ell^{t_1} G\{t_2\}$.

6.3 Proof of Corollary 2

As already discussed, Theorem 3 can be applied with $f_n = d(\cdot, \mathbb{X}_n)$ and f the null function on the manifold \mathbb{X} . Take $p = \infty$, $d < \alpha$ and $0 < a < \min(1, \alpha - d)$:

$$\begin{aligned} \|\Phi(\text{Dgm}_q[f_n]) - \Phi(\text{Dgm}_q[f])\|_{\mathcal{B}} &\leq \text{Lip}(\phi) \frac{A}{\alpha} G\{\alpha\} W_{\infty}(\text{Dgm}_q[f_n], \text{Dgm}_q[f]) \\ &\quad + 2\|\phi\|_{\infty} A G\{\alpha - a\} W_{\infty}(\text{Dgm}_q[f_n], \text{Dgm}_q[f])^a. \end{aligned} \quad (28)$$

It is mentioned at the end of Section 2 in [14] that, for $m > d$, $\text{Pers}_m(\text{Dgm}_q[f_n]) \leq m C_{\mathbb{X}} \|f_n\|_{\infty}^{m-d} / (m - d)$ for some constant $C_{\mathbb{X}}$ depending only on \mathbb{X} . Moreover, the stability theorem for the bottleneck distance ensures that $W_{\infty}(\text{Dgm}_q[f_n], \text{Dgm}_q[f]) \leq \|f_n\|_{\infty}$. Therefore,

$$\begin{aligned} \|\Phi(\text{Dgm}_q[f_n]) - \Phi(\text{Dgm}_q[f])\|_{\mathcal{B}} &\leq \text{Lip}(\phi) \frac{\alpha A C_{\mathbb{X}}}{\alpha - d} \|f_n\|_{\infty}^{\alpha-d+1} + \|\phi\|_{\infty} \frac{2 A C_{\mathbb{X}} (\alpha - a)}{\alpha - a - d} \|f_n\|_{\infty}^{\alpha-a-d+a} \\ &\leq \text{Lip}(\phi) \frac{\alpha A C_{\mathbb{X}}}{\alpha - d} \|f_n\|_{\infty}^{\alpha-d+1} + \|\phi\|_{\infty} \frac{2 A C_{\mathbb{X}} \alpha}{\alpha - d} \|f_n\|_{\infty}^{\alpha-d}, \end{aligned} \quad (29)$$

where, in the last line, the second term was minimized over a . The quantity $\|f_n\|_{\infty}$ is the Hausdorff distance between \mathbb{X}_n and \mathbb{X} . Elementary techniques of geometric probability (see for instance [15]) show that if \mathbb{X} is a compact d -dimensional manifold, then $E[\|f_n\|_{\infty}^{\beta}] \leq c \left(\frac{\ln n}{n}\right)^{\beta/d}$ for $\beta \geq 0$, where c is some constant depending on \mathbb{X} , $\inf \kappa$ and $\sup \kappa$. Therefore, the first term of the sum (29) being negligible,

$$\begin{aligned} E \left[\|\Phi(\text{Dgm}_q[f_n]) - \Phi(\text{Dgm}_q[f])\|_{\mathcal{B}} \right] &\leq \|\phi\|_{\infty} \frac{2 A C_{\mathbb{X}} \alpha}{\alpha - d} c \left(\frac{\ln n}{n}\right)^{(\alpha-d)/d} + o \left(\left(\frac{\ln n}{n}\right)^{(\alpha-d)/d} \right). \end{aligned}$$

In particular, the conclusion holds for any $C > 2 A C_{\mathbb{X}} c$, for n large enough. \square

We now prove the propositions of Section 4. In the following proofs, c is a constant, depending on κ , d and q , which can change from line to line (or even represent two different constants in the same line). A careful read can make all those constants explicit. If a constant depends also on some additional parameter x , it is then denoted by $c(x)$.

6.4 Proof of Proposition 1

First, as the right hand side of inequality (24) does not depend on n , one may safely assume that μ_n is built with the binomial process. The proof is based on two observations.

(i) Let $r(\sigma) := \min\{r > 0, \sigma \in K^r(\mathbb{X}_n)\}$ denote the *filtration time* of σ . A simplex σ is said to be negative in the filtration $\mathcal{K}(\mathbb{X}_n)$ if σ is not included in any cycle of $K^{r(\sigma)}(\mathbb{X}_n)$. A basic result of persistent homology states that points in $\text{Dgm}_q[\mathcal{K}(\mathbb{X}_n)]$ are in bijection with pair of simplexes, one negative and one positive (i.e. non-negative). Moreover, the death time r_2 of a point $r = (r_1, r_2)$ of the diagram is exactly $r(\sigma)$ for some negative $(q+1)$ -simplex σ . Therefore, $n\mu_n(U_M)$ is equal to $N_q(\mathbb{X}_n, M)$, the number of negative $(q+1)$ -simplexes in the filtration $\mathcal{K}(\mathbb{X}_n)$ appearing after $M_n := n^{-1/d}M$.

(ii) The number of negative simplexes in the Čech and Rips filtration can be efficiently bounded thanks to elementary geometric arguments.

6.4.1 Geometric arguments for the Rips filtration

We have

$$N_q(\mathbb{X}_n, M) = \frac{1}{q+2} \sum_{i=1}^n \#\Xi(X_i, \mathbb{X}_n), \quad (30)$$

where, for $x \in \mathbb{X}$, with \mathbb{X} a finite set, $\Xi(x, \mathbb{X})$ is the set of negative $(q+1)$ -simplexes (and therefore of size $q+2$) in $\mathcal{R}(\mathbb{X})$ that are containing x , and have a filtration time larger than M_n . The following construction is inspired by the proof of Lemma 2.4 in [23].

The angle (with respect to 0) of two vectors $x, y \in \mathbb{R}^d$ is defined as

$$\widehat{xy} := \arccos \left(\frac{\langle x, y \rangle}{\|x\| \|y\|} \right).$$

The angular section of a cone A is defined as $\sup_{x, y \in A} \widehat{xy}$. Denote by $C(x, r)$ the cube centered at x of side length $2r$. For $0 < \delta < 1$, and for each face of the cube

$C(x, r)$, consider a regular grid with spacing δr , so that the center of each face is one of the grid points. This results in a partition of the boundary of the cube $C(x, r)$ into $(d-1)$ -dimensional cubes $(C_j^\delta(x, r))_{j=1\dots Q}$ of side length δr . Using this partition of the boundary of $C(x, r)$, we construct a partition of $C(x, r)$ into closed convex cones $(A_j^\delta(x, r))_{j=1\dots Q}$, where each cone $A_j^\delta(x, r)$ is defined as a d -simplex spanned by x and one of the $(d-1)$ -dimensional cubes $C_j^\delta(x, r)$ of side length δr on a face of $C(x, r)$. In other words, the point x is the apex of each $A_j^\delta(x, r)$, and $C_j^\delta(x, r)$ is its base. Two such cones $A_j^\delta(x, r)$ and $A_{j'}^\delta(x, r)$ are *adjacent*, if $A_j^\delta(x, r) \cap A_{j'}^\delta(x, r) \neq \emptyset$.

Fix $0 < \eta < 1$ and define $R_{\delta, \eta}(x, \mathbb{X}_n)$ to be the smallest radius r so that each cone $A_j^\delta(x, \eta r)$ in $C(x, \eta r)$ either contains a point of \mathbb{X}_n other than x , or is not a subset of $(0, 1)^d$.

Lemma 2. *Let $x \in [0, 1]^d$. Fix $\delta > 0$, and $0 < r \leq \frac{1}{2}$, and let $A_j^\delta(x, r)$ be a cone of $C(x, r)$ whose base $C_j^\delta(x, r)$ intersects $[0, 1]^d$. Then, either $A_j^\delta(x, r)$ is a subset of $[0, 1]^d$, or there exists a cone $A_{j'}^\delta(x, r)$ of $C(x, r)$ adjacent to $A_j^\delta(x, r)$ that is a subset of $[0, 1]^d$.*

Proof. A necessary and sufficient condition for a cone $A_j^\delta(x, r)$ to be a subset of $[0, 1]^d$ is that $C_j^\delta(x, r) \subset [0, 1]^d$. Suppose that this is not the case, i.e. we have $C_j^\delta(x, r) \setminus [0, 1]^d \neq \emptyset$. For each coordinate $i = 1, \dots, d$ for which $C_j^\delta(x, r)$ extends beyond a face of $[0, 1]^d$, move one step in the ‘opposite’ direction, and find the corresponding adjacent cone. The fact that $r \leq 1/2$ ensures that these (at most d) steps, each of size $r\delta$, do not make the exterior boundary of the corresponding adjacent cone extend beyond any of the opposite faces of the cube corresponding to the directions of the steps. \square

Note that the angular section (with respect to x) of the union of a cone $A_j^\delta(x, r)$ and its adjacent cones is bounded by $c\delta$ for some constant c .

Lemma 3. *Let $\eta = \min\{1/\sqrt{d}, 1/2\}$. There exists a $\delta = \delta(d) > 0$, such that each simplex σ of $\Xi(x, \mathbb{X}_n)$ is included in $C(x, R_{\delta, \eta}(x, \mathbb{X}_n))$. Furthermore, $\Xi(x, \mathbb{X}_n)$ is empty if $R_{\delta, \eta}(x, \mathbb{X}_n) > M_n$.*

Proof. To ease notation, denote $R_{\delta, \eta}(x, \mathbb{X}_n)$ by R . We are going to prove that all negative simplexes containing x are included in $C(x, R)$, a fact that proves the two assertions of the lemma. First, if $\eta R \geq 1/2$, then $C(x, R)$ contains $[0, 1]^d$ and the result is trivial. Otherwise, consider a negative $(q+1)$ -simplex $\sigma = \{x, x_1, \dots, x_{q+1}\}$. Assume without loss of generality that x_1 is the point in σ maximizing the distance to x , which in particular means that x_1 is not in $C(x, R)$. The line $[x, x_1]$ hits $C(x, \eta R)$ at some cone $A_j^\delta(x, \eta R)$. If $A_j^\delta(x, \eta R)$ is included in $[0, 1]^d$, then, by definition of R ,

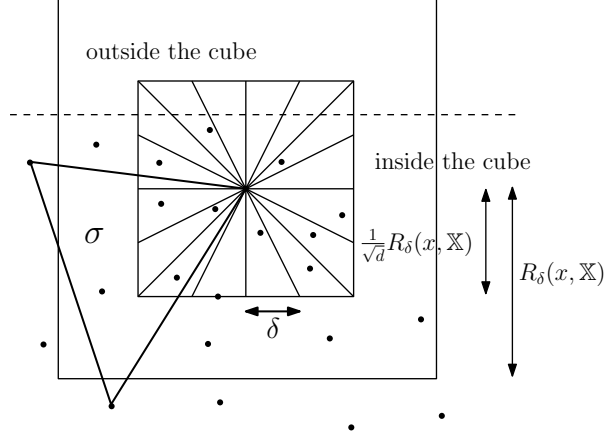


Figure 3: Definition of $R_\delta(x, \mathbb{X}_n)$ for some point cloud \mathbb{X}_n . The simplex σ is positive for the Rips filtration, as it is not included in $C(x, R_\delta(x, \mathbb{X}_n))$.

there exists a point z of \mathbb{X}_n in this cone. Otherwise, apply Lemma 2 to this cone and denote by A_δ the union of $A_j^\delta(x, \eta R)$ and its adjacent cones in $C(x, \eta R)$. By definition of R , there exists a point z of \mathbb{X}_n in $A_\delta \subset C(x, \eta R)$. The angle formed by $[z, x]$ and $[z, x_1]$ is in both cases smaller than $c\delta$. Let us prove that all the $(q+1)$ -simplexes σ_t of the form $(\sigma \setminus \{t\}) \cup \{z\}$, for $t \in \sigma$, have a filtration time smaller than $r(\sigma)$. If this is the case, then the cycle formed by the σ_t 's and σ is contained in the complex at time $r(\sigma)$, meaning that σ is not negative, concluding the proof. Therefore, it suffices to prove that $\|z - x_i\| \leq r(\sigma)$ for all i . First consider the case $i = 1$.

- $\|z - x\| \leq \sqrt{d}\eta R \leq \|x - x_1\|$.
- If $c\delta \leq \frac{\pi}{3}$,

$$\begin{aligned} \|z - x_1\|^2 &= \|z - x\|^2 + \|x_1 - x\|^2 - 2\langle z - x, x_1 - x \rangle \\ &\leq \|z - x\|^2 + \|x_1 - x\|^2 - \|z - x\|\|x_1 - x_0\| \leq \|x_1 - x\|^2 \leq r(\sigma)^2. \end{aligned}$$

For $i \geq 2$, we have $\|x - x_i\| \leq \|x - x_1\|$ by assumption. Let $I(z)$ denote the set of all $t \in \mathbb{R}^d$ with $\|z - t\| \geq \|x - t\|$ and $\|z - t\| \geq \|x_1 - t\|$, i.e. $I(z)$ is the intersection of two half spaces (see Figure 4). Let $F_x(z) = d(I(z), x)$. If we find a δ with $F_x(z) > \|x - x_1\|$ for all $z \in A_\delta$, then no x_i is in $I(z)$, whatever the position of $z \in A_\delta$ is, meaning that all x_i 's satisfy $\|z - x_i\| \leq \max\{\|x_0 - x_i\|, \|x_1 - x_i\|\} \leq r(\sigma)$, concluding the proof. The method of Lagrange multipliers shows that $F_x(z)^2$ is a continuous function of z , with a known (but complex) expression. A straightforward study of this expression shows that for δ small enough, the minimum of F_x on A_δ can be made arbitrarily large: therefore, there exists δ such that $F_x(z) > \sqrt{d} \geq \|x - x_1\|$, for all $z \in A_\delta$. \square

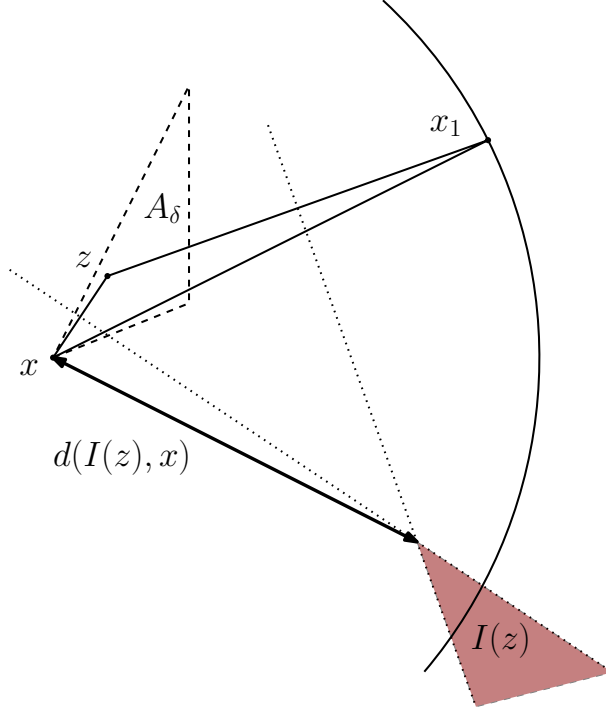


Figure 4: *The geometric construction used in the proof of Lemma 3.*

6.4.2 Construction for the Čech filtration

A similar construction works for the Čech filtration, but the arguments are slightly different. First, note that each negative simplex σ in the Čech filtration is such that there exists a subsimplex σ' of σ that enters in the filtration at the same time $r(\sigma)$ as σ , and so that $r(\sigma)$ is the circumradius of σ' . Then

$$\begin{aligned}
 N_q(\mathbb{X}_n, M) &= \sum_{\sigma \in \mathcal{C}_{q+1}(\mathbb{X}_n)} \mathbf{1}\{\sigma \text{ negative, } r(\sigma) \geq M\} \\
 &= \sum_{\sigma \in \mathcal{C}_{q+1}(\mathbb{X}_n)} \frac{1}{\#\sigma'} \sum_{i=1}^n \mathbf{1}\{\sigma \text{ negative, } r(\sigma) \geq M, X_i \in \sigma'\} \\
 &\leq \sum_{i=1}^n \#\Xi'(X_i, \mathbb{X}_n),
 \end{aligned}$$

where $\Xi'(x, \mathbb{X}_n)$ the set of negative $(q+1)$ -simplexes σ in the Čech filtration $\mathcal{C}(\mathbb{X}_n)$ with $r(\sigma) \geq M$ and $x \in \sigma'$.

Lemma 4. For $\eta = \min\{1/\sqrt{d}, 1/2\}$ and some $\delta = \delta(d) > 0$, each simplex σ of $\Xi'(x, \mathbb{X}_n)$ is included in $C(x, R_{\delta, \eta}(x, \mathbb{X}_n))$. Furthermore, $\Xi(x, \mathbb{X}_n)$ is empty if $R_{\delta, \eta}(x, \mathbb{X}_n) > M_n$.

Proof. Recall the definition of $C(x, r)$ and the partition of $C(x, r)$ into the cones $(A_j^\delta(x, r))_{j=1\dots Q}$ with corresponding bases $(C_j^\delta(x, r))_{j=1\dots Q}$. As above, denote $R_{\delta, \eta}(x, \mathbb{X}_n)$ by R . Let $\sigma = \{x_0, \dots, x_{q+1}\}$ denote a $(q+1)$ -simplex not included in $C(x, R)$, with $x = x_0$, say, and $r(\sigma) \geq M$. As in the Rips case, the result is trivial if $\eta R \geq 1/2$. By definition of the Čech filtration, the intersection $\bigcap_{i=0}^{q+1} B(x_i, r(\sigma))$ consists of a singleton $\{y\}$. If there is a point z of \mathbb{X}_n in $B(y, r(\sigma))$, then, by the nerve theorem applied to $\sigma \cup \{z\}$, we can conclude with similar arguments as in Lemma 3 that σ is positive in the filtration, meaning that every negative $\sigma \in \Xi(x, \mathbb{X}_n)$ has to be included in $C(x, R)$.

Let us prove the existence of such a z . As $x_0 \in \sigma'$, the distance between x_0 and y is equal to $r(\sigma) \geq R$. Therefore, the line $[x_0, y]$ hits $C(x, \eta R)$ in some cone $A_j^\delta(x, \eta R)$, whose base $C_j^\delta(x, r)$ intersects $[0, 1]^d$, as it intersects $[x_0, y]$. Lemma 2 can be applied to $A_j^\delta(x, \eta R)$. As in the Rips case, there exists a point z of \mathbb{X}_n in $C(x, \eta R)$ such that the angle made by z , x and y is smaller than $c\delta$. Then,

$$\begin{aligned} \|y - z\|^2 &= \|y - x\|^2 + \|x - z\|^2 + 2\langle y - x, x - z \rangle \\ &\leq \|y - x\|^2 + \|x - z\|^2 - \|y - x\|\|x - z\| \text{ if } c\delta \leq \frac{\pi}{3} \\ &\leq \max\{\|y - x\|^2, \|x - z\|^2\} \leq \max\{r(\sigma)^2, (\sqrt{d}\eta R)^2\} \leq r(\sigma)^2 \text{ if } \sqrt{d}\eta \leq 1. \end{aligned}$$

This concludes the proof of Lemma 4. \square

Remark 3. Note that the fact that the support of κ is the cube only enters the picture through the geometric arguments used here and in the above proof. Some more refine work is needed to show that a similar construction holds when the cube is replaced by a convex body.

In the following, fix $\eta = \min\{1/\sqrt{d}, 1/2\}$, choose δ sufficiently small, and let $R_{\delta, \eta}(x, \mathbb{X}_n)$ and $A_j^\delta(x, r)$ be denoted by $R(x, \mathbb{X}_n)$ and $A_j(x, r)$, respectively. Both $\Xi(x, \mathbb{X}_n)$ and $\Xi'(x, \mathbb{X}_n)$ are included in the set of $(q+1)$ -tuples of $\mathbb{X}_n \cap C(x, R(x, \mathbb{X}_n))$, so that the following inequality holds for either the Rips or the Čech filtration:

$$N_q(\mathbb{X}_n, M) \leq \sum_{i=1}^n \mathbf{1}\{R(X_i, \mathbb{X}_n) > M_n\} (\#\{\mathbb{X}_n \cap C(X_i, R(X_i, \mathbb{X}_n))\})^{q+1}. \quad (31)$$

Denote $R(X_1, \mathbb{X}_n)$ by R_n . As we will see, an estimate of the tail of R_n is sufficient to get a control of $N_q(\mathbb{X}_n, M)$. The probability $P(R_n > t)$ is bounded by the probability

that one of the cones pointing at X_1 , of radius $t/2$, wholly included in the cube $[0, 1]^d$, is empty. Conditionally on X_1 , this probability is exactly the probability that a binomial process with parameters $n - 1$ and κ does not intersect this cone. Therefore,

$$P(R_n > t) \leq c \exp(-cnt^d), \quad (32)$$

and we obtain, for $\lambda > 0$,

$$\begin{aligned} E \left[e^{\lambda n R_n^d} \right] &= \int_1^\infty P(\lambda n R_n^d > \ln(t)) dt \\ &\leq \int_1^\infty c \exp\left(-c \frac{\ln(t)}{\lambda}\right) dt < \infty \text{ if } \lambda < c/2. \end{aligned} \quad (33)$$

Lemma 5. *The random variable $\#(\mathbb{X}_n \cap C(X_1, R_n))$ has exponential tail bounds: for $t > 0$,*

$$P(\#(\mathbb{X}_n \cap C(X_1, R_n)) > t) \leq c \exp(-ct).$$

Proof. Conditionally on X_1 and R_n , two possibilities may occur. In the first one, the cube centered at X_1 of radius ηR_n contains a point on its boundary, in the cone $A_{j_0}(X_1, \eta R_n)$. Denote this event E and let Q_0 be the number of cones wholly included in the support. The configuration of \mathbb{X}_n is a binomial process conditioned to have at least one point in the cones $A_j(X_1, \eta R_n)$ wholly included in the cube, except for $j = j_0$, and a point on the outer boundary of $A_{j_0}(X_1, \eta R_n)$. In this case, $\#(\mathbb{X}_n \cap C(X_1, R_n))$ is equal to $Q_0 + Z$, where Z is a binomial variable of parameters $n - Q_0$ and

$$\int_{C(X_1, R_n) \setminus A_{j_0}(X_1, \eta R_n)} \kappa(x) dx \leq c R_n^d.$$

Therefore, for $\beta > 0$, using a Chernoff bound and a classical bound on the moment generating function of a binomial variable:

$$\begin{aligned} P(\#(\mathbb{X}_n \cap C(X_1, R_n)) > t | R_n, Q_0, E) &\leq P(Q_0 + Z > t | R_n, Q_0) \\ &\leq \frac{e^{\beta Q_0} E[e^{\beta Z} | R_n]}{e^{\beta t}} \\ &\leq \frac{e^{\beta Q_0} e^{cn R_n^d (e^\beta - 1)}}{e^{\beta t}}, \end{aligned}$$

where Q is the number of elements in the partition of $C(x, R)$. Take β sufficiently small so that $E[e^{cn R_n^d (e^\beta - 1)}] < \infty$ (such a β exists by equation (33)). We have the conclusion in this first case.

The other possibility is that there exists a cone not wholly included in the cube containing no point of \mathbb{X}_n . In this case, the configuration of \mathbb{X}_n is a binomial process conditioned on having at least one point in the cones $A_j(X_1, R_n)$ wholly included in cube and no point in a certain cone not wholly included in the cube. Likewise, a similar bound is shown. \square

We are now able to finish the proof of Proposition 1: for $p \geq 1$,

$$\begin{aligned} E[\mu_n(U_M)^p] &= E \left[\left(\frac{N_q(\mathbb{X}_n, M)}{n} \right)^p \right] \\ &\leq \frac{1}{(q+2)^p} E \left[\left(\frac{\sum_{i=1}^n \mathbf{1}\{R(X_i, \mathbb{X}_n) > M_n\} (\#\mathbb{X}_n \cap C(X_i, R(X_i, \mathbb{X}_n)))^{q+1}}{n} \right)^p \right] \\ &\leq \frac{1}{(q+2)^p} E \left[\mathbf{1}\{R_n > M_n\} \#\mathbb{X}_n \cap C(X_1, R_n)^{p(q+1)} \right] \text{ by Jensen's inequality} \\ &\leq \frac{1}{(q+2)^p} \left(P(R_n > M_n) E[\#\mathbb{X}_n \cap C(X_1, R_n)^{2p(q+1)}] \right)^{1/2}. \end{aligned}$$

Lemma 5 implies that, for $p' > 0$,

$$E \left[\#\mathbb{X}_n \cap C(X_1, R_n)^{p'} \right] \leq \int_1^\infty c e^{-ct^{1/p'}} dt = \frac{p'c}{C^{p'}} \int_1^\infty u^{p'-1} e^{-u} du = \frac{c}{C^{p'}} (p')!.$$

Therefore, for $q \geq 1$,

$$E[\mu_n(U_M)^{p/(q+1)}] \leq c \frac{1}{(q+2)^{p/(q+1)}} \exp(-cM^d) \left((2p)! C^{-2p} \right)^{1/2} \leq \exp(-cM^d) c^p p!.$$

To finish the proof, we use a simple lemma relating the moments of a random variable to its tail.

Lemma 6. *Let X be a positive random variable such that there exists constants $A, C > 0$ with*

$$E[X^k] \leq AC^k k!. \quad (34)$$

Then, there exists a constant $c > 0$ such that $\forall x > 0, P(X > x) \leq A \exp(-cx)$.

Proof. Fix $\lambda = \frac{1}{2C}$. The moment generating function of X in λ is bounded by:

$$E \left[e^{\lambda X} \right] = \sum_{k \geq 0} \frac{\lambda^k E[X^k]}{k!} \leq A \sum_{k \geq 0} \lambda^k C^k = A.$$

Therefore, using a Chernoff bound, $P(X > x) \leq A \exp(-\lambda x)$. \square

Apply Lemma 6 to $X = \mu_n(U_M)^{1/(q+1)}$ to obtain the assertion of Proposition 1. \square

6.5 Proof of Theorem 6

We prove the various statements of the theorem in several steps.

6.5.1 Step 1: Vague convergence in the non-uniform case

The first intermediate result we prove is the vague convergence v_c of diagrams in the non-uniform case. We need several preliminary lemmas.

Lemma 7. *Let $0 \leq r \leq s \leq \infty$ and $q \geq 0$. If $\mathbb{B}_n = \{X_1, \dots, X_n\}$ is a binomial process of intensity κ and $\mathbb{P}_n = \{X_1, \dots, X_{N_n}\}$ is a Poisson process of intensity κ , then, with probability one:*

$$\begin{aligned} n^{-1} \left(\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{B}_n)) - E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{B}_n))] \right) &\xrightarrow[n \rightarrow \infty]{} 0, \\ n^{-1} \left(\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{P}_n)) - E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{P}_n))] \right) &\xrightarrow[n \rightarrow \infty]{} 0 \text{ and} \\ n^{-1} \left(\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{B}_n)) - \beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{P}_n)) \right) &\xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

Proof. The proof of Lemma 7 is *mutatis mutandis* the proof of the analogous Theorem 4.6 in [33] for Betti numbers: the geometric lemma used for Betti numbers in the proof still holds for persistent Betti numbers (see Lemma 1). Note that the third limit is not stated as a theorem in [33], but appears in the proof of Theorem 4.6. Therefore, we do not provide further details here. \square

Lemma 7 implies that it is equivalent to show convergence of the persistent Betti numbers either in expectation or almost surely, and either in the Poisson model or in the binomial model. Our next goal is to show that $E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{X}_n))]$ converges in the binomial model. To this end, we show that small vertical or horizontal bands in persistence diagrams contains few points. Recall that $r(\sigma) := \min\{r > 0, \sigma \in K^r(\mathbb{X}_n)\}$ denotes the filtration time of σ .

Lemma 8. *Let $q \geq 0$. For $r_1 < r_2$, let $F_q(\mathbb{X}_n, r_1, r_2)$ be the number of q -simplexes σ in the filtration $\mathcal{K}(\mathbb{X}_n)$ with $r(\sigma) \in [r_1, r_2]$. Assume that \mathbb{X}_n is a binomial n -sample of density κ . Then,*

$$n^{-1} F_q(n^{1/d}\mathbb{X}_n, r_1, r_2) \xrightarrow[n \rightarrow \infty]{L_2} F_q(r_1, r_2), \quad (35)$$

where $F_q(r_1, r_2) \leq cr_2^{2dq-1}|r_2 - r_1|$.

Proof. For a finite set $\mathbb{X} \subset \mathbb{R}^d$, define

$$\xi^{r_1, r_2}(x, \mathbb{X}) = \frac{1}{q+1} \sum_{\sigma \in \mathcal{K}_q(\mathbb{X})} \mathbf{1}\{r(\sigma) \in [r_1, r_2] \text{ and } x \in \sigma\}.$$

Then, $F_q(\mathbb{X}_n, r_1, r_2) = \sum_{x \in \mathbb{X}} \xi^{r_1, r_2}(x, \mathbb{X}_n)$. The paper [26] shows convergence in L_2 of such functionals $\xi(x, \mathbb{X})$ under two conditions. The first one of them is called stabilization. Let \mathbb{P} be a homogeneous Poisson process in \mathbb{R}^d . A quantity $\xi(x, \mathbb{X})$ is stabilizing if, with probability one, there exists some random radius $R < \infty$ such that, for all finite sets A which are equal to \mathbb{P} on $B(0, R)$,

$$\xi(0, \mathbb{P} \cap B(0, R)) = \xi(0, A),$$

Denote this quantity by $\xi_\infty(\mathbb{P})$. In our case, ξ^{r_1, r_2} is stabilizing with $R = 2r_2$. The second condition is a moment condition: there exists some number $\beta > 2$ such that

$$\sup_n E[\xi(n^{1/d} X_1, n^{1/d} \mathbb{X}_n)^\beta] < \infty.$$

Once again, ξ^{r_1, r_2} possesses this property: the random variable $\xi^{r_1, r_2}(n^{1/d} X_1, n^{1/d} \mathbb{X}_n)$ is bounded by the number of q -simplexes of $\mathcal{K}(\mathbb{X}_n)$ containing X_1 and being included in $B(X_1, 2n^{-1/d} r_2)$. This number of q -simplexes is bounded by $\#(\mathbb{X}_n \cap B(X_1, 2n^{-1/d} r_2))^q$, which, in turn, is stochastically dominated by a binomial random variable with parameters n and $cn^{-1} r_2^d$. In particular, its moment of order $3q$ is smaller than a constant independent of n . This means that the moment condition is satisfied. Applying the main theorem of [26], convergence (35) is obtained, with $F_q(r_1, r_2) = E[\xi_\infty^{r_1, r_2}(\mathbb{P})]$, where $\xi_\infty^{r_1, r_2}(\mathbb{P}) = \xi^{r_1, r_2}(0, \mathbb{P} \cap B(0, 2r_2))$. The set $\mathbb{P} \cap B(0, 2r_2)$ can be expressed as $\{X_1, \dots, X_N\}$, where $(X_i)_{i \geq 0}$ is a sequence of i.i.d. uniform random variables on $B(0, 2r_2)$, and N is an independent Poisson variable with parameter cr_2^d . Therefore,

$$\begin{aligned} E[\xi_\infty^{r_1, r_2}(\mathbb{P})] &= E \left[\sum_{i_1, \dots, i_q} \mathbf{1}\{r(\{0, X_{i_1}, \dots, X_{i_q}\}) \in [r_1, r_2]\} \right] \\ &= E \left[\frac{N!}{(N-q)!} \right] P(r(\{0, X_1, \dots, X_q\}) \in [r_1, r_2]) \\ &\leq cr_2^{2dq-1} |r_2 - r_1|. \end{aligned}$$

The last inequality is a consequence of (i) the fact that the q -th factorial moment of N equals cr_2^{dq} , and (ii) results of [9]. Indeed, write

$$P(r(\{0, X_1, \dots, X_q\}) \in [r_1, r_2]) = \int_{B(0, 2r_2)^q} \mathbf{1} \left\{ r \left(\left\{ 0, \frac{x_1}{r_2}, \dots, \frac{x_q}{r_2} \right\} \right) \in \left[\frac{r_1}{r_2}, 1 \right] \right\} dx$$

$$= r_2^{dq} \int_{B(0,2)^q} \mathbf{1} \left\{ r(\{0, u_1, \dots, u_q\}) \in \left[\frac{r_1}{r_2}, 1 \right] \right\} du.$$

It is shown in [9] that for either the Rips or the Čech filtration built on a random point cloud \mathbb{X} having a smooth density with respect to the Lebesgue measure, the corresponding expected diagrams have a smooth density with respect to the Lebesgue measure on $\mathbb{R}_{>}^2$. With $\mathbb{X} = \{0, X_1, \dots, X_q\}$, a q -sample of uniform random variables on $B(0, 2)$, the diagram $\text{Dgm}_q[\mathcal{K}(\mathbb{X})]$ consists of the single point $r = (r_1, r_2)$, with r_2 the filtration time of the simplex \mathbb{X} . Therefore, $r(\{0, X_1, \dots, X_{q-1}\})$ has a smooth density with respect to the Lebesgue measure on \mathbb{R} , which implies the inequality. \square

Lemma 9. *Let $0 < r \leq s < \infty$. Then*

$$n^{-1} E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{X}_n))] \xrightarrow{n \rightarrow \infty} E \left[b_q \left(r\kappa(X)^{1/d}, s\kappa(X)^{1/d} \right) \right], \quad (36)$$

where $b_q(r, s)$ depends on r, s and q and X has density κ in the right hand side.

Note that $b_q(r, s)$ is the limit of $n^{-1} E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{X}_n))]$ when \mathbb{X}_n is a homogeneous binomial process in the cube. The existence of the limit in this case is a consequence of Theorem 5 which states the convergence in the Poisson setting, coupled with Lemma 7 which shows the equivalence between convergences in the Poisson and binomial models. Classical techniques (see for instance Chapter 7 of [35]), which we now detail, are then used to generalize the convergence to the non-uniform case.

Proof. A density κ is said to be *blocked* if it is of the form

$$\sum_{i=1}^{m^d} \alpha_i \mathbf{1}\{Q_i\} \quad (37)$$

where m is a power of 2 and the set $(Q_i)_{i=1 \dots m^d}$ forms a partition of $[0, 1]^d$ into isometric cubes of side length m^{-1} . The following lemma is an important part of the proof of Lemma 9.

Lemma 10. *Assume that whenever κ is a blocked density satisfying (16), the convergence (36) holds. Then, the convergence (36) holds for any distribution having a density κ satisfying (16).*

Proof of Lemma 10. Fix κ a density on $[0, 1]^d$ satisfying (16). There exists κ_ε , a blocked density which is an ε -approximation of κ in the L_1 -distance:

$$\|\kappa - \kappa_\varepsilon\|_1 \leq \varepsilon.$$

It is also possible to choose κ_ε such that $\forall x \in [0, 1]^d$, $\kappa_\varepsilon(x) \leq \sup \kappa$. As the L_1 -distance equals twice the total variation between measures, there exists a coupling (X, Y) such that X has a density κ , Y has a density κ_ε and $P(X \neq Y) \leq 2\varepsilon$. Define two sequences of i.i.d. variables $((X_i, Y_i))_{n \geq 0}$, and let $\mathbb{X}_n = \{X_1, \dots, X_n\}$ and $\mathbb{Y}_n = \{Y_1, \dots, Y_n\}$. By the geometric Lemma 1,

$$\begin{aligned} & |E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{X}_n))] - E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{Y}_n))]| \\ & \leq \sum_{j=q}^{q+1} \left(E[|K_j^s(n^{1/d}(\mathbb{X}_n \cup \mathbb{Y}_n)) \setminus K_j^s(n^{1/d}\mathbb{X}_n)|] + E[|K_j^s(n^{1/d}(\mathbb{X}_n \cup \mathbb{Y}_n)) \setminus K_j^s(n^{1/d}\mathbb{Y}_n)|] \right). \end{aligned}$$

The two terms are bounded in a similar fashion. Therefore, we only bound the first one. Simple geometric considerations show that if $F \subset G$ are two finite subsets of \mathbb{R}^d , then

$$|K_j^s(G) \setminus K_j^s(F)| \leq \sum_{x \in G \setminus F} |B(x, 2s) \cap G|^j. \quad (38)$$

Denote $G_n = \mathbb{X}_n \cup \mathbb{Y}_n$ and $r_n = 2n^{-1/d}s$. Thanks to this last inequality, we have

$$\begin{aligned} E[|K_j^s(n^{1/d}(\mathbb{X}_n \cup \mathbb{Y}_n)) \setminus K_j^s(n^{1/d}\mathbb{X}_n)|] & \leq E \left[\sum_{i=1}^n \mathbf{1}\{X_i \neq Y_i\} |B(Y_i, r_n) \cap G_n|^j \right] \\ & = nE \left[\mathbf{1}\{X_1 \neq Y_1\} |B(Y_1, r_n) \cap G_n|^j \right]. \end{aligned}$$

We have $|B(Y_1, r_n) \cap G_n| = \sum_{i=1}^n (\mathbf{1}\{X_i \in B(Y_1, r_n)\} + \mathbf{1}\{Y_i \in B(Y_1, r_n)\})$. If $a_i = \max\{\mathbf{1}\{X_i \in B(Y_1, r_n)\}, \mathbf{1}\{Y_i \in B(Y_1, r_n)\}\}$, then $|B(Y_1, r_n) \cap G_n| \leq 2 + 2 \sum_{i=2}^n a_i = 2 + 2Z$ where, conditionally on Y_1 , Z is binomial variable with parameters $n-1$ and $p(Y_1) \leq cs^d n^{-1}$. Therefore, conditionally on Y_1 , the j -th moment of $|B(Y_1, r_n) \cap G_n|$ is bounded by some constant (depending on j and s). Finally,

$$\begin{aligned} E[|K_j^s(n^{1/d}(\mathbb{X}_n \cup \mathbb{Y}_n)) \setminus K_j^s(n^{1/d}\mathbb{X}_n)|] & \leq nE[\mathbf{1}\{X_1 \neq Y_1\} E[|B(Y_1, r_n) \cap G_n|^{j-1} | Y_1]] \\ & \leq 2nc(j, s)\varepsilon. \end{aligned}$$

As we assumed that the convergence holds for blocked distribution, it is now sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} E[b_q(r\kappa_\varepsilon(X_\varepsilon)^{1/d}, s\kappa_\varepsilon(X_\varepsilon)^{1/d})] = E[b_q(r\kappa(X)^{1/d}, s\kappa(X)^{1/d})],$$

where X_ε has density κ_ε and X has density κ . A continuity of the constant $b_q(r, s)$ is needed to conclude. It is the limit of $n^{-1}E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{X}_n))]$ when κ is the uniform density on $[0, 1]^d$. We now study this quantity.

Fix $0 \leq r_1 \leq s_1 < \infty$ and $0 \leq r_2 \leq s_2 < \infty$ with $r_1 \leq r_2$ and $s_1 \leq s_2$. The quantity $|\beta_q^{r_1, s_1}(\mathcal{K}(n^{1/d}\mathbb{X}_n)) - \beta_q^{r_2, s_2}(\mathcal{K}(n^{1/d}\mathbb{X}_n))|$ is bounded by the number of points in $\text{Dgm}_q[\mathcal{K}(n^{1/d}\mathbb{X}_n)]$ that either lie in the vertical band $[r_1, r_2]$ or in the horizontal band $[s_1, s_2]$. The number of points in the vertical line $[r_1, r_2]$ is equal to the number of positive q -simplices σ in the filtration \mathcal{K} which verify $r(\sigma) \in [r_1, r_2]$, which is in turn bounded by $F_q(n^{1/d}\mathbb{X}_n, r_1, r_2)$. Using a similar bound for the number of points in the horizontal band:

$$\begin{aligned} n^{-1} |E[\beta_q^{r_1, s_1}(\mathcal{K}(n^{1/d}\mathbb{X}_n))] - E[\beta_q^{r_2, s_2}(\mathcal{K}(n^{1/d}\mathbb{X}_n))]| \\ \leq n^{-1} (F_q(n^{1/d}\mathbb{X}_n, r_1, r_2) + F_{q+1}(n^{1/d}\mathbb{X}_n, s_1, s_2)). \end{aligned}$$

Taking the limit and using Lemma 8,

$$\begin{aligned} |b_q(r_1, s_1) - b_q(r_2, s_2)| &\leq F_q(r_1, r_2) + F_{q+1}(s_1, s_2) \\ &\leq cr_2^{2dq} \left(1 - \frac{r_1}{r_2}\right) + cs_2^{2d(q+1)} \left(1 - \frac{s_1}{s_2}\right) \end{aligned} \quad (39)$$

$$\leq c \max\{s_2^{2d(q+1)}, 1\} \left(2 - \frac{r_1}{r_2} - \frac{s_1}{s_2}\right). \quad (40)$$

In particular, with $r_1, s_1 = 0$, one obtains the control $b_q(r, s) \leq 2c \max\{s^{2d(q+1)}, 1\}$. Define $f_\varepsilon(x) = b_q(r\kappa_\varepsilon(x)^{1/d}, s\kappa_\varepsilon(x)^{1/d})$ and $f(x) = b_q(r\kappa(x)^{1/d}, s\kappa(x)^{1/d})$. Define $\kappa_-(x)$ to be the minimum of $\kappa(x)$ and $\kappa_\varepsilon(x)$ and $f_-(x)$ to be equal to $f(x)$ or $f_\varepsilon(x)$ whether $\kappa_-(x)$ is equal to $\kappa(x)$ or $\kappa_\varepsilon(x)$. Define $\kappa^+(x)$ and $f^+(x)$ in a similar way. We have

$$\begin{aligned} &|E[b_q(r\kappa_\varepsilon(X_\varepsilon)^{1/d}, s\kappa_\varepsilon(X_\varepsilon)^{1/d}) - b_q(r\kappa(X)^{1/d}, s\kappa(X)^{1/d})]| \\ &= \left| \int (\kappa_\varepsilon(x)f_\varepsilon(x) - \kappa(x)f(x)) dx \right| \\ &\leq \int \kappa_+(x) |f_\varepsilon(x) - f(x)| dx + \int f_-(x) |\kappa(x) - \kappa_\varepsilon(x)| dx \\ &\leq 2c \max\{s^{2d(q+1)} \sup \kappa^{2(q+1)}, 1\} \int \kappa_+(x) \left(1 - \left(\frac{\kappa_-(x)}{\kappa_+(x)}\right)^{1/d}\right) dx \\ &\quad + 2c \max\{s^{2d(q+1)} \sup \kappa^{2(q+1)}, 1\} \varepsilon \\ &\leq 2c \max\{s^{2d(q+1)} \sup \kappa^{2(q+1)}, 1\} \int \kappa_+(x)^{1-1/d} |\kappa(x)^{1/d} - \kappa_\varepsilon(x)^{1/d}| dx \\ &\quad + 2c \max\{s^{2d(q+1)} \sup \kappa^{2(q+1)}, 1\} \varepsilon \\ &\leq 2c \max\{s^{2d(q+1)} \sup \kappa^{2(q+1)}, 1\} (\sup \kappa^{1-1/d} L\varepsilon + \varepsilon), \end{aligned}$$

where $L = d^{-1}(\inf \kappa)^{-1+1/d}$ is equal to $\sup_{t \geq \inf \kappa, s \geq 0} \left| \frac{t^{1/d} - s^{1/d}}{t-s} \right|$. We have the conclusion by letting $\varepsilon \rightarrow 0$. \square

Therefore, it is sufficient to prove (36) for blocked distributions. Let κ be of the form

$$\sum_{i=1}^{m^d} \alpha_i \mathbb{1}\{Q_i\},$$

where m is a power of 2 and the set (Q_i) forms a partition of $[0, 1]^d$ into isometric cubes, which are of side length m^{-1} . Let \mathbb{P}_n be a Poisson distribution of intensity $n\kappa$. Put

$$\mathring{\mathcal{K}}(n^{1/d}\mathbb{P}_n) = \bigsqcup_{i=1}^{m^d} \mathcal{K}(n^{1/d}(\mathbb{P}_n \cap Q_i)).$$

Using the geometric Lemma 1, we get

$$|\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{P}_n)) - \beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}\mathbb{P}_n))| \leq \sum_{j=q}^{q+1} \sum_{i=1}^{m^d} A_{i,j},$$

where $A_{i,j}$ is the number of j -simplexes of $K_s(n^{1/d}\mathbb{P}_n)$ that have at least one point in $\partial(n^{1/d}Q_i)^{2s}$, the $(2s)$ -neighborhood of the boundary of $n^{1/d}Q_i$. This random variable is stochastically dominated by $\tilde{A}_{i,j}$, the same variable with \mathbb{P}_n replaced by $\tilde{\mathbb{P}}_n$, an homogeneous Poisson variable of intensity $\sup \kappa \cdot n$. It is shown in [32] that $E[\tilde{A}_{i,j}] \leq c(s)\ell(\partial(n^{1/d}Q_i)^{2s})$, where ℓ is the Lebesgue measure on \mathbb{R}^d . Hence,

$$\begin{aligned} E[|\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{P}_n)) - \beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}\mathbb{P}_n))|] &\leq \sum_{j=q}^{q+1} \sum_{i=1}^{m^d} c(s)n^{1-1/d}m^{-d+1} \\ &\leq c(s)n^{1-1/d}m \end{aligned} \quad (41)$$

Moreover, $\beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}\mathbb{P}_n)) = \sum_{i=1}^{m^d} \beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}(\mathbb{P}_n \cap Q_i)))$. The process $n^{1/d}(\mathbb{P}_n \cap Q_i)$ is a homogeneous Poisson point process of intensity α_i on $n^{1/d}Q_i$, which is a cube of side length $n^{1/d}m^{-1}$. Hence, $E[\beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}(\mathbb{P}_n \cap Q_i)))]/(nm^{-d}\alpha_i)$ converges to $b_q(\alpha_i^{1/d}r, \alpha_i^{1/d}s)$. Therefore,

$$\begin{aligned} \frac{E[\beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}(\mathbb{P}_n \cap Q_i)))]}{n} &= \sum_{i=1}^{m^d} \frac{E[\beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/d}(\mathbb{P}_n \cap Q_i)))]}{nm^{-d}\alpha_i} m^{-d}\alpha_i \\ &\xrightarrow{n \rightarrow \infty} \sum_{i=1}^{m^d} m^{-d}\alpha_i b_q(\alpha_i^{1/d}r, \alpha_i^{1/d}s) \end{aligned}$$

$$= E \left[b_q \left(r\kappa(X)^{1/d}, s\kappa(X)^{1/d} \right) \right].$$

Combining this with (41) concludes the proof. \square

Up to this point, we have proven that there exists a function $b_q(r, s)$ such that for any density κ ,

$$n^{-1} E[\beta_q^{r,s}(\mathcal{K}(n^{1/d}\mathbb{X}_n))] \xrightarrow{n \rightarrow \infty} E \left[b_q \left(r\kappa(X)^{1/d}, s\kappa(X)^{1/d} \right) \right],$$

when \mathbb{X}_n is a binomial variable. By Lemma 7, the same convergence holds a.s. and for a Poisson sample. Proposition 3.4 in [17] shows that the convergence of the persistent Betti numbers imply the vague convergence of the diagrams: we have shown that there exists a Radon measure μ_q^κ such that $E[\mu_q^n] \xrightarrow[n \rightarrow \infty]{v_c} \mu_q^\kappa$ and, with probability one, $\mu_q^n \xrightarrow[n \rightarrow \infty]{v_c} \mu_q^\kappa$. Moreover, equation (36) implies the point (iii) of Theorem 6.

Remark 4. *It is straightforward to adapt Step 1 to densities κ whose supports are not the cube, but any open set.*

6.5.2 Step 2: Convergence for functions vanishing on the diagonal

The next step of the proof is to show that the convergence holds for continuous bounded functions vanishing of the diagonal. Let us denote $C_0(\mathbb{R}_>^2)$ this class of function. The crucial part of the proof is Proposition 1, which bounds the total number of points in the diagrams. An elementary lemma from measure theory is then used to show that it implies the a.s. convergence for vanishing functions. We say that a sequence of measures (μ_n) converges C_0 -vaguely to μ if $\mu_n(\phi) \rightarrow \mu(\phi)$ for all functions ϕ in $C_0(\mathbb{R}_>^2)$.

Lemma 11. *Let E be a locally compact Hausdorff space. Let $(\mu_n)_{n \geq 0}$ be a sequence of Radon measure on E which converges C_c -vaguely to some measure μ . If $M := \sup_n |\mu_n| < \infty$, then $(\mu_n)_{n \geq 0}$ converges C_0 -vaguely to μ .*

Proof. Let (h_q) be a sequence of functions with compact support converging to 1 and let $\phi \in C_0(E)$. Fix $\varepsilon > 0$. By definition of $C_0(E)$, there exists a compact set K_ε such that f is smaller than ε outside of K_ε . For q large enough, the support of h_q includes K_ε . Let $\phi_q = \phi \cdot h_q$. Then,

$$\begin{aligned} |\mu_n(\phi) - \mu(\phi)| &\leq |\mu_n(\phi) - \mu_n(\phi_q)| + |\mu_n(\phi_q) - \mu(\phi_q)| + |\mu(\phi_q) - \mu(\phi)| \\ &\leq (M + |\mu|)\varepsilon + |\mu_n(\phi_q) - \mu(\phi_q)|. \end{aligned}$$

As $(\mu_n)_n$ converges vaguely to μ , the last term of the sum converges to 0 when ε is fixed. Hence, $\limsup_{n \rightarrow \infty} |\mu_n(\phi) - \mu(\phi)| \leq (M + |\mu|)\varepsilon$. As this holds for all $\varepsilon > 0$, $\mu_n(\phi)$ converges to $\mu(\phi)$. \square

Taking $M = 0$ in Proposition 1, we see that $\sup_n E[|\mu_n|] < \infty$. Therefore, the C_0 -vague convergence of $E[\mu_n]$ is shown in the binomial setting. To show that the convergence also holds almost surely for $|\mu_n|$, we need to show that $\sup_n |\mu_n| < \infty$. For this, we use concentration inequalities. We do not show concentration inequalities for $|\mu_n|$ directly. Instead, we derive concentration inequalities for $\sum_{i=1}^n \#(\mathbb{X}_n \cap C(X_i, R(X_i, \mathbb{X}_n)))^{q+1}$, which is a majorant of $|\mu_n|$. Recall that $R(X_i, \mathbb{X}_n)$ is defined as the smallest radius R such that for each $j = 1 \dots Q$, $A_j(X_i, \frac{1}{\sqrt{d}}R)$, either contains a point of \mathbb{X}_n different than X_i , or is not contained in the cube. To ease the notations, we denote $R(X_i, \mathbb{X}_n)$ by R_{in} .

Lemma 12. *Fix $M \geq 0$ and define $Z_n^M = \sum_{i=1}^n \#(\mathbb{X}_n \cap C(X_i, R_{in}))^q \mathbb{1}\{R_{in} \geq M\}$. Then, for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that*

$$P(|Z_n^M - E[Z_n^M]| > t) \leq \frac{n^{\frac{3}{2}+\varepsilon}}{t^3} \exp(-c_\varepsilon n^{-1} M^d).$$

The constant c_ε depends on ε, d, q and κ .

As a consequence of the concentration inequality, $n^{-1}Z_n^0$ is almost surely bounded. Indeed, choose $\varepsilon < 1/2$:

$$P(n^{-1}|Z_n^0 - E[Z_n^0]| > t) \leq \frac{n^{\frac{3}{2}+\varepsilon}}{(nt)^3} = \frac{n^{-\frac{3}{2}+\varepsilon}}{t^3}.$$

By Borel-Cantelli lemma, almost surely, for n large enough, $n^{-1}|Z_n^0 - E[Z_n^0]| \leq 1$. Moreover, $\sup_n n^{-1}E[Z_n^0] < \infty$. As a consequence, $\sup_n n^{-1}Z_n^0$ is almost surely finite. As this is an upper bound of $\sup_n |\mu_n|$, we have proven that $\sup_n |\mu_n| < \infty$ almost surely. By Lemma 11, the sequence μ_n converges C_0 vaguely to μ . The proof of Lemma 12 is based on an inequality of the Efron-Stein type and is rather long and technical. It can be found in Section 6.7.

We now briefly consider the Poisson setting. Define $\mu'_n = \mu_{N_n} \times (N_n/n)$, where $(N_i)_{i \geq 1}$ is some sequence of independent Poisson variables of parameter n , independent of $(X_i)_{i \geq 1}$.

- $E[|\mu'_n|] = E\left[\frac{N_n}{n} E[|\mu_{N_n}| | N_n]\right] \leq \sup_n E[|\mu_n|] < \infty$. Therefore, C_0 -convergence of the expected diagram holds in the Poisson setting.
- Likewise, it is sufficient to show that $\sup_n \frac{N_n}{n} < \infty$ to conclude to the C_0 -convergence of the diagram in the Poisson setting. Fix $t > 1$. It is shown in the chapter 1 of the monograph [25] that $P(N_n > nt) \leq \exp(-nH(t))$, where $H(t) = 1 - t + t \ln t$. This gives us

$$P\left(\sup_n \frac{N_n}{n} \leq t\right) = \prod_{n \geq 0} (1 - P(N_n > nt))$$

$$\geq \prod_{n \geq 0} (1 - \exp(-nH(t))) = \exp\left(\sum_{n \geq 0} \ln(1 - \exp(-nH(t)))\right) \quad (42)$$

The series $\sum_n \ln(1 - x^n)$ is equal to $-\sum_n \sigma(n)/nx^n$ when $|x| < 1$, and where $\sigma(n)$ is the sum of the proper divisors of n . Therefore it is a power series, and is continuous on $] -1, 1[$. Since t tends to infinity, $\exp(-H(t))$ converges to 0, and thus the quantity (42) converges to 1 as t tends to infinity.

6.5.3 Step 3: Convergence for functions with polynomial growth

The last step consists in extending the convergence to functions $\phi \in C_{poly}(\mathbb{R}_{>}^2)$. We only show the result for binomial processes. The proof can be adapted to the Poisson case using similar techniques as at the end of Step 2. The core of the proof is a bound on the number of points in a diagram with high persistence. For $M > 0$, define $T_M = \{r = (r_1, r_2) \in \mathbb{R}_{>}^2 \text{ s.t. } pers(r) \geq M\}$. Let $P_n(M) = n\mu_n(T_M)$ denote the number of points in the diagram with persistence larger than M .

First, we show that the expectation of $P_n(M)$ converges to 0 at an exponential rate when M tends to ∞ . The random variable $P_n(M)$ is bounded by $n\mu_n(U_M)$. By Proposition 1, recalling that q is the degree of homology,

$$\begin{aligned} E[P_n(M)] &\leq \int_0^\infty P(n\mu_n(U_M) \geq t) dt \\ &\leq \int_0^\infty c \exp\left(-c(M^d + (t/n)^{1/(q+1)})\right) dt \\ &\leq cn \exp(-cM^d) \int_0^\infty \exp(-u) qu^q du = cn \exp(-cM^d) \end{aligned}$$

Fix a sequence (g_M) of continuous functions with support inside the complement of T_M taking their values in $[0, 1]$, equal to 1 on T_{M-1} . Let ϕ be a function with polynomial growth, i.e. satisfying (22) for some $A, \alpha > 0$. Define $\phi_M = \phi \cdot g_M$. We have the decomposition:

$$E[\mu_n(\phi)] = (E[\mu_n(\phi)] - E[\mu_n(\phi_M)]) + E[\mu_n(\phi_M)].$$

The first term on the right is bounded by

$$\begin{aligned} E[\mu_n(A(1 + pers^\alpha))(1 - g_M)] &\leq AE[P_n(M)]/n + AE[\text{Pers}_\alpha(M; \text{Dgm}[\mathbb{X}_n])]/n \\ &\leq cA \exp(-cM^d) + AE[\text{Pers}_\alpha(M; \text{Dgm}[\mathbb{X}_n])]/n. \end{aligned} \quad (43)$$

It is shown in [14] that

$$\text{Pers}_\alpha(M; \text{Dgm}[\mathbb{X}_n]) \leq M^\alpha P_n(M) + \alpha \int_M^\infty P_n(\varepsilon) \varepsilon^{\alpha-1} d\varepsilon.$$

Hence, by Fubini's theorem:

$$(43) \leq c \exp(-cM^d) (A + M^\alpha) + c\alpha \int_M^\infty \exp(-c\varepsilon^d) \varepsilon^{\alpha-1} d\varepsilon. \quad (44)$$

and this quantity goes to 0 as M goes to infinity. As $\phi_M \in C_0(\mathbb{R}_>^2)$, $E[\mu_n(\phi_M)]$ converges to $\mu(\phi_M)$. Moreover, applying this inequality to $M = 0$, we get that $C_0 = \sup_n E[\mu_n(\phi)] < \infty$. Therefore, $\mu(\phi_M) \leq C_0$. By the monotone convergence theorem, $\mu(\phi_M)$ converges to $\mu(\phi)$ when ϕ is non negative, with $\mu(\phi)$ finite by the latter inequality. If ϕ is not always non negative, we conclude by separating its positive and negative parts. Finally,

$$\limsup_{n \rightarrow \infty} |E[\mu_n(\phi)] - \mu(\phi)| \leq (43) + |\mu(\phi_M) - \mu(\phi)| \xrightarrow{M \rightarrow \infty} 0.$$

We now prove that $\mu_n(\phi) - E[\mu_n(\phi)]$ converges a.s. to 0. Similar to the above, it is enough to show that $P_n(M)$ is almost surely bounded by a quantity independent of n , which converges to 0 at an exponential rate when M goes to ∞ . The random variable $(q+2)P_n(M)$ is bounded by Z_n^M , which is defined in Lemma 12, and whose expectation is controlled. Therefore, it remains to show that Z_n^M is close to its expectation. Let $M_n = n^{-1/d}M$.

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\mu_n(\phi) - E[\mu_n(\phi)]| \\ \leq \limsup_{n \rightarrow \infty} (|\mu_n(\phi - \phi_M)| + |(\mu_n - E[\mu_n])(\phi_M)| + |E[\mu_n](\phi - \phi_M)|) \\ \leq \limsup_{n \rightarrow \infty} |\mu_n(\phi - \phi_M)| + 0 + (44). \end{aligned}$$

The random variable $|\mu_n(\phi - \phi_M)|$ is bounded by

$$\begin{aligned} An^{-1} \left(P_n(M) + M^\alpha P_n(M) + \alpha \int_M^\infty P_n(\varepsilon) \varepsilon^{\alpha-1} d\varepsilon \right) \\ \leq An^{-1} \left(Z_n^{M_n} + M^\alpha Z_n^{M_n} + \alpha \int_M^\infty Z_n^{\varepsilon_n} \varepsilon^{\alpha-1} d\varepsilon \right), \end{aligned} \quad (45)$$

where $M_n = n^{-1/d}M$ and $\varepsilon_n = n^{-1/d}\varepsilon$. As a consequence of Lemma 12, by choosing ε so that $-3/2 + \varepsilon < -1$,

$$P \left(\sup_n n^{-1} |Z_n^{M_n} - E[Z_n^{M_n}]| > t \right) \leq c \frac{\exp(-cM^d)}{t^3}.$$

Fixing $t = \exp(-(c/6)M^d)$ and using Borel-Cantelli lemma, for $M \in \mathbb{N}$ large enough, $\sup_n n^{-1} |Z_n^{M_n} - E[Z_n^{M_n}]| < \exp(-(c/6)M^d)$. Also, $E[Z_n^{M_n}] \leq nc \exp(-cM^d)$. Therefore, for $\alpha \geq 0$,

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{-1} M^\alpha Z_n^{M_n} = 0 \text{ a.s.}$$

The third term in the sum (45) is less straightforward to treat. As Z_n^M is a decreasing function of M , for $M \in \mathbb{N}$ large enough and with $k_n = n^{-1/d}k$:

$$\begin{aligned} \int_M^\infty n^{-1} Z_n^{\varepsilon_n} \varepsilon^{\alpha-1} d\varepsilon &\leq \sum_{k \geq M} n^{-1} Z_n^{k_n} \int_k^{k+1} \varepsilon^{\alpha-1} d\varepsilon \\ &\leq \sum_{k \geq M} c \exp(-ck^d) \frac{1}{\alpha} ((k+1)^\alpha - k^\alpha) \\ &\leq \sum_{k \geq M} c \exp(-ck^d) \frac{k^\alpha 2^\alpha}{\alpha} = o(1). \end{aligned}$$

As a consequence, $\lim_{M \rightarrow \infty} \limsup_n |\mu_n(\phi - \phi_M)| = 0$. As the quantity appearing in inequality (44) also converges to 0 when M goes to infinity, we have proven that $\mu_n(\phi) - E[\mu_n(\phi)]$ converges a.s. to 0. Therefore, $\mu_n(\phi)$ converges a.s. to $\mu(\phi)$.

Finally, we have to prove assertion (ii) in Theorem 6, i.e. that the convergence holds in L_p . As the convergence holds in probability, it is sufficient to show that $(\mu_n(\phi)^p)_n$ is uniformly integrable. Observing that $E[\mu_n(\phi)^p \mathbf{1}\{\mu_n(\phi)^p > M\}] \leq (E[\mu_n(\phi)^{2p}] P(\mu_n(\phi)^p > M))^{1/2}$, uniform integrability follows from $\sup_n E[\mu_n(\phi)^p] < \infty$ for any $p > 1$. We have

$$\mu_n(\phi)^p \leq \mu_n(A(1 + pers^\alpha))^p \leq 2^{p-1}(A^p |\mu_n|^p + \mu_n(pers^\alpha)^p),$$

and from Proposition 1 we easily obtain that $E[|\mu_n|^p]$ is uniformly bounded. We treat the other part by assuming without loss of generality that p is an integer:

$$\begin{aligned} &\leq \frac{1}{n^p} E \left[\left(\int_0^\infty P_n(\varepsilon) \varepsilon^{\alpha-1} d\varepsilon \right)^p \right] \\ &= \frac{1}{n^p} E \left[\int_{[0, \infty]^p} P_n(\varepsilon_1) \cdots P_n(\varepsilon_p) (\varepsilon_1 \cdots \varepsilon_p)^{\alpha-1} d\varepsilon_1 \cdots d\varepsilon_p \right] \\ &= \frac{1}{n^p} \int_{[0, \infty]^p} E[P_n(\varepsilon_1) \cdots P_n(\varepsilon_p)] (\varepsilon_1 \cdots \varepsilon_p)^{\alpha-1} d\varepsilon_1 \cdots d\varepsilon_p \\ &\leq \frac{1}{n^p} \int_{[0, \infty]^p} (E[P_n(\varepsilon_1)^p] \cdots E[P_n(\varepsilon_p)^p])^{1/p} (\varepsilon_1 \cdots \varepsilon_p)^{\alpha-1} d\varepsilon_1 \cdots d\varepsilon_p \\ &\leq \frac{1}{n^p} \int_{[0, \infty]^p} (n^{p^2} c \exp(-c(\varepsilon_1^d + \cdots + \varepsilon_p^d)))^{1/p} (\varepsilon_1 \cdots \varepsilon_p)^{\alpha-1} d\varepsilon_1 \cdots d\varepsilon_p \\ &= c \int_{[0, \infty]^p} \exp\left(-\frac{c}{p}(\varepsilon_1^d + \cdots + \varepsilon_p^d)\right) (\varepsilon_1 \cdots \varepsilon_p)^{\alpha-1} d\varepsilon_1 \cdots d\varepsilon_p \\ &= c \left(\int_0^\infty \exp\left(-\frac{c}{p}\varepsilon^d\right) \varepsilon^{\alpha-1} d\varepsilon \right)^p < \infty. \end{aligned}$$

6.6 Proof of Proposition 2

The two ingredients of the proof are the vague convergence of (μ_n) to μ and Lemma 8. Fix $0 < r \leq s$. We wish to show that, as r and s get closer, $\pi_1^* \mu([r, s])$ goes to 0. By the Portemanteau Theorem, $\pi_1^* \mu([r, t]) \leq \limsup \pi_1^* \mu_n([r, s])$. It is shown in Lemma 5 that this quantity is smaller than $cs^{2dq-1}|r-s|$, a quantity which converges to 0 when r goes to s . A similar proof holds for $\pi_2^* \mu$.

6.7 Proof of Lemma 12

The lemma is based on an inequality of the Efron-Stein type, combined with Markov's inequality.

Theorem 7 (Theorem 2 in [5]). *Let \mathbb{X} be a measurable set and $F : \mathbb{X}^n \rightarrow \mathbb{R}$ a measurable function. Define a n -sample $\mathbb{X}_n = \{X_1, \dots, X_n\}$ and let $Z = F(\mathbb{X}_n)$. If \mathbb{X}'_n is an independent copy of \mathbb{X}_n , denote $Z'_i = F(X_1, \dots, X_{i-1}, X'_i, X_{i+1}, \dots, X_n)$. Define*

$$V = \sum_{i=1}^n E[(Z - Z'_i)^2 | \mathbb{X}_n].$$

Then, for $p \geq 2$, there exists a constant C_p depending only on p such that

$$E[|Z - E[Z]|^p] \leq C_p E[V^{p/2}].$$

Denote $\mathbb{X}_n^i = \mathbb{X}_n \setminus \{X_i\}$ and $S(X_i, \mathbb{X}_n) = \#(\mathbb{X}_n \cap C(X_i, R_{in}))^q \mathbb{1}\{R(X_i, \mathbb{X}_n) \geq M\}$. We will apply Theorem 7 to $F(\mathbb{X}_n) = \sum_{i=1}^n S(X_i, \mathbb{X}_n)$. The quantity $(Z - Z'_i)^2$ is bounded by $2(Z - Z_i)^2 + 2(Z'_i - Z_i)^2$, where $Z_i = F(\mathbb{X}_n^i)$. For most X_j 's, $S(X_j, \mathbb{X}_n) = S(X_j, \mathbb{X}_n^i)$, and therefore V can be efficiently bounded. More precisely,

$$\begin{aligned} E[V^{p/2}] &= E \left[\left(\sum_{i=1}^n E[(Z - Z'_i)^2 | \mathbb{X}_n] \right)^{p/2} \right] \\ &\leq n^{p/2} E[(Z - Z'_n)^p] && \text{(by Jensen's inequality)} \\ &\leq n^{p/2} 2^{p-1} E[(Z - Z_n)^p + (Z'_n - Z_n)^p] \\ &= n^{p/2} 2^p E[(Z - Z_n)^p] \text{ as } (Z, Z_n) \sim (Z'_n, Z_n) \\ &= 2^p n^{p/2} E \left[\left(S(X_n, \mathbb{X}_n) + \sum_{j=1}^{n-1} (S(X_j, \mathbb{X}_n) - S(X_j, \mathbb{X}_{n-1})) \right)^p \right] \end{aligned}$$

$$\leq 2^p n^{p/2} 2^{p-1} \left(E[S(X_n, \mathbb{X}_n)^p] + E \left[\left(\sum_{j=1}^{n-1} (S(X_j, \mathbb{X}_n) - S(X_j, \mathbb{X}_{n-1})) \right)^p \right] \right). \quad (46)$$

Fix $p = 3$. Lemma 5 shows that for $p \geq 1$, $B_p = \sup_n E[S(X_n, \mathbb{X}_n)^p] < \infty$. Define $Y_j = (S(X_j, \mathbb{X}_n) - S(X_j, \mathbb{X}_{n-1}))$. Denote G_j the event that $X_n \in C(X_j, R_{j_n})$. If G_j is not realized, then $Y_j = 0$. Expanding the product,

$$\begin{aligned} E \left[\left(\sum_{j=1}^{n-1} (S(X_j, \mathbb{X}_n) - S(X_j, \mathbb{X}_{n-1})) \right)^3 \right] &\leq \sum_{j_1, j_2, j_3} E[\mathbf{1}\{G_{j_1} \cap G_{j_2} \cap G_{j_3}\} Y_{j_1} Y_{j_2} Y_{j_3}] \\ &\leq \sum_{j_1, j_2, j_3} P(G_{j_1} \cap G_{j_2} \cap G_{j_3})^{1/q'} E[(Y_{j_1} Y_{j_2} Y_{j_3})^{p'}]^{1/p'}, \end{aligned} \quad (47)$$

where $\frac{1}{p'} + \frac{1}{q'} = 1$ and $p' \geq 1$ is some quantity to be fixed later.

- We first bound $E[(Y_{j_1} Y_{j_2} Y_{j_3})^{p'}]$. If $Y_{j_1} \neq 0$, then $R(X_j, \mathbb{X}_{n-1}) > M$. Therefore,

$$E[(Y_{j_1} Y_{j_2} Y_{j_3})^{p'}] \leq \sqrt{P(R(X_j, \mathbb{X}_{n-1}) > M) E[(Y_{j_1} Y_{j_2} Y_{j_3})^{2p'}]}.$$

Also, $E[(Y_{j_1} Y_{j_2} Y_{j_3})^{2p'}] \leq E[Y_{j_1}^{6p'}] \leq B_{6p'}$, as $0 \leq Y_{j_1} \leq S(X_{j_1}, \mathbb{X}_n)$. Therefore, using inequality (32):

$$E[(Y_{j_1} Y_{j_2} Y_{j_3})^{p'}] \leq c \exp(-cM^d).$$

- We now bound the probability $P(G_{j_1} \cap G_{j_2} \cap G_{j_3})$.

If $j_1 = j_2 = j_3$, then it is clear that $P(G_{j_1} \cap G_{j_2} \cap G_{j_3}) \leq c/n$. However, in the general case, the joint law of the different $R_{j_i, n-1}$ becomes of interest. To ease the notation, assume that $j_i = i$ and denote $R_{i, n-1}$ simply by R_i . Also, define D_{ij} the distance between X_i and X_j . The fact that inequality (32) still holds conditionally on X_1, X_2 and X_3 , and with the joint laws of R_1, R_2 and R_3 , will be repeatedly used.

Lemma 13. *The following bound holds:*

$$P(R_1 \geq t_1, R_2 \geq t_2, R_3 \geq t_3 | X_1, X_2, X_3) \leq c \exp\left(-cn(t_1^d + t_2^d + t_3^d)\right) \quad (48)$$

Proof. Suppose that $\max t_i = t_1$. Inequality (32) states that $P(R_1 \geq t_1) \leq c \exp(-cnt_1^d)$. Likewise, it is straightforward to show that a similar bound holds conditionally on X_1, X_2 and X_3 . As $t_1^d \geq \frac{t_1^d + t_2^d + t_3^d}{3}$, the result follows. \square

Let us prove that $P(G_1 \cap G_2) \leq c/n^2$. If the event is realized, then X_n is in the intersection of $C(X_1, R_1)$ and $C(X_2, R_2)$. Therefore, this intersection is non empty and $D_{12} \leq \sqrt{d}(R_1 + R_2)$. Hence,

$$\begin{aligned}
P(G_1 \cap G_2) &\leq P(D_{12} \leq \sqrt{d}(R_1 + R_2) \text{ and } X_n \in C(X_1, R_1) \cap C(X_2, R_2)) \\
&\leq 2P(D_{12} \leq 2\sqrt{d}R_1 \text{ and } X_n \in C(X_1, R_1)) \\
&= 2E \left[\mathbf{1}\{D_{12} \leq 2\sqrt{d}R_1\} P(X_n \in C(X_1, R_1) | \mathbb{X}_{n-1}) \right] \\
&\leq 2cE \left[\mathbf{1}\{D_{12} \leq 2\sqrt{d}R_1\} R_1^d \right] \\
&\leq 2cE \left[\int_{(D_{12}/(2\sqrt{d}))^d} P(R_{1,n-1}^d \geq t | X_1, X_2) dt \right] \\
&\leq 2cE \left[\int_{(D_{12}/(2\sqrt{d}))^d} \exp(-cnt) dt \right] \\
&\leq \frac{c}{n} E \left[\exp(-cnD_{12}^d) \right] \\
&= \frac{c}{n} \int_0^1 P(cnD_{12}^d \leq -\ln(t)) dt \\
&\leq \frac{c}{n} \int_0^1 \frac{-\ln(t)}{cn} dt = \frac{c}{n^2}.
\end{aligned}$$

Finally, we bound $P(G_1 \cap G_2 \cap G_3)$. If the event is realized, then

$$\begin{cases} D_{12} \leq \sqrt{d}(R_1 + R_2) \\ D_{23} \leq \sqrt{d}(R_2 + R_3) \\ D_{13} \leq \sqrt{d}(R_1 + R_3) \end{cases} \implies \begin{cases} D_{12} \leq 2\sqrt{d}R_1 \text{ or } D_{12} \leq 2\sqrt{d}R_2 \\ D_{23} \leq 2\sqrt{d}R_2 \text{ or } D_{23} \leq 2\sqrt{d}R_3 \\ D_{13} \leq 2\sqrt{d}R_1 \text{ or } D_{13} \leq 2\sqrt{d}R_3 \end{cases}$$

There are therefore 8 possibilities. Each of these event is either bounded by an event of the form $(D_{12} \leq 2\sqrt{d}R_1 \text{ and } D_{13} \leq 2\sqrt{d}R_1)$ (six events), or by a event of the form $(D_{12} \leq 2\sqrt{d}R_1 \text{ and } D_{23} \leq 2\sqrt{d}R_2 \text{ and } D_{13} \leq 2\sqrt{d}R_3)$ (two events). Using this, we obtain

$$\begin{aligned}
P(G_1 \cap G_2 \cap G_3) &\leq 6P(X_n \in C(X_1, R_1) \text{ and } D_{12} \leq 2\sqrt{d}R_1 \text{ and } D_{13} \leq 2\sqrt{d}R_1) \\
&\quad + 2P(X_n \in C(X_1, R_1) \text{ and } D_{12} \leq 2\sqrt{d}R_1 \text{ and } D_{23} \leq 2\sqrt{d}R_2 \text{ and } D_{13} \leq 2\sqrt{d}R_3) \\
&\leq cE[R_1^d \mathbf{1}\{D_{12} \leq 2\sqrt{d}R_1 \text{ and } D_{13} \leq 2\sqrt{d}R_1\}] \\
&\quad + cE[R_1^d \mathbf{1}\{D_{12} \leq 2\sqrt{d}R_1 \text{ and } D_{23} \leq 2\sqrt{d}R_2 \text{ and } D_{13} \leq 2\sqrt{d}R_3\}]
\end{aligned}$$

$$\begin{aligned}
&= cE \left[\int_{\left(\frac{\max(D_{12}, D_{13})}{2\sqrt{d}}\right)^d}^{\infty} P(R_1^d \geq u | X_1, X_2, X_3) du \right] \\
&+ cE \left[\int_{\left(\frac{D_{12}}{2\sqrt{d}}\right)^d}^{\infty} P \left(R_1^d \geq u \text{ and } R_2 \geq \frac{D_{23}}{2\sqrt{d}} \text{ and } R_3 \geq \frac{D_{13}}{2\sqrt{d}} \middle| X_1, X_2, X_3 \right) du \right] \\
&\leq cE \left[\int_{\left(\frac{\max(D_{12}, D_{13})}{2\sqrt{d}}\right)^d}^{\infty} e^{-cnu} du \right] + cE \left[\int_{\left(\frac{D_{12}}{2\sqrt{d}}\right)^d}^{\infty} e^{-cn(u+D_{23}^d+D_{13}^d)} du \right] \\
&= \frac{c}{n} E \left[e^{-cn \max(D_{12}, D_{13})^d} \right] + \frac{c}{n} E \left[e^{-cn(D_{12}^d+D_{23}^d+D_{13}^d)} \right] \\
&\leq 2\frac{c}{n} E \left[e^{-cn \max(D_{12}, D_{13})^d} \right] = \frac{c}{n} \int_0^1 P \left(\max(D_{12}, D_{13})^d \leq \frac{-\ln(t)}{cn} \right) dt \\
&\leq \frac{c}{n} \int_0^1 \left(\frac{-\ln(t)}{cn} \right)^2 dt = \frac{c}{n^3}.
\end{aligned}$$

Finally, inequality (47) becomes

$$\begin{aligned}
E[V^{3/2}] &\leq cn^{3/2}(\exp(-cM^d) + \exp(-cM^d)(n^{1-1/q'} + n^{2-2/q'} + n^{3-3/q'})) \\
&\leq cn^{3/2+3(1-1/q')} \exp(-cM^d).
\end{aligned}$$

Choose $p' = 3/\varepsilon$ and apply Markov inequality to conclude.

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