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ON THE LOWER PART OF THE LATTICE OF PARTIAL CLONES

MIGUEL COUCEIRO, LUCIEN HADDAD, AND KARSTEN SCHÖLZEL

Abstract. Let $\mathbf{k}$ be a k-element set. We show that the lattice of all strong partial clones on $\mathbf{k}$ has no minimal elements. Moreover, we show that if C is a strong partial clone, then the family of all partial subclones of C is of continuum cardinality. Finally we show that every non-trivial strong partial clone contains a family of continuum cardinality of strong partial subclones.

To the memory of Professor Ivan Stojmenovic

1. Preliminaries

In this section we recall basic concepts and terminology needed throughout the paper. For further background, see [2, 5, 6, 7].

Let A be a finite set with $|A| \geq 2$. If $|A| = k \geq 2$, then we assume that $A = \mathbf{k} := \{0, \ldots, k-1\}$. For a positive integer n, an *n-ary partial function* on $\mathbf{k}$ is a map

$$f : \mathrm{dom}(f) \to \mathbf{k}$$

where $\mathrm{dom}(f)$ is a subset of $\mathbf{k}^n$ called the *domain* of f. We denote by $\mathrm{Par}^{(n)}(\mathbf{k})$ the set of all n-ary partial functions on $\mathbf{k}$ and we set

$$\mathrm{Par}(\mathbf{k}) := \bigcup_{n \geq 1} \mathrm{Par}^{(n)}(\mathbf{k}).$$

For positive integers $n, m \geq 1$, and $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ and $g_1, \ldots, g_n \in \mathrm{Par}^{(m)}(\mathbf{k})$, the *composition of f with $g_1, \ldots, g_n$* is the partial function $f[g_1, \ldots, g_n] \in \mathrm{Par}^{(m)}(\mathbf{k})$ whose domain $\mathrm{dom}(f[g_1, \ldots, g_n])$ is the set of all

$$\vec{a} \in \bigcap_{i=1}^{m} \mathrm{dom}(g_i) \quad \text{such that} \quad (g_1(\vec{a}), \ldots, g_m(\vec{a})) \in \mathrm{dom}(f),$$

and whose values are given by

$$f[g_1, \ldots, g_n](\vec{a}) := f(g_1(\vec{a}), \ldots, g_n(\vec{a}))$$

for all $\vec{a} \in \mathrm{dom}(f[g_1, \ldots, g_n])$. For every positive integer n and each $1 \leq i \leq n$, we denote by e_i^n the *n-ary i-th projection function* (or, simply, *projection*) defined by

$$e_i^n(a_1, \ldots, a_n) = a_i$$

for all $(a_1, \ldots, a_n) \in \mathbf{k}^n$. Furthermore, we use the notation

$$J_{\mathbf{k}} := \{e_i^n : 1 \leq i \leq n, n \in \mathbb{N}\}$$

for the set of all projections.

Definition 1. *A* partial clone *on* $\mathbf{k}$ *is a subset of* $\mathrm{Par}(\mathbf{k})$ *closed under composition and containing the set of all projection functions* $J_{\mathbf{k}}$.

Remark 2. Partial clones can be equivalently defined in the formalism of Mal'cev; see, e.g., Chapter 20 of [7].

The set of partial clones on $\mathbf{k}$ forms a lattice $\mathcal{L}_{P_{\mathbf{k}}}$ under inclusion, in which the infimum is the set-theoretical intersection. Thus the intersection of arbitrarily many partial clones on $\mathbf{k}$ is itself a partial clone on $\mathbf{k}$. Clearly, the set $J_{\mathbf{k}}$ of all projections on $\mathbf{k}$ is the smallest element in the lattice $\mathcal{L}_{P_{\mathbf{k}}}$ of partial clones.
For $f, g \in \mathrm{Par}^{(n)}(\mathbf{k})$, the partial function g is said to be a *subfunction of* f, fact that we denote by $g \leq f$, if

$$g = f|_{\mathrm{dom}(g)},$$

i.e., if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $g(\vec{a}) = f(\vec{a})$ for all $\vec{a} \in \mathrm{dom}(g)$.

Definition 3. *A partial clone* C *on* $\mathbf{k}$ *is said to be* strong *if it is closed under taking subfunctions, i.e., if the condition*

$$g \leq f \Longrightarrow g \in C$$

holds for every $f \in C$ *and* $g \in \mathrm{Par}(\mathbf{k})$. *For* $F \subseteq \mathrm{Par}(\mathbf{k})$, *we denote its* strong closure *by* $\mathrm{Str}(F)$, *i.e.,*

$$\mathrm{Str}(F) := \{g \in \mathrm{Par}(\mathbf{k}) \ : \ g \leq f \textit{ for some } f \in F\}.$$

In what follows, by an n-ary *partial projection* we mean a subfunction of a projection function whose domain is a proper subset of $\mathbf{k}^n$.

Example 4. It is easy to see that the set

$$\{f \in \mathrm{Par}(\mathbf{k}) \ : \ (0,\dots,0) \in \mathrm{dom}(f) \Longrightarrow f(0,\dots,0) = 0\}$$

is a strong partial clone on $\mathbf{k}$.

Strong partial clones have been widely studied in the literature, see, e.g., [2, 5, 6] and [7] for a long list of references. The recent paper [8] is dedicated to the study of strong partial clones of Boolean functions. It is well known that a partial clone C is strong if and only if $\mathrm{Str}(J_{\mathbf{k}}) \subseteq C$ (see, e.g., [2], Lemma 2.11). Moreover, it is easy to see that the intersection of an arbitrary family of strong partial clones on $\mathbf{k}$ is also a strong partial clone. In fact, the set of all strong partial clones, ordered by inclusion, also forms a lattice $\mathcal{L}_{\mathrm{Str}(P_{\mathbf{k}})}$ whose smallest element is $\mathrm{Str}(J_k)$.
For $F \subseteq \mathrm{Par}(\mathbf{k})$, let $\mathrm{pclone}(F)$ denote the *strong partial clone generated by* F, i.e., the intersection of all strong partial clones on $\mathbf{k}$ containing F. As it is well-known, strong partial clones coincide exactly with the sets of polymorphisms of relations on $\mathbf{k}$, that we now briefly describe. For $m \geq 1$, let ρ be an m-ary relation on $\mathbf{k}$ (i.e., a subset of $\mathbf{k}^m$). Denote by $\mathcal{M}(\rho, \mathrm{dom}(f))$ the set of all $m \times n$ matrices M on $\mathbf{k}$ whose columns $M_{*j} \in \rho$, for $j = 1,\dots,n$ and whose rows $M_{i*} \in \mathrm{dom}(f)$ for $i = 1,\dots,m$. A partial function $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ is said to *preserve* $\rho \subseteq \mathbf{k}^m$ if $(f(M_{1*}),\dots,f(M_{m*})) \in \rho$ for all $M \in \mathcal{M}(\rho, \mathrm{dom}(f))$. Set

$$\mathrm{pPol}\,\rho := \{f \in \mathrm{Par}(\mathbf{k}) \ : \ f \text{ preserves } \rho\}.$$

It is well known that $\mathrm{pPol}\,\rho$ is a strong partial clone on $\mathbf{k}$ called the partial clone *determined by* ρ. Note that $f \in \mathrm{pPol}\rho$ if there is no matrix M with columns in ρ and rows in $\mathrm{dom}(f)$.

Main achievements: The main contributions of this paper are 3-fold. First, we show that

(I) *minimal strong partial clones* (i.e., atoms of $\mathcal{L}_{\mathrm{Str}(P_{\mathbf{k}})}$) do not exist, and

(II) the family $\{D \ : \ D$ is a partial clone and $D \subseteq \mathrm{Str}(J_{\mathbf{k}})\}$ is of continuum cardinality.

Since every strong partial clone contains $\mathrm{Str}(J_{\mathbf{k}})$, it follows that any strong partial clone contains a family of partial subclones of continuum cardinality.

However, partial clones strictly included in $\mathrm{Str}(J_{\mathbf{k}})$ are not strong. Hence, in the last section of the paper, we show that

(III) every strong partial clone $C \neq \mathrm{Str}(J_{\mathbf{k}})$ contains a family of strong partial subclones of continuum cardinality.

Some results established in this paper have been presented at the 47th IEEE *International Symposium on Multiple-Valued Logic* (Novi Sad, Serbia) and have already appeared in the conference paper [4].

2. Minimal Strong Partial Clones

A *minimal partial clone* on $\mathbf{k}$ is an atom of the lattice $\mathcal{L}_{P_{\mathbf{k}}}$. In other words, a partial clone C is minimal if there is no partial clone C_0 such that $J_k \neq C_0 \subset C \subset \mathrm{Par}(\mathbf{k})$. Minimal partial clones consisting of partial functions have been completely described in [1], none of which is strong. The natural question is then to ask whether *strong minimal partial clones* exist, i.e., whether atoms of $\mathcal{L}_{\mathrm{Str}(P_{\mathbf{k}})}$ exist.

In this section we show that such partial clones do not exist. This result is not surprising, but to our knowledge it has never been published. The idea behind the proof we present here comes from a discussion between the second author and Ferdinand Börner just after the paper [1] has appeared. We will show that, given a partial function g that is not a partial projection function, we can construct a partial function $h \in \mathrm{pclone}(g)$, with $h \notin \mathrm{Str}(J_{\mathbf{k}})$ and such that $g \notin \mathrm{pclone}(h)$.

Let $g \notin \mathrm{Str}(J_{\mathbf{k}})$ be an n-ary partial function and $C := \mathrm{pclone}(g)$. We distinguish two cases:

Case 1. For every $(a_1, \ldots, a_n) \in \mathrm{dom}(g), g(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.

Let f be a restriction of g so that any further restriction of the domain of f results in a partial projection. Let $m := |\mathrm{dom}(f)|$ and let $\alpha^1, \ldots, \alpha^m \in \mathrm{dom}(f) \subseteq \mathbf{k}^n$ with $\alpha^i := (a_1^i, \ldots, a_n^i)$ and such that

$$f \begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ a_1^2 & \ldots & a_n^2 \\ \vdots & \vdots & \vdots \\ a_1^m & \ldots & a_n^m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \tag{1}$$

Since f is not a partial projection, the right tuple $(b_1, \ldots, b_m)^T$ is not a column of the matrix on the left. Let M be the $m \times n$ matrix in (1) and let ρ_M be the m-ary relation on $\mathbf{k}$ consisting of all columns of M. Clearly $f \notin \mathrm{pPol}\rho_M$, and thus $g \notin \mathrm{pPol}\rho_M$.
Let $\overline{f}$ be the $(n+2)$-ary partial function given by

$$\overline{f}(x_1, \ldots, x_{n+2}) := f(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_{n+2}) \in \mathrm{dom}(f) \times \mathbf{k}^2$. Note that

$$\overline{f} \in \mathrm{pclone}(f) \subseteq \mathrm{pclone}(g).$$

Now take a $c \in \mathbf{k}, c \neq b_1$, and consider the subfunction h of $\overline{f}$ whose domain is the set of all rows of the following matrix:

$$\begin{pmatrix} a_1^1 & \ldots & a_n^1 & c & b_1 \\ a_1^1 & \ldots & a_n^1 & b_1 & c \\ a_1^2 & \ldots & a_n^2 & b_2 & b_2 \\ \vdots & & \vdots & \vdots & \vdots \\ a_1^m & \ldots & a_n^m & b_m & b_m \end{pmatrix}$$

where $b_1, b_2, \ldots, b_m$ are the values of the partial function f. We therefore have

$$h \begin{pmatrix} a_1^1 & \ldots & a_n^1 & c & b_1 \\ a_1^1 & \ldots & a_n^1 & b_1 & c \\ a_1^2 & \ldots & a_n^2 & b_2 & b_2 \\ \vdots & & \vdots & \vdots & \vdots \\ a_1^m & \ldots & a_n^m & b_m & b_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Note that $|\mathrm{dom}(h)| = m + 1$.

We show that any restriction of h to a set with at most m tuples is a partial projection. Let h' be a restriction of h with $|\mathrm{dom}(h')| \leq m$. If $(a_1^1, \ldots, a_n^1, c, b_1) \notin \mathrm{dom}(h')$, then $h' \leq e_{n+1}^{n+2}$, i.e., h' is a partial projection on its $(n+1)$-st coordinate. Similarly, if $(a_1^1, \ldots, a_n^1, b_1, c) \notin \mathrm{dom}(h')$, then $h' \leq e_{n+2}^{n+2}$. On the other hand if $(a_1^i, \ldots, a_n^i, b_i, b_i) \notin \mathrm{dom}(h')$ for some $i = 2, \ldots, m$, then h' is a partial projection function on the same coordinate as the subfunction of f whose domain is contained in the set of rows of M except the i-th row.
Now we show that $h \in \mathrm{pPol}\rho_M$. Let N be an $m \times (n+2)$ matrix with all columns in ρ_M and all rows in $\mathrm{dom}(h)$. Then as N has m rows, the restriction of h on the rows of N is a partial projection function and so it preserves any relation, including the relation ρ_M. Thus $h \in \mathrm{pPol}\rho_M$, i.e., $\mathrm{pclone}(h) \subseteq \mathrm{pPol}\rho_M$. Since $g \notin \mathrm{pPol}\rho_M$ we conclude that $g \notin \mathrm{pclone}(h)$ and thus $\mathrm{pclone}(h) \subsetneq C$.

Case 2. There is an n-tuple $(a_1, \ldots, a_n) \in \mathrm{dom}(g)$ such that $g(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$.
Let f be the restriction of g to the domain $\{(a_1, \ldots, a_n)\}$ and let $f(a_1, \ldots, a_n) = b$ with $b \notin \{a_1, \ldots, a_n\}$. Choose $c \neq b$ and define the $(n+2)$-ary partial function h by

$$\mathrm{dom}(h) = \{(a_1, \ldots, a_n, c, b), (a_1, \ldots, a_n, b, c)\},$$

and $h(a_1, \ldots, a_n, c, b) = h(a_1, \ldots, a_n, b, c) = b$ and proceed as in **Case 1.** Thus we have shown the following result.

Theorem 5. *Let $C \neq \mathrm{Str}(J_{\mathbf{k}})$ be a strong partial clone. Then there is a strong partial clone C' such that $\mathrm{Str}(J_{\mathbf{k}}) \subsetneq C' \subsetneq C$. Hence there are no strong minimal partial clones on $\mathbf{k}$.*

3. Clones of Partial Projections

In this section we consider partial clones consisting only of projections and partial projection functions. For $F \subseteq \mathrm{Par}(\mathbf{k})$, let $\langle F \rangle$ denote the *partial clone generated by F*, i.e., the intersection of all partial clones

on $\mathbf{k}$ containing F. Notice that $\langle F\rangle$ is not necessarily a strong partial clone.

Let R_n denote the set of all n-tuples on $\mathbf{k}$ of weight 1, i.e.,

$$R_n := \{(x_1,\dots,x_n)\in \mathbf{k}^n \ : \ \sum_{i=1}^{n} x_i = 1\}.$$

Furthermore, let p_n be the n-ary partial projection given by

$$p_n(x_1,\dots,x_n) := \begin{cases} x_1 & \text{if } (x_1,\dots,x_n)\in R_n \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let $\mathbf{P} := \{p_n \ : \ n\geq 2\}$. We will show that for $t\geq 2$, $p_t \notin \langle \mathbf{P}\setminus\{p_t\}\rangle$. Recall Lemma 2.10 in [2].

Lemma 6. *Let $F\subseteq \mathrm{Par}(\mathbf{k})$ and set $\mathcal{D}_0 := F\cup J_{\mathbf{k}}$. For $\ell\geq 0$, define*

$$\begin{aligned}\mathcal{D}_{\ell+1} \ &:= \mathcal{D}_\ell \cup \{f[g_1,\dots,g_n] \ \mid \ f\in \mathcal{D}_0^{(n)} \ \text{ and} \\ &\qquad g_1,\dots,g_n\in \mathcal{D}_\ell^{(m)}, \text{ for some } n,m\geq 1\}.\end{aligned}$$

Then $\langle F\rangle = \bigcup_{\ell\geq 0}\mathcal{D}_\ell$. ∎

Fix $t\geq 2$ and $F := \mathbf{P}\setminus\{p_t\}$. Observe that, since $F\subseteq \mathrm{Str}(J_{\mathbf{k}})$, $\langle F\rangle \subseteq \mathrm{Str}(J_{\mathbf{k}})$, i.e., $\langle F\rangle$ consists only of projections and partial projections.

Lemma 7. *Let $\mathcal{D}_0,\mathcal{D}_1,\dots$ be as in Lemma 6. Then, for each $\ell\geq 0$, $\mathcal{D}_\ell$ does not contain any partial projection p of arity t with $R_t\subseteq \mathrm{dom}(p)$.*

Proof. We prove Lemma 7 by induction on $\ell\geq 0$. For $\ell=0$, we have $\mathcal{D}_0 = (\mathbf{P}\setminus\{p_t\})\cup J_{\mathbf{k}}$, which does not contain any partial projection of arity t.

Now let $\ell\geq 0$ and suppose that $\mathcal{D}_\ell$ does not contain any partial projection p of arity t with $R_t\subseteq \mathrm{dom}(p)$. We show that the same holds for $D_{\ell+1}$. By the induction hypothesis and the definition of $D_{\ell+1}$, we need to show that if $f\in \mathcal{D}_0^{(n)}$, $n\geq 1$, $g_1,\dots,g_n\in \mathcal{D}_\ell^{(t)}$ and $p = f[g_1,\dots,g_n]$, then either p is a projection or a partial projection with $R_t\not\subseteq \mathrm{dom}(p)$. Since $g_1,\dots g_n\in \mathcal{D}_\ell$, it follows from the induction hypothesis that for every $i=1,\dots,n$, either g_i is a total projection or g_i is a partial projection with $R_t\not\subseteq \mathrm{dom}(g_i)$.

Suppose first that g_i is a partial projection for some $i = 1, \dots, n$. Then $R_t \not\subseteq \mathrm{dom}(g_i)$ and thus there is a $\vec{v} \in R_t \setminus \mathrm{dom}(g_i)$. But then, by the definition of the composition of partial functions $p := f[g_1, \dots, g_n]$ is not defined on the tuple $\vec{v}$, and thus $\vec{v} \notin \mathrm{dom}(p)$. This shows that $R_t \not\subseteq \mathrm{dom}(p)$.

Suppose now that each $g_1, \dots, g_n$ is a total projection of arity t. If $f \in J_{\mathbf{k}}$ is a projection, then $p = f[g_1, \dots, g_n]$ is a total projection and we are done.

So suppose that f is a partial projection. Since $f \in \mathcal{D}_0$, we have $f = p_n$ with $n \neq t$.

Case 1. Let $n < t$. Then there is a $j \in \{1, \dots, t\}$ such $e_j^t \notin \{g_1, \dots, g_n\}$. Therefore

$$\vec{v}_j := (0, \dots, 0, \underbrace{1}_{\text{j−th position}}, 0, \dots 0) \notin \mathrm{dom}(p) \tag{2}$$

since $g_1(\vec{v}_j) = \cdots = g_n(\vec{v}_j) = 0$ and $(0, \dots, 0) \notin \mathrm{dom}(f)$.

Case 2. Let $t < n$. Then there are $j, k \in \{1, \dots, n\}$, $j \neq k$, such that $g_j = g_k$. Without loss of generality, suppose that $g_j = g_k = e_j^t$. In this case, the t-tuple $\vec{v}_j$ defined in (2) does not belong to $\mathrm{dom}(p)$, since the n-tuple $(g_1(\vec{v}_j), \dots, g_n(\vec{v}_j))$ contains at least two entries equal to 1, and thus $(g_1(\vec{v}_j), \dots, g_n(\vec{v}_j)) \notin \mathrm{dom}(f)$. This completes the proof of Lemma 7. ∎

From Lemma 7, it follows that $p_t \notin \langle \mathbf{P} \setminus \{p_t\} \rangle$ for every $t \geq 2$, and thus we have the following corollary.

Corollary 8. *Let $X, Y \subseteq \{2, 3, \dots\}$. If $X \neq Y$, then*

$$\langle \{p_i \ : \ i \in X\} \rangle \neq \langle \{p_j \ : \ j \in Y\} \rangle.$$

Let $\mathcal{F}_{J_{\mathbf{k}}}$ be the set of all partial clones contained in $\mathrm{Str}(J_{\mathbf{k}})$.

Corollary 9. *The set $\mathcal{F}_{J_{\mathbf{k}}}$ is of continuum cardinality on $\mathbf{k}$.*

Now as every strong partial clone on $\mathbf{k}$ contains the set $\mathrm{Str}(J_{\mathbf{k}})$ we immediately have the following result.

Theorem 10. *Let C be a strong partial clone on $\mathbf{k}$. Then the interval of partial clones $[J_{\mathbf{k}}, C]$ is of continuum cardinality.*

Note that none of the partial clones contained in $\mathrm{Str}(J_{\mathbf{k}})$ is strong. Therefore, Theorem 10 does not provide any insight into the interval of strong partial clones $[\mathrm{Str}(J_{\mathbf{k}}), C]$. Hence, for a strong partial clone C, it is natural to ask what is the size of the interval

$$\mathcal{C}lone(C) := \{D : \ D \text{ is a strong partial subclone of } \ C\}.$$

This is discussed in the next section.

4. Strong partial clones generated by a single partial function

Let f be an n-ary partial function not in $\mathrm{Str}(J_{\mathbf{k}})$. Then there are m tuples $\alpha^1, \ldots, \alpha^m \in \mathbf{k}^n$, with $\alpha^i := (a_1^i, \ldots, a_n^i)$, such that

$$f \begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ \vdots & \vdots & \vdots \\ a_1^i & \ldots & a_n^i \\ \vdots & \vdots & \vdots \\ a_1^m & \ldots & a_n^m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix}$$

and such that $(b_1, \ldots, b_m)^T$ is not a column of the matrix on the left hand side of the equation. Denote this $m \times n$ matrix by A. In the sequel we assume that the n-ary partial function f, the matrix A and the number m of rows of A are fixed.

In this section we show that the strong partial clone $\mathrm{pclone}(f)$ contains a family of strong partial subclones of continuum cardinality.

We will distinguish two cases:

Case 1. The following condition is satisfied by the matrix A:

(*) there is $\alpha \in \mathbf{k}$ such that the m-tuple $(\alpha, \ldots, \alpha)^T$ is not a column of A.

Case 2. The partial function f is not constant.

We will explain in next subsection why those two cases cover all possibilities.

4.1. **Matrix A satisfies condition (*).** For notational ease, we assume in this section that $\alpha = 0$, i.e., we assume that $(0, \ldots, 0)^T$ is not a column of A. Denote by I_r the *identity matrix* of size $r \geq 2$ on $\mathbf{k}$, i.e., I_r is the $r \times r$ matrix with symbol 1 on its diagonal and with 0 elsewhere. Furthermore, consider the following infinite set of positive integers

$$E := \{2m^r + \sum_{1 \leq i \leq r} m^{i-1} \ : \ r \geq 0\}.$$

Remark 11. *Notice that, for each $r \geq 1$, we have*

$$2m^r + m^{r-1} + \cdots + m + 1 = m(2m^{r-1} + \cdots + m + 1) + 1.$$

Consequently, if $t > \ell$ with $t, \ell \in E$, then $t \geq m\ell + 1$.

We will construct an infinite set of partial functions

$$\{f_\ell \ : \ \ell \in E\}$$

each of which is in pclone(f), and with the property that for every $\ell \in E$, $f_\ell \notin$ pclone$(\{f_k \ : \ k \neq \ell, \, k \in E\})$.
For $\ell \in E$, let M_ℓ be the matrix whose first n columns are

$$(a_i^1, \ldots, a_i^1, a_i^2, \ldots, a_i^2, \ldots, a_i^m, \ldots, a_i^m)^T, \ i = 1, \ldots, n,$$

and the last $m \times \ell$ columns are the columns of the identity matrix $I_{m \times \ell}$. Thus, M_ℓ is a matrix with $m \times \ell$ rows and $n + m \times \ell$ columns. Note also that each of the tuples $(a_1^i, \ldots, a_n^i)$ is repeated ℓ times in M_ℓ.

$$M_\ell = \left(\begin{array}{ccc|ccccc} a_1^1 & \ldots\ldots & a_n^1 & 1 & 0 & \ldots\ldots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_1^1 & \ldots\ldots & a_n^1 & 0 & & \ldots\ldots & & 0 \\ a_1^2 & \ldots\ldots & a_n^2 & 0 & & \ldots\ldots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_1^2 & \ldots\ldots & a_n^2 & 0 & & \ldots\ldots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_1^m & \ldots\ldots & a_n^m & 0 & & \ldots\ldots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_1^m & \ldots\ldots & a_n^m & 0 & & \ldots\ldots & & 1 \end{array} \right)$$

Let ρ_ℓ be the relation that consists of all columns of M_ℓ. Note that ρ_ℓ is a relation of arity $m \times \ell$ over $\mathbf{k}$.

Now let f_ℓ be the partial function of arity $n + m \times \ell$ whose domain is the set of all rows of the matrix M_ℓ, and that is given by $f_\ell(\vec{v}) = f(a_1^i, \ldots, a_n^i)$, where $(a_1^i, \ldots, a_n^i)$ are the first n coordinates of the tuple $\vec{v}$.

Example 12. *Let* $\mathbf{k} = \{0, 1, 2\}$*, and let* f *be the ternary partial function defined by*

$$f \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Here $m = n = 3$ *and for* $7 = 2\dot{3} + 1 \in E$*, we get*

$$f_7 \left(\begin{array}{ccc|cccc} 0 & 0 & 1 & 1 & 0 & \ldots & 0 \;\; 0 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 0 \;\; 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \;\; \vdots \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 \;\; 0 \\ 0 & 1 & 1 & 0 & 0 & \ldots & 0 \;\; 0 \\ 0 & 1 & 1 & 0 & 0 & \ldots & 0 \;\; 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \;\; \vdots \\ 0 & 1 & 1 & 0 & 0 & \ldots & 0 \;\; 0 \\ 2 & 0 & 1 & 0 & 0 & \ldots & 0 \;\; 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \;\; \vdots \\ 2 & 0 & 1 & 0 & 0 & \ldots & 1 \;\; 0 \\ 2 & 0 & 1 & 0 & 0 & \ldots & 0 \;\; 1 \end{array} \right) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{I_{21}}$$

Note that this matrix has 7 *rows starting with* $(0, 0, 1)$*,* 7 *rows starting with* $(0, 1, 1)$*, and* 7 *rows starting with* $(2, 0, 1)$*.*

Lemma 13. *For each* $\ell \in E$*, we have that* $f_\ell \in \mathrm{pclone}(f)$ *and that* f_ℓ *is not a partial projection.*

Proof. It is easy to see that $f_\ell \in \mathrm{pclone}(f)$. Since f is not a partial projection, none of the first n columns of M_ℓ is equal to

$$(b_1, \ldots, b_1, b_2, \ldots, b_2, \ldots, b_n, \ldots, b_n)^T.$$

Moreover, each of the last $m \times \ell$ columns of M_ℓ consists of one 1 and $m\ell - 1$ 0's, and thus none of those columns can be equal to

$$(b_1, \ldots, b_1, b_2, \ldots, b_2, \ldots, b_n, \ldots, b_n)^T.$$

■

Lemma 14. *If $\ell, t \in E$, then $f_\ell \in \mathrm{pPol}\rho_t$ if and only if $\ell \neq t$.*

Proof. Clearly, $f_\ell \notin \mathrm{pPol}\rho_\ell$. We show that if $t \neq \ell$, then there is no matrix whose rows are rows of the matrix M_ℓ and whose columns belong to the relation ρ_t.

For the sake of contradiction, suppose that such a matrix B exists. Then B has $m \times t$ rows and $n + m \times \ell$ columns. Let $B_{[i,*]}$, $i = 1 \ldots m \times t$, be all the rows of B and $B_{[*,j]}, j = 1, \ldots, n + m \times \ell$, be all its columns.

Case 1. $t < \ell$.

For $i = 1, \ldots, m \times t$, the i-th row of B has the form

$$B_{[i,*]} = (a_1^s, \ldots, a_n^s, x_1, \ldots, x_{m\times\ell})$$

where $s \in \{1, \ldots, m\}$ and exactly one of $x_1, \ldots, x_{m\times\ell}$ is equal to 1 and all other are 0. Given that B consists of $m \times t$ such rows and given that $mt < m \times \ell$, there is a $j \in \{n+1, \ldots, n + m \times \ell\}$ such that the column $B_{[*,j]}$ consists of only 0's. Hence, it does not belong to ρ_t, which yields the desired contradiction.

Case 2. $\ell < t$.

As $\ell, t \in E$, we must have $t \geq m \times \ell + 1$, and

$$M_{\ell[i,*]} = (a_1^s, \ldots, a_n^s, 0, \ldots, 0, \underbrace{1}_{i-\text{th position}}, 0, \ldots, 0)$$

where $1 \leq s \leq m$. Therefore, if for some $i \in \{1, \ldots, m \times \ell\}$, the i-th row $M_{\ell[i,*]}$ of M_ℓ does not appear in B, then $B_{[*,n+i]}$ will contain only 0's, and it will not belong to the relation ρ_t. Thus every row of M_ℓ

must appear in the matrix B. Since $\ell < t$, some of the rows of M_ℓ are repeated in B.

Now if $M_{\ell[i,*]}$ appears repeated exactly r times in B, $2 \leq r \leq m \times t$, then $B_{[*,n+i]}$ will contain r 1's. Given that this column belongs to ρ_t and given that each row of A occurs t times in M_t, we conclude that r is a multiple of t.

For $i = 0, 1, \ldots, s$ let r_i be the number of rows of M_ℓ that are repeated $i \times t$ times in B, with r_0 being the number of rows in M_ℓ that appear only once in B. Then counting the rows of B we get the equality

$$r_0 + t \times r_1 + 2t \times r_2 + 3t \times r_3 + \cdots + st \times r_s = m \times t,$$

i.e., r_0 is a multiple of t. As $m \times \ell$ is the total number of rows in M_ℓ, and since at least one row of M_ℓ is repeated in B, we have that $r_0 < m \times \ell$. Furthermore, as $t \geq m \times \ell + 1$, we conclude that $r_0 < m \times \ell < t$ and thus $r_0 = 0$. Hence, every row of M_ℓ appears $i \times t$ times in B with $i \geq 1$. It thus follows that B has no column with one 1 and $m \times t - 1$ 0's, i.e., every column of B is one of the first n columns of M_t.

Let B_R be the block matrix consisting of the last $m \times \ell$ columns to the right of the matrix B. Let us count the number of 0's in B_R. Since the relation ρ_t contains no zero tuple, every column of B_R contains at least t 1's, i.e., every column in B_R contains at most $mt - t$ 0's. Given that there are $m \times \ell$ columns in B_R, there are at most $m \times \ell \times (m \times t - t)$ 0's in B_R. Furthermore, every row of B is a row of M_ℓ, and thus every row of B_R contains exactly one 1 and $m \times \ell - 1$ 0's.

Hence, there are $m \times t \times (m \times \ell - 1)$ 0's in B_R. Since

$$m \times t \times (m \times \ell - 1) > m \times \ell \times (m \times t - t),$$

we conclude that at least one column of B_R must contain only 0's, which again yields a contradiction. ∎

Corollary 15. *If $X \subseteq E$ and $\ell \notin X$, then $f_\ell \notin \mathrm{pclone}(\{f_t \ : \ t \in X\})$.*

Proof. Since $\ell \notin X$, by Lemma 14 we have that $f_t \in \mathrm{pPol}\rho_\ell$ for every $t \in X$, and thus $\mathrm{pclone}(\{f_t \ : \ t \in X\}) \subseteq \mathrm{pPol}\rho_\ell$. Since $f_\ell \notin \mathrm{pPol}\rho_\ell$, we have $f_\ell \notin \mathrm{pclone}(\{f_t \ : \ t \in X\})$. ∎

Now let C be a strong partial clone and suppose that C contains an n-ary partial function $f \notin \mathrm{Str}(J_{\mathbf{k}})$ such that $|\mathrm{dom}(f)| = m$. Suppose also that f satisfies condition $(*)$. Reconsider the sets E and $\{f_\ell \ : \ \ell \in E\}$ defined above. By Corollary 15, for every $X, Y \subseteq E$ with $X \neq Y$, the two strong partial clones $\mathrm{pclone}(\{f_t \ : \ t \in X\})$ and $\mathrm{pclone}(\{f_t \ : \ t \in Y\})$ are distinct and both are contained in $\mathrm{pclone}(f) \subseteq C$. Therefore, we have the following corollary.

Corollary 16. *Let C be a strong partial clone on $\mathbf{k}$ and suppose that C contains a partial function $f \notin \mathrm{Str}(J_{\mathbf{k}})$ whose matrix A satisfies $(*)$. Then the set of strong partial clones contained in C is of continuum cardinality.*

Example 17. *Let C be a strong partial clone on $\mathbf{k}$ and suppose that C contains a unary partial function g such that $1 \in \mathrm{dom}(g)$ and $g(1) = 0$. Let $f := g|_{\{1\}}$, i.e., f is given by $\mathrm{dom}(f) = \{1\}$ and $f(1) = 0$. Since C is a strong partial clone, we have $f \in C$.*
Consider the set E defined above with $m = 1$, i.e.,

$$E = \{2, 3, \dots\}.$$

For $\ell \in E$, let f_ℓ be the $(\ell+1)$-ary partial function whose domain is the set of rows of the matrix

$$M_\ell = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 \end{array}\right)$$

and let $f_\ell(\vec{v}) := 0$ for all $\vec{v} \in \mathrm{dom}(f_\ell)$. Then f_ℓ is a subfunction of $f[e_1^{\ell+1}]$ and so $f_\ell \in C$. By Corollary 15 we have that for every $X, Y \subseteq E$, $X \neq Y$,

$$\mathrm{pclone}(\{f_t \ : \ t \in X\}) \neq \mathrm{pclone}(\{f_t \ : \ t \in Y\}).$$

Hence the strong partial clone C contains each member of the family of strong partial clones $\{\mathrm{pclone}(\{f_t \ : \ t \in X\}) \ : \ X \subseteq E\}$, which is of continuum cardinality.

Remark 18. The construction presented in the above section cannot be used to show that pclone(f) contains a family of strong partial clones of continuum cardinality if the matrix A contains all constant columns. Indeed in such a case the proof of Lemma 14 fails. The following example illustrates this fact.

Example 19. *Let* $\mathbf{k} = \{0, 1, 2\}$ *and let* f *be the 4-ary partial function given by*

$$f\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Note that here $m = 3$ *and* $n = 4$*. Thus* $E = \{2, 7, 22, \dots\}$*. For* $\ell = 2$ *we have*

$$M_2 = \left(\begin{array}{cccc|cccccc} 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

We want to show that $f_7 \notin \mathrm{pPol}\rho_2$*, however the matrix* M_7 *associated with the partial function* f_7 *is of size* 21×25*. So for the sake of brevity we will use* f_3 *instead of* f_7*. Here the partial function* f_3 *is given by*

$$f_3\left(\begin{array}{cccc|ccccccccc} 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let ρ_2 be the relation of arity 6 that consists of all columns of M_2. The matrix below shows that $f_3 \notin \mathrm{pPol}\rho_2$.

$$f_3 \left(\begin{array}{cccc|ccccccccc} 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

Therefore the construction presented in this subsection is not applicable if we start with an n-ary partial function f whose matrix A does not satisfy the condition (*).

4.2. **The partial function f is non-constant.** Notice first that any partial function whose matrix does not satisfy the above condition (*) is included in this case. Indeed, let f be such an n-ary partial function. Then each of the m-tuples $(\alpha, \ldots, \alpha)^T$, $\alpha \in \mathbf{k}$, is a column of A. If f was a constant function, then f would be a partial projection function, contradicting $f \notin \mathrm{Str}(J_{\mathbf{k}})$.

Without loss of generality we may assume that the domain of the n-ary partial function f is repetition free. Indeed if there are $1 \leq i < j \leq n$ such that $x_i = x_j$ for all $\vec{x} = (x_1, \ldots, x_n) \in \mathrm{dom}(f)$, then we can define the $(n-1)$-ary partial function g by

$$g(x_1, \ldots, x_i, \ldots, x_{j-1}, x_{j+1}, x_n) := f(x_1, \ldots, x_i, \ldots, x_{j-1}, x_i, x_{j+1} \ldots, x_n)$$

and use g instead of f. Thus we assume that all n columns of A are distinct. We shall make use of the following definitions and notation. Let $\mathrm{Col}(A)$ be the set of all the columns of A, $d_f := |\mathrm{dom}(f)|$ and let $\vec{v}_f := f(A) \in \mathbf{k}^{d_f}$. For $\vec{x} := (x_1, \ldots, x_h) \in \mathbf{k}^h$ and $\ell \geq 1$, we define $\vec{x}^{\times \ell} \in \mathbf{k}^{h\ell}$ by

$$\vec{x}^{\times \ell} = (\underbrace{x_1, \ldots, x_1}_{\ell \text{ times}}, \underbrace{x_2, \ldots, x_2}_{\ell \text{ times}}, \ldots, \underbrace{x_h, \ldots, x_h}_{\ell \text{ times}}),$$

and let $[\vec{x}] := \{x_1, \ldots, x_h\}$. For a set $X \subseteq \mathbf{k}$ with $X = \{x_1 < \cdots < x_{|X|}\}$ and $a \in X$ we define $\mathrm{next}_X(a) \in X$ by

$$\mathrm{next}_X(a) := \begin{cases} x_{i+1} & \text{if } a = x_i \text{ and } i < |X|, \\ x_1 & \text{if } a = x_{|X|}. \end{cases}$$

Furthermore, for $\vec{x} = (x_1, \ldots, x_h) \in [\vec{v}_f]^h$ and $1 \leq i \leq h$ we define $c_i(\vec{x})$ by

$$c_i(\vec{x}) := (x_1, \ldots, x_{i-1}, \mathrm{next}_{[\vec{v}_f]}(x_i), x_{i+1}, \ldots, x_h).$$

Since the partial function f is non-constant, the set $[\vec{v}_f]$ contains at least two different elements, and so $c_i(\vec{x}) \neq \vec{x}$ for all $\vec{x} \in [\vec{v}_f]^h$ and all $i = 1, \ldots, h$.

Let $\alpha \geq 0$ be the number of columns $\vec{u}$ in the matrix A that satisfy $[\vec{u}] = [\vec{v}_f]$. Without loss of generality we may assume that all those α columns (if any) are the first columns to the left of the matrix A.

Let $\ell \geq 1$. We define the relation ρ_ℓ of arity ℓd_f by

$$\rho_\ell := \{c_i(\vec{v}_f^{\times \ell}) \mid 1 \leq i \leq \ell d_f\} \cup \{\vec{u}^{\times \ell} \mid \vec{u} \in \mathrm{Col}(A)\}.$$

Notice that $|\rho_\ell| = \ell d_f + n$. We denote by M_ℓ the matrix with ℓd_f rows and whose $\ell d_f + n$ columns are the tuples of ρ_ℓ written in the following order:

$$c_1(\vec{v}_f^{\times \ell}), \ldots, c_{\ell d_f}(\vec{v}_f^{\times \ell}), \vec{u}_1^{\times \ell}, \ldots, \vec{u}_n^{\times \ell},$$

where $\vec{u}_1, \ldots, \vec{u}_n$ are the columns of A written in the same order as they appear in A. We denote by $f^{\times \ell}$ the $(\ell d_f + n)$-ary partial function whose domain is the set of all rows of M_ℓ and defined by

$$f^{\times \ell}(M_\ell) := \vec{v}_f^{\times \ell}.$$

Notice that for every $\mathbf{x} := (x_1, \ldots, x_{\ell d_f + n}) \in \mathrm{dom}(f^{\times \ell})$, we have that $x_1, \ldots, x_{\ell d_f + \alpha} \in [\vec{v}_f]$.

Example 20. *Let* $\mathbf{k} = \{0, 1, 2\}$, $\ell = 3$ *and*

$$f \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here $\vec{v}_f = (0,0,1)$, $[\vec{v}_f] = \{0,1\}$, $\vec{v}_f^{\times 3} = (0,0,0,0,0,0,1,1,1)$,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

and $\mathrm{Col}(A) = \{(0,1,0)^T, (0,0,0)^T, (0,1,2)^T\}$. *In this case we have*

$$f^{\times 3}(M_3) = f^{\times 3}\left(\begin{array}{ccccccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \end{array}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Remark 21. *Let* $\ell \geq 1$. *Then for* $i = 1, \ldots, \ell d_f$, *the tuples* $\vec{v}_f^{\times \ell}$ *and* $c_i(\vec{v}_f^{\times \ell})$ *differ only by their* i*-th coordinate. Therefore, for* $i = 1, \ldots, \ell d_f$, *the restriction of the partial function* $f^{\times \ell}$ *to a set of tuples not containing the* i*-th row of* M_ℓ *is the partial projection function on the* i*-th coordinate.*

Lemma 22. *Let* $\ell \geq 1$. *Then* $f^{\times \ell} \in \mathrm{pclone}(f)$ *and* $f^{\times \ell} \notin \mathrm{pPol}\rho_\ell$.

Proof. Let h be the $(\ell d_f + n)$-ary partial function such that $\mathrm{dom}(h) := \mathbf{k}^{\ell d_f} \times \mathrm{dom}(f)$ and $h(x_1, \ldots, x_{\ell d_f}, y_1, \ldots, y_n) := f(y_1, y_2, \ldots, y_n)$. Then $f^{\times \ell}$ is the restriction of h on the rows of the matrix M_ℓ, thus $f^{\times \ell} \in \mathrm{pclone}(f)$.

Now since the set $[\vec{v}_f]$ contains at least two different elements, we have that $c_i(\vec{v}_f^{\times \ell}) \neq \vec{v}_f^{\times \ell}$ for all $1 \leq i \leq \ell d_f$. Morever since f is not a partial projection, we have that $\vec{u}^{\times \ell} \neq \vec{v}_f^{\times \ell}$ for all $\vec{u} \in \mathrm{Col}(A)$ and so $\vec{v}_f^{\times \ell}$ is not a column of M_ℓ proving that $f^{\times \ell} \notin \mathrm{pPol}\rho_\ell$. ∎

Next we want to show that $f^{\times \ell} \in \mathrm{pPol}\rho_m$ whenever $\ell \neq m$. In order to do so, we need the following definition. Let M_ℓ^* be the submatrix of M_ℓ whose columns are $c_1(\vec{v}_f^{\times \ell}), \ldots, c_{\ell d_f}(\vec{v}_f^{\times \ell})$ together with all columns $\vec{u}^{\times \ell}$ such that $[\vec{u}^{\times \ell}] = [\vec{v}_f]$. Thus the matrix M_ℓ^* has exactly $\ell d_f + \alpha$

columns. Furthermore, let R_ℓ be the set of all columns of M_ℓ^*, thus $R_\ell = \{\omega \in \rho_\ell \mid [\omega] = [v_f]\}$.
Denote by F_ℓ the $(\ell d_f + \alpha)$-ary partial function with domain the set of all rows of M_ℓ^* and defined by $F_\ell(M_\ell^*) := v_f^{\times\ell}$. Now by construction of $f^{\times\ell}$, we have that the value of $f^{\times\ell}(\mathbf{x})$ is uniquely determined by the first ℓd_f entries of $\mathbf{x}$ for every row $\mathbf{x}$ of the matrix M_ℓ. Therefore the value of $f^{\times\ell}$ is determined by the first ℓd_f columns of M_ℓ and thus the partial function F_ℓ is well defined.
Summing up, we have the following result.

Lemma 23. *For all $\ell, m \geq 2$, if $F_\ell \in \mathrm{pPol}R_m$ then $f^{\times\ell} \in \mathrm{pPol}\rho_m$.*

Proof. Suppose that $f^{\times\ell} \notin \mathrm{pPol}\rho_m$. So there is a matrix P whose $\ell d_f + n$ columns are all from ρ_m and whose rows are from $\mathrm{dom}(f^{\times\ell})$, and such that $f^{\times\ell}(P) \notin \rho_m$. Let P' be the submatrix of P consisting of the first $\ell d_f + \alpha$ columns of P.
Since the first $\ell d_f + \alpha$ entries of every tuple $\mathbf{x} \in \mathrm{dom}(f^{\times\ell})$ belong to $[\mathbf{v}_f]$, we have that all columns of P' belong to R_m, and since $F_\ell(P') = f^{\times\ell}(P) \notin \rho_m$ we have that $F_\ell \notin \mathrm{pPol}R_m$. ∎

Lemma 24. *Let $\ell, m \geq 2$ with $\ell \neq m$. Then $f^{\times\ell} \in \mathrm{pPol}\rho_m$.*

Proof. We distinguish two cases.
Case 1. $m < \ell$. Let M be a matrix whose rows are in $\mathrm{dom}(f^{\times\ell})$ and whose columns are in ρ_m. Thus M has md_f rows all of which are rows of M_ℓ and M has $\ell d_f + n$ columns all of which are columns of M_m. As $m < \ell$ the matrix M has less rows than M_ℓ and so by Remark 21, $f^{\times\ell}$ restricted to the rows of M is a partial projection function. Thus $f^{\times\ell}(M) \in \rho_m$ proving that $f^{\times\ell} \in \mathrm{pPol}\rho_m$.
Case 2. $\ell < m$. We argue the contrapositive. Suppose that $f^{\times\ell} \notin \mathrm{pPol}\rho_m$. Then there is a matrix M whose rows are in $\mathrm{dom}(f^{\times\ell})$ and whose columns are in ρ_m and such that $f^{\times\ell}(M) \notin \rho_m$. If some row of M_ℓ is missing in M, then by Remark 21, $f^{\times\ell}$ restricted to the rows of M is a partial projection function, but this contradicts our assumption that $f^{\times\ell}(M) \notin \rho_m$.

Therefore we assume that all rows of M_ℓ appear in M and consequently the matrix M contains $md_f - \ell d_f$ repeated rows. Also, as all columns of M_ℓ are distinct, we have that all $\ell d_f + n$ columns of M are distinct. Let M^* be the submatrix of M consisting of the first $\ell d_f + \alpha$ columns of M. Since $f^{\times\ell}(M) \notin \rho_m$, we have, by Lemma 23, that $F_\ell(M^*) \notin R_m$. Since all columns of M are distinct, we have that all $\ell d_f + \alpha$ columns of M^* are distinct. Furthermore, since the matrix M contains $md_f - \ell d_f$ repeated rows we have that M^* contains at least $md_f - \ell d_f$ repeated rows.

Set $\vec{v}_f^{\times m} := (v_1, \ldots, v_{md_f})$. Let $j_1 \neq j_2 \in \{1, \ldots, md_f\}$ and suppose that the j_1-th row and the j_2-th row of M^* are identical.

Now if $v_{j_1} \neq v_{j_2}$, then none of the tuples $c_i(\vec{v}_f^{\times m})$ ($i = 1, \ldots, \ell d_f$, $i \neq j_1$ and $i \neq j_2$) can appear as a column of M^* because the j_1-th (j_2-th) coordinate of $c_i(\vec{v}_f^{\times m})$ is v_{j_1} (is v_{j_2}). This contradicts the fact that all $\ell d_f + \alpha$ columns of the matrix M^* are distinct.

Therefore we assume that if $j_1, j_2 \in \{1, \ldots, md_f\}$ are such that the j_1-th row and the j_2-th row of M^* are identical, then $v_{j_1} = v_{j_2}$ and in such a case neither $c_{j_1}(\vec{v}_f^{\times m})$ nor $c_{j_2}(\vec{v}_f^{\times m})$ appear as columns in M^*. Since at least $md_f - \ell d_f$ rows of M^* are repeated, we conclude that at least $md_f - \ell d_f + 1$ tuples of the form $c_i(\vec{v}_f^{\times m})$ cannot appear as columns of M^*. Consequently, the number of all possible distinct columns in M^* is at most

$$md_f + \alpha - (md_f - \ell d_f + 1) = \ell d_f - 1 < \ell d_f,$$

and thus the matrix M^* cannot have all its ℓd_f columns distinct. This contradiction shows that $F_\ell \in \mathrm{pPol} R_m$, and by Lemma 23 $f^{\times\ell} \in \mathrm{pPol}\rho_m$. ∎

The proof of the following result is similar to that of Corollary 16.

Corollary 25. *Let C be a strong partial clone on $\mathbf{k}$ and suppose that C contains a partial function $f \notin \mathrm{Str}(J_{\mathbf{k}})$ that is not a constant function. Then the set of strong partial clones contained in C is of continuum cardinality.*

By combining Corollaries 16 and 25, we obtain the third main result of this paper.

Theorem 26. *Let $C \neq \mathrm{Str}(J_{\mathbf{k}})$ be a strong partial clone over $\mathbf{k}$. Then the family of all strong partial clones contained in C is of continuum cardinality.*

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